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COMPACTIFICATIONS, $C(X)$ AND RING EPIMORPHISMS

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ABSTRACT. Given a topological space X, K(X) denotes the upper semi-lattice of its (Hausdorff) compactifications. Recent studies have asked when, for $\alpha X \in$ K(X), the restriction homomorphism $\rho\colon C(\alpha X) \to C(X)$ is an epimorphism in the category of commutative rings. This article continues this study by examining the sub-semilattice, $\mathrm{K_{epi}(X)}$, of those compactifications where ρ is an epimorphism along with two of its subsets, and its complement $\mathrm{K_{nepi}(X)}$. The role of $\mathrm{K_z(X)} \subseteq \mathrm{K(X)}$ of those αX where X is z-embedded in αX, is also examined. The cases where X is a P-space and, more particularly, where X is discrete, receive special attention.

1. Introduction

Throughout, "topological space" will be taken to mean a completely regular Hausdorff topological space. For a topological space X, $C(X)$ denotes, as usual, the ring of continuous real valued functions on X. A *compactification* of a space X will be a compact *Hausdorff* space αX along with a continuous injection $X \to \alpha X$ whose image is dense in αX. The space X will be identified with its image in αX. Following the terminology of [C], the complete upper semi-lattice of equivalence classes of (Hausdorff) compactifications of X will be denoted by K(X). When we write $\alpha X \in$ K(X) we mean that αX is a representative of a class in K(X). The maximal element of K(X) is the Tychonoff compactification, usually known as the Stone-Čech compactification, βX.

Several recent articles (e.g., [BBR], [BRW], [HM2], [S]) have discussed the question of when the restriction mapping

$$\rho\colon C(\alpha X) \to C(X)\,,\ \text{for } \alpha\mathrm{X} \in \mathrm{K(X)}\ ,$$

is an epimorphism. (Recall that in a category $\mathcal{C}$, a morphism f is an *epimorphism* if given $gf = hf$ then $g = h$.) Of course, the answer depends on the category involved: the objects $C(Y)$, Y a topological space, live in many important categories. Some of these are: **CR**, the category of commutative rings, the category **R/N** of reduced commutative rings (i.e., rings with no non-zero nilpotent elements), and various categories of partially ordered groups and rings, such as that of archimedean f-rings. Epimorphisms in this last

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category are called $\mathcal{C}$-*epics* in [HM2]. If $C(\alpha X) \to C(X)$ is a **CR**-epimorphism then it is also a **R/N**-epimorphism and a $\mathcal{C}$-epic.

It is easily seen that the question of when an inclusion $X \to Y$ of topological spaces, X dense in Y, induces a **CR**-epimorphism can be reduced to the case where Y is a compactification of X. To see this, suppose X is dense in Y. Then, βY is a compactification of X which we call αX. Now, $C(\alpha X) \to C(Y)$ is a **CR**-epimorphism since Y is C^*-embedded in $\alpha X = \beta Y$. The composition $C(\alpha X) \to C(Y) \to C(X)$ shows that $C(Y) \to C(X)$ is a **CR**-epimorphism if and only if $C(\alpha X) \to C(X)$ is.

The aim of this article will be to look at when, for $\alpha X \in \mathrm{K(X)}$, $\rho\colon C(\alpha X) \to C(X)$ is a **CR**-epimorphism. The word "epimorphism" without qualifier will always mean **CR**-epimorphism in what follows.

The collection of those $\alpha X \in \mathrm{K(X)}$ *for which* X *is* z*-embedded in* αX *is denoted* $\mathrm{K_z(X)}$*; those for which* $C(\alpha X) \to C(X)$ *is an epimorphism is called* $\mathrm{K_{epi}(X)}$*.* The main theme is to discuss $\mathrm{K_z(X)}$ and $\mathrm{K_{epi}(X)}$, along with two subsets of the latter. Of particular interest is the case where X is a P-space and, more specially, an uncountable discrete space. More details are found at the end of this introduction.

Before outlining the results, we establish some notation and terminology to be used throughout.

I. Compactifications. If $\alpha X \in \mathrm{K(X)}$ for some space X, then the canonical continuous surjection $\beta X \to \alpha X$, fixing X, is denoted σ_α, or just σ; its restriction to $\beta X - X$ is τ_α or $\tau\colon \beta X - X \to \alpha X - X$. We denote by $I_\alpha = \{a \in \alpha X - X \mid |\tau^{-1}(a)| > 1\}$ and $M_\alpha = \tau^{-1}(I_\alpha)$. Note that X is C^*-embedded in $\alpha X - I_\alpha$.

As usual, $C^*(X)$ is the subring of $C(X)$ of bounded functions. If $\alpha X \in \mathrm{K(X)}$ and $f \in C^*(X)$ extends to αX, the unique extension is denoted f^α. When $\alpha X \in \mathrm{K(X)}$, $C_\alpha = \{g \in C^*(X) \mid g^\beta \text{ factorsas } \mathrm{g}^\beta = \tilde{\mathrm{g}}\sigma, \tilde{\mathrm{g}} \in \mathrm{C}(\alpha\mathrm{X})\} = \{\mathrm{g} \in \mathrm{C}^*(\mathrm{X}) \mid \mathrm{g}^\alpha \text{ exists}\}$. Then, C_α is a uniformly closed algebra which separates points from closed sets of X (see, e.g., [Wa]); it is the restriction of $C(\alpha X)$ to X. We will also encounter $S_\alpha = \{h \in C_\alpha \mid \operatorname{coz} h = X\}$, a multiplicatively closed subset of the non-zero divisors of C_α.

The complete upper semi-lattice $\mathrm{K(X)}$ is a complete lattice exactly when X is locally compact ([C, Theorem 2.19]). In this case, the minimal element is the one-point compactification, ωX.

As already mentioned, $\mathrm{K_z(X)} = \{\alpha\mathrm{X} \in \mathrm{K(X)} \mid X \text{ } z\text{-embedded in } \alpha X\}$ (i.e., every zero-set of X is the restriction of one in αX) and $\mathrm{K_{epi}(X)} = \{\alpha\mathrm{X} \in \mathrm{K(X)} \mid \mathrm{C}(\alpha\mathrm{X}) \to \mathrm{C(X)} \text{ is an epimorphism}\}$. Both are easily seen to be complete upper sub-semi-lattices of $\mathrm{K(X)}$ and both contain βX. The complement of $\mathrm{K_{epi}(X)}$ in $\mathrm{K(X)}$, when non-empty, is denoted $\mathrm{K_{nepi}(X)}$. When X is locally compact and $\omega X \in \mathrm{K_{epi}(X)}$, then $\mathrm{K(X)} = \mathrm{K_{epi}(X)}$. The situation when $\mathrm{K(X)} = \mathrm{K_{epi}(X)}$ is studied in detail in [BRW]. In particular, when $X = \mathbf{N}$, the countable discrete space, then $\omega\mathbf{N} \in \mathrm{K_{epi}}(\mathbf{N})$. In contrast, if X is uncountable and discrete, $\omega X \notin \mathrm{K_{epi}(X)}$, an observation which motivated the question: When X is uncountable discrete, to what extent can $\mathrm{K_{epi}(X)}$ be described? This question was the starting point of this article.

II. Rings of quotients. There is an extensive discussion about rings of quotients of rings of the form $C(Y)$ in [FGL]. We follow the notation of that monograph and write $Q(Y)$ for the complete ring of quotients $Q(C(Y))$, and $\mathrm{Q_{cl}}(Y)$ for the classical ring of quotients $\mathrm{Q_{cl}}(C(Y))$. Of particular importance for us is the fact that if S is a multiplicatively closed set of non-zero divisors of a commutative ring R, the homomorphism $R \to RS^{-1}$ is a epimorphism. Of course, $R \to \mathrm{Q_{cl}}(R)$ is an instance. Another example is the embedding $C^*(X) \to C(X)$ which is always an epimorphism since, for any $f \in C(X)$,

$$f = \frac{f}{1+f^2}\left(\frac{1}{1+f^2}\right)^{-1},$$

showing that $C(X) = C^*(X)S^{-1}$, where S is the set of bounded continuous functions non-zero everywhere on X.

III. Epimorphisms and zig-zags. If $A \subseteq B$ is an inclusion of commutative rings, then the inclusion is an epimorphism if and only if, for each $b \in B$ there is, for some $n \in \mathbf{N}$, an equation $b = GHK$, where (i) G, H and K are matrices over B of size $1 \times n$, $n \times n$ and $n \times 1$, respectively, and (ii) GH, H and HK are matrices over A. Such a matrix equation is called an $n \times n$ *zig-zag* over A. (This is due to Mazet and quoted in [BBR] and [BRW].) It follows that if A is infinite then $|B| = |A|$; a fact which will be used below.

Any undefined terminology about $C(X)$ conforms with that of the text by Gillman and Jerison ([GJ]) and about compactifications with that of the monograph by Chandler ([C]). Finally, if $V \subseteq X$, then χ_V is the characteristic function of V.

IV. A summary of results. **Section 2** prepares the way for the paper by presenting tools concerning: (I) Reducing properties to C^*-embedded subsets of X; (II) Constructing compactifications with desirable properties; (III) Relating $\alpha X \in \mathrm{K_z(X)}$ to $C_\alpha S_\alpha^{-1}$; (IV) Relating $\alpha X \in \mathrm{K_z(X)}$ to the Hewitt realcompactification υX of X, showing, in particular (2.8), that $\mathrm{K_z(X)} = \mathrm{K_z(\upsilon X)}$; (V) Describing zero-sets of βX lying in $\beta X - X$ (these zero-sets are an important feature in everything else). *For* $\alpha X \in \mathrm{K(X)}$*, a zero-set of* αX *lying in* $\alpha X - X$ *will be called an* α*-zero-set*; these are the zero-sets $z(g^\alpha)$, for $g \in S_\alpha$.

Section 3 specializes X to a P-space (a space in which zero-sets are open). The two key facts established here are that (i) $\mathrm{K_{epi}(X)} \subseteq \mathrm{K_z(X)}$ (3.1), and (ii) $\alpha X \in \mathrm{K_{epi}(X)}$ if and only if $C(X) = \mathrm{Q_{cl}}(\alpha X)$, a regular ring, (3.4), a result to be strengthened in Section 4. As a consequence, when X is a P-space, $\mathrm{K_{epi}(X)} = \mathrm{K_{epi}(\upsilon X)}$. It is also shown (3.5) that when X is a P-space and $\alpha X \in \mathrm{K(X)}$, $\rho\colon C(\alpha X) \to C(X)$ is an $\mathbf{R/N}$-epimorphism if and only if it is a $\mathbf{CR}$-epimorphism. It is not known if this statement is true for arbitrary spaces.

Section 4 looks at special subsets of $\mathrm{K_{epi}(X)}$ related to fractions. The easiest case is where $\alpha X \in \mathrm{K(X)}$ has the property that for some $h \in S_\alpha$, $M_\alpha \subseteq z(h^\beta)$. When this happens, $\alpha X \in \mathrm{K_{epi}(X)}$ (4.1) and we call the set of all such compactifications $\mathrm{K^1_{epi}(X)}$. Another, potentially larger, family of elements of $\mathrm{K_{epi}(X)}$, called $\mathrm{K^f_{epi}(X)}$, is the set of compactifications αX so that $C(X) = C_\alpha S_\alpha^{-1}$. We always have $\mathrm{K^f_{epi}(X)} \subseteq \mathrm{K_z(X)}$. When

X is a P-space, $\mathrm{K_{epi}(X)} = \mathrm{K^f_{epi}(X)}$ (4.5 (C)). The first of several results which give situations where $\alpha X \in \mathrm{K^f_{epi}(X)}$ if and only if $\alpha X \in \mathrm{K^1_{epi}(X)}$ is (4.9); in particular, if I_α is separable, then $\alpha X \in \mathrm{K^f_{epi}(X)}$ if and only if it is in $\mathrm{K^1_{epi}(X)}$.

Section 5 looks at lattice properties of the various subsets of $\mathrm{K(X)}$ which are related to epimorphisms. It is shown (5.2) that, except in some trivial (from this point of view) cases the upper semi-lattices $\mathrm{K_z(X)}$, $\mathrm{K^f_{epi}(X)}$ and $\mathrm{K_{epi}(X)}$ do not have minimal elements. However, (5.3) presents a situation, with a locally compact X, in which a family of elements $\mathrm{K^1_{epi}(X)}$ has its meet also in $\mathrm{K^1_{epi}(X)}$. The section ends with a construction showing that, for X uncountable and discrete, $\mathrm{K_{nepi}(X)}$ is not closed under finite joins (5.6).

Section 6 deals mostly with the special case where X is uncountable and discrete. It starts with a description (6.2) of the β-zero-sets as closures of unions of sets of the form $\mathrm{cl}_{\beta X} V - V$, $V \subseteq X$, countable. This tool allows us to show (6.3) that the join of a *countable* family from $\mathrm{K^1_{epi}(X)}$ is again in $\mathrm{K^1_{epi}(X)}$; examples ((5.4) and (6.5)) show that "countable" is neither necessary nor sufficient. The strongest statement about joins of elements of $\mathrm{K^1_{epi}(X)}$ is (6.6), a characterization of when a union of β-zero-sets is contained in a β-zero-set. It is followed by corollaries and examples. The section ends with a description of some elements of $\mathrm{K_{nepi}(X)}$ using cardinalities in various ways.

Section 7 asks some of the many questions that remain about $\mathrm{K_z(X)}$, $\mathrm{K_{epi}(X)}$, $\mathrm{K^f_{epi}(X)}$ and $\mathrm{K_{nepi}(X)}$.

2. On constructing compactifications, on fractions and on the realcompactification.

This section is divided into five subsections; it has various items which will be used as tools later in the article. Before giving some remarks on the construction of compactifications of general spaces and of discrete spaces, we show that the property of a compactification being in $\mathrm{K_z(X)}$ or in $\mathrm{K_{epi}(X)}$ is inherited by C^*-embedded subsets.

