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TAKING CATEGORIES SERIOUSLY

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ABSTRACT. The relation between teaching and research is partly embodied in simple general concepts which can guide the elaboration of examples in both. Notions and constructions, such as the spectral analysis of dynamical systems, have important aspects that can be understood and pursued without the complication of limiting the models to specific classical categories. Pursuing that idea leads to a dynamical objectification of Dedekind reals which is particularly suited to the simple identification of metric spaces as enriched categories over a special closed category. Rejecting the complacent description of that identification as a mere analogy or amusement, its relentless pursuit [8] is continued, revealing convexity and geodesics as concepts having a definite meaning over any closed category. Along the way various hopefully enlightening exercises for students (and possible directions for research) are inevitably encountered: (1) an explicit treatment of the contrast between multiplication and divisibility that, in inexorable functorial fashion, mutates into the adjoint relation between autonomous and non-autonomous dynamical systems; (2) the role of commutation relations in the contrast between equilibria and orbits, as well as in qualitative distinctions between extensions of Heyting logic; (3) the functorial contrast between translations and rotations (as appropriately defined) in an arbitrary non symmetric metric space.

The theory of categories originated [1] with the need to guide complicated calculations involving passage to the limit in the study of the qualitative leap from spaces to homotopical/homological objects. Since then it is still actively used for those problems but also in algebraic geometry [2], logic and set theory [3], model theory [4], functional analysis [5], continuum physics [6], combinatorics [7], etc. In all these the categorical concept of adjoint functor has come to play a key role.

Such a universal instrument for guiding the learning, development, and use of advanced mathematics does not fail to have its indications also in areas of school and college mathematics, in the most basic relationships of space and quantity and the calculations based on those relationships. In saying "take categories seriously", I advocate noticing, cultivating,

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After the original manuscript was stolen, I was able to reconstruct, in time for inclusion in the proceedings of the 1983 Bogotá Workshop, this"sketch" (as Saunders Mac Lane's review was to describe it). Interested teachers can elaborate this and similar material into texts for beginning students. For the present opportunity, I am grateful to Xavier Caicedo who gave permission to reprint, to Christoph Schubert for his expert transcription into TEX, and to the editors of *TAC*.

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and teaching of helpful examples of an elementary nature.

1. Elementary mutability of dynamical systems

Already in [1] it was pointed out that a preordered set is just a category with at most one *morphism* between any given pair of objects, and that functors between two such categories are just order-preserving maps; at the opposite extreme, a monoid is just a category with exactly one *object*, and functors between two such categories are just homomorphisms of monoids. But category theory does not rest content with mere classification in the spirit of Wolffian metaphysics (although a few of its practitioners may do so); rather it is the *mutability* of mathematically precise structures (by morphisms) which is the essential content of category theory. If the structures are themselves categories, this mutability is expressed by functors, while if the structures are functors, the mutability is expressed by natural transformations. Thus if Λ is a preordered set and $\mathcal{X}$ is any category (for example the category of sets and mappings, the category of topological spaces and continuous mappings, the category of linear spaces and linear transformations, or the category of bornological linear spaces and bounded linear transformations) then there are functors

$$\Lambda \longrightarrow \mathcal{X}$$

sometimes called "direct systems" in $\mathcal{X}$, and the natural transformations

$$\begin{tikzcd} \Lambda \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathcal{X} \end{tikzcd}$$

between two such functors are the appropriate morphisms for the study of such direct systems as objects.

An important special case is that where $\Lambda = \boxed{0 \to 1}$, the ordinal number $\mathbf{2}$, then the functors $\mathbf{2} \longrightarrow \mathcal{X}$ may be identified with the morphisms in the category $\mathcal{X}$ itself; likewise if $\Lambda = \boxed{0 \to 1 \to 2 \to \cdots}$ is the ordinal number ω, functors

$$\omega \xrightarrow{X} \mathcal{X}$$

are just sequences of objects and morphisms

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \ldots$$

in $\mathcal{X}$, and a natural transformation $X \xrightarrow{f} Y$ between two such is a sequence $X_n \xrightarrow{f_n} Y_n$ of morphisms in $\mathcal{X}$ for which all squares

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \\ X_{n+1} \arrow[r, "f_{n+1}"'] & Y_{n+1} \end{tikzcd}$$

commute in $\mathcal{X}$ (here the vertical maps are the ones given as part of the structure of X and Y).

Similarly, if M is a monoid then the functors $M \longrightarrow \mathcal{X}$ are extremely important mathematical objects often known as *actions* of M on objects of $\mathcal{X}$ (or *representations* of M by $\mathcal{X}$-endomorphisms, or ...) and the natural transformations between such actions are known variously as M-equivariant maps, intertwining operators, homogeneous functions, etc. depending on the traditions of various contexts. Historically the notion of monoid (or of group in particular) was abstracted from the actions, a pivotally important abstraction since as soon as a particular action is constructed or noticed, the demands of learning, development, and use mutate it into: 1) other actions on the same object, 2) actions on other related objects, and 3) actions of related monoids. For if $M \xrightarrow{X} \mathcal{X}$ is an action and $M' \xrightarrow{h} M$ is a homomorphism, then (composition of functors!) Xh is an action of M', while if $\mathcal{X} \xrightarrow{C} \mathcal{Y}$ is a functor, then CX is an action of M on objects of $\mathcal{Y}$. To exemplify, if M is the additive group of time-translations, then a functor $M \longrightarrow \mathcal{X}$ is often called a dynamical system (continuous-time and autonomous) in $\mathcal{X}$, but if we are interested in observing the system only on a daily basis we could consider a homomorphism $M' \xrightarrow{h} M$ where $M' = \mathbb{N}$ is the additive monoid of natural numbers, and concentrate attention on the predictions of the discrete-time, autonomous, future-directed dynamical system Xh. In other applications we might have $M' = M =$ the multiplicative monoid of real numbers, but consider the homomorphism $M \xrightarrow{(\ )^p} M$ of raising to the pth power; then if we are given two actions $M \rightrightarrows \mathcal{X}$ on objects of $\mathcal{X}$, a natural transformation $X \xrightarrow{f} (Y)^p$ is just a morphism of the underlying $\mathcal{X}$-objects which satisfies

$$f(\lambda x) = \lambda^p f(x)$$

for all λ in M and all $T \xrightarrow{x} X$ in $\mathcal{X}$, i.e. a function homogeneous of degree p. An extremely important example of the second mutation of action mentioned above is that in which $\mathcal{Y}$ is the opposite of an appropriate category of algebras and the functor C assigns to each object (domain of variation) of $\mathcal{X}$ an algebra of functions (= intensive quantities) on it. Then the induced action CX of M describes the evolution of intensive quantities which results from the evolution of "states" as described by the action X. A frequently-occurring example of the third type of mutation of action arises from the surjective homomorphisms $M' \longrightarrow M$ from the additive monoid of time-translations M' to the circle group M. Then a dynamical system $M' \xrightarrow{X} \mathcal{X}$ is said to be "periodic of period h" if there exists a commutative diagram of functors as follows:

$$\begin{tikzcd}
M' \arrow[rr, "X"] \arrow[dr, "h"'] & & X \\
& M \arrow[ur, dotted] &
\end{tikzcd}$$

2. Kan quantifiers in spectral analysis

Most dynamical systems are only partly periodic, and such an analysis can conveniently be expressed by "Kan-extensions" as follows (we do not assume that M, M' are monoids):

