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HOMOTOPY IS NOT CONCRETE

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Theorem:

Let $\mathcal{T}$ be a category of base-pointed topological spaces including all finite-dimensional CW*-complexes. Let $T : \mathcal{T} \to \mathcal{S}$ be any set-valued functor that is homotopy-invariant. There exists $f : X \to Y$ such that f is not null-homotopic but $T(f) = T(\star)$, where $\star$ is null-homotopic.*

Corollary:

Let κ be any cardinal number. There exist finite-dimensional CW*-complexes X, Y and a map $f : X \to Y$ not null-homotopic but such that $f|X'$ is null-homotopic whenever $X' \subset X$ is a sub-*CW*-complex with fewer than κ cells.*

The corollary follows from the theorem as follows: let Z be the wedge[1] of all CW-complexes with fewer than κ cells. The theorem says that there must exist $f : X \to Y$ not null-homotopic such that $[Z, X] \xrightarrow{[Z, f]} [Z, Y]$ is constant. For any $X' \subset X$ with fewer than κ cells there exists $X' \to Z \to X' = 1_{X'}$, from which we may conclude that $[X', X] \xrightarrow{[X', f]} [X', Y]$ is constant and in particular that $f|X'$ is null-homotopic.

Let $\mathcal{H}$ be the HOMOTOPY CATEGORY obtained from $\mathcal{T}$. Its objects are the objects of $\mathcal{T}$, its maps are homotopy-classes of maps. The theorem says that $\mathcal{H}$ may not be faithfully embedded in the category of sets—or in the language of Kurosh— $\mathcal{H}$ is not CONCRETE. There is no interpretation of the objects of $\mathcal{H}$ so that the maps may be interpreted as functions (in a functorial way, at least). $\mathcal{H}$ has always been the best example of an abstract category, historically and philosophically. Now we know that it was of necessity abstract, mathematically.

The theorem says a bit more: $\mathcal{H}$ has a ZERO-OBJECT, that is, an object 0 such that for any X there is a unique $0 \to X$ and a unique $X \to 0$, and consequently for any X, Y a unique $X \to 0 \to Y$, the ZERO-MAP from X to Y.

We shall shortly restrict our attention to zero-preserving functors between categories with zero. Instead of functors into the category of sets, $\mathcal{S}$ we'll consider functors into the category of base-pointed sets $\mathcal{S}_\star$ and only those functors that preserve zero. But first:

Proposition:

If $\mathcal{C}$ is a category with zero and $T : \mathcal{C} \to \mathcal{S}$ any functor, then there exists a zero-preserving functor $T_\star : \mathcal{C} \to \mathcal{S}_\star$ such that

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[1] "Wedge" is the topologist's word for the coproduct in $\mathcal{T}$.

for all $f, g : A \to B$ in $\mathcal{C}$ it is the case that $T(f) = T(g)$ iff $T_\star(f) = T_\star(g)$

Proof:

Let $\mathcal{G}$ be the category of abelian groups and $F : \mathcal{S} \to \mathcal{G}$ the functor that assigns free groups. F is faithful, hence $T(f) = T(g)$ iff $FT(f) = FT(g)$. Let Z be the constant functor valued $FT(0)$. There exist transformations $Z \to FT \to Z = 1_Z$, and FT splits as $Z \oplus Z'$ (remember that the category of functors from $\mathcal{G}$ to $\mathcal{G}$ is an abelian category). Z' preserves zero and $FT(f) = FT(g)$ iff $Z'(f) = Z'(g)$. Finally, let $U : \mathcal{G} \to \mathcal{S}_\star$ be the forgetful functor and define $T_\star = UZ'$. ∎ [2]

In light of this proposition the main theorem is equivalent with

For any zero-preserving functor $T : \mathcal{H} \to \mathcal{S}_\star$, there exists f in $\mathcal{H}$ such that $f \neq \star$ and $T(f) = \star$.