I. Hereditary properties.

2.1. Proposition. *For any space X and $V \subseteq X$, suppose that V is C^*-embedded in X.*
(i) If $\alpha X \in \mathrm{K_z(X)}$ then V is z-embedded in $\mathrm{cl}_{\alpha X} V$.
(ii) If $\alpha X \in \mathrm{K_{epi}(X)}$ and we write $\mathrm{cl}_{\alpha X} V = \gamma V \in \mathrm{K(V)}$ then, $\gamma V \in \mathrm{K_{epi}(V)}$.

Proof. (i) is clear since a zero-set in V is the intersection of a zero-set of X with V since V is C^*-embedded in X.

To show (ii), we look at the diagram (with restriction maps):

$$\begin{array}{ccc} C(\alpha X) & \to & C(X) \\ \downarrow & & \downarrow \\ C(\gamma V) & \to & C(V) \end{array}$$

The upper arrow is an epimorphism and we need that the arrow on the right be an epimorphism to show that $C(\gamma V) \to C(V)$ is one. However, $C^*(X) \to C^*(V) \to C(V)$ is a surjection followed by an epimorphism, while this same homomorphism can also be written $C^*(X) \to C(X) \to C(V)$. Hence, $C(X) \to C(V)$ is an epimorphism, as required. ∎

II. Constructions of compactifications. One of the ways in which compactifications are constructed is via subalgebras of $C^*(X)$, as described in [C, Chapter 2]. We call a subalgebra C of $C^*(X)$ a *comp-algebra* ("suitable for building a compactification") if for each closed subset $V \subseteq X$ and $x \in X - V$ there is $f \in C$ with $f(x) \notin \mathrm{cl}_{\mathbf{R}} f(V)$. (The algebra C "separates points from closed sets".) Such algebras are easy to find when X is discrete but we first look at general spaces.

Finite collections of compact sets in $\beta X - X$ (and, even, in $\alpha X - X$, for $\alpha X \in \mathrm{K(X)}$) give rise to compactifications, as we will see. It follows from the next proposition that when T is a compact set in $\beta X - X$ then there is a compactification so that $\sigma(T)$ is one point and σ is one-to-one on $\beta X - T$; this compactification will be called $\alpha_T X$. When the compact sets are β-zero-sets the compactifications constructed below will be in $\mathrm{K_{epi}(X)}$ (see Proposition 4.1). In the next proposition, "$\wedge$" means "meet" in the poset $\mathrm{K(X)}$.

2.2. Proposition. *Let X be any space and $\alpha X \in \mathrm{K(X)}$. Suppose $T_i \subseteq \alpha X$, $i = 1, \ldots, k$, are pairwise disjoint compact subsets of $\alpha X - X$. (i) There is a compactification, $\gamma X \leq \alpha X$, which identifies only the points of each T_i. (ii) Set $K_i = \tau^{-1}(T_i)$, $i = 1, \ldots k$; then, $\gamma X = \alpha X \wedge \bigwedge_{i=1}^{k} \alpha_{K_i} X$. (iii) Moreover, when each T_i is a zero-set and $\alpha X \in \mathrm{K_z(X)}$, then $\gamma X \in \mathrm{K_z(X)}$.*

Proof. (i) is essentially [C, Lemma 5.18]. It follows by repeated application of the fact that αX is a normal space.

(ii) is [C, Theorem 2.18] since $C_\gamma = C_\alpha \cap \bigcap_{i=1}^{k} C_{K_i}$.

(iii). We let $T_i = z(f_i)$, $f_i \in C(\alpha X)$, $i = 1, \ldots, k$. If $V \subseteq X$ is a zero-set then there exists $g \in C(\alpha X)$ with $z(g) \cap X = V$. Moreover, $(gf_1 \cdots f_k)|_X \in C_\gamma$ and $z(((gf_1 \cdots f_k)|_X)^\gamma) \cap X = V$, since $f_1 \cdots f_k$ is non-zero on X,. ∎

Given a compactification $\alpha X \in \mathrm{K(X)}$ there is a way of constructing some compactifications above αX in $\mathrm{K(X)}$. Special cases of it will be used below.

2.3. Construction. *For any space X, given $\alpha X \in \mathrm{K(X)}$ and a subset $B \subseteq I_\alpha$, there is a compactification $\alpha_{(B)} X$ with $\alpha X \leq \alpha_{(B)} X$ and a factorization $\tau_\alpha = \mu \tau_{\alpha_{(B)}}$ such that (i) μ is one-to-one on $\mu^{-1}(B)$ and, for each $b \in B$, $\tau_{\alpha_{(B)}}(\tau_\alpha^{-1}(b)) = \{\mu^{-1}(b)\}$; and (ii) $\mu^{-1}(B)$ is dense in $I_{\alpha_{(B)}}$.*

Proof. Let $C = \{f \in C^*(X) \mid f^\beta \text{ is constant on } \tau_\alpha^{-1}(b) \text{ for each } b \in B\}$. It follows that C is a comp-algebra because $C_\alpha \subseteq C$. Let $\alpha_{(B)} X$ be the compactification corresponding to C (in fact $C = C_{\alpha_{(B)}}$, since it is uniformly closed). We have $C_\alpha \subseteq C$ and, hence, $\alpha X \leq \alpha_{(B)} X$ in $\mathrm{K(X)}$. Hence, there is a factorization $\tau_\alpha = \mu \tau_{\alpha_{(B)}}$.

By choice of C, for each $b \in B$, $\tau_{\alpha_{(B)}}(\tau^{-1}(b))$ is a single point. Moreover, if $b \neq b'$ in B, there is $f \in C_\alpha$ with $f^\alpha(b) \neq f^\alpha(b')$. Hence, $\tau_{\alpha_{(B)}}(\tau_\alpha^{-1}(b)) \neq \tau_{\alpha_{(B)}}(\tau_\alpha^{-1}(b'))$. In other words, μ is one-to-one on $\mu^{-1}(B)$.

Let $c \in I_{\alpha_{(B)}} - \mu^{-1}(B)$. Suppose there is an open neighbourhood U of c in $\alpha_{(B)}X$ which does not meet $\mu^{-1}(B)$. Consider $\tau_{\alpha_{(B)}}^{-1}(U)$, an open set of βX containing $\tau_{\alpha_{(B)}}^{-1}(c)$ and not meeting $L = \bigcup_{b\in B} \tau_{\alpha_{(B)}}^{-1}(\mu^{-1}(b))$. Pick $p \neq q$ in $\tau_{\alpha_{(B)}}^{-1}(c)$. There is $g \in C(\beta X)$ such that $g(p) = 1$, $g(q) = 2$ and $g(\mathrm{cl}_{\beta X} L) = \{0\}$. Then, $g|_X \in C$ but g is not constant on $\tau_{\alpha_{(B)}}^{-1}(c)$, which is impossible. ∎

In what follows we have in mind the case where X is infinite discrete but some more generality is available. We will look at spaces X which have a base of compact open sets; for example, an infinite sum (disjoint union) of copies of $\omega\mathbf{N}$.

2.4. Proposition. *Let X be an infinite space which has a basis of compact open sets. If C is any subalgebra of $C^*(X)$ which contains the characteristic functions of the compact open subsets of X, then C is a comp-algebra. Conversely, if $\alpha X \in \mathrm{K(X)}$, C_α contains the characteristic functions of the compact open subsets of X.*

Proof. Given a closed subset $V \subseteq X$ and $x \in X - V$, then there is a compact open subset $U \subseteq X - V$ containing x; its characteristic function, χ_U, separates V from x.

In the converse, for any clopen U in X, U is a clopen subset of αX since X is locally compact. Hence, $\chi_U \in C(\alpha X)$. ∎

At one extreme in Proposition 2.4, the subalgebra C of $C^*(X)$ generated by the characteristic functions of the compact open sets of X yields the compactification ωX, while $C^*(X)$ itself gives rise to βX. The proposition has a useful corollary. A family $\{S_\nu \mid \nu \in E\}$ of non-empty subsets of a space Y is called *separated* ([KV, page 688]) if there is a family of pairwise disjoint open sets $\{U_\nu \mid \nu \in E\}$ with $S_\nu \subseteq U_\nu$ for $\nu \in E$.

2.5. Corollary. *Let X be an infinite space as in Proposition 2.4. Let $\{S_\nu \mid \nu \in E\}$ be a separated family of closed sets in βX with each $S_\nu \subseteq \beta X - X$ and $|S_\nu| > 1$. Then there is $\alpha X \in \mathrm{K(X)}$ such that $I_\alpha = \{a_\nu \mid \nu \in E\}$ and $\tau^{-1}(a_\nu) = S_\nu$, for $\nu \in E$.*

Proof. Put $C = \{f \in C^*(X) \mid f^\beta$ is constant on each $S_\nu\}$. This algebra satisfies the criterion of Proposition 2.4. However, it must be shown that the resulting compactification, αX, behaves as predicted. Let $\{U_\nu \mid \nu \in E\}$ be a family a open sets as in the definition of a separated family. Given $\mu \in E$, the disjoint closed sets S_μ and $\beta X - U_\mu$ can be separated in the normal space βX, say by f_μ which is 1 on S_μ and zero on $\beta X - U_\mu$; $f_\mu|_X \in C$. Hence, the images of the S_ν are all distinct in αX. Next, if $p \notin \bigcup_E S_\nu$, for each $\mu \in E$ we can separate p from S_μ. Indeed, if $p \notin U_\mu$, then $f_\mu \in C$ has $f_\mu(p) = 0$ and $f_\mu(S_\mu) = \{1\}$. If $p \in U_\mu$, then U_μ can be replaced by $U_\mu - \{p\}$ to get a function. Similarly, any two elements of βX not in $\bigcup_E S_\nu$ can be separated by an element of C, showing that $M_\alpha = \bigcup_E S_\nu$. ∎

If X is as in Proposition 2.4 then any *discrete* family in βX of closed sets in $\beta X - X$ ([E, page 193]) will work in Corollary 2.5 since βX is *collectionwise normal* ([E, page 214, definition and Theorem 6]).

III. Fractions and $\mathrm{K_z(X)}$. We will see that the ring $\mathrm{Q_{cl}}(\alpha X)$ and its subring $C_\alpha S_\alpha^{-1}$ play a role in our study of $\mathrm{K_{epi}(X)}$. However, fractions also show up when dealing with $\mathrm{K_z(X)}$, a topic we take up now. It will be seen in Section 3 that when X is a P-space, $\alpha X \in \mathrm{K_{epi}(X)}$ implies $\alpha X \in \mathrm{K_z(X)}$.

For any space X, there is a characterization of when $\alpha X \in \mathrm{K(X)}$ is in $\mathrm{K_z(X)}$: [HM2, Theorem 8.2 (b) $\Leftrightarrow$ (c)]. We suppose, as usual, that the compactification is given by $\tau\colon \beta X - X \to \alpha X - X$. Recall that $S_\alpha = \{g \in C_\alpha \mid \operatorname{coz} g = X\}$.

2.6. Lemma. [HM2, Theorem 8.2] *A compactification αX is in $\mathrm{K_z(X)}$ if and only if for each $h \in C(\beta X)$ there is a countable subset $D_h \subseteq S_\alpha$ so that if for $p \neq q$ in βX, $h(p) \neq h(q)$ while $\tau(p) = \tau(q)$, then, for some $g \in D_h$, $\{p, q\} \subseteq z(g^\beta)$.*

Proof. Note that the countable family in [HM2, Theorem 8.2] has been expanded to be closed under finite products. ∎

This result will now be translated into a statement about fractions.

2.7. Lemma. *Let X be any space and suppose that $\alpha X \in \mathrm{K_z(X)}$. For any $f \in C(\beta X)$ and any finite subset $\{a_1, \dots, a_m\}$ of I_α, there is $g \in S_\alpha$ such that fg^β is constant on each $\tau^{-1}(a_i)$, $i = 1 \dots, m$. In particular, there is $g \in S_\alpha$ such that $\bigcup_{i=1}^m \tau^{-1}(a_i) \subseteq z(g)$. Hence, if $B = \{a_1, \dots, a_m\}$, as in (2.3), there is $h \in C_{\alpha(B)}$ with $f|_X = hg^{-1}$.*

Proof. If f is already constant on each $\tau^{-1}(a_i)$, we can take $g = \mathbf{1}$. Otherwise, let $F = \{i \mid 1 \leq i \leq m$ and f is not constant on $\tau^{-1}(a_i)\}$. For $i \in F$, let $g_i \in S_\alpha$ be such that g_i^β is zero on $\tau^{-1}(a_i)$ (using the criterion quoted above). Then $g = \prod_{i \in F} g_i$ does what is required.

For the second part, it suffices to choose pairs of distinct elements p_i, q_i from $\tau^{-1}(a_i)$, $i = 1, \dots, m$, and $f \in C(\beta X)$ so that $f(p_i) \neq f(q_i)$, $i = 1, \dots, m$. ∎

IV. The Hewitt realcompactification and $\mathrm{K_z(X)}$. The following are tools to help us avoid requiring that a space be realcompact. Lemma 2.6 will play a role. A space X is C-embedded in its Hewitt realcompactification υX. If $f \in S_\beta$ then $f^{-1} \in C(X)$ and, hence, f^{-1} extends to υX. This shows that for any $f \in S_\beta$, $z(f^\beta) \cap \upsilon X = \emptyset$ ([GJ, Theorem 8.4]).

2.8. Proposition. *Let X be any space and $\alpha X \in \mathrm{K_z(X)}$. (i) For any $a \in I_\alpha$, $\tau^{-1}(a) \subseteq \beta X - \upsilon X$. (ii) There is Υ, $X \subseteq \Upsilon \subseteq \alpha X$ so that $X \to \Upsilon$ is a copy of the Hewitt realcompactification of X in αX. (iii) The restriction of σ_α to υX is a homeomorphism onto Υ. Hence, Υ can be identified with υX. With this identification, $\alpha X \in \mathrm{K_z(\upsilon X)}$. Moreover, $\mathrm{K_z(X)} = \mathrm{K_z(\upsilon X)}$.*

PROOF. (i) Suppose, for some $a \in I_\alpha$, that $\tau^{-1}(a) \cap \upsilon X \neq \emptyset$. Pick $p \in \tau^{-1}(a) \cap \upsilon X$ and $q \in \tau^{-1}(a)$, $q \neq p$. By Lemma 2.6, there is a $g \in S_\alpha$ with $\{p, q\} \subseteq z(g^\beta)$. However, this is impossible.