For a functor $M' \xrightarrow{h} M$ and a category $\mathcal{X}$, the induced functor $\mathcal{X}^M \longrightarrow \mathcal{X}^{M'}$ will often have a left adjoint $X \longmapsto h\sum X$ and a right adjoint $X \longmapsto h\prod X$. These Kan adjoints vastly generalize the existential and universal quantifiers of logic, special cases arising when all the objects of $\mathcal{X}$ are truth-values. (Usually this just means that the objects of $\mathcal{X}$ are canonically idempotent with respect to cartesian product or coproduct.) Kan adjoints also generalize the induced representations frequently considered when $\mathcal{X}$ is a *linear* category (i.e. finite coproduct is canonically isomorphic to finite product) and when $M' \longrightarrow M$ is of "finite index", in which case there is a strong tendency for $h\sum(\ )$ and $h\prod(\ )$ to coincide. The defining property of these as adjoints are the natural bijections

$$\frac{h\sum X \longrightarrow Y}{X \longrightarrow Yh} \qquad\qquad \frac{T \longrightarrow h\prod X}{Th \longrightarrow X}$$

between M'-natural, respectively M-natural transformations, where Y, T are functors $M \longrightarrow \mathcal{X}$ (that is objects of the category $\mathcal{X}^M$ whose morphisms are the M-natural transformations) and X is a functor $M' \longrightarrow \mathcal{X}$ (that is an object of the category $\mathcal{X}^{M'}$ whose morphisms are M'-natural transformations). Since these refined "rules of inference" uniquely characterize the adjoints up to unique natural isomorphism, if $M'' \xrightarrow{k} M' \xrightarrow{h} M$ are two functors for each of which the two Kan adjoints exist, then from the associativity of substitution, $Y(hk) = (Yh)k$, follow the two rules

$$(hk)\sum Z \cong h\sum(k\sum Z)$$
$$(hk)\prod Z \cong h\prod(k\prod Z).$$

If M' is a discrete category $\mathcal{I}$ with I objects (and no morphisms except the identity morphisms) and if M is the single morphism category $\mathbf{1}$, then there is a unique functor $\mathcal{I} \longrightarrow \mathbf{1}$, often also called I and the Kan adjoints are just the coproduct and product functors respectively:

$$I\sum X = \textstyle\sum_{i\in I} X_i$$
$$I\prod X = \textstyle\prod_{i\in I} X_i$$

where a functor $I \xrightarrow{X} \mathcal{X}$ is just a family of objects. It is chiefly in regard to the existence of Kan extensions that questions of "largeness" and "smallness" enter category theory. The class of all categories $\mathcal{X}$ for which $h\sum(\ )$ and $h\prod(\ )$ exist in $\mathcal{X}$ can be called the "smallness" of $M' \xrightarrow{h} M$, while dually (in the sense of Galois connections) the class of all functors h for which these exist over a given $\mathcal{X}$ can be called the degree of "(bi) completeness" of $\mathcal{X}$, with obvious refinements for left completeness where only $\sum$ is considered and for right completeness where only $\prod$ is considered. Informally we may just say that $M' \xrightarrow{h} M$ is sufficiently small for $\mathcal{X}$ or that $\mathcal{X}$ is sufficiently complete for h when these constructions can be carried out.

Returning to the example of a "period", i.e. a surjective homomorphism $M' \xrightarrow{h} M$ from the additive group of time-translations to the circle group, the induced functor

$$\mathcal{X}^M \hookrightarrow \mathcal{X}^{M'}$$

is just the full inclusion, into the category of all $\mathcal{X}$-dynamical systems (continuous, autonomous), of the subcategory of those that happen to have period h. Then the construction $h \prod X$ just gives the part of X consisting of the h-periodic states. More precisely the following adjunction morphism (derived from the rule of inference for $\prod$)

$$(h \prod X)h \longrightarrow X$$

will typically be the inclusion of the h-periodic part.

[Of course there is also

$$X \longrightarrow (h \sum X)h$$

obtained by forcing the arbitrary dynamical system X into the h-periodic mold, with an accompanying collapse of states, whose detailed understanding depends on a detailed understanding of the "collapsing" or quotient process in $\mathcal{X}$. The quotient process is just in general $\mathbf{\Delta}_1^{\mathrm{op}} \sum(\ )$. Here $\mathbf{\Delta}_1^{\mathrm{op}}$ is the finite category $\boxed{E \leftleftarrows\!\!\rightarrow V}$ in which the two composites at V are both the identity (implying that the two composites at E are idempotents which absorb one another in a non-commutative way) and functors $\mathbf{\Delta}_1^{\mathrm{op}} \longrightarrow \mathcal{X}$ are often referred to as (reflexive) graphs in $\mathcal{X}$; $\mathbf{\Delta}_1$ itself can be concretely represented as the full category of the category of categories consisting of the two objects $V = \mathbf{1}$ and $E = \mathbf{2} = \boxed{0 \to 1}$ in which representation the two arrows $V \rightrightarrows E$ in $\mathbf{\Delta}_1$ are the two *adjoints* of the unique $E \longrightarrow V$. The reflexive graph in $\mathcal{X}$ arising from a period h (homomorphism of monoids) and a particular dynamical system X is just $\check{X}$ given by

$$\begin{tikzcd} E_h \cdot X \arrow[r, shift left] \arrow[r, shift right] & X \arrow[l, bend left] \end{tikzcd}$$

where E_h is the set of all pairs m_1', m_2' for which $h(m_1') = h(m_2')$ and $E_h \cdot X$ is the coproduct of E_h copies of X. The detailed properties of $\mathbf{\Delta}_1^{\mathrm{op}} \sum \check{X}$ depend sensitively on the nature of the category $\mathcal{X}$, usually in concrete examples more so than do the detailed properties of the dual construction $\mathbf{\Delta}_1 \prod \hat{X}$, where $\hat{X}$ is the $\mathbf{\Delta}_1 \longrightarrow X$ given by

$$\begin{tikzcd} X \arrow[r, shift left] \arrow[r, shift right] & X^{E_h} \arrow[l, bend left] \end{tikzcd}$$

(where X^{E_h} denotes the product of E_h copies of X, and where, as throughout this bracket, we have followed the usual abuse of notation of using the same letter X to denote also the object $X(0)$ of $\mathcal{X}$ that underlies the action of the monoid M'); thus $\mathbf{\Delta}_1 \prod \hat{X}$ is the $(h \prod X)h$ outside this bracket.]

The full period spectrum of a dynamical system $M' \xrightarrow{X} \mathcal{X}$ can be regarded as a single functor as follows. Say that a period h_2 divides a period h_1 if there exists an endomorphism q of the circle group M such that

$$\begin{tikzcd} & M' \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ M \arrow[rr, "q"] & & M \end{tikzcd} \qquad h_2 = qh_1$$

(then q is unique because h_1 is assumed surjective and q itself is surjective because h_2 is assumed to be). Denote by $\mathcal{Q}$ the category (actually a pre-ordered set) whose objects are the periods and whose morphisms are the q as indicated. By the definitions, if h_2 divides h_1 and if X is (part of) a dynamical system of period h_2, then it is (part of) a dynamical system of period h_1 as well:

$$\begin{tikzcd} & M' \arrow[r, "X"] \arrow[dl, "h_1"'] \arrow[d, "h_2"] & \mathcal{X} \\ M \arrow[r] \arrow[urr, dotted, "Y_1"] & M \arrow[ur, dotted, "Y_2"'] & \end{tikzcd} \qquad \exists Y_2 \Longrightarrow \exists Y_1$$

Similar reasoning shows that for any dynamical system X, q induces in a functorial way an inclusion

$$(h_2 \textstyle\prod X)h_2 \longrightarrow (h_1 \textstyle\prod X)h_1$$

of the h_2-periodic part of X into the h_1-periodic part of X, whenever q is the reason for h_2 dividing h_1. Thus we get a functor $\bar{X} : \mathcal{Q}^{\mathrm{op}} \longrightarrow \mathcal{X}$ (where $\bar{X}(h) = (h \prod X)h$) that in turn depends functorially on X so that $X \longmapsto \bar{X}$ defines the "periodic pre-spectrum" functor

$$\mathcal{X}^{M'} \xrightarrow{\bar{(\,)}} \mathcal{X}^{\mathcal{Q}^{\mathrm{op}}}$$

from dynamical systems in any sufficiently $\prod$-complete category $\mathcal{X}$ into the category of direct systems in $\mathcal{X}$ indexed by the poset $\mathcal{Q}$ of periods.