$\mathcal{H}$ is thus worse than non-concrete: not only must any $T : \mathcal{H} \to \mathcal{S}_\star$ confuse two distinct maps, it must confuse two maps one of which is a zero-map. Such failure to be concrete is easier to work with than the more general. We will say that a functor $T : \mathcal{C} \to \mathcal{S}_\star$ is FAITHFUL-AT-ZERO if $T(f) = \star$ iff $f = \star$ and $\mathcal{C}$ is CONCRETE-AT-ZERO if there exists faithful-at-zero $T : \mathcal{C} \to \mathcal{S}_\star$.[3] We wish to show that $\mathcal{H}$ is not concrete-at-zero; we shall isolate a property that any concrete-at-zero category must possess and then demonstrate its failure in $\mathcal{H}$.

We shall work for a while in an arbitrary category $\mathcal{C}$ with zero. Given $A \in \mathcal{C}$ we may define an equivalence relation on the monomorphisms into A as follows: $(B_1 \to A) \equiv (B_2 \to A)$ if there exists an isomorphism $B_1 \to B_2$ such that $B_1 \to B_2 \to A = B_1 \to A$.

A SUBOBJECT of A is defined to be an equivalence class of monomorphisms. A kernel of a map $A \to B$ is usually defined as a monomorphism into A satisfying the well-known universal property. We note here that "the" kernel of $A \to B$ may be defined as a subobject, removing completely the ambiguity. (Every monomorphism in the equivalence class must of necessity be a kernel.) A NORMAL SUBOBJECT is one that appears as a kernel. The following will be a corollary of a later theorem:

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[2]There are two other lemmas of a somewhat similar nature that should, perhaps, be pointed out here. I had assumed—until I learned otherwise—that each of these lemmas went without saying. The first is that we can easily replace any pointed-set-valued functor with an equivalent functor that is an injection as far as objects go: given T define T' by $T'(A) = T(A) \times \{A\}$. The second is that we can replace any set-valued "pre-functor" (one that preserves composition but not identities) with a functor: given T define T' with the slogan:

$$T'(A \xrightarrow{f} B) = \mathrm{Image}(T(1_A)) \xrightarrow{T(f)} \mathrm{Image}(T(1_B)).$$

Actually, I did know that this last lemma needed saying: when, at the original exposition, I explained what I meant by "homotopy is not concrete" the most honored member of the audience interrupted with a putative faithful functor. At that time, the best I could do was point out that the putative functor did not preserve identity maps. Later, when I was writing the paper, I decided to forget the whole thing.

[3]To see that not being concrete-at-zero is, indeed, strictly worse than not being concrete, take any non-concrete category $\mathcal{C}$ and formally adjoin a zero object. (To the objects add a new object, 0, and for each ordered pair of objects, old or new, A, B add a new map, $A \xrightarrow{0} B$.) There's an obvious faithful-at-zero functor to the category of pointed-sets-with-at-most-two-elements. (Obvious or not it's what the construction below yields: every old object has exactly two GNSs.)

If $\mathcal{C}$ is concrete-at-zero then each of its objects has only a set of normal subobjects. Moreover, if every map in G has a kernel then the converse holds.

This theorem—as it stands—is not useful for $\mathcal{H}$. There are very few kernels, indeed there are very few monomorphisms in $\mathcal{H}$. We therefore introduce another equivalence relation, this time on *all* the maps into a fixed object A.

$(X \to A) \equiv (X' \to A)$ if they kill the same maps coming out of A, that is, if for all $A \to Y$ it is the case that $X \to A \to Y = \star$ iff $X' \to A \to Y = \star$. We shall call the equivalence classes GENERALIZED NORMAL SUBOBJECTS, of A. We'll abbreviate the notion as GNS.[4] The connection with normal subobjects is this:

Proposition:

If each map in $\mathcal{C}$ has a kernel and a cokernel then each GNS *uniquely contains a unique normal subobject.*

Proof:

One may first check that two normal monomorphisms are equivalent in the previous (the "subobject" sense) iff they are equivalent in the new sense (the "generalized normal subobject" sense). If $\mathcal{C}$ has kernels and cokernels then given $f : X \to A$ we note that $\mathrm{Ker}(\mathrm{Cok}(f))$ is equivalent to f. ∎

Theorem:

$\mathcal{C}$ is concrete-at-zero iff each object has only a set of generalized normal subobjects.