(ii) By [BH, Corollary 2.6 (a)], the intersection of all the cozero-sets of αX which contain X, call it Υ for now, is a copy of the realcompactification.

(iii) We give names to the various inclusions: $X \xrightarrow{j_1} \upsilon X \xrightarrow{i_1} \beta X$ and $X \xrightarrow{j_2} \Upsilon \xrightarrow{i_2} \alpha X$. By [GJ, Theorem 8.7], there is a homeomorphism $\zeta\colon \upsilon X \to \Upsilon$ so that $\zeta j_1 = j_2$. Since $\beta X = \beta(\upsilon X)$, there is $\xi\colon \beta X \to \alpha X$ with $\xi i_1 = i_2 \zeta$. Thus, $\xi i_1 j_1 = \sigma_\alpha i_1 j_1$. The density of X in βX shows that $\xi = \sigma_\alpha$ and, thus, that ζ is the restriction of σ_α to υX.

The next part of (iii) is clear from [GJ, 8D 1.]. The last statement follows in the same way. ∎

2.9. REMARK. *Let X be any space and $\alpha X \in \mathrm{K(X)}$. If there is a subspace Y, $X \subseteq Y \subseteq \alpha X$, so that X is C-embedded in Y, then, $\alpha X \in \mathrm{K_{epi}(X)}$ if and only if $\alpha X \in \mathrm{K_{epi}(Y)}$. If $\alpha X \in \mathrm{K_z(X)}$ there is a copy of $\upsilon X \subseteq \alpha X$ to play the role of Y.*

PROOF. Consider the homomorphisms $C(\alpha X) \to C(Y) \to C(X)$. The second homomorphism is an isomorphism because X is C-embedded in Y. The second statement is from Proposition 2.8 (ii). ∎

V. ZERO-SETS IN $\beta X - X$. Zero-sets of βX lying in $\beta X - X$ (i.e., β-zero-sets) will appear many times in what follows and thus it would be helpful to know more about them. The following is an adaptation of [Ca, Corollary 4.5].

2.10. PROPOSITION. *Let X be any space. Then, there is a one-to-one correspondence between the β-zero-sets of βX and the z-filters $\mathcal{F}$ of X such that (i) $\mathcal{F}$ has a countable base, (ii) $\mathcal{F}$ is the intersection of the z-ultrafilters containing it, and (iii) all the z-ultrafilters containing $\mathcal{F}$ are free.*

In other words, the key players are countable families $\mathcal{Z} = \{Z_n\}_{\mathbf{N}}$ of zero-sets of X which have the finite intersection property and $\bigcap_{\mathbf{N}} Z_n = \emptyset$. In one direction, with such $\mathcal{Z}$ where $Z_n = z(f_n)$, $\mathbf{0} \leq f_n \leq \mathbf{1}$, we can associate $f = \sum_n (1/2^n) f_n$ and f^β whose zero-set is non-empty and is in $\beta X - X$. In the other direction, if $f \in C^*(X)$ is such that $\emptyset \neq z(f^\beta) \subseteq \beta X - X$, we can find $\mathcal{Z} = \{Z_n\}_{\mathbf{N}}$, where $Z_n = \{x \in X \mid |f_n(x)| \leq 1/n\}$.

The easiest case is where X is a P-space, then, the β-zero-sets can be attached to partitions of X. The following proposition will be used without mention, especially in Section 6.

2.11. PROPOSITION. *Let X be a P-space. Then, there is one-to-one correspondence between non-empty β-zero-sets of βX and partitions of X into countably many clopen sets.*

PROOF. First assume that $\{T_n\}_{\mathbf{N}}$ is a partition of X into clopen sets and let $m_1 < m_2 < \cdots$ be a sequence from $\mathbf{N}$. Then $f \in C^*(X)$ may be defined by $f(x) = 1/m_n$, if $x \in T_n$. Then, $z(f^\beta) \subseteq \beta X - X$. In the other direction suppose, for $g \in C(\beta X)$ that $\emptyset \neq z(g) \subseteq \beta X - X$. We may assume $\mathbf{0} < g \leq \mathbf{1}$. Set $U_n = \{x \in X \mid g(x) \leq 1/n\}$. Put

$T_n = U_n - U_{n+1}$. Each T_n is a clopen set of X and infinitely many of them are non-empty. Suppose that $T_{m_1}, T_{m_2}, \ldots$ are the non-empty ones, $m_1 < m_2 < \cdots$. We, thus, have a partition of X into countably many clopen sets. ∎

We know from [GJ, Theorem 9.5] that a non-empty β-zero-set contains a copy of $\beta\mathbf{N}$. However, these zero-sets can have the same cardinality as βX. Indeed, by [C, Lemma 5.2], if X is infinite and discrete, $\beta X - X$ contains a copy of βX. It is clear from the proof of that lemma that the set $\overline{T}$, homeomorphic to βX, is in a β-zero-set. (If $|X|$ is a measurable cardinal, the elements a_i chosen in the construction, need to be taken from $\beta X - \upsilon X$ by [GJ, Theorem 12.2].)

When X is discrete, the β-zero-sets are studied in Proposition 6.2, below.

3. $\mathrm{K_{epi}(X)}$ vs $\mathrm{K_z(X)}$ in P-spaces.

Lemma 2.7 established, for general X, a connection between $\mathrm{K_{epi}(X)}$ and $\mathrm{K_z(X)}$. However, for general spaces, we have neither implication: "$\alpha X \in \mathrm{K_z(X)} \Rightarrow \alpha \mathrm{X} \in \mathrm{K_{epi}(X)}$" nor "$\alpha X \in \mathrm{K_{epi}(X)} \Rightarrow \alpha \mathrm{X} \in \mathrm{K_z(X)}$". For the first we can use [BRW, Corollary 2.3] and the space $\mathbf{Q}$: for the second, [BRW, Theorem 3.3] supplies examples. This section contains information when X is required to be a P-space. (See [GJ, 4J] for many equivalent conditions: we use that $C(X)$ is a regular ring and that zero-sets in X are open.) Fractions will again play an important role and we will see that for a P-space X, $\alpha X \in \mathrm{K_{epi}(X)}$ implies $C(X) \cong \mathrm{Q_{cl}}(X)$. The first part of the next proposition could also be deduced from [Wa, Theorem 2.8].

3.1. Proposition. *Let X be a P-space and $\alpha X \in \mathrm{K(X)}$. Then: (i) $\alpha X \in \mathrm{K_z(X)}$ if and only if the idempotents in $C(X)$ are in $\mathrm{Q_{cl}}(C_\alpha)$. In particular, if X is infinite and discrete then $\alpha X \in \mathrm{K_z(X)}$ if and only if the characteristic functions of subsets of X are in $\mathrm{Q_{cl}}(C_\alpha)$.*
(ii) If $\alpha X \in \mathrm{K_{epi}(X)}$, then X is z-embedded in αX.

Proof. (i) Since X is a P-space, the zero-sets of X are open and the complement of a zero-set is a zero-set.

Suppose now that the idempotents of $C(X)$ are in $\mathrm{Q_{cl}}(C_\alpha)$. We first note that if $g \in C_\alpha$ is a non-zero divisor, then $\operatorname{coz} g = X$. To see this, suppose $\operatorname{coz} g = V$. Write $\chi_{X-V} = hk^{-1}$, $h, k \in C_\alpha$ and k a non-zero divisor. Then $\chi_{X-V} \cdot k = h$ shows that $V \subseteq z(h)$, implying that $gh = 0$. Hence, $h = 0$ and $V = X$. Now let V be any zero-set of X. We write $\chi_V = hk^{-1} \in \mathrm{Q_{cl}}(C_\alpha)$. The equation $\chi_V \cdot k = h$ says that $z(h) = z(\chi_V) = V$ and $h \in C_\alpha$.

In the other direction, if V is a zero-set in X then there is $f \in C_\alpha$ with $z(f) = V$ and $g \in C_\alpha$ with $z(g) = X - V$. Then $f/(f+g) = \chi_V$.

(ii) This part is [BRW, Lemma 5.1]. ∎

In [HM2], the authors define the notion of "C-epic" which means epimorphism in the category of archimedean f-rings. As an illustration of the difference between this and our version of epimorphism, [HM2, Theorem 6.3] shows how to construct, in particular, a compactification αX of an uncountable discrete space X, where X is not z-embedded in αX but $C(\alpha X) \to C(X)$ is C-epic. It suffices to take, in the construction, Y uncountable discrete. In fact, the criteria of [HM2, Theorems 7.1 and 8.2] show that "z-embedded" implies "C-epic" (explicitly [HM2, Corollary 2.5]), whereas this fails for **CR**-epimorphisms. As an example we take the space $\mathbf{Q}$ because $\mathbf{Q}$ is z-embedded in all spaces ([BH, Theorem 4.4]) but $\mathrm{K_z(X)} = \mathrm{K(X)} \neq \mathrm{K_{epi}(X)}$ by [BRW, Corollary 2.23].

3.2. Lemma. *Let X be a P-space and $\alpha X \in \mathrm{K_z(X)}$. Then $\mathrm{Q_{cl}}(\alpha X)$ is regular.*

Proof. Consider $f \in C(\alpha X)$: then, $V = z(f) \cap X$ is a zero-set of X, and is, hence, clopen. It follows that $X - V$ is also a zero-set and there is $g \in C(\alpha X)$ with $z(g) \cap X = X - V$. Consider $f + g$. Since fg is zero on X, $fg = \mathbf{0}$. If, for some $h \in C(\alpha X)$, $(f + g)h = \mathbf{0}$, then h is zero on X, showing $h = \mathbf{0}$. Thus $\mathrm{coz}(f + g)$ is dense in αX and contains X. Put $l = (f + g)^{-1} \in \mathrm{Q_{cl}}(C(\alpha X)$. It follows that $f^2 l$ coincides with f on X and extends continuously to $\mathrm{coz}(f + g)$. This shows that $f = f^2 l$ in $\mathrm{Q_{cl}}(\alpha X)$ ([FGL, page 14]). ∎

3.3. Proposition. *Let X be a realcompact space. Suppose $\alpha X \in \mathrm{K_z(X)}$. Then X is an intersection of cozero-sets of αX.*

Proof. By [BH, Corollary 2.4], each $f \in C(X)$ can be extended to a countable intersection of cozero-sets of αX, each containing X. Let Y be the intersection of all such cozero-sets of αX. Then, $C(X)$ is C-embedded in $C(Y)$. This says ([GJ, 8.14 and 8.9] along with [GJ, 8.10 (a)]) that Y is the realcompactification of X. However, X is realcompact and so $Y = X$. ∎

Lemma 3.2 and Proposition 3.3 are used in the proof of the following theorem which says that for a P-space X, if $\alpha X \in \mathrm{K_{epi}(X)}$ then $C(X) = \mathrm{Q_{cl}}(\alpha X)$. (Theorem 4.5 (C) gives a sharper version of this.)

3.4. Theorem. *Let X be a P-space. Suppose $\alpha X \in \mathrm{K_{epi}(X)}$. Then, $C(X) \cong \mathrm{Q_{cl}}(\alpha X)$ via the natural inclusion $C(\alpha X) \to C(X)$.*

Proof. We first assume that X is realcompact. By Proposition 3.3, X is an intersection of cozero-sets of αX. From [RW, Definition 4.8], gY is the intersection of all dense cozero-sets of a space Y. By [RW, Lemma 4.9(3)], $X \subseteq g(\alpha X)$, and, hence, $X = g(\alpha X)$. Then [RW, Lemma 4.7] says that the regular ring $C(X)$ is a ring of quotients of $C(\alpha X)$. It follows that we have embeddings $C(\alpha X) \to \mathrm{Q_{cl}}(\alpha X) \to H(\alpha X) \to C(X)$, where (as in [RW, Section 2] and its references) $H(\alpha X)$ is the smallest regular subring between $C(\alpha X)$ and $Q(\alpha X)$, the complete ring of quotients of $C(\alpha X)$. However, Lemma 3.2 says that $\mathrm{Q_{cl}}(\alpha X)$ is regular, showing that $\mathrm{Q_{cl}}(\alpha X) = H(\alpha X)$.

Since $C(\alpha X) \to C(X)$ is an epimorphism, so is $H(\alpha X) \to C(X)$. However, $H(\alpha X)$ is regular and so it has no proper epimorphic extensions ([St, Korollar 5.4]). This shows that $\mathrm{Q_{cl}}(\alpha X) = C(X)$.

We now drop the assumption of realcompactness. By Proposition 3.1, $\alpha X \in \mathrm{K_{epi}(X)}$ implies that $\alpha X \in \mathrm{K_z(X)}$. Then Proposition 2.8 (ii) says that there is a copy of the realcompactification $X \to \upsilon X \to \alpha X$. However, υX is also a P-space (because $C(\upsilon X) \cong C(X)$ is regular) and the first part of the proof says that $C(\upsilon X) \cong \mathrm{Q_{cl}}(\alpha X)$, via restriction, because $\alpha X \in \mathrm{K_{epi}}(\upsilon \mathrm{X})$ by Remark 2.9. Restricting further to X gives $C(X) \cong \mathrm{Q_{cl}}(\alpha X)$. ∎

When X is a P-space and $\alpha X \in \mathrm{K_z(X)}$, then, [HW, Theorem 1.4 (d)] shows that, αX is *cozero complemented*. Hence, the minimum spectrum of $\mathrm{Q_{cl}}(C_\alpha) = \mathrm{Q_{cl}}(\alpha X)$ is compact and all the equivalent conditions of [HW, Theorem 1.3] apply to αX.