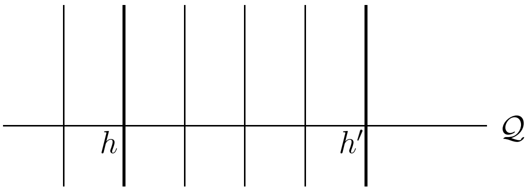
More closely corresponding to the usual notion of spectrum is the following: the weight attached to a given period h is not so much the space $(h \prod X)h$ of states having that period as it is the smaller space of orbits of such states, where in general the notion of orbit space is the left adjoint

$$\mathcal{X}^{M'} \xrightarrow{M' \sum (\,)} \mathcal{X}$$

to the functor induced by the unique $M' \longrightarrow \mathbf{1}$. Since for any h, the space $(h \prod X)h$ of h-periodic points is itself a dynamical system in its own right, hence combining these as h varies through $\mathcal{Q}$ we get a lifted pre-spectrum functor indicated by the dotted arrow below; the latter can be composed with the functor induced by the orbit space functor (upon parameterizing by $\mathcal{Q}$ its inputs and outputs):

$$\begin{tikzcd}
\mathcal{X}^{M'} \arrow[r, dotted] \arrow[rr, bend left, "\bar{\bar{(\ )}}"] \arrow[dr, "\bar{(\ )}"'] & (\mathcal{X}^{M'})^{\mathcal{Q}^{\mathrm{op}}} \arrow[r, "(M'\sum(\ ))^{\mathcal{Q}}"] \arrow[d] & \mathcal{X}^{\mathcal{Q}^{\mathrm{op}}} \\
 & \mathcal{X}^{\mathcal{Q}^{\mathrm{op}}} &
\end{tikzcd}$$

to yield what could be called the periodic spectrum $X \longmapsto \bar{\bar{X}}$. The periodic spectrum $\bar{\bar{X}}$ of a dynamical system X can in many cases be pictured as



where the darkness of the line at a period $h \in \mathcal{Q}$ is proportional to the size of the space $M \sum((h \prod X)h)$ of equivalence classes of states of period h, where two states are equivalent if the dynamical action moves one through the other.

There is one other point (not in $\mathcal{Q}$) which may also be considered part of the spectrum, namely the map $M' \longrightarrow \mathbf{1}$ whose corresponding $M' \prod X$ is the space of fixed states of the dynamical system X. If M' is a group, then the fixed point space is usually a subspace of the orbit space, for example if $\mathcal{X}$ is the category of sets and mappings. The same conclusion follows if M' is any commutative monoid. However, for the three element monoid (essentially equivalent, insofar as actions are concerned, to the category $\mathbf{\Delta}_1$ mentioned above) consisting of the morphisms 1, δ_0, δ_1 with the multiplication table $\delta_i \delta_j = \delta_i$ one can find examples of (right) actions $\mathbf{\Delta}_1^{\mathrm{op}} \xrightarrow{X} \mathcal{S}$ on sets (i.e. reflexive graphs) having any given number of fixed points (i.e. vertices) but only one orbit so that the map $\mathbf{\Delta}_1^{\mathrm{op}} \prod X \longrightarrow \mathbf{\Delta}_1^{\mathrm{op}} \sum X$ is not at all a monomorphism.

3. Enhanced algebraic structure in dynamics and logic

Incidentally, the above remark that both groups and commutative monoids share a property not true for all monoids can be made more explicitly algebraic by the following exercise. If $\mathcal{C}$ is a category that is *either* a group, i.e. every morphism in $\mathcal{C}$ has a two-sided inverse, *or* a commutative monoid, then $\mathcal{C}$ acts on its endomorphisms in the following way: for any morphism $X' \xrightarrow{a} X$ in $\mathcal{C}$, and for any endomorphism x of X, we can define an endomorphism x^a of X' such that

$$\boxed{xa = ax^a} \qquad \begin{tikzcd} X' \arrow[loop above, "x^a"] \arrow[r, "a"] & X \arrow[loop above, "x"] \end{tikzcd}$$

and moreover this is an action in the sense that $\boxed{1^a = 1}$ and

$$\boxed{x^{ab} = (x^a)^b} \qquad \begin{tikzcd} X'' \arrow[r, "b"] \arrow[loop above, dotted] & X' \arrow[r, "a"] \arrow[loop above] & X \arrow[loop above, "x"] \end{tikzcd}$$

and even an action by monoid homomorphisms in that

$$\boxed{(xy)^a = x^a y^a}$$
$$\boxed{1^a = 1}$$

for any two endomorphisms x, y of the codomain of a. Of course if $\mathcal{C}$ is itself a monoid then all its morphisms are endomorphisms, and if all morphisms in $\mathcal{C}$ are monomorphisms (a cancellation law) then there could be at most one operation $x, a \longmapsto x^a$ with the crucial property $xa = ax^a$. In the intersection of the two claimed cases, (i.e. for abelian groups) the two formulas for x^a reduce to the same (trivial) thing. If we restrict consideration to the full subcategory of monoids determined by groups and commutative monoids (i.e. the union of the two kinds of objects but containing all four types of homomorphisms between them) we get an example of a *natural operation* on the underlying-set functor that does not extend (from the full subcategory) to all monoids; note that the "algebraic structure" of a full subcategory of an equationally defined algebraic category may have additional operations as well as additional identities between the given operations. In our example the subcategory is the union (made full) of two full subcategories which are themselves equationally defined (in the sense that each consists of *all* algebras satisfying all the identities on all its natural operations). If we take the algebraic category equationally defined by the identities listed above, we get

$$\begin{tikzcd} & \text{Monoids with commutation rule} \arrow[dr, dotted] & \\ \mathrm{Gr} \cup \mathrm{Comm} \arrow[ur, dotted] \arrow[rr, hook, "\text{full}"] \arrow[dr, "\text{underlying set functor}"'] & & \text{Monoids} \arrow[dl] \\ & \text{Sets} & \end{tikzcd}$$

where the descending dotted arrow is a faithful functor which however does not reflect isomorphisms. That is, there exists a monoid (necessarily *not* satisfying the monomorphic cancellation rule) on which there exist two *different* self actions satisfying the commutation rule $xa = ax^a$. Of course, one interest for operations of this sort on a monoid $\mathcal{C}$ is the strong properties it implies for the category $\mathcal{S}^{\mathcal{C}^{\mathrm{op}}}$ of right actions on sets (or any Boolean topos $\mathcal{S}$) in particular with regard to the properties of the intrinsically defined "intuitionistic" negation operator defined on the sub-actions A of any action X by

$$\neg A = \{x \in X | \, \forall r \in \mathcal{C}[xr \notin A]\}.$$

Namely, if the monoid $\mathcal{C}$ admits a self-action with a commutation rule as above, then any non-empty A contained in $X = T$ (= the action of $\mathcal{C}$ on itself by right multiplication) satisfies $\neg\neg A = T$. By contrast, if $\mathcal{C}$ is so non-grouplike and non-commutative as the free monoid on two generators, then every "principal" $A = w\mathcal{C} \subset T$ satisfies $\neg\neg A = A$.