Proof:

If $TX \to TA$ and $TX' \to TA$ have the same image and if $T : \mathcal{C} \to \mathcal{S}_\star$ is faithful-at-zero then necessarily $(X \to A) \equiv (X' \to A)$, hence there could not be more GNSs of A then there are subsets of TA.

For the converse, define $T : \mathcal{C} \to \mathcal{S}_\star$ by letting TA be the set of GNSs of A. Given $f : A \to B$ note that if $(X \to A) \equiv (X' \to A)$ then $(X \to A \to B) \equiv (X' \to A \to B)$; thus $A \to B$ induces a function $TA \to TB$, clearly seen to be functorial. If $TA \to TB$ were constant then $(A \xrightarrow{1} A \to B) \equiv (0 \to A \to B)$ and from $A \xrightarrow{1} A \to B = \star$ iff $0 \to A \to B = \star$ we conclude that $A \to B = \star$. ∎

The previous assertion for categories in which each map has a kernel: the fact that concreteness-at-zero is equivalent with each object having only a set of normal subobjects may be seen by looking at the dual category, $\mathcal{C}^\circ$, and noticing that in general $(X \to A) \equiv (X' \to A)$ iff $\mathrm{C}ok(X \to A) = \mathrm{C}ok(X' \to A)$. This would yield $T : \mathcal{C}^\circ \to \mathcal{S}_\star$. However, the contravariant functor represented by the two-point set is faithful and we would obtain $\mathcal{C} \to \mathcal{S}_\star$.

A WEAK-KERNEL of $A \to Y$ is a map $X \to A$ such that

WK1: $X \to A \to Y = \star$.

WK2: If $Z \to A \to Y = \star$ then there exists $Z \to X$ such that $Z \to X \to A = Z \to A$.

$$\begin{array}{ccc} Z & & \\ \downarrow & \searrow & \\ & & A \\ & \nearrow & \\ X & & \end{array}$$

(no uniqueness condition).

If both $X \to A$ and $X' \to A$ are weak-kernels (of possibly different things) then $(X \to A) \equiv (X' \to A)$ iff there exist

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow & \\ & & A \\ & \nearrow & \\ X' & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X' & & \\ \downarrow & \searrow & \\ & & A \\ & \nearrow & \\ X & & \end{array}$$

by direct application of the definitions of $\equiv$ and weak-kernels.

In $\mathcal{H}$ we have many weak-cokernels (indeed, the suspension of any map is always such). We are directed, therefore, to look

[4] The term I used in 1970 was "abstract normal subobject". The reason for changing it here is given below in Footnote 10.

at the dual side, keeping in mind that a contravariant faithful-at-zero $T : \mathcal{H} \to \mathcal{S}_\star$ yields a covariant functor if followed by the faithful $(-, 2)$. We wish to find a space A, a proper class of maps $\{A \to X_i\}_{i \in I}$ all of which are weak-cokernels such that for $i \neq j$ not both

$$\begin{tikzcd} & X_j \arrow[d] \\ A \arrow[ur] \arrow[r] & X_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X_i \arrow[d] \\ A \arrow[ur] \arrow[r] & X_j \end{tikzcd}$$

exist.[5]

From the theory of abelian groups:

<u>Lemma</u>:

For any prime integer p there exists a family of p-primary torsion abelian groups $\{G_\alpha\}$, for α running through the ordinal numbers and for each G_α a special element $x_\alpha \in G_\alpha$ with the properties that $x_\alpha \neq 0$, $px_\alpha = 0$ and for any homomorphism $f : G_\beta \to G_\alpha$ with $\beta > \alpha$ it is the case that $f(x_\beta) = 0$.

<u>Proof</u>:

We recall the theory of "height" in torsion groups. Let $\mathcal{G}_p$ be the category of p-primary torsion abelian groups, let I be its identity functor. For each ordinal α we define a subfunctor inductively by:

$I_0 = I$;

$I_{\alpha+1} = \mathrm{Image}(I_\alpha \xrightarrow{p} I_\alpha)$;

$I_\alpha = \bigcap_{\beta<\alpha} I_\beta$ for α a limit ordinal.