When X is infinite discrete and $\alpha X \in \mathrm{K_z(X)}$, then $\mathrm{Q_{cl}}(\alpha X)$ and $C(X)$ have the same idempotents. However, $Q(\alpha X) = C(X)$ (since X is the unique smallest dense open set of αX) so that $\mathrm{Q_{cl}}(\alpha X)$ and $Q(\alpha X)$ have the same idempotents. Then the language of [HM1] applies and we can say that αX is *fraction dense* ([HM1, Theorem 1.1]). Moreover, when $\alpha X \in \mathrm{K_{epi}(X)}$, Theorem 3.4, in this special case, shows that $\mathrm{Q_{cl}}(\alpha X) = Q(\alpha X)$, which implies that αX is *strongly fraction dense* ([HM1, page 979]).

The next topic is somewhat apart from the main theme of the article but it underlines the fact that, when talking about epimorphisms, P-spaces are easier to deal with than general spaces. Let X be a space and $\alpha X \in \mathrm{K(X)}$. It was asked in [BBR, Section 3D] whether, in our context, $\mathbf{R}/\mathbf{N}$-epimorphisms were $\mathbf{CR}$-epimorphisms. We cannot answer the question in general but can say that it is "yes" when X is a P-space. (See [BBR, Theorem 3.21] for another partial answer.)

3.5. THEOREM. *Let X be a P-space and $\alpha X \in \mathrm{K(X)}$. Then, $\rho\colon C(\alpha X) \to C(X)$ is an $\boldsymbol{R}/\boldsymbol{N}$-epimorphism if and only if it is $\boldsymbol{CR}$-epimorphism.*

PROOF. We always have that ρ a $\mathbf{CR}$-epic implies it is an $\mathbf{R}/\mathbf{N}$-epic.

By a result of Lazard, quoted in [S, page 351] and [BBR, Proposition 1.1], ρ is an $\mathbf{R}/\mathbf{N}$-epimorphism if and only if $\rho^{-1}\colon \operatorname{Spec} C(X) \to \operatorname{Spec} C(\alpha X)$ is injective and, for each $P \in \operatorname{Spec} C(X)$ and $Q = \rho^{-1}(P)$, $\mathrm{Q_{cl}}(C(\alpha X)/Q) = \mathrm{Q_{cl}}(C(X)/P)$, via ρ. We now assume that ρ is an $\mathbf{R}/\mathbf{N}$-epimorphism.

Now let $T(\alpha X)$ stand for $T(C(\alpha X))$, the universal regular ring of $C(\alpha X)$, and $\mu\colon C(\alpha X) \to T(\alpha X)$ the canonical injection (which is $\mathbf{CR}$-epic). See [BBR, Section 3] for details. Since $C(X)$ is a regular ring, the universal property of $T(\alpha X)$ defines a canonical $\zeta\colon T(\alpha X) \to C(X)$ extending ρ. Hence, $\rho = \zeta\mu$ and it will suffice to show that ζ is a surjection.

Recall that $\operatorname{Spec} T(\alpha X) = \operatorname{Spec} C(\alpha X)$ as sets but $\operatorname{Spec} T(\alpha X)$ has the constructible topology. In a regular ring S, for $s \in S$, s' denotes the unique element where $s^2 s' = s$ and $(s')^2 s = s'$. Moreover, $\operatorname{Spec} S$ is a totally disconnected compact Hausdorff space.

Fix $f \in C(X)$. We will show that f is in the image of ζ. For $P \in \operatorname{Spec} C(X)$, let $Q = \rho^{-1}(P)$. By Lazard's criterion, there are $u_Q, v_Q \in C(\alpha X)$ with $v_Q \notin Q$, such that $f + P = (u_Q + P)(v_Q + P)^{-1}$ (we identify elements of $C(\alpha X)$ with their images under ρ). Then, $\zeta(\mu(u_Q)\mu(v_Q)')$ and f coincide module P. Because we are dealing with regular rings there is a clopen neighbourhood U_P of P in $\operatorname{Spec} C(X)$ so that, for all $R \in U_P$, $v_Q \notin R$ and $(u_Q + R)(v_Q + R)^{-1} = f + R$.

For $Q \in \operatorname{Spec} C(\alpha X) - \rho^{-1}(\operatorname{Spec} C(X)) = \mathcal{U}$, an open subset of $\operatorname{Spec} T(\alpha X)$, there is a clopen neighbourhood U_Q of Q in $\mathcal{U}$. Then, $\operatorname{Spec} T(\alpha X)$ is covered by $\bigcup_{P \in \operatorname{Spec} C(X)} \rho^{-1}(U_P) \cup \bigcup_{Q \in \mathcal{U}} U_Q$. There is, by compactness, a finite subcover and since the elements of the finite subcover are clopen, there is a refinement $\{U_1, \dots, U_k\}$ where the clopen sets are now disjoint. Let the corresponding idempotents of $T(\alpha X)$ be $e_1, \dots, e_k$.

Divide this cover into two parts (the second part may be empty), $\{U_1, \dots, U_m\}$ and $\{U_{m+1}, \dots, U_k\}$, where $U_i \cap \rho^{-1}(\operatorname{Spec} C(X)) \neq \emptyset$, $i = 1, \dots, m$, and $U_j \subseteq \mathcal{U}$, $j = m+1, \dots, k$. Each U_i, $i = 1, \dots, m$, lies in one of the U_P, say U_{P_i} with $\rho^{-1}(P_i) = Q_i$. We now consider $l = \sum_{i=1}^{m} e_i \mu(u_{Q_i}) \mu(v_{Q_i})' + \sum_{j=m+1}^{k} e_j$. By construction, $\zeta(l) = f$, since the two elements coincide modulo each $P \in \operatorname{Spec} C(X)$. ∎

4. On $\mathrm{K}_{\mathrm{epi}}(\mathrm{X})$

This section contains tools to be used in later parts of the article; however, its main theme is to look at when $C(X) = C_\alpha S_\alpha^{-1}$ (which implies that $\alpha X \in \mathrm{K}_{\mathrm{epi}}(\mathrm{X})$). *The collection of such compactifications is denoted* $\mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$ ("f" for "fractions"). These are the easiest sorts of elements of $\mathrm{K}_{\mathrm{epi}}(\mathrm{X})$ to study. We always have $\mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X}) \subseteq \mathrm{K}_{\mathrm{z}}(\mathrm{X})$ and we will show that for P-spaces, $\mathrm{K}_{\mathrm{epi}}(\mathrm{X}) = \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$.

We begin with a special kind of compactification defined as follows: $\mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X}) = \{\alpha\mathrm{X} \in \mathrm{K}(\mathrm{X}) \mid \text{there is } h \in S_\beta \text{ with } z(h^\beta) \supseteq M_\alpha\}$. The next proposition will show that $\mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X}) \subseteq \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$. The existence of compactifications of this type is guaranteed by Proposition 2.2 but other methods for finding them will be discussed later in this section and in Sections 5 and 6.

4.1. Proposition. *Let X be any space and $\alpha X \in \mathrm{K}(\mathrm{X})$. Suppose that $\alpha X \in \mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$. Then $C(X)$ is the localization $C_\alpha S_\alpha^{-1}$; i.e., $\alpha X \in \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$.*

Proof. (The method is close to that of [BBR, Proposition 2.1(ii)].) Since h^β is zero on each $\tau^{-1}(a)$, $a \in I_\alpha$, $h \in S_\alpha$. Given $f \in C^*(X)$, $f^\beta h^\beta$ is zero on all of M_α and, hence, $l = fh \in C_\alpha$. Then $f = lh^{-1} \in \mathrm{Q}_{\mathrm{cl}}(C_\alpha)$. For general $f \in C(X)$, we write

$$f = \frac{f}{1+f^2}\left(\frac{1}{1+f^2}\right)^{-1} = l_1 h^{-1}(l_2 h^{-1})^{-1} = l_1 l_2^{-1}\ ,$$

for some $l_1, l_2 \in C_\alpha$ and $\operatorname{coz} l_2 = X$. It follows that $C(X)$ is an epimorphic extension of C_α. ∎

The denominator set used in the proof is, in fact, smaller than that described since the only elements which are used are of the form $l \in S_\alpha$ where $z(l^\beta) \supseteq M_\alpha$.

[BRW, Theorem 3.3] shows that, in general, $\mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X}) \neq \mathrm{K}_{\mathrm{epi}}(\mathrm{X})$ and that $\mathrm{K}_{\mathrm{epi}}(\mathrm{X}) \not\subseteq \mathrm{K}_{\mathrm{z}}(\mathrm{X})$. It is clear that if $\alpha X \in \mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$, then so is every compactification above $\alpha X \in \mathrm{K}(\mathrm{X})$. Recall ([Wk, Theorem, page 31]) that if X is realcompact then every $p \in \beta X - X$ is contained in a zero-set, $z(g^\beta)$, of some $g \in S_\beta$. The following is an immediate consequence. (See also Proposition 4.9 (iii).)

4.2. Corollary. *Let X be any space. Suppose $\alpha X \in \mathrm{K(X)}$ with M_α finite and $M_\alpha \subseteq \beta X - \upsilon X$. Then $\alpha X \in \mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$. In particular, $\beta X \in \mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$.*

Proof. Let $M_\alpha = \{p_1, \ldots, p_k\}$. Each $p_i \in z(h_i^\beta)$ for some $h_i \in C^*(\upsilon X)$ which are nowhere zero. Hence, $h = h_1|_X \cdots h_k|_X \in S_\beta$ can be used in Proposition 4.1. ∎

There is another observation about υX and $\mathrm{K_{epi}(X)}$. A related result, [HM2, Corollary 2.7 (a)], is about C-epics; it requires that υX be Lindelöf.

4.3. Proposition. *Let X be a space and $\alpha X \in \mathrm{K_{epi}(X)}$. Then, $\tau|_{\upsilon X}$ is one-to-one, i.e., for each $a \in I_\alpha$, $|\tau^{-1}(a) \cap \upsilon X| \leq 1$.*

Proof. Suppose that, for some $a \in I_\alpha$, there are $p \neq q$ in $\tau^{-1}(a) \cap \upsilon X$. Let $f \in C^*(X)$ be such that $f^\beta(p) \neq f^\beta(q)$. By assumption, there is an $n \times n$ zig-zag for f over C_α, for some n, say $f = GHK$. However, all the functions in the zig-zag extend to υX. We use the same symbols for the extensions to υX. Then, $0 \neq f(p) - f(q) = GH(p)K(p) - GH(q)K(q) = GH(p)(K(p) - K(q)) = G(p)(H(p)K(p) - H(q)K(q)) = G(p)(HK(p) - HK(q)) = 0$, a contradiction. ∎

4.4. Examples. *There are examples of a space X and $\alpha X \in \mathrm{K_{epi}(X)}$ so that for some $a \in I_\alpha$, $|\tau^{-1}(a) \cap \upsilon X| = 1$.*

Proof. One can use any of the examples found in [BRW, Theorem 3.3], where X is a sum of a Lindelöf absolute **CR**-epic space L (i.e., a space such that $\mathrm{K_{epi}(L)} = \mathrm{K(L)}$) and an almost compact space A. Indeed, $\mathrm{K(X)} = \mathrm{K_{epi}(X)}$ by the quoted theorem. The single point s of $\beta A - A$ is in $\upsilon X - X$ and, when it is in M_α for some $\alpha X \in \mathrm{K(X)}$, will be identified with some closed set of $\beta L - L \subseteq \beta X - \upsilon X$. As a specific example, we take any closed $\emptyset \neq Y \subseteq \beta L - L$, $T = Y \cup \{s\}$ and $\alpha_T X$. ∎

The following theorem contains observations about fractions. Some of the arguments can also be found in [BBR] and [BRW].

4.5. Theorem. *Let X be any space and $\alpha X \in \mathrm{K(X)}$. Then,*
(A) The following are equivalent.
(i) $C^*(X) \subseteq C_\alpha S_\alpha^{-1}$.
(ii) Each $f \in C^*(X)$ extends to a cozero-set of αX containing X.
(iii) $C(X) = C_\alpha S_\alpha^{-1}$; i.e., $\alpha X \in \mathrm{K^f_{epi}(X)}$.
(B) If the conditions of (A) are satisfied then, $\alpha X \in \mathrm{K_{epi}(X)} \cap \mathrm{K_z(X)}$.
(C) Let X be a P-space. Then the following are equivalent for $\alpha X \in \mathrm{K(X)}$:
(a) $\alpha X \in \mathrm{K_{epi}(X)}$.
(b) $\alpha X \in \mathrm{K^f_{epi}(X)}$.

PROOF. (A) Assume (i). Then, each $f \in C^*(X)$ can be written $f = gh^{-1}$, where $g \in C_\alpha$ and $h \in S_\alpha$. Since both g and h extend to αX, gh^{-1} extends to $\operatorname{coz} h^\alpha$.

Assume (ii). Suppose that $f \in C^*(X)$ extends to $\operatorname{coz} h$, $h \in C(\alpha X)$, $X \subseteq \operatorname{coz} h$ with extension $\tilde{f}$. Then, $\tilde{f}h|_{\operatorname{coz} h}$ extends to, say, $l \in C(\alpha X)$. Then $f = l|_X (h|_X)^{-1} \in C_\alpha S_\alpha^{-1}$. For any $f \in C(X)$, we write $f/(1+f^2) = g_1 h_1^{-1}$ and $1/(1+f^2) = g_2 h_2^{-1}$, expressions from $C_\alpha S_\alpha^{-1}$. Then, $f = g_1 h_2 (g_2 h_1)^{-1} \in C_\alpha S_\alpha^{-1}$, since $g_2 \in S_\alpha$.

(iii) $\Rightarrow$ (i) is obvious.

(B) Clearly, $\alpha X \in \mathrm{K_{epi}(X)}$. Moreover, if $V = z(f)$ is a zero-set of X, $f \in C^*(X)$ then, we write $f = gh^{-1} \in C_\alpha S_\alpha^{-1}$ and note that $z(g^\alpha) \cap X = V$.