4. Functors from school mathematics mutate monoids into ordered sets, and back

Before passing to the discussion of non-autonomous dynamical systems let us point out a crucial example of a functor which occurs in school mathematics: suppose $x \leq y \leq z$ are non-negative integers or non-negative reals, then the differences $y - x$, $z - y$, $z - x$ are also non-negative and satisfy

$$\boxed{z - x = (z - y) + (y - x)}$$
$$\boxed{0 = x - x}$$

Even though this theorem is a very familiar and useful identity, it cannot be explained by either the monotonicity property nor as a homomorphism property in the usual sense, for in fact it is a structure-preserving property of a process (namely difference) that goes from a pre-ordered set to a monoid. As we have already pointed out, posets and monoids are on the face of it very "opposite" kinds of categories, thus it appears that once we have recognized the necessity for giving a rational status to something as basic as the difference operation discussed above, we are nearly compelled to accept the category of categories, since it is the only reasonable category broad enough to include objects as disparate as posets and monoids *and* hence to include the above difference operator as one of the concomitant structural mutations. To be perfectly explicit, let us denote the relation $x \leq y$ (in the poset of quantities in question) by f, and similarly $y \leq z$ by g. Define

$$\phi(f) = y - x$$

and similarly $\phi(g) = z - y$ and $\phi(1_x) = 1_0$ for any x where 1_x denotes $x \leq x$. 0 may be identified with (the identity morphism of) the unique object of the monoid of quantities where composition is addition. Then ϕ satisfies

$$\phi(gf) = \phi(g)\phi(f)$$
$$\phi(1_x) = 1_0$$

and hence is precisely a functor from a poset to a monoid. We can be still more explicit. In our example, what does $f : x \leq y$ mean? We could identify f with the proof that $x \leq y$ holds, that is with the non-negative quantity f such that $x + f = y$, or in other words with the morphism f in the monoid for which

$$\begin{tikzcd} & 0 \arrow[dl, "x"'] \arrow[dr, "y"] & \\ 0 \arrow[rr, "f"'] & & 0 \end{tikzcd}$$

commutes. That is f is a morphism in the so called "comma category" $0/\mathcal{C}$ where 0 is the unique object of the monoid $\mathcal{C}$. Of course, f is just the difference, so that ϕ is identified as the forgetful functor

$$\begin{tikzcd} 0/\mathcal{C} \arrow[d, "\phi"'] \\ \mathcal{C} \end{tikzcd}$$

well-defined for any comma category. Note that the comma category will be a poset only in case $\mathcal{C}$ satisfies a cancellation law.

This construction can also be applied to $\mathcal{P}$ = multiplicative monoid whose morphisms are positive whole numbers. Then $1/\mathcal{P}$ is isomorphic to the poset whose *objects* are positive whole numbers ordered by divisibility, and under that identification, the functoriality of the forgetful functor back to the monoid is expressed by

$$n|m \ \& \ m|r \implies \frac{r}{n} = \frac{r}{m} \cdot \frac{m}{n}.$$

There are non-trivial consequences of these observations, for any forgetful functor (such as $1/\mathcal{P} \longrightarrow \mathcal{P}$) on a comma category satisfies the "unique lifting of factorizations" property: $\phi(f) = vu$ implies there are unique $\bar{v}$, $\bar{u}$ such that $f = \bar{v}\bar{u}$, $\phi(\bar{v}) = v$, $\phi(\bar{u}) = u$. If $\mathcal{P}$ is a category satisfying a suitable local finiteness condition then we can define an algebra structure on the set $a(\mathcal{P})$ of all complex-valued functions on the set of morphisms of $\mathcal{P}$ by the convolution formula

$$(\beta * \alpha)(f) = \sum_{ba=f} \beta(b)\alpha(a).$$

Then the "unique lifting of factorizations" property of a functor $\mathcal{P}' \xrightarrow{\phi} \mathcal{P}$ is just what is needed to induce a convolution-preserving homomorphism $a(\mathcal{P}) \longrightarrow a(\mathcal{P}')$. In case $\mathcal{P}$ has cancellation, we thus get an inclusion of the Dirichlet algebra $a(\mathcal{P})$ into the algebra $a(1/\mathcal{P})$ associated to a *poset*; in particular the μ-function (defined as the $*$-inverse, when it exists, of the constantly 1 function) of P becomes the μ-function of the poset $1/\mathcal{P}$. Since the ordering in $1/\mathcal{P}$ is by divisibility, one thus sees how the functions μ, μ^2, etc. in $a(\mathcal{P})$ can be related to counting primes.

5. Non-autonomous systems, monoidal categories, Dedekind completeness, and the construction of the real numbers as an abstraction of dynamical waiting-times

Because right actions (= contravariant functors) of $\mathcal{C}$ are more directly related to the analysis of $\mathcal{C}$ itself (for by the Cayley-Dedekind-Grothendieck-Yoneda lemma, there is a canonical full embedding $\mathcal{C} \hookrightarrow \mathcal{S}^{\mathcal{C}^{\mathrm{op}}}$) we are led to repeat the above discussion also for the comma categories $\mathcal{C}/0 \longrightarrow \mathcal{C}$ whose morphisms are commutative triangles

$$\begin{tikzcd} \bullet \arrow[rr, dotted] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \\ & 0 & \end{tikzcd}$$

In case $\mathcal{C}$ is an additive monoid with cancellation, we would thus naturally write $x \geq y$ to denote the existence of a morphism $x \to y$ in $\mathcal{C}/0$. Since for any object X of a category of the form $\mathcal{S}^{\mathcal{C}^{\mathrm{op}}}$ (where $\mathcal{S}$ is the category of sets and $\mathcal{C}$ is any small category) there is an equivalence of categories

$$\mathcal{S}^{\mathcal{C}^{\mathrm{op}}}/X \cong \mathcal{S}^{(\mathcal{C}/X)^{\mathrm{op}}}$$

where $\mathcal{C}/X$ is the "category of elements" of X, we get in particular for a commutative and cancellative monoid $\mathcal{C}$ that

$$\boxed{\mathcal{S}^{\mathcal{C}^{\mathrm{op}}}/T \xrightarrow{\sim} \mathcal{S}^{(\mathcal{C}/0)^{\mathrm{op}}}}$$

where T is $\mathcal{C}$ acting on itself on the right and $\mathcal{C}/0$ is the poset described above. How is the equation in the box to be interpreted in terms of dynamical systems? Well, T is the simple autonomous dynamical system whose "states" reduce to just the instants of time themselves, and an object of the left hand category is just an arbitrary autonomous dynamical system X equipped with an equivariant morphism $X \longrightarrow T$; (of course many X, for example any periodic one, will not admit any such further structure $X \longrightarrow T$). The nature of the functor from left to right is just to consider the family of fibers (another instance of a $\prod$ construction) X_t of the given map $X \longrightarrow T$ as t varies through T, and whenever $t' \geq t$ the global dynamics of X induces a map $X_t \longrightarrow X_{t'}$, which completes the specification of a functor $(\mathcal{C}/0)^{\mathrm{op}} \longrightarrow \mathcal{S}$ corresponding to X. A natural interpretation of the objects of $\mathcal{S}^{(\mathcal{C}/0)^{\mathrm{op}}}$ is that they are *non-autonomous* dynamical systems, such as arise from the solution of ordinary differential equations which contain "forcing" terms or whose inertial or frictional terms depend on time in some manner (such as usury or heating) external to the self-interaction modeled by the differential equation itself. In general for a non-autonomous system the space of states X_t available at time t may itself depend on t. The (left) adjoint functor is, as already remarked, actually an equivalence of categories; it assigns to any non-autonomous system the single state space

$$\sum_{t \in T} X_t$$

with the single autonomous dynamics naturally induced by the non-autonomous dynamics given as $X_t \longrightarrow X_{t'}$ for $t' \geq t$. In case all the instantaneous state spaces are identifiable as a single Y

$$X_t \cong Y \quad \text{all} \quad t$$

then we see that the associated autonomous system is identifiable with

$$T \times Y$$

and this is just the universally-used construction for making a system autonomous: augment the state space by adding one dimension.