We must show that this descending sequence continues to descend forever. Given α we shall find G_α such that $I_\alpha(G_\alpha) \neq 0$, $I_{\alpha+1}(G_\alpha) = 0$. By letting x_α be a non-zero element in $I_\alpha(G_\alpha)$ we will achieve the announced end, because if $\beta > \alpha$ then $I_\beta(G_\beta) \to I_\beta(G_\alpha)$ but $I_\beta(G_\alpha) \subset I_{\alpha+1}(G_\alpha) = 0$.

Given α let W_α be the set of finite words of ordinals $\langle \gamma_1\gamma_2 \cdots \gamma_n \rangle$ where $\gamma_1 < \gamma_2 < \cdots < \gamma_n \leq \alpha$, including the empty word $\langle\rangle$. Let G_α be the group generated by W_α subject to the relations $p\langle \gamma_1\gamma_2 \cdots \gamma_n \rangle = \langle \gamma_2 \cdots \gamma_n \rangle$ and $\langle\rangle = 0$. Then G_α is p-primary torsion. Note that every non-zero element in G_α is expressible uniquely as something of the form $a_1w_1 + a_2w_2 + \cdots + a_nw_n$, where $0 < q_i < p$, $w_i \in W_\alpha - \langle\rangle$. We may then show, inductively, that $I_\gamma(G_\alpha)$ is generated by elements of the form $\langle \gamma_1\gamma_2 \cdots \gamma_n \rangle$ where $\gamma \leq \gamma_1$. Hence $I_\alpha(G_\alpha)$ is isomorphic to $\mathbb{Z}_p$, the cyclic group with p elements, and $I_{\alpha+1}(G_\alpha) = O$. ■ [6]

Let $M(G)$ denote the Moore space, $H_1(M(G)) \simeq G$.[7] Choose a prime p, a generator x for $H_1(M(\mathbb{Z}_2))$ and for each ordinal α a map $f_\alpha : H_1(M(\mathbb{Z}_2)) \to H_1(M(\mathcal{G}_\alpha))$ such that $(H_1(f_\alpha))(x) = x_\alpha$. We use Σ to denote the suspension functor.

For $\beta > \alpha$ there is no

$$\begin{tikzcd} & \Sigma M(G_\beta) \arrow[d] \\ \Sigma M(\mathbb{Z}_p) \arrow[ur, "\Sigma f_\beta"] \arrow[r, "\Sigma f_\alpha"'] & \Sigma M(G_\alpha) \end{tikzcd}$$

because application of H_2 would contradict the choice of x_α, G_α. Each Σf_α is a weak-cokernel. Hence each Σf_α represents a different generalized normal quotient object. Hence $\Sigma M(\mathbb{Z}_2)$ has more than a set of generalized normal quotient objects. Hence $\mathcal{H}$ is not concrete-at-zero.

We may be more specific: for any $n > 0$

[5]Going to the dual is not needed. The maps into A for which these maps out of A are weak cokernels would necessarily represent different GNSs.

[6]For a real proof see the first addendum below.

[7]This material was written for a crowd of topologists. The second addendum below is an attempt to broaden the audience. (John Moore, as it happened, was in the original audience.)

consider the mapping-cone sequence

$$\Sigma^{n-1}M(\mathbb{Z}_p)\xrightarrow{\Sigma^{n-1}f_\alpha}\Sigma^{n-1}M(G_\alpha)$$

$$\xrightarrow{\Sigma^{n-1}f'_\alpha}\Sigma^{n-1}M(G_\alpha/\mathbb{Z}_p)\xrightarrow{\Sigma^{n-1}f''_\alpha}$$

$$\Sigma^{n}M(\mathbb{Z}_p)\xrightarrow{\Sigma^{n}f_\alpha}\Sigma^{n}M(G_\alpha)$$

(The mapping cone of $M(\mathbb{Z}_p) \to M(G_\alpha)$ is a Moore space because $\mathbb{Z}_p \to G_\alpha$ is a monomorphism.) For $\beta > \alpha$ the composition $\Sigma^n f_\alpha \Sigma^{n-1} f''_\beta$ is not null-homotopic.