(C) Theorem 3.4 shows that (a) implies that $C(X) = \mathrm{Q_{cl}}(\alpha X)$. To get (b) we need to verify that if $g \in C(\alpha X)$ is a non-zero divisor then $g|_X \in S_\alpha$. Thus, $\operatorname{coz} g$ is dense in αX and we put $V = \operatorname{coz} g \cap X$, a clopen set of X. If $V \neq X$ there is an open U in αX with $U \cap X = X - V$. We can find $\mathbf{0} \neq h \in C(\alpha X)$ with $\operatorname{coz} h \cap U \neq \emptyset$. Then, $\operatorname{coz} h \cap U \cap X \neq \emptyset$. We get $(g|_X)(h|_X) = \mathbf{0}$ and, hence, $gh = \mathbf{0}$, which is impossible. Hence, $V = X$ and $g|_X \in S_\alpha$. ■

Notice also that when X is locally compact and Lindelöf, *all* compactifications fall under Proposition 4.1 since $\omega X \in \mathrm{K^1_{epi}(X)}$ by [BRW, Theorem 2.15] and the proof of [BRW, Lemma 2.28]. The most obvious examples are $X = \mathbf{N}$ and $X = \mathbf{R}$. Lemma 2.7 says that when X is locally compact and $\omega X \in \mathrm{K_z(X)}$ then, in fact, $\mathrm{K(X)} = \mathrm{K^1_{epi}(X)}$. To see this, by Lemma 2.7 there is a $g \in S_\omega$ with $\operatorname{coz} g^\omega = X$, which is what Proposition 4.1 requires. Now, [BRW, Theorem 2.29] implies that X is Lindelöf or almost compact. We summarize.

4.6. COROLLARY. [BRW] *Let X be a locally compact space. Then the following are equivalent.*

(i) $\omega X \in \mathrm{K_z(X)}$. *(ii) X is Lindelöf or almost compact. (iii)* $\mathrm{K(X)} = \mathrm{K^1_{epi}(X)}$.

The existence of compactifications in $\mathrm{K_{epi}(X)}$ where 1×1 zig-zags do not suffice ([BRW, Theorem 3.3] or Examples 4.4, above) shows that, in general, $\mathrm{K^f_{epi}(X)}$ can be strictly included in $\mathrm{K_{epi}(X)}$. However, we have seen that when X is a P-space, the two coincide. Moreover, we always have $\beta X \in \mathrm{K^1_{epi}(X)} \subseteq \mathrm{K^f_{epi}(X)}$.

4.7. REMARK. cf. [HM2, Corollary 2.9 (a)] *A space X is pseudocompact space if and only if* $\mathrm{K_{epi}(X)} = \{\beta \mathrm{X}\}$. *In particular, if $|\beta X - X| < 2^{\mathfrak{c}}$ then* $\mathrm{K_{epi}(X)} = \{\beta \mathrm{X}\}$.

PROOF. If X is not pseudocompact then ([GJ, 6I 1.]) there is a non-empty β-zero-set Z, which is infinite by [GJ, Theorem 9.5]. Then, Proposition 4.1 says that $\alpha_Z X \in \mathrm{K^1_{epi}(X)}$. On the other hand, if X is pseudocompact then the only C-epic compactification is βX by [HM2, Corollary 2.9 (a)], showing, *a fortiori*, that $\mathrm{K_{epi}(X)} = \{\beta \mathrm{X}\}$.

The second statement follows from [GJ, 9D 3.], which says that, under the hypothesis, X is pseudocompact. ■

We next find a situation in which $\alpha X \in \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$ if and only if $\alpha X \in \mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$, and one where $\alpha X \in \mathrm{K}_{\mathrm{z}}(\mathrm{X})$ if and only if $\alpha X \in \mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$.

4.8. Lemma. [C, Theorem 5.32] *Let X be any space. Suppose $Y = \{y_n \mid n \in \mathbf{N}\}$ is a countable subset. Then, there is $f \in C^*(X)$, $\mathbf{0} \leq f$, such that $f(y_i) \neq f(y_j)$, for all $i \neq j$.*

4.9. Proposition. *Let X be any space. (i) Let $\alpha X \in \mathrm{K}(\mathrm{X})$ be such that there is $D \subseteq I_\alpha$ which is dense in I_α and $f \in C^*(X)$ such that f^β is not constant on any $\tau^{-1}(a)$, $a \in D$. Then, $\alpha X \in \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$ implies $\alpha X \in \mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$. (ii) If I_α is separable then $\alpha X \in \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$ implies $\alpha X \in \mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$. (iii) Let $\alpha X \in \mathrm{K}(\mathrm{X})$ be such that I_α is finite. If $\alpha X \in \mathrm{K}_{\mathrm{z}}(\mathrm{X})$ then $\alpha X \in \mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$.*

Proof. (i) Since $\alpha X \in \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$, there are $g, h \in C_\alpha$, $h \in S_\alpha$ so that $f = gh^{-1}$. If follows from $f^\beta h^\beta = g^\beta$ that h^β is zero on $\tau^{-1}(a)$, for each $a \in D$ because g^β and h^β are both constant on each $\tau^{-1}(a)$, $a \in D$. Then $h^\alpha(D) = \{0\}$. Since D is dense in I_α, $h^\alpha(I_\alpha) = \{0\}$, showing that $M_\alpha \subseteq z(h^\beta)$ and that $\alpha X \in \mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$ using the function h in Proposition 4.1.

(ii) Lemma 4.8 shows that there is $f \in C^*(X)$ as required in part (i). More precisely, we write $D = \{a_1, a_2, \ldots\}$, dense in I_α, and pick $p_n \neq q_n$ from $\tau^{-1}(a_n)$, $n \in \mathbf{N}$. Then $l \in C(\beta X)$ is chosen so that $l(p_n) \neq l(q_n)$, for all $n \in \mathbf{N}$. Hence, $f = l|_X$ will work.

(iii) We can find $f \in C(\beta X)$ such that f is not constant on any $\tau^{-1}(a)$, $a \in I_\alpha$, since I_α is finite. By Lemma 2.7, there is $h \in S_\alpha$ so that h^β is zero on $\bigcup_{a \in I_\alpha} \tau^{-1}(a)$, showing that $z(h^\beta) \supseteq M_\alpha$. ∎

Construction 2.3 shows how to get spaces of the type found in Proposition 4.9 (ii). It suffices to start with $\alpha X \in \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$ and apply the construction with B countable.

It can be seen that Proposition 4.5 (A) contains a stronger statement than [BBR, Theorem 2.6] since, if $\alpha X \in \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$, the proof of (A) (ii) $\Rightarrow$ (iii) shows that, here, every $f \in C(X)$ extends to a cozero-set of αX containing X. We can now list three facts about extending functions.

1. If $\alpha X \in \mathrm{K}_{\mathrm{epi}}(\mathrm{X})$, each $f \in C(X)$ extends to an open set of αX containing X ([BBR, Proposition 2.6]).

2. If $\alpha X \in \mathrm{K}_{\mathrm{z}}(\mathrm{X})$, for each $f \in C(X)$ and finite $B \subseteq I_\alpha$, f extends to a cozero-set of $\alpha_{(B)} X$ containing X (Lemma 2.7 and Proposition 4.9 (iii)).

3. If $\alpha X \in \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$, each $f \in C(X)$ extends to a cozero-set of αX containing X (Theorem 4.5 (A)).

5. Lattice properties

In this section it will be shown that while the subsets $\mathrm{K}_{\mathrm{z}}(\mathrm{X})$ and $\mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$ of $\mathrm{K}(\mathrm{X})$ have some lattice closure properties (they are both complete upper semilattices), they are not closed under all meets unless υX is locally compact and $\omega(\upsilon X) \in \mathrm{K}_{\mathrm{z}}(\mathrm{X})$, in which case $\mathrm{K}_{\mathrm{z}}(\mathrm{X}) = \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X}) = \mathrm{K}(\upsilon \mathrm{X})$. In fact, if X is locally compact and realcompact, then Corollary 4.2 says that ωX is always a meet of elements of $\mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$, but it need be neither in $\mathrm{K}_{\mathrm{epi}}(\mathrm{X})$ nor in $\mathrm{K}_{\mathrm{z}}(\mathrm{X})$. We will see that, in general, $\mathrm{K}_{\mathrm{z}}(\mathrm{X})$ and $\mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$ have no minimal elements. Moreover, $\mathrm{K}_{\mathrm{nepi}}(\mathrm{X})$ need not be closed under finite joins. Recall the definition of $\alpha_T X$ from before Proposition 2.2.

5.1. LEMMA. *Let X be any space, $\alpha X \in \mathrm{K}(\mathrm{X})$ and, for some $g \in S_\alpha$, $T = z(g^\beta)$. Let $\gamma X = \alpha X \wedge \alpha_T X$.*

(i) If $\alpha X \in \mathrm{K}_{\mathrm{z}}(\mathrm{X})$ then $\gamma X \in \mathrm{K}_{\mathrm{z}}(\mathrm{X})$.

(ii) If $\alpha X \in \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$, then $\gamma X \in \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$.

(iii) If $\alpha X \in \mathrm{K}_{\mathrm{epi}}(\mathrm{X})$, then $\gamma X \in \mathrm{K}_{\mathrm{epi}}(\mathrm{X})$.

PROOF. In each case, the existence of γX is guaranteed by Proposition 2.2.

(i) This is a special case of Proposition 2.2 (iii).

(ii) If $f \in C(X)$ and we write $f = uv^{-1}$, $u \in C_\alpha$, $v \in S_\alpha$, then $f = ug(vg)^{-1}$, where $ug \in C_\gamma$ and $vg \in S_\gamma$.

(iii) The method here is like that of (ii) but we need to use zig-zags. Suppose $f \in C(X)$ and we write f as a $n \times n$ zig-zag over C_α:

$$f = \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} [h_{ij}] \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}, \quad [h_{ij}] \text{ an } n \times n \text{ matrix}.$$

Then, we can get an $n \times n$ zig-zag for f over C_γ as follows:

$$f = \begin{bmatrix} g_1 g^{-1} & \cdots & g_n g^{-1} \end{bmatrix} \begin{bmatrix} h_{ij} g^2 \end{bmatrix} \begin{bmatrix} k_1 g^{-1} \\ \vdots \\ k_n g^{-1} \end{bmatrix}$$

∎

5.2. THEOREM. *Let X be any space.*

(i) If $\mathrm{K}(\upsilon\mathrm{X}) \neq \mathrm{K}_{\mathrm{z}}(\mathrm{X})$, then $\mathrm{K}_{\mathrm{z}}(\mathrm{X})$ has no minimal elements.

(ii) If $\mathrm{K}(\upsilon\mathrm{X}) \neq \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$, then $\mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$ has no minimal elements.

(iii) If $\mathrm{K}(\upsilon\mathrm{X}) \neq \mathrm{K}_{\mathrm{epi}}(\upsilon\mathrm{X})$, then $\mathrm{K}_{\mathrm{epi}}(\upsilon\mathrm{X})$ has no minimal elements.

PROOF. (i) (By [BH, Theorem 4.1], we have that υX is neither Lindelöf nor almost compact.) Let $\alpha X \in \mathrm{K}_{\mathrm{z}}(\mathrm{X}) = \mathrm{K}_{\mathrm{z}}(\upsilon\mathrm{X})$ (by Proposition 2.8). If $\alpha X - \upsilon X$ were finite then υX would be locally compact and, by Lemma 5.1 (i) and Proposition 3.3, $\omega(\upsilon X) \in \mathrm{K}_{\mathrm{z}}(\upsilon\mathrm{X})$, contradicting the hypothesis. Hence, $\alpha X - \upsilon X$ is infinite and, given $a_1 \neq a_2$ in $\alpha X - \upsilon X$, there is, by Proposition 3.3 again, some $g \in S_\alpha$ with $\{a_1, a_2\} \subseteq z(g^\alpha)$. Put $T = z(g^\beta)$. Then Lemma 5.1 (i) says that $\alpha X \wedge \alpha_T X$, which is strictly smaller than αX, is in $\mathrm{K}_{\mathrm{z}}(\mathrm{X}) = \mathrm{K}_{\mathrm{z}}(\upsilon\mathrm{X})$.

In part (ii), since $\mathrm{K}_{\mathrm{z}}(\mathrm{X}) \subseteq \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X})$, we see that $\mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X}) = \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\upsilon\mathrm{X})$. With this observation, parts (ii) and (iii) have proofs like that for part (i) using the corresponding parts of Lemma 5.1. ∎

If X is locally compact and $\alpha X \in \mathrm{K(X)}$, the set of compactifications containing αX, $\mathrm{K}_{\geq\alpha}(\mathrm{X})$, of $\mathrm{K(X)}$ forms a complete sublattice. Clearly, if αX is in one of $\mathrm{K_z(X)}$, $\mathrm{K_{epi}(X)}$, $\mathrm{K^f_{epi}(X)}$ or $\mathrm{K^1_{epi}(X)}$, then all of $\mathrm{K}_{\geq\alpha}(\mathrm{X})$ is. Moreover, if $\alpha_1 X, \dots, \alpha_k X \in \mathrm{K^1_{epi}(X)}$, then, $\alpha_1 X \wedge \cdots \wedge \alpha_k X \in \mathrm{K^1_{epi}(X)}$ because there is some $\alpha X \in \mathrm{K^1_{epi}(X)}$ contained in all of them.

Another situation is worthy of note. It shows a case where a meet of elements of $\mathrm{K^1_{epi}(X)}$ is in $\mathrm{K^1_{epi}(X)}$, if it is in $\mathrm{K_z(X)}$.

5.3. Proposition. *Let X be a locally compact space. For some index set M, let $\{T_\mu\}_{\mu\in M}$ be a family of closed sets in $\beta X - X$ with each $|T_\mu| > 1$. We write $\alpha_\mu X$ for $\alpha_{T_\mu} X$ and C_μ for $C_{\alpha_{T_\mu}}$. Put $\alpha X = \bigwedge_{\mu\in M} \alpha_\mu X$ and $T = \mathrm{cl}_{\beta X}(\bigcup_{\mu\in M} T_\mu)$. Suppose, for some $\emptyset \neq N \subseteq M$, that (a) if $\nu_1, \nu_2 \in N$ then $T_{\nu_1} \cap T_{\nu_2} \neq \emptyset$, and (b) for $\mu \in M - N$, $T_\mu \cap [\mathrm{cl}_{\beta X}(\bigcup_{\nu\in N} T_\nu)] \neq \emptyset$. Then,*

(i) $C_\alpha = \{f \in C^*(X) \mid f$ is constant on $T\}$, showing $\alpha X = \alpha_T X$.