In case $\mathcal{C}$ is a commutative monoid, the category $\mathcal{C}/0$ becomes a "symmetric monoidal" category in the sense that there is a "tensor" *functor*

$$\mathcal{C}/0 \times \mathcal{C}/0 \longrightarrow \mathcal{C}/0$$

induced by the composition (which we will also write as $+$) and having the terminal object $\begin{smallmatrix}0\\ \downarrow\\ 0\end{smallmatrix}$ of $\mathcal{C}/0$ as "unit" object. In many cases this "tensor" has a right adjoint "Hom" $(\mathcal{C}/0)^{\mathrm{op}} \times (\mathcal{C}/0) \longrightarrow \mathcal{C}/0$ that can be naturally denoted by "subtraction"; it is characterized (assuming $\mathcal{C}$ has cancellation) by the logical equivalence

$$\frac{t + a \geq s}{t \geq s - a}.$$

In the fundamental example where $\mathcal{C}$ is the monoid of non-negative real (or rational) numbers, the meaning of "subtraction" is forced by this adjointness to be *truncated* subtraction. This adjointness persists after *completing*, as discussed below.

For a poset (using $\geq$ for $\rightarrow$) to be complete means that for any functor $M' \xrightarrow{h} M$, $h\sum(\ )$ exists in the poset (these are essentially *infima*) and also that $h\prod(\ )$ exists in the poset (these are essentially arbitrary *suprema*). To "complete" the poset of non-negative reals (or rationals) means roughly to adjoin (reals and) ∞, but since the precise meaning of this in terms of one-sided Dedekind cuts is sensitive to the precise nature of the internal cohesiveness/variation of the "sets" in $\mathcal{S}$, it is fortunate that there is a precise analysis of this process that goes back to the *monoid* $\mathcal{C}$. Namely, define $\mathrm{Pos} \hookrightarrow T$ to be the intersection of all the subdynamical systems P of T which are large enough so that, given any family $f(p) \in T$ indexed by $p \in P$ and satisfying $f(p) + t = f(p + t)$ for all $p \in P$, $t \in T$, there exists a unique $s \in T$ such that $f(p) = s + p$ for all $p \in P$. Then in favorable examples $\mathcal{C}$ is "continuous" (not necessarily complete) in the sense that

$$p \in \mathrm{Pos} \implies \exists p_1, p_2 \in \mathrm{Pos}[p = p_1 + p_2]$$

and Pos itself is the smallest such P. It is then reasonable to consider the subcategory $\mathcal{A} \xrightarrow{i_*} \mathcal{S}^{\mathcal{C}^{\mathrm{op}}}$ of "semicontinuous dynamical systems" defined to consist of those X for which every "possible future" Pos $\xrightarrow{f} X$ comes from a unique present state, or in other words for which the inclusion Pos $\hookrightarrow T$ induces a bijection $(T, X) \longrightarrow (\mathrm{Pos}, X)$ between the indicated sets of $\mathcal{C}$-equivariant (= natural) morphisms. Then the inclusion i_* has a left adjoint i^* that in turn has a left adjoint $\mathcal{A} \xrightarrow{i_!} \mathcal{S}^{\mathcal{C}^{\mathrm{op}}}$ including the "identical" category $\mathcal{A}$ as an "opposite" (to i_*) subcategory of $\mathcal{S}^{\mathcal{C}^{\mathrm{op}}}$. This implies that $\mathcal{A}$ is a topos whose truth value object $\Omega_{\mathcal{A}}$ has as elements all the $\mathcal{A}$-subobjects of T; essentially the same is true of $\mathcal{R} \underset{\mathrm{def}}{=} \mathcal{A}/i^*T$, the category of semicontinuous non-autonomous dynamical systems. But its truth values are also the subobjects of the terminal object 1_T because $\mathcal{R}$ is the topos of sheaves on a topological space $[0, \infty]$ topologized in such a way that there are as many open sets as points, with ∞ corresponding to the empty set (or truth value "false"). A dynamical system X in $\mathcal{R}$ has a real number $|X|$ as its support, namely $\inf\{t |\, X_t \neq 0\}$, and this construction can also be viewed as a functor

$$\mathcal{R} \longrightarrow \mathcal{V}$$

from the topos $\mathcal{R}$ to the poset $\mathcal{V} = [0, \infty]$ (the latter having $\geq$ as arrows) and both $\mathcal{R}$, $\mathcal{V}$ have tensor and Hom operations related to addition and truncated subtraction, that are compared by $\mathcal{R} \longrightarrow \mathcal{V}$. Because this discussion can be viewed as a deep objective version of Dedekind's constructions of $\mathcal{V}$, it appears that $\mathcal{R}$ as well as the $\mathcal{A}$ that gives rise to it as a comma category, should be taken seriously.

6. Metric spaces as enriched categories clarify rotations as "homotopy invariants"

It was extensively discussed in a 1973 seminar in Milan [8] that categories enriched in $\mathcal{V}$ are just metric spaces and hence that a detailed mutual clarification of enriched category theory [9] and metric space theory can be exploited. Continuing to take that remark seriously between 1973 and the Bogotá meeting of 1983 led me to several additional points of mutual clarification, that I will now explain. For a metric space A we have

$$\left.\begin{array}{c} 0 = A(a, a) \\ A(a, b) + A(b, c) \geq A(a, c) \end{array}\right. \qquad \text{in } \mathcal{V}$$

for any triple a, b, c of objects (points) of $\mathcal{A}$; in general as explained in the cited article, it is better, both for the theory and for the examples, not to insist on further axioms of finiteness or symmetry. $\mathcal{V}$-functors $A \longrightarrow B$ turn out to be just distance-non-increasing maps, and the $\mathcal{V}$-object of $\mathcal{V}$-natural transformations between two such f, g is easily proved to be

$$B^A(f, g) = \sup_{a \in A} B(fa, ga).$$

More general than $\mathcal{V}$-functors are the $\mathcal{V}$-modules (= $\mathcal{V}$-relations = profunctors) $A \xrightarrow{\phi} B$ which may be viewed as $\mathcal{V}$-functors $B^{\mathrm{op}} \times A \xrightarrow{\phi} \mathcal{V}$; the composition of such arising from the enriched category notion of Trace (= "coend" = tensor product of modules) can be shown in this case to reduce to

$$(\psi \circ \phi)(c, a) = \inf_{b \in B} [\psi(c, b) + \phi(b, a)]$$

for $A \xrightarrow{\phi} B \xrightarrow{\psi} C$.

Note that if ϕ, ψ happened to have only $\infty =$ false and $0 =$ true as values, then $\psi \circ \phi$ would reduce to the usual $\exists$, $\wedge$ composition of relations as a special case of the above "least-cost" composition that arises when all of $\mathcal{V} = [0, \infty]$ is admitted. This relationship can be made even more explicit as follows:

Let $\mathcal{V}_0$ be the two-object closed category $\boxed{\text{false} \to \text{true}}$ with conjunction as tensor and logical implication as internal Hom. Then the inclusion $\mathcal{V}_0 \hookrightarrow \mathcal{V}$ defined in the previous paragraph is actually a closed functor that has a right adjoint $\mathcal{V} \longrightarrow \mathcal{V}_0$ inducing the poset structure on any metric space; that exemplifies the kind of de-enrichment process which is a universal possibility in enriched category theory, giving the underlying ordinary category for categories enriched in any closed category $\mathcal{V}$. However, in our example there is moreover also a left adjoint

$$\mathcal{V} \xrightarrow{\pi_0} \mathcal{V}_0$$

to the inclusion that is also a closed functor. [I have called it π_0 because of the close analogy with other graphs of adjoint functors

$$\mathcal{E} \quad \begin{array}{c} \xrightarrow{\text{components}} \\ \xleftarrow{\text{discrete}} \\ \xrightarrow{\text{points}} \end{array} \quad \mathcal{E}_0$$

that occur, such as $\mathcal{E} =$ simplicial sets, $\mathcal{E}_0 =$ sets, and because of the tradition in topology of calling the components functor π_0 because it is sometimes part of a sequence in which the next term is the Poincaré groupoid π_1]. Because π_0 is closed, if we consider for any metric space A the relation defined by

$$\pi_0 A(a, b)$$

we get another (usually very coarse) preordering. This trivial construction is the key to a pedagogical problem as follows:

I wanted to give, for a beginning course in abstract algebra, the basic example of a normal subgroup and a quotient group.

$$\text{Translations} \hookrightarrow \text{Motions} \twoheadrightarrow \text{Rotations}$$

(where each point of the underlying space should give rise to a splitting and hence to a concrete representation of the abstract rotation group as a subgroup of motions, namely those motions which are rotations about the point in question). However, it is desirable to be able to make this basic construction before assuming detailed axioms on the structure of the underlying metric space A. Motions should of course be defined as invertible $\mathcal{V}$-functors f; these will then in particular be distance-preserving

$$A(fa, fb) = A(a, b).$$

But how to define "translations"? At first is seems reasonable to say that they are motions t which moreover satisfy $A(ta, a) =$ constant, for it is not hard to show that the set of such t is "normal", i.e.

$$A(ftf^{-1}a, a) = A(tf^{-1}a, f^{-1}a) = \text{same constant}.$$

But they may *not* form a sub*group* of the motions, for (theoretically) knowing nothing about the structure of A, how could we know which constant should result from composing t_1, t_2 having constants c_1, c_2? In this case the theoretical worry is substantiated by the practical fact that we can construct an example of a five point metric space, indeed embeddable in three-dimensional space as three equidistant points on the rim of a wheel and two judiciously placed points on the axis through the center of the wheel, such that the "translations" as defined are *not* closed under composition. But we shouldn't give up.

Define a translation of an arbitrary metric space A to be an automorphism t such that

$$\sup_{a\in A} A(ta, a) < \infty.$$

Now it is easy (assuming the metric is symmetric) to prove *both* parts of the statement that the translations form a normal subgroup of the motions, as desired. Even better, there are many examples of metric spaces A, such as the ordinary Euclidean plane, which have the property that every translation does in fact move all points through the same distance due to the "searchlight effect": if $A(ta, a) \neq A(tb, b)$ then $A(tx, x)$ can be arbitrarily large, for if there are enough strict translations we can assume that $ta = a$ and construct

$$\begin{tikzcd}
& b \arrow[dd] \arrow[r, dash, dashed] & x \arrow[dd] \\
a \arrow[ur, dash, dashed] \arrow[dr, dash, dashed] & & \\
& tb \arrow[r, dash, dashed] & tx
\end{tikzcd}$$

Why is the left adjoint to the inclusion

$$\{\text{false}, \text{true}\} \hookrightarrow [0, \infty]$$

the key to this problem? Because it (in contrast to the right adjoint, which seems to admit as truly possible only those projects that cost no effort) is given by

$$\pi_0(s) = \text{true} \quad \text{iff} \quad s < \infty$$

as is easily verified. Thus it appears we should take seriously the idea that the homotopy theory of metric spaces, that is the 2-functor

$$\mathcal{V}\text{-cat} \longrightarrow \mathcal{V}_0\text{-cat}$$

induced by π_0, is in large part the theory of rotations.

7. Convexity, Isbell conjugacy, closed balls, and radii in enriched categories

The Cayley-Dedekind-Grothendieck-Yoneda embedding

$$A \longrightarrow \mathcal{V}^{A^{\mathrm{op}}}$$

for $\mathcal{V}$-enriched categories A, reduces in the case $\mathcal{V} = [0, \infty]$ under discussion to the fact that $A(-, a)$ is a distance-non-increasing real function for any point of any metric space A, and that the sup-distance between two such functions is equal to the distance between the given points or, more generally, that if $A^{\mathrm{op}} \xrightarrow{f} \mathcal{V}$ is any distance-non-decreasing nonnegative real function (not necessarily of the special form indicated) on the opposite of the metric space A, then

$$\begin{array}{rl}
A(a', a) + fa \geq fa' & \quad \text{all } a' \\
\hline
fa \geq fa' - A(a', a) & \quad \text{all } a' \\
\hline
fa \geq \mathcal{V}(A(a', a), fa') & \quad \text{all } a' \\
\hline
fa \geq \sup\limits_{a'} \mathcal{V}(A(a', a), fa') & \\
\hline
fa \geq \mathcal{V}^{A^{\mathrm{op}}}(A(-, a), f) &
\end{array}$$

but the last inequality is *actually an equality* because the sup is achieved at $a' = a$.

Now in between we can insert the space of *closed subsets* of A

$$A \hookrightarrow \mathcal{F}(A) \hookrightarrow \mathcal{V}^{A^{\mathrm{op}}}$$

by assigning to each $F \hookrightarrow A$ the function

$$F(a') = \inf_{a \in F} A(a', a)$$

which vanishes on (by definition) the closure of F. The sup metric on $\mathcal{V}^{A^{\mathrm{op}}}$, restricted to $\mathcal{F}(A)$ is a refined non-symmetric version of the Hausdorff metric; that is, its symmetrization is the Hausdorff metric, but it itself reduces via $\mathcal{V} \longrightarrow \mathcal{V}_0$ to an ordering which reflects the *inclusion* of the closed sets

$$F_1 \subseteq F_2 \quad \text{as closed sets iff} \quad F_1 \geq F_2 \quad \text{in } \mathcal{V}^{A^{\mathrm{op}}}.$$

Since any $f \in \mathcal{V}^{A^{\mathrm{op}}}$ does have a zero-set, therefore a *right adjoint* to the inclusion

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, hook] & \mathcal{V}^{A^{\mathrm{op}}} \arrow[l, bend right, dotted, "Z"'] \end{tikzcd}$$

can be constructed. This leads to the idea that the objects in $\mathcal{V}^{A^{\mathrm{op}}}$ might be considered as "refined" closed sets; a point of view already forced upon researchers in constructive analysis and variational calculus by the stringent requirements of their proofs and calculations thus receives also a conceptual support from enriched category theory.

Now an extremely fundamental construction in enriched category theory is the adjoint pair known as *Isbell conjugation*

$$\mathcal{V}^{A^{\mathrm{op}}} \underset{(\ )^{\#}}{\overset{(\ )^{*}}{\rightleftarrows}} (\mathcal{V}^{A})^{\mathrm{op}}$$

which is defined in both directions by a similar formula

$$\begin{aligned} \xi^{*}(a) &= \mathcal{V}^{A^{\mathrm{op}}}(\xi, a) \\ \alpha^{\#}(a) &= \mathcal{V}^{A}(\alpha, \bar{a}) \end{aligned}$$

where we have followed the usual practice of letting the Yoneda lemma justify the abuses of notation

$$\begin{aligned} A(-, a) &= a \text{ in } \mathcal{V}^{A^{\mathrm{op}}} \\ A(a, -) &= \bar{a} \text{ in } \mathcal{V}^{A}. \end{aligned}$$

The general significance of this construction is somewhat as follows: if $\mathcal{V}$ is the category of sets, or simplicial sets, or (properly construed) topological spaces, or bornological linear spaces, and if the $\mathcal{V}$-category A is construed as a category of basic geometrical figures, then $\mathcal{V}^{A^{\mathrm{op}}}$ is a large category which includes very general geometrical objects that can be probed with help of A, but that would inevitably come up in a thorough study of A itself. On the other hand, $\mathcal{V}^{A}$ includes very general algebras of quantities whose operations (à la Descartes) mirror the geometric constructions and incidence relations in A itself. Then the conjugacies are the first step toward expressing the duality between space and quantity fundamental to mathematics: $(\ )^{*}$ assigns to each general space the algebra of functions on it, whereas $(\ )^{\#}$ assigns to each algebra its "spectrum" which is a general space. Of course neither of the conjugacies is usually surjective; the second step in expressing the fundamental duality is to find subcategories with reasonable properties that still include the images of the conjugacies:

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
\mathcal{X} \arrow[rr, shift left] \arrow[d, hook] & & \mathcal{A}^{\mathrm{op}} \arrow[ll, shift left] \arrow[d, hook] \\
\mathcal{V}^{A^{\mathrm{op}}} \arrow[rr, shift left] \arrow[urr, dotted] & & (\mathcal{V}^{A})^{\mathrm{op}} \arrow[ll, shift left] \arrow[ull, dotted]
\end{tikzcd}$$

For example if $\mathcal{V}$ is the category of sets and A is a small category with finite products, we could take $\mathcal{X}$ to be the *topos* of "canonical sheaves" on A, and $\mathcal{A}$ to be the *algebraic category* of all finite-product-preserving functors.