(ii) $\alpha X \in \mathrm{K_z(X)}$ if and only if $\alpha X \in \mathrm{K^1_{epi}(X)}$.

Proof. We have $C_\alpha = \bigcap_{\mu\in M} C_\mu = \{f \in C^*(X) \mid f^\beta$ is constant on each $T_\mu\}$. However, if $f \in C_\alpha$ and f^β has value r on T_ν, some $\nu \in N$, then it has value r on $\mathrm{cl}_{\beta X}(\bigcup_{\nu\in N} T_\nu)$, and, hence, on all T_μ, $\mu \in M$, by the hypotheses on the intersections. In other words, $C_\alpha = \{f \in C^*(X) \mid f^\beta$ is constant on $T\}$. This gives (i).

Having established (i), part (ii) is an instance of Proposition 4.9 (iii). ■

The following example shows that Proposition 5.3 (ii) can fail. It is a case where the T_μ form an uncountable chain.

5.4. Example. [HM1, page 988] *There is an uncountable discrete space X where $\mathrm{K^1_{epi}(X)}$ is not closed under taking the meet of a descending chain of elements. The same example shows that neither $\mathrm{K_z(X)}$ nor $\mathrm{K_{epi}(X)}$ is closed under the meet of a descending chain.*

Proof. In the cited example, $|X| = \omega_1$ and the compactifications, there called $L(\alpha_i)$, are in $\mathrm{K^1_{epi}(X)}$, while their meet is not even in $\mathrm{K_z(X)}$. ■

The situation is different when the meet takes place in some $\mathrm{K}_{\geq\alpha}(\mathrm{X})$, where $\alpha X \in \mathrm{K_z(X)}$.

5.5. Proposition. *Let X be any space and let $\alpha X \in \mathrm{K_z(X)}$. Then αX is a meet of elements of $\mathrm{K^1_{epi}(X)}$ in $\mathrm{K}_{\geq\alpha}(\mathrm{X})$.*

Proof. We use Construction 2.3 using each of the finite subsets F of I_α; let the resulting compactifications be $\alpha_{(F)} X$. By Lemma 2.7, each $\alpha_{(F)} X \in \mathrm{K^1_{epi}(X)}$ (since each is above an element of $\mathrm{K^1_{epi}(X)}$). Recall that $C_{\alpha_{(F)}} = \{f \in C^*(X) \mid f$ is constant on each $\tau^{-1}(a), a \in F\}$. Hence, $C_\alpha = \bigcap_{F\subseteq I_\alpha, F \text{ finite}} C_{\alpha_{(F)}}$. Then, [C, Theorem 2.18] shows that $\alpha X = \bigwedge_{F\subseteq I_\alpha, F\text{ finite}} \alpha_{(F)} X$. ■

Recall that the complement of $\mathrm{K}_{\mathrm{epi}}(\mathrm{X})$ in $\mathrm{K}(\mathrm{X})$ is called $\mathrm{K}_{\mathrm{nepi}}(\mathrm{X})$. If $\varphi\colon A \to B$ is any epimorphism in **CR** which is one-to-one and A is infinite then A and B have the same cardinality. This is one way to see that for X discrete and uncountable, $\omega X \in \mathrm{K}_{\mathrm{nepi}}(\mathrm{X})$ (although it is also a consequence of Corollary 4.6). Indeed, if X is uncountable and discrete, $f \in C(\omega X)$ is constant on a co-countable subset of X. Hence, $|C(\omega X)| \leq |X| \cdot \mathfrak{c}^{\aleph_0} < |C(X)| = \mathfrak{c}^{|X|}$.

5.6. Proposition. *When X is uncountable and discrete then $\mathrm{K}_{\mathrm{nepi}}(\mathrm{X})$ is not closed under finite joins in $\mathrm{K}(\mathrm{X})$.*

Proof. We first use Proposition 2.1 to show that if $V \subseteq X$ is uncountable and $T = \mathrm{cl}_{\beta X} V - X = \mathrm{cl}_{\beta X} V - V$, then $\alpha_T X \in \mathrm{K}_{\mathrm{nepi}}(\mathrm{X})$. Indeed, $\mathrm{cl}_{\alpha_T X} V$ is the one-point compactification of V. (This observation will be generalized in Proposition 6.9.)

Now let V_1 and V_2 be uncountable subsets of X with $|V_1 \cap V_2| \leq \aleph_0$. Put $T_i = \mathrm{cl}_{\beta X} V_i - X$, $i = 1, 2$. Let $\alpha X = \alpha_{T_1} X \vee \alpha_{T_2} X$. In the construction of the join ([C, Theorem 2.7]) it is clear that for $p \neq q$ in $\beta X - X$, $\tau_\alpha(p) = \tau_\alpha(q)$ if and only if $\{p, q\} \subseteq T_1 \cap T_2$. In other words, $\alpha X = \alpha_{T_1 \cap T_2} X$.

When $V_1 \cap V_2$ is finite, $T_1 \cap T_2 = \emptyset$ giving $\alpha X = \beta X$. When $V_1 \cap V_2$ is infinite, $T_1 \cap T_2$ is a zero-set. Indeed, if we write $V_1 \cap V_2 = \{x_1, x_2, \ldots\}$ and let $f \in C^*(X)$ be defined by $f(x_n) = 1/n$ and $f(x) = 1$ for $x \notin V_1 \cap V_2$, then $T_1 \cap T_2 = z(f^\beta)$. In either case, $\alpha X \in \mathrm{K}_{\mathrm{epi}}(\mathrm{X})$ while $\alpha_{T_1} X$ and $\alpha_{T_2} X$ are in $\mathrm{K}_{\mathrm{nepi}}(\mathrm{X})$. ∎

6. The special case where X is discrete

When X is uncountable and discrete, we know that $\mathrm{K}_{\mathrm{epi}}(\mathrm{X}) = \mathrm{K}^{\mathrm{f}}_{\mathrm{epi}}(\mathrm{X}) \subseteq \mathrm{K}_{\mathrm{z}}(\mathrm{X})$ (because X is a P-space) but we do not know if $\mathrm{K}_{\mathrm{epi}}(\mathrm{X}) = \mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$ or if $\mathrm{K}_{\mathrm{z}}(\mathrm{X}) = \mathrm{K}_{\mathrm{epi}}(\mathrm{X})$. The theme of the first part of this section will be to examine when a meet of compactifications of the form $\alpha_Z X$, Z a zero-set of some $g \in S_\beta$, is again in $\mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$. The second part finds elements of $\mathrm{K}_{\mathrm{nepi}}(\mathrm{X})$.

For general spaces, if X is a dense subset of Y and $\emptyset \neq U$ is open in Y, then U is contained in $\mathrm{cl}_Y(U \cap X)$. In particular, if $T \subseteq \beta X - X$ and U is open in βX with $T \subseteq U$, $T \subseteq \mathrm{cl}_{\beta X}(U \cap X)$. This fact will often be used below. We recall also that if $V \subseteq X$, then $\mathrm{cl}_{\beta X} V = \{p \in \beta X \mid V \in A^p\}$, where A^p is the ultrafilter attached to p ([GJ, 6.5]).

We begin by showing that a join of *countably* many elements of $\mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$ is again in $\mathrm{K}^{1}_{\mathrm{epi}}(\mathrm{X})$. (Example 5.4 shows that "countable" is essential.) A preliminary result about zero-sets of elements of S_β is needed; it relates them to almost disjoint families of countable subsets of X. We start with a lemma.

6.1. Lemma. *Let X be an infinite discrete space, $h \in S_\beta$. Suppose that P a countable subset of X such that $\inf_{x \in P} h(x) = 0$ and, for each $n \in \mathbf{N}$, $|h|^{-1}([1/n, \infty)) \cap P$ is finite. Then, (i) $\mathrm{cl}_{\beta X} P - X \subseteq z(h^\beta)$; and, (ii) if, in addition, $|h|$ is bounded away from 0 on $X - P$, then $\mathrm{cl}_{\beta X} P - X = z(h^\beta)$.*

PROOF. (i) For $q \in \mathrm{cl}_{\beta X} P - X$ and any $V \in A^q$, $P \cap V$ is infinite since $P \in A^q$ and the ultrafilter is free. By hypothesis, $\inf_{x \in V \cap P} h(x) = 0$ and, thus, $h^\beta(q) = \sup_{V \in A^q} \inf_{x \in V} h(x) = 0$ ([GJ, 6C]).

(ii) If $q \in z(h^\beta)$, then $\sup_{V \in A^q} \inf_{x \in V} h(x) = 0$, which implies that $V \cap P$ is infinite for each $V \in A^q$. It follows that $P \in A^q$ and, hence, $q \in \mathrm{cl}_{\beta X} P$. ∎

6.2. THEOREM. *(A) Let X be an uncountable discrete space and $h \in S_\beta$ with $z(h^\beta) \neq \emptyset$. Then there is a family of countable subsets of X, $\mathcal{P} = \{P_\gamma\}_{\gamma \in \Gamma}$, such that (i) for $\gamma, \lambda \in \Gamma$, $\gamma \neq \lambda$, $P_\gamma \cap P_\lambda$ is finite (i.e., the family $\mathcal{P}$ is almost disjoint); (ii) for each $\gamma \in \Gamma$, $\inf_{P_\gamma} h(x) = 0$; and (iii) $(\bigcup_\Gamma \mathrm{cl}_{\beta X} P_\gamma) - X$ is dense in $z(h^\beta)$.*

(B) Let $\mathcal{P} = \{P_\gamma\}_{\gamma \in \Gamma}$, Γ well-ordered, be an almost disjoint family of countable subsets of X. Then, there is a function $h \in S_\beta$ such that for all $\gamma \in \Gamma$, $\inf_{x \in P_\gamma} h(x) = 0$.

PROOF. (A) We may assume that $\mathbf{0} < h \leq \mathbf{1}$. Put $V_n = \{x \in X \mid 1/(n+1) < h(x) \leq 1/n\}$. Notice that $\mathcal{V} = \{\bigcup_{n \geq m} V_n \mid m \in \mathbf{N}\}$ is a filter base and that $z(h^\beta)$ is the set of ultrafilters containing $\mathcal{V}$.

In order to find an appropriate set $\mathcal{P}$, we look at two cases. (a) If all but finitely many of the V_n are finite, then h is bounded away from zero on the union of those subsets V_n which are infinite. Put $P = \bigcup_{n \in \mathbf{N}, V_n \text{ finite}} V_n$. Then, $\mathcal{P} = \{P\}$ gives the required set since $z(h^\beta) = \mathrm{cl}_{\beta X} P - X$, by Lemma 6.1 (ii).

(b) In the contrary case, fix a subsequence $\{n(j)\}_{j \in \mathbf{N}}$ of $\mathbf{N}$ so that for each $j \in \mathbf{N}$, $V_{n(j)}$ is infinite and if there is i, $n(j) < i < n(j+1)$, then V_i is finite. For any sequence $\{x_{n(j)}\}_{j \in \mathbf{N}}$, $x_{n(j)} \in V_{n(j)}$, we have that $\mathrm{cl}_{\beta X}\{x_{n(j)}\}_{j \in \mathbf{N}} \subseteq z(h^\beta)$, by Lemma 6.1 (i). Any such choice of a sequence will serve as the starting point of an induction.

Suppose now that we have a well-ordered family of countable subsets of X, P_γ, $\gamma < \mu$, μ an ordinal, satisfying conditions (i) and (ii). Then put $T = (\bigcup_{\gamma < \mu} \mathrm{cl}_{\beta X} P_\gamma) - X$; $T \subseteq z(h^\beta)$ since h^β is zero on each $\mathrm{cl}_{\beta X} P_\gamma - X$, $\gamma < \mu$. Assume that T is not dense in $z(h^\beta)$. Let $q \in z(h^\beta) - \mathrm{cl}_{\beta X} T$. There is a clopen set U of βX with $q \in U$ and $U \cap \mathrm{cl}_{\beta X} T = \emptyset$. Put $U = \mathrm{cl}_{\beta X} V$, for $V \subseteq X$. Then $W_j = V \cap (\bigcup_{k \geq j} V_{n(k)}) \in A^q$, for all $j \in \mathbf{N}$.

Pick a sequence $Q = \{x_j\}_{j \in \mathbf{N}}$, where $x_j \in V_{n(j)} \cap V$; some indices may be missing if the intersection is empty, but infinitely many of the intersections must be non-empty because $W_j \in A^q$, for all $j \in \mathbf{N}$. Suppose, for some $\gamma < \mu$, that $Q \cap P_\gamma$ is infinite. Then $\emptyset \neq \mathrm{cl}_{\beta X}(Q \cap P_\gamma) - X \subseteq \mathrm{cl}_{\beta X} P_\gamma - X \subseteq T$; however, $\mathrm{cl}_{\beta X}(Q \cap P_\gamma) - X \subseteq U$ because $Q \subseteq V$. This is impossible. Hence, $Q \cap P_\gamma$ is finite for all $\gamma < \mu$. This means that the collection $\mathcal{P}$ may be enlarged to satisfy (i); condition (ii) is satisfied by the choice of elements of Q. We, thus, have found $P_\mu = Q$.

When the process above terminates, we have condition (iii).

(B) We build the function h by induction. The starting point is to well-order $P_1 = \{x_{1,1}, x_{1,2}, \ldots\}$ and assign $h(x_{1,n}) = 1/n$.

Suppose now that $\gamma \in \Gamma$, $\gamma > 1$ and that values of h (in $\{1, 1/2, 1/3, \ldots\}$) have been assigned to the elements of P_ζ for $\zeta < \gamma$ so that the condition on infima is satisfied. Put $Q = P_\gamma \cap \bigcup_{\zeta < \gamma} P_\zeta$. There are two cases to look at.

First suppose that Q is not cofinite in P_γ. Then we well-order the elements of $P_\gamma - Q$ as $q_1, q_2, \ldots$ and assign $h(q_n) = 1/n$, while leaving h unchanged on $P_\gamma \cap Q$.