Our poset $\mathcal{V} = [0, \infty]$ appears rather puny compared to the grand examples mentioned in the previous paragraph, but seriousness eventually leads us to try the Isbell conjugacy on it as well, and in particular to ask which closed sets $F \in \mathcal{F}(A)$ are fixed by the composed Isbell conjugacies for a metric space A.

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, hook] & \mathcal{V}^{A^{\mathrm{op}}} \arrow[l, bend right, "Z"'] \arrow[loop above, "(\,)^{*\#}"] \arrow[r, shift left, "(\,)^*"] & (\mathcal{V}^A)^{\mathrm{op}} . \arrow[l, shift left, "(\,)^{\#}"] \end{tikzcd}$$

Note that from the adjointness, $\xi \geq \xi^{*\#}$ for all ξ in $\mathcal{V}^{A^{\mathrm{op}}}$, so that the idempotent operation $(\,)^{*\#}$ gives a kind of lower envelope for functions and hence a kind of hull $Z(\xi) \subseteq Z(\xi^{*\#})$ for the corresponding closed zero-sets; the question is what kind of hull?

To answer this it is relevant to explicitly introduce the following parameterized family of special elements of $\mathcal{V}^{A^{\mathrm{op}}}$:

$$\mathcal{V}^{\mathrm{op}} \otimes A \xrightarrow{B} \mathcal{V}^{A^{\mathrm{op}}}$$

defined by

$$B(r, c)(a') = \mathcal{V}(r, A(a', c)),$$

where $\mathcal{V}(x, y) = y - x$ is the truncated subtraction in our example $\mathcal{V} = [0, \infty]$; but for any closed category $\mathcal{V}$ denotes its internal Hom, and where the tensor product of $\mathcal{V}$-categories (defined in our case as the metric which is the *sum* of the coordinate distances) is used instead of the cartesian product because it guarantees that B itself is also a $\mathcal{V}$-functor. The letter B stands for *closed ball* of a given radius and center since

$$0 \geq B(r, c)(a') \quad \text{iff} \quad r \geq A(a', c).$$

An amazing example of the seriously-pursued study of the mutual relationship of a key example with general philosophy is that these "closed balls" occur and are useful over many apparently quite diverse closed categories $\mathcal{V}$, for example in homological algebra. Moreover, let us denote by $\bigcap$ the supremum operation on $\mathcal{V}^{A^{\mathrm{op}}}$

$$\left(\bigcap \xi_i\right)(a') = \sup_i \xi_i(a')$$

since that is what it corresponds to under the operation Z of taking zero-sets.

With the above-introduced notation, we see that $\xi = \xi^{*\#}$ iff $\xi^{*\#} \geq \xi$, and also that

$$\begin{aligned} \alpha^{\#}(a'') &= \mathcal{V}^A(\alpha, \bar{a}'') \\ &= \sup_{a'} \mathcal{V}(\alpha(a'), A(a'', a')) \end{aligned}$$

so that

$$\alpha^{\#} = \bigcap_{a'} B(\alpha(a'), a')$$

is the intersection of all the closed balls, centered at all points a', of specified radius $\alpha(a')$. [Naturally this construction, under the name of "end" or "center", also comes up for general $\mathcal{V}$]. Now what if α is of the form ξ^*? The number $\xi^*(a')$ is a radius (about an arbitrary center a') that ξ somehow prefers:

$$\xi^*(a') = \sup_a \mathcal{V}(\xi(a), A(a, a'))$$

so that

$$\begin{aligned} r \geq \xi^*(a') \quad &\text{iff} \quad r + \xi(a) \geq A(a, a') \quad \text{for all } a \\ &\text{iff} \quad \xi(a) \geq A(a, a') - r \quad \text{for all } a \\ &\text{iff} \quad \xi \geq B(r, a'). \end{aligned}$$

Thus (since throwing some larger balls into the family won't change the intersection)

$$\xi^{*\#} = \bigcap \{B(r, a') \,|\, \xi \geq B(r, a')\}$$

is a geometrical description of the double-conjugate lower envelope of ξ. In the case where ξ is fixed under the other idempotent operation on $\mathcal{V}^{A^{\mathrm{op}}}$ coming from Z, i.e. if ξ is determined by its closed zero-set F as explained before

$$\xi(a'') = \inf_{a \in F} A(a'', a)$$

then the condition $\xi \geq B(r, a')$ reduces to

$$\begin{aligned} \forall a \in F \; \forall a'' \quad A(a'', a) &\geq B(r, a')(a'') \\ &= A(a'', a') - r \end{aligned}$$

i.e. to

$$A(a'', a) + r \geq A(a'', a') \quad \text{for} \quad a \in F, \text{ arbitrary } a''.$$

In particular this means that $F \subseteq$ the ball (= zero set of) $B(r, a')$, so we see that

$$F \subseteq F^{*\#} \subseteq \text{the intersection of all closed balls that contain } F.$$

But in fact that intersection of balls is equal to $F^{*\#}$, since the right adjointness of Z says in particular that for any closed set F and any ball we have the equivalence

$$\frac{F \subseteq ZB(r, a')}{F \geq B(r, a')}.$$

Putting it differently, since we have in general that

$$A(a'', a) + A(a, a') \geq A(a'', a'),$$

if $a \in F$ implies $0 \geq B(r, a')(a)$, i.e. $r \geq A(a, a')$, then for any a'' and any $a \in F$

$$A(a'', a) + r \geq A(a'', a) + A(a, a') \geq A(a'', a')$$

so that

$$A(a'', a) \geq B(r, a')(a'')$$

meaning that

$$F(a'') \geq B(r, a')(a'') \quad \text{as functions of } a'',$$

because the right hand side is independent of a and the function F was defined as the *infimum* (adjointness of "$\sum$ along a diagonal"). Thus

$$\begin{aligned} F^{*\#} &= \text{intersection of all closed balls which contain } F \\ &= \text{closed convex hull of } F, \end{aligned}$$

at least in many metric spaces of geometric importance, and for the others the proposed use of the term "convex" for those closed sets F satisfying $F = F^{*\#}$ should be as good or better than other proposals because of the apparent importance of adjointness in calculations.

As noticed above, the numbers $\xi^*(a')$ have in many connections the concrete significance of radii for balls about a'. Using the obvious "direct limit" functor $\mathcal{V}^A \xrightarrow{\inf} \mathcal{V}$ (which exists because $A \longrightarrow 1$ is a $\mathcal{V}$-functor), we can define a single "radius" for ξ itself by applying the composite

$$\begin{tikzcd} \mathcal{V}^{A^{\mathrm{op}}} \arrow[r, "(\,)^*"] \arrow[rr, bend right, "\mathrm{rad}"'] & (\mathcal{V}^A)^{\mathrm{op}} \arrow[r, "\inf^{\mathrm{op}}"] & \mathcal{V}^{\mathrm{op}} \end{tikzcd}$$

In particular, the radius of a closed set $F \in \mathcal{F}(A) \subseteq \mathcal{V}^{A^{\mathrm{op}}}$ is

$$\mathrm{rad}(F) = \inf\{r \mid \exists a' \, [F \subseteq ZB(r, a')]\}$$

where the candidate centers a' are not themselves necessarily in F. (Note that to say we have a functor $\mathcal{F}(A) \longrightarrow \mathcal{V}^{\mathrm{op}}$ from a poset to $\mathcal{V}^{\mathrm{op}}$ is to say that the values *increase* as the objects in $\mathcal{F}(A)$ increase). The radius is a more functorial quantity than the habitually-used "diameter"; for a symmetric metric space there is the estimate diam ≤ 2 rad whereas for certain reasonably well behaved spaces one may also have a converse estimate rad $\leq$ diam. There is a strong tendency for the radius to be realized at a *unique* center a'. Note that $\mathrm{rad}(F^{*\#}) = \mathrm{rad}(F)$.

8. Geodesic remetrization as an adequacy comonad

Now let us say a few words about the important role of *paths* in metric spaces. The comma categories $d/\mathcal{V} = [0, d]$ with their canonical $d/\mathcal{V} \hookrightarrow \mathcal{V}$ have a retraction given again by "double dualization":

$$x \longmapsto \mathcal{V}(\mathcal{V}(x, d), d) = d - (d - x)$$

is always in the interval; moreover single d-dualization is an invertible duality

$$(d/\mathcal{V}) \rightleftarrows (d/\mathcal{V})^{\mathrm{op}}$$

when restricted, provided $d < \infty$. Denote by $\mathcal{V}' \subset \mathcal{V}$ all those d such that $\pi_0(d) = \text{true}$, i.e. for which $d < \infty$. Let $\mathcal{V}(d)$ be the *symmetric* metric space determined by $d/\mathcal{V}$, so that in particular $d - (\ )$ becomes (for $d < \infty$) a self-motion of the interval $\mathcal{V}(d)$, because symmetrizing is a functor. Indeed, $d \longmapsto \mathcal{V}(d)$ defines a functor

$$\mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{V}\text{-cat}$$

using $d' \geq d \Longrightarrow \mathcal{V}(d) \subseteq \mathcal{V}(d')$, but also a functor $\mathcal{V}' \longrightarrow \mathcal{V}$-cat by invoking the retractions. The essential question we want to understand is: to what extent is the structure of an arbitrary metric space $A \in \mathcal{V}$-cat analyzable in terms of the paths (= $\mathcal{V}$-functors)

$$\mathcal{V}(d) \longrightarrow A$$

of duration d, with d variable? Since all constants are paths, such analysis easily maintains the points of A. If there is such a path, passing a_0 at time 0 and a_1 at time d, then

$$d \geq A(a_0, a_1).$$

This leads to the idea of *geodesic distance*:

$$(\Gamma A)(a_0, a_1) = \inf\{d \in \mathcal{V}' \,|\, \exists\, \sigma : V(d) \longrightarrow A \text{ with } \sigma(0) = a_0, \sigma(d) = a_1\}$$

which can be seen to be a new metric since

$$\begin{array}{ccc}
\mathcal{V}(0) & \longrightarrow & \mathcal{V}(d') \\
\downarrow & & \downarrow \\
\mathcal{V}(d) & \hookrightarrow & \mathcal{V}(d+d')
\end{array}$$

is a pushout in $\mathcal{V}$-cat. That is, if σ is a path in A of duration d, and σ' of duration d', and if

$$\sigma(d) = \sigma'(0) = a_1$$

we must show that if $d \geq t$, $d + d' \geq s \geq d$, then

$$A(\sigma(t), \sigma'(s-d)) \overset{?}{\leq} s - t.$$

But we have

$$\begin{aligned}
A(\sigma(t), \sigma'(s-d)) &\leq A(\sigma(t), a_1) + A(a_1, \sigma'(s-d)) \\
&= A(\sigma(t), \sigma(d)) + A(\sigma'(0), \sigma'(s-d)) \\
&\leq d - t + s - d = s - t
\end{aligned}$$

because $d \geq t$ and $s \geq d$. The other cases of the $\mathcal{V}$-functorality of $(d + d') \xrightarrow{\sigma' * \sigma} A$ are obvious. Thus (using the logicians' symbol for "proves")

$$\left.\begin{array}{r} \sigma \Vdash d \geq \Gamma A(a_0, a_1) \\ \sigma' \Vdash d' \geq \Gamma A(a_1, a_2) \end{array}\right\} \Longrightarrow \sigma' * \sigma \Vdash d + d' \ \geq \ \Gamma A(a_0, a_2).$$

Because direct limits in metric spaces are essentially computable in terms of direct limits of sets and infima of distances, it can be seen that

$$\Gamma A = \varinjlim_{\sigma \in \mathcal{V}'/A} \mathcal{V}(\operatorname{dom} \sigma)$$

is an endofunctor of $\mathcal{V}$-cat having a natural distance-non-increasing map

$$\begin{array}{c} \Gamma A \\ \downarrow \\ A \end{array}$$

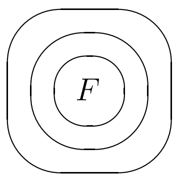
back to the identity functor; it is in fact the "adequacy comonad" of $\mathcal{V}' \longrightarrow \mathcal{V}$-cat, a notion defined for any small subcategory of any category having direct limits. Note for example that path-connectedness of A becomes π_0-connectedness of ΓA, for if two points of A are not connectable by a path, then (empty inf) their geodesic distance in ΓA is infinite.

9. History still has much to teach, and raises fresh questions

Finally, I believe that we should take seriously the historical precursors of category theory, such as Grassmann, whose works contain much clarity, contrary to his reputation for obscurity. For example, I read there a statement of the sort "diversity can be added" whereas "unity can be multiplied" together with quite convincing geometrical and algebraic substantiation of these principles. The first of them suggests the following:

If $\mathcal{F}(A)$ is a *poset* of parts of a space and $\mathcal{C}$ is a suitable additive *monoid*, then the amount by which F must be extended to achieve a diverse $G \supseteq F$ might be given by $\mu(F, G) \in \mathcal{C}$; that should again give a functor $\mathcal{F}(A) \longrightarrow \mathcal{C}$ in that

$$F \subseteq G \subseteq H \Longrightarrow \mu(F, H) = \mu(F, G) + \mu(G, H).$$



Of course, if $0 \in \mathcal{F}(A)$ and if $\mathcal{C}$ has cancellation then $\mu(G, H)$ will be determined by $\mu H \underset{\text{def}}{=} \mu(0, H)$ and μF. To further express the "quantitative" nature of such a measurement of extension μ, we can consider the further condition that μ "only depends on the difference". Since the crucial property of difference is again that it is a Hom, adjoint this time to union as $\otimes$,

$$\frac{S \supseteq H \setminus G}{G \cup S \supseteq H}$$

we can express this invariance of the functor μ using only the union structure on $F(A)$:

$$\{S|\, G \cup S \supseteq H\} = \{S|\, G_1 \cup S \supseteq H_1\} \Longrightarrow \mu(G, H) = \mu(G_1, H_1).$$

It would be interesting to determine for which upper semilattices $\mathcal{F}$, 0, $\cup$ there exists a commutative monoid $\mathcal{C}$ and a functor μ with this invariance property which moreover has the "unique-lifting-of-factorizations" property previously discussed, i.e., for which the object X in a process of intermediate expansion $F \subseteq X \subseteq H$ is uniquely determined by sufficiently many quantitative measurements of its size. For example, area alone is not sufficient but $\mathcal{C}$ can be a cartesian product of many different kinds of quantities. Again adjointness makes at least an initial contribution to the problem: for each $\mathcal{F}$ there is a well-defined universal $\mathcal{C}$ and μ, so that one need only study the lifting question for that.

10. Dialectical relation between teaching and research can be exemplified by metrical development of enriched category theory

We have seen that the application of some simple general concepts from category theory leads from a clarification of basic constructions on dynamical systems to a construction of the real number system with its structure as a closed category; applied to that particular closed category, the general enriched category theory leads inexorably to embedding theorems and to notions of Cauchy completeness, rotation, convex hull, radius, and geodesic distance for arbitrary metric spaces. In fact, the latter notions present themselves in such a form that the calculations in elementary analysis and geometry can be explicitly guided by the experience that is concentrated in adjointness. It seems certain that this approach, combined with a sober appreciation of the historical origin of all notions, will apply to many more examples, thus unifying our efforts in the teaching, research, and application of mathematics.

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