If Q is cofinite in P_γ, there are again two cases to consider. If $\inf_{x\in Q} h(x) = 0$, the value 1 is assigned to each element of $P_\gamma - Q$. Then $\inf_{x\in P_\gamma} h(x) = 0$. If $\inf_{x\in Q} h(x) \neq 0$ then Q is well-ordered as $Q = \{q_1, q_2, \ldots\}$ and h is redefined on Q (and we recall that this changes only finitely many values on each P_ζ, $\zeta < \gamma$) by $h(q_n) = \min\{1/n, \text{existing value}\}$.

In either situation, h has now been defined on each P_ζ, $\zeta \leq \gamma$, so that h is never 0 but $\inf_{x\in P_\zeta} h(x) = 0$. The construction continues until Γ has been exhausted. The remaining elements of X are assigned value 1. ∎

We can now deal with the meet of a countable set of elements of $\mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$.

6.3. Corollary. *Let X be an infinite discrete space and suppose that $h_n \in S_\beta$, $n \in \mathbf{N}$, are such that h_n^β, $n \in \mathbf{N}$, have non-empty and distinct zero-sets. Then, there is $h \in S_\beta$ with $T = z(h^\beta) \supseteq z(h_n^\beta)$, for all $n \in \mathbf{N}$. Hence, if $T_n = z(h_n^\beta)$, $n \in \mathbf{N}$, then, $\bigwedge_{n\in\mathbf{N}} \alpha_{T_n} X \in \mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$.*

Proof. Theorem 6.2 is applied repeatedly where we assume, without loss of generality since we are dealing with zero-sets, that each h_n satisfies $\mathbf{0} < h_n \leq \mathbf{1}$. The case where all the zero-sets are in a union of finitely many of them is easy. Hence, we assume this is not the case and we delete any h_n, $n > 1$ for which $z(h_n^\beta) \subseteq \bigcup_{i<n} z(h_i^\beta)$. With renumbering, we then have, for $n > 1$, $z(h_n^\beta) \not\subseteq \bigcup_{i<n} z(h_i^\beta)$.

We begin by finding a well-ordered family of countable subsets as in Theorem 6.2 (A) for the zero-set T_1, say $\mathcal{P}_1 = \{P_\gamma\}_{\gamma\in\Gamma_1}$. Then $T_1 = \mathrm{cl}_{\beta X}(\bigcup_{\gamma\in\Gamma_1} \mathrm{cl}_{\beta X} P_\gamma - X)$. Since $T_1 \neq T_2$, we look at $T_1 \cup T_2 = z((h_1h_2)^\beta)$ and continue the induction in the proof of the proposition to enlarge the set $\mathcal{P}_1$ to $\mathcal{P}_2 = \{P_\gamma\}_{\gamma\in\Gamma_2}$, where $\Gamma_2 \supseteq \Gamma_1$ and the conditions of Theorem 6.2 (A) are satisfied for $T_1 \cup T_2$. Notice that $\mathcal{P}_1$ satisfies condition (A) (ii) in Theorem 6.2 for h_1 and, hence, it also satisfies that condition for h_1h_2 because $h_1h_2 \leq h_1$.

This process continues, giving index sets $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_n$ and $\mathcal{P}_n = \{P_\gamma\}_{\gamma\in\Gamma_n}$ satisfying the conditions for $T_1 \cup \cdots \cup T_n$, for each $n \in \mathbf{N}$. Put $\Gamma = \bigcup_{n\in\mathbf{N}} \Gamma_n$ and $\mathcal{P} = \bigcup_{n\in\mathbf{N}} \mathcal{P}_n$. We now apply Theorem 6.2 (B) to $\mathcal{P}$ to get $h \in S_\beta$.

The set $R = (\bigcup_{\gamma\in\Gamma} \mathrm{cl}_{\beta X} P_\gamma) - X$ contains subsets dense in each $z(h_n^\beta)$, $n \in \mathbf{N}$ and, moreover, by the construction of h in Theorem 6.2 B, $T = z(h^\beta) \supseteq \mathrm{cl}_{\beta X} R$. Hence, $\alpha_T X \leq \alpha_{T_n} X$, for each $n \in \mathbf{N}$. The conclusion follows. ∎

The following is in the same vein as Proposition 4.9. When X is discrete we can go from $\mathrm{K_z(X)}$ to $\mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$; however, a stronger condition on the countable subset of I_α is needed.

6.4. Corollary. *Let X be an uncountable discrete space and $\alpha X \in \mathrm{K_z(X)}$. Let $f \in C(\beta X)$ and $B = \{b \in I_\alpha \mid f \text{ is not constant on } \tau^{-1}(b)\}$. Suppose that $\bigcup_{b\in B} \tau^{-1}(b)$ is dense in M_α. Then, $\alpha X \in \mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$.*

In particular, if there is a countable subset D of I_α with $\tau^{-1}(D)$ dense in M_α, then $\alpha X \in \mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$.

PROOF. By Lemma 2.6 there is a countable subset $G = \{g_n\}_{n\in\mathbf{N}}$ of S_α so that for each $b \in B$, there is $m \in \mathbf{N}$ with $\tau^{-1}(b) \subseteq z(g_m^\beta)$. According to Corollary 6.3 there is $g \in S_\beta$ so that $z(g_n^\beta) \subseteq z(g^\beta)$, for each $n \in \mathbf{N}$. Then, $cl_{\beta X}(\bigcup_{b\in B} \tau^{-1}(b)) \subseteq z(g^\beta)$. In other words, $M_\alpha \subseteq z(g^\beta) = Z$. Hence, $\alpha_Z X \leq \alpha X$ and so $\alpha X \in \mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$.

For the second part, from each $\tau^{-1}(d), d \in D$, choose a pair of distinct points and then apply Lemma 4.8 to the resulting countable set to get $f \in C(\beta X)$ not constant on any $\tau^{-1}(d), d \in D$. Then, the first part of the statement applies. ∎

We know from Example 5.4 that the condition "countable" cannot simply be dropped from the second part of Corollary 6.4; however, we will next show that the condition is not necessary either.

6.5. EXAMPLE. *Let X be an uncountable discrete space which is partitioned into disjoint countable subsets, $\mathcal{P} = \{Q_\gamma\}_{\gamma\in\Gamma}$, where $Q_\gamma = \{x_{\gamma,1}, x_{\gamma,2}, \ldots\}$, $\gamma \in \Gamma$, and we put $T_\gamma = \mathrm{cl}_{\beta X} Q_\gamma$. Then, $\alpha X = \bigwedge_{\gamma\in\Gamma} \alpha_{T_\gamma} X \in \mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$ while no countable subset of I_α satisfies the requirements of Corollary 6.4.*

PROOF. We note first that each T_γ is a zero-set, namely of f_γ^β, where $f_\gamma(x_{\lambda,n}) = 1/n$, if $\lambda = \gamma$, and 1, otherwise. Then, Corollary 2.5 shows that I_α has one element for each $\gamma \in \Gamma$, say $\tau(T_\gamma) = \{a_\gamma\}$. Let $f \in C^*(X)$ be defined by $f(x_{\gamma,n}) = 1/n$, for all $\gamma \in \Gamma$ and $n \in \mathbf{N}$. Then, $Z = z(f^\beta)$ contains each T_γ, showing that $\alpha_Z(X) \leq \alpha X$ in K(X). However, since $\alpha_Z X \in \mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$, so is αX and the last statement is clear. ∎

The next theorem and its corollary look at arbitrary families of β-zero-sets. The aim is still to find meets of families of elements of $\mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$ which are in $\mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$.

6.6. THEOREM. *Let X be an uncountable discrete space and $\{h_\nu\}_{\nu\in E}$ a family of elements of S_β. Then, the following statements are equivalent.*

(i) There is an $h \in S_\beta$ such that $z(h^\beta) \supseteq \bigcup_{\nu\in E} z(h_\nu^\beta)$.

(ii) There is a family $\{g_\nu\}_{\nu\in E}$ of positive elements of S_β, so that (a) $z(g_\nu^\beta) = z(h_\nu^\beta)$ for all $\nu \in E$, and (b) the function defined by $\tilde{h}(x) = \inf_{\nu\in E} g_\nu(x)$ is in S_β and $z(\tilde{h}^\beta) \supseteq \bigcup_{\nu\in E} z(h_\nu^\beta)$.

(iii) $\bigwedge_{\nu\in E} \alpha_{T_\nu} X \in \mathrm{K}^1_{\mathrm{epi}}(\mathrm{X})$.

PROOF. The equivalence of (i) and (iii) is clear, as is the implication (ii) $\Rightarrow$ (i).

Assume (i). As usual, we assume that, for $\nu \in E$, $\mathbf{0} < h_\nu \leq \mathbf{1}$ and $\mathbf{0} < h \leq \mathbf{1}$. Define, for each $n \in \mathbf{N}$ and each $\nu \in E$, $V_n = \{x \in X \mid h(x) \leq 1/n\}$ and $V_{\nu,n} = \{x \in X \mid h_\nu(x) \leq 1/n\}$. Let $\mathcal{H} = \{V_n\}_{n\in\mathbf{N}}$ and $\mathcal{H}_\nu = \{V_{\nu,n}\}_{n\in\mathbf{N}}$; these are filter bases. Then, $z(h^\beta) = \{p \in \beta X \mid A^p \supseteq \mathcal{H}\}$ and, for $\nu \in E$, $z(h_\nu^\beta) = \{p \in \beta X \mid A^p \supseteq \mathcal{H}_\nu\}$.

For each $\nu \in E$, define $g_\nu = h + h_\nu$. Since $z(h_\nu^\beta) \subseteq z(h^\beta)$, $z(g_\nu^\beta) = z(h_\nu^\beta)$, for each $\nu \in E$, because all the functions are non-negative. Now $\tilde{h} \in C^*(X)$ is defined as follows: for each $x \in X$, $\tilde{h}(x) = h(x) + \inf_{\nu\in E} h_\nu(x) = \inf_{\nu\in E} g_\nu(x)$. Since $h > \mathbf{0}$, $\tilde{h} > \mathbf{0}$ (i.e., $\tilde{h} \in S_\beta$) and so it remains to be shown that, for each $\nu \in E$, $z(h_\nu^\beta) \subseteq z(\tilde{h}^\beta)$.

For some $\nu \in E$, let $p \in z(h_\nu^\beta) = z(g_\nu^\beta)$. Then $A^p \supseteq \mathcal{H} \cup \mathcal{H}_\nu$. Hence, for each $V \in A^p$ and every $m, n \in \mathbf{N}$, $V \cap V_m \cap V_{\nu,n} \neq \emptyset$. Then,

$$\inf_{x \in V} \tilde{h}(x) \leq \inf_{x \in V \cap V_m \cap V_{\nu,n}} \tilde{h}(x) \leq 1/m + \inf_{\mu \in E} h_\mu(x) \leq 1/m + 1/n .$$

Since this is true for all $m, n \in \mathbf{N}$, $\inf_{x \in V} \tilde{h}(x) = 0$. Thus, $\tilde{h}^\beta(p) = 0$, as required. ∎

It can be remarked that, in the proof of (i) $\Rightarrow$ (ii) in the proposition, each $g_\nu > h_\nu$. The pointwise infimum of the original family of functions is not necessarily in S_β, while that of the new family is.

The next result presents two situations where Theorem 6.6 can be applied.

6.7. COROLLARY. *Let X be uncountable and discrete and $\{h_\nu\}_{\nu \in E}$ be a family of elements of S_β. Put $L = \{x \in X \mid \inf_{\nu \in E} |h_\nu|(x) = 0\}$. If one of the following holds then there is an $h \in S_\beta$ such that $z(h^\beta) \supseteq \bigcup_{\nu \in E} z(h_\nu^\beta)$ so that Theorem 6.6 applies.*

(i) $A = \{\nu \in E \mid z(h_\nu^\beta) \cap \mathrm{cl}_{\beta X} L \neq \emptyset\}$ is countable.

(ii) L is countable.

Moreover, if there is $J \subseteq L$ so that, for each $\nu \in E$, $\inf_{x \in J} |h_\nu|(x) > 0$; then, there is an $h \in S_\beta$ with $Z(h^\beta) \supseteq \bigcup_{\nu \in E} z(h_\nu^\beta)$ if and only if the conclusion holds for $\{h_\nu|_{L-J}\}_{\nu \in E}$ in the space $\beta(L - J)$.

PROOF. We need the following remark: if $f \in C^*(X)$ and $W \subseteq X$ we define $f_W = f \cdot \chi_W$ and $f_{W'} = f \cdot \chi_{X-W}$, then $z(f^\beta) = z(f_W^\beta) \cup z(f_{W'}^\beta)$.

We normalize to get $\mathbf{0} < h_\nu \leq \mathbf{1}$, for all $\nu \in E$.

(i) Assume that L is uncountable since the contrary case is covered in part (ii). We use Corollary 6.3 applied to $\{h_\nu|_L\}_{\nu \in A}$. There is $u \in C^*(L)$ with $\mathrm{coz}\, u = L$ and $\mathbf{0} < u \leq \mathbf{1}$, so that $z(u^\beta) \supseteq \bigcup_{\nu \in A} z((h_\nu|_L)^\beta)$ (in $\mathrm{cl}_{\beta X} L = \beta L$). We extend u to X, without changing its name, by making it zero on $X - L$. We now define, for $\nu \in E$, $g_\nu = h_\nu + u$ and $h = \inf_{\nu \in E} g_\nu$. The first thing to note is that on $X - L$, g_ν and h_ν coincide, and that, for $x \in X - L$, $h(x) = \inf_{\nu \in E} h_\nu(x) \neq 0$, while for $x \in L$, $h(x) = u(x) \neq 0$.

For $\nu \in E - A$, $z(g_\nu^\beta) = z(h_\nu^\beta)$ because the zero-set is in $\mathrm{cl}_{\beta X}(X - L)$. For $\nu \in A$,

$$z(g_\nu^\beta) = (z(g_\nu^\beta) \cap \mathrm{cl}_{\beta X} L) \cup (z(g_\nu^\beta) \cap \mathrm{cl}_{\beta X}(X - L))$$

$$= (z(h_\nu^\beta + u^\beta) \cap \mathrm{cl}_{\beta X} L) \cup (z(h_\nu^\beta) \cap \mathrm{cl}_{\beta X}(X - L)) = z(h_\nu^\beta) ,$$

since $z(u^\beta) \supseteq z((h_\nu|_L)^\beta)$ on $\mathrm{cl}_{\beta X} L$.

Since $h \leq g_\nu$, for each $\nu \in E$, $z(h^\beta) \supseteq \bigcup_{\nu \in E} z(h_\nu^\beta)$.

(ii) This part is similar except that we order $L = \{x_1, x_2, \ldots\}$ and define $u(x_n) = 1/n$ on L and 0 elsewhere. Then, $z(u^\beta) = \beta X - L$. Again g_ν is defined as $h_\nu + u$, for each $\nu \in E$ and $h = \inf_{\nu \in E} g_\nu$.

For the last part, it suffices to notice that the zero-sets do not change if one makes each h_ν constantly 1 on J. ∎

Examples can be constructed to illustrate each of the parts of Corollary 6.7.

6.8. EXAMPLES. *Let X be an uncountable discrete space. Then there are examples of families of functions satisfying Corollary 6.7 (i) and families satisfying Corollary 6.7 (ii). The last part of the corollary can be illustrated by using one of the examples in a subset of X.*

PROOF. (i) We partition X into uncountably many uncountable subsets $\{Y_\nu\}_{\nu\in E}$ and single out special elements $\nu_0, \nu_1, \nu_2, \ldots$ of E. Partition each Y_ν into countably many infinite subsets $\{Y_{\nu,n}\}_{n\in\mathbf{N}}$. For ν not in the special subset of E, we define $h_\nu\colon X \to \mathbf{R}$ by

$$h_\nu(x) = \begin{cases} 1/n & \text{for } x \in Y_{\nu,n} \\ 1 & \text{for } x \notin Y_\nu \end{cases}.$$

For $m > 0$, we define h_{ν_m} by

$$h_{\nu_m}(x) = \begin{cases} 1/n & \text{for } x \in Y_{\nu_m,n} \\ 1/(m+k) & \text{for } x \in Y_{\nu_0,k} \\ 1 & \text{for } x \notin Y_{\nu_m} \cup Y_{\nu_0} \end{cases}.$$

Here, $L = Y_{\nu_0}$ is uncountable but $A = \{h_{\nu_m}\}_{m>0}$ is countable.

(ii) We divide X into uncountably many disjoint uncountable subsets Y_ν, $\nu \in E$, and one countably infinite set $P = \{x_1, x_2, \ldots\}$. Each Y_ν is further partitioned into countably many infinite subsets $\{Y_{\nu,n}\}_{n\in\mathbf{N}}$. We need a family of bijections $\theta_\nu\colon \mathbf{N} \to \mathbf{N}$, $\nu \in E$, which may be chosen in any way but with the proviso that for each pair $m, n \in \mathbf{N}$, there is $\nu \in E$ so that $\theta_\nu(n) = m$.

For each $\nu \in E$ we define $h_\nu\colon X \to \mathbf{R}$ by

$$h_\nu(x) = \begin{cases} 1/n & \text{if } x \in Y_{\nu,n} \\ 1/\theta_\nu(n) & \text{if } x = x_n \\ 1 & \text{if } x \notin Y_\nu \cup P \end{cases}.$$

In this case $L = P$ is countable. ∎

Not much has been said so far about $\mathrm{K_{nepi}(X)}$, except that it is not closed under finite joins (Proposition 5.6); we give here some of its elements. The example from [HM1] quoted as Example 5.4 gives an instance of the following result which will give compactifications in $\mathrm{K_{nepi}(X)}$. Notice that the cardinality of $\alpha X - X$ is not involved since in these examples we can have $|\alpha X - X| = |\beta X - X|$.

6.9. PROPOSITION. *Let X be an uncountable discrete space and $\alpha X \in \mathrm{K(X)}$. Suppose that for some uncountable $V \subseteq X$ there is $a \in I_\alpha$ such that for every neighbourhood U of a in αX, $|V - U| < |V|$. Then, $\alpha X \notin \mathrm{K_z(X)}$, and, hence, $\alpha X \in \mathrm{K_{nepi}(X)}$.*

PROOF. By Proposition 2.1, it suffices to show that V is not z-embedded in $V \cup \{a\}$ (as a subspace of $\mathrm{cl}_{\alpha X} V \subseteq \alpha X$). To establish this, consider $W \subseteq V$ with $|W| = |V - W| = |V|$ and $f \in C(V \cup \{a\})$ so that $W = z(f) \cap V$. There are two cases. (i) If $f(a) = 0$ then $W = \bigcap_{n\in\mathbf{N}} f^{-1}((-1/n, 1/n)) \cap V$ and $V - W = \bigcup_{n\in\mathbf{N}}(V - f^{-1}((-1/n, 1/n))$, a set of cardinality $< |V|$. This is not possible. (ii) If $f(a) = r \neq 0$. For any $n \in \mathbf{N}$ with $1/n < |r|$, put $f^{-1}((r - 1/n, r + 1/n)) \cap V = Y$ and repeat the argument from (i). Hence, V is not z-embedded in $\mathrm{cl}_{\alpha X} V$. ∎

Proposition 6.9 can be used to give a $\mathrm{K}_{\mathrm{nepi}}$ counterpart to Corollary 6.7. The notation of that corollary is used.

6.10. COROLLARY. *Let X be uncountable and discrete and $\{h_\nu\}_{\nu\in E}$ be a family of elements of S_β, E uncountable. Suppose that (i) $L = \{x \in X \mid \inf_{\nu\in E} h_\nu(x) = 0\}$ is uncountable and (ii) for every $W \subseteq L$ with $|W| = |L - W|$ there is some $\nu \in E$ with $\inf_{x\in W} h_\nu(x) = 0$. Then, for any closed set K, $\bigcup_{\nu\in E} z(h_\nu^\beta) \subseteq K \subseteq \beta X - X$, $\alpha_K X \notin \mathrm{K_z(X)}$. In particular, if, for each $\nu \in E$, we write $T_\nu = z(h_\nu^\beta)$, then $\bigwedge_{\nu\in E} \alpha_{T_\nu} X \notin \mathrm{K_{epi}(X)}$.*

PROOF. We use Proposition 2.1 to assume that $X = L$. Let K be a closed set as in the statement. Suppose, if possible, that for some open neighbourhood U of K, $|U \cap X| = |X - U|$. We have that $K \subseteq \mathrm{cl}_{\beta X}(U \cap X)$. However, $|X - U| = |X|$ and so, by hypothesis, there is $\nu \in E$ so that $\inf_{x\in X-U} h_\nu(x) = 0$, and, hence, $z(h_\nu^\beta)$ meets $\mathrm{cl}_{\beta X}(X - U)$. This is impossible. Hence, for each neighbourhood U of K, $|X - U| < |X|$, showing, by Proposition 6.9, that $\alpha_K X \notin \mathrm{K_z(X)}$ and, hence, not in $\mathrm{K_{epi}(X)}$. ∎

The "size" of $\alpha X - X$ may, however, come into play. Recall that if $\alpha X \in \mathrm{K_{epi}(X)}$ then $|C_\alpha| = |C(X)|$ and, when X is discrete, this is $\mathfrak{c}^{|X|}$.

[C, Lemma 7.6] says that given a compact space Y with a dense subset D where $|D| \le |X|$ and a map $\rho\colon X \to D$ such that $\rho(V)$ is dense in Y for each cofinite subset of X, there is an explicit construction of a compactification αX so that $\alpha X - X = Y$. This construction gives examples of the compactifications which appear in the following.

6.11. PROPOSITION. *Let X be an infinite P-space and $\alpha X \in \mathrm{K(X)}$.*

(A) Suppose (i) there is D dense in $\alpha X - X$ with $|D| < |X|$, and (ii) $|C(X)| > \mathfrak{c}^{|D|} \cdot |X|$. Then, $\alpha X \notin \mathrm{K_z(X)}$, and, thus, $\alpha X \in \mathrm{K_{nepi}(X)}$.

(B) Let X be uncountable and discrete and suppose there is D dense in $\alpha X - X$ with $2^{|D|} < 2^{|X|}$. Then, $\alpha X \notin \mathrm{K_z(X)}$.

PROOF. (A) We first need an observation about elements of $C(\beta X)$. Suppose $f, g \in C(\beta X)$ and they agree on $\beta X - X$. For $n \in \mathbf{N}$, put $U_n = \{x \in X \mid |(f - g)|(x) \ge 1/n\}$. If U_n were in a free z-ultrafilter, p, then $|f - g|(p) \ge 1/n$. Since this is impossible, U_n must be compact and, therefore, finite ([GJ, 4 K 2.]). Hence, $\mathrm{coz}(f - g)$ is countable. This same remark then applies to elements of $C(\gamma X)$, for any $\gamma X \in K(X)$, by lifting to βX.

We now get an upper bound on $|C(\alpha X)|$: In the light of the above, an element $f \in C(\alpha X)$ is completely determined by its action on D and on a countable subset of X. Thus, $|C(\alpha X)| \le \mathfrak{c}^{|D|} \cdot |X| \cdot \mathfrak{c}^\omega$, since X is uncountable. On the other hand, $\mathfrak{c}^{\omega+|D|} \cdot |X| = \mathfrak{c}^{|D|}|X| < |C(X)|$, by hypothesis. Since, for any infinite space Y, $|C(Y)|$

equals the cardinality of the set of zero-sets of Y ([DZ, Corollary 1,3])), we see that αX cannot be in $\mathrm{K_z(X)}$.

(B) We show that part (A) applies here. This is easy since X has $2^{|X|}$ zero-sets (since every subset of X is a zero-set) and $|C(\alpha X)| \leq \mathfrak{c}^{|D|}|X| = 2^{\omega|D|}|X| = 2^{|D|}|X| < 2^{|X|}$. Hence, X has more zero-sets than there are elements of $C(\alpha X)$. ■

The cardinality of $C(\alpha X)$ is too coarse a measure to determine elements of $\mathrm{K_z(X)}$. When X is uncountable and discrete, it suffices to partition it into two parts, V_1 and V_2, of equal cardinality and put $T = \mathrm{cl}_{\beta X} V_1 - X$. Then, $|C(\alpha_T X)| = |C(X)|$, but, by Proposition 6.9, $\alpha_T X \notin \mathrm{K_z(X)}$.

Let us apply the construction of [C, Lemma 7.6] (using its terminology) to the case where X is uncountable and discrete, $K = \omega X$, $D = X \subset \omega X = K$ and $f\colon X \to D$ described as follows. We partition $X = \bigcup_{d \in D} X_d$, where $|X_d| = |X|$, for all $d \in D$ and set $f(x) = d$ when $x \in X_d$. Then, for finite $F \subseteq X$, $f(X - F) = D$ is dense in K, as required. Hence, there is a compactification $\alpha X \in \mathrm{K(X)}$ with $\alpha X - X$ homeomorphic to ωX. Proposition 6.11 does not apply here. But the next result will show that $\alpha X \in \mathrm{K_{nepi}(X)}$. Recall that a *scattered space* is one in which every subspace has an isolated point; ωX is an example.

6.12. Proposition. *Let X be a scattered space and let $\alpha X \in \mathrm{K(X)}$ be such that $\alpha X - X$ is scattered. If X is not functionally countable then $\alpha X \in \mathrm{K_{nepi}(X)}$. The result applies, in particular, if X is uncountable and discrete.*

Proof. We prove this separately for X a P-space because in this case the proof just uses Theorem 4.5. Since X is scattered and $\alpha X - X$ is scattered, then αX is scattered. By [LR, Proposition 3.1], αX is functionally countable (every $f \in C(\alpha X)$ has countable range). If $\alpha X \in \mathrm{K_{epi}(X)} = \mathrm{K^f_{epi}(X)}$, we would have $C(X) = C_\alpha S_\alpha^{-1}$ (Theorem 4.5 (C)). This would say that every element of $C(X)$ had countable range, since it is true of elements of C_α. This is not the case.

The general case is done using the material from [BBR, Section 3] and the regular ring $G(X)$. By [BBR, Proposition 3.3], if αX is functionally countable, so is $G(X)$. If $\alpha X \in \mathrm{K_{epi}(X)}$, then the same result says that X is functionally countable ■

7. Questions

Many questions remain unanswered and we only list a few of them. The case where X is uncountable and discrete remains a challenge.

7.1. Question. *(i) Let X be an uncountable discrete space. We know that $\mathrm{K_z(X)} \subseteq \mathrm{K_{epi}(X)} = \mathrm{K^f_{epi}(X)}$. (a) Does $\mathrm{K_z(X)} = \mathrm{K^f_{epi}(X)}$? (b) Does $\mathrm{K^f_{epi}(X)} = \mathrm{K^1_{epi}(X)}$. In more general form: Is there a P-space X where $\mathrm{K_z(X)} \neq \mathrm{K^f_{epi}(X)}$? (ii) Is there a space where $\mathrm{K^f_{epi}(X)} \neq \mathrm{K^1_{epi}(X)}$?*

Various results above restrict where one might look for a positive answer to Question 7.1 (i), (a) or (b). In particular, the examples of elements of $\mathrm{K_{nepi}(X)}$ in Section 6 are found by showing that $\alpha X \notin \mathrm{K_z(X)}$. Another constraint is given in [HVW, Proposition 3.2] which says that if αX is a quasi-F space then $\alpha X \in \mathrm{K_z(X)}$ if and only if $\alpha X = \beta X$. Moreover, [BBR, Proposition 4.1 (iii)] and Theorem 4.5 (C) show that if X is a P-space and X is functionally countable then $\mathrm{K_z(X)} = \mathrm{K_{epi}(X)} = \mathrm{K^f_{epi}(X)}$.

7.2. Question. *Is there a proof of Lemma 2.7 (ii) ([BRW, Lemma 5.1]) just using the sorts of methods in this paper? Is there a theorem in the spirit of [HM2, Theorem 8.2] for* $\alpha X \in \mathrm{K_{epi}(X)}$ *(rather than* $\alpha X \in \mathrm{K_z(X)}$*)?*

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