Let $T : \mathcal{H} \to \mathcal{S}_\star$ be any functor. Let $\beta > \alpha$ be such that $T(\Sigma^{n-1}f''_\beta)$ and $T(\Sigma^{n-1}f''_\alpha)$ have the same image in $T(\Sigma^n M(\mathbb{Z}_p))$. Then because $T(\Sigma^n f_\alpha \Sigma^{n-1} f''_\alpha) = \star$ it must be the case that $T(\Sigma^n f_\alpha \Sigma^{n-1} f''_\beta) = \star$.

Note that for each n we have shown that the homotopy category of $(n+3)$-dimensional, n-connected CW-complexes is not concrete-at-zero. With $n \geq 1$ we know that it is not the basepoints that prevent concreteness.[8] For $n \geq 3$ we know that the stable category is not concrete.[9]

On Concreteness in General

When we move away from zero, the notion of normal subobjects is not enough. A REGULAR SUBOBJECT is one that appears as an equalizer. Accordingly define yet another equivalence relation on maps into A: $(X \to A) \equiv (X' \to A)$ if they equalize the same pairs of maps coming out of A, that is, if for all $f, g : A \to Y$ it is the case that $X \to A \xrightarrow{f} Y = X \to A \xrightarrow{g} A$ iff $X' \to A \xrightarrow{f} Y = X' \to A \xrightarrow{g} A$. We'll call the equivalence classes GENERALIZED REGULAR SUBOBJECTS.[10]

A necessary condition for concreteness is that every object have only a set of generalized regular subobjects, and I have just recently proved that for categories with finite products this is a sufficient condition. For categories without products a different condition is available, discovered by John Isbell (1963): fixing A, B define

$$\begin{tikzcd}[row sep=small] & A \arrow[dd] \\ X \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd} \equiv \begin{tikzcd}[row sep=small] & A \arrow[dd] \\ X' \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd} \text{ iff for all } \begin{tikzcd}[row sep=small] A \arrow[dr] & \\ & Y \\ B \arrow[ur] & \end{tikzcd}$$

it is the case that

$$\begin{tikzcd}[row sep=small] & A \arrow[dr] & \\ X \arrow[ur] \arrow[dr] & & Y \\ & B \arrow[ur] & \end{tikzcd} \text{ commutes iff}$$

$$\begin{tikzcd}[row sep=small] & A \arrow[dr] & \\ X' \arrow[ur] \arrow[dr] & & Y \\ & B \arrow[ur] & \end{tikzcd} \text{ commutes.}$$

The condition, then, is that for any A, B only a set of equivalence classes arise. This condition allows us formally to adjoin finite products in a way to get the generalized-regular-subobject condition. That condition allows us formally to adjoin equalizers (while preserving the products) to get the condition that every object has only a set of regular subobjects. Now the hard part. A

[8] That is, both free and strict homotopy fail to be concrete.

[9] In his review in the *Mathematical Reviews* John Isbell wrote "The author asserts also that the stable category is not concrete." He apologized when he next saw me saying "I didn't know a stable result when I saw one." As do most of us category people, he had been thinking of the stable category as what one obtains when one forces Σ, the suspension functor, to be an automorphism of the category. What he was forgetting—mostly because the author in question didn't take the trouble to point it out—was that the subject of stable homotopy began with the Freudenthal theorem that if X and Y are of dimension n and at least $(n/2)$-connected then the suspension functor induces an isomorphism $[X, Y] \to [\Sigma X, \Sigma Y]$.

[10] The term I used in 1970 was "abstract regular subobject" which yielded the acronym "ARS". Alas in my 1973 paper, Concreteness, *Journal of Pure and Applied Algebra*, Vol. 3, 1973, that acronym, in its repeated use, became a distraction. So I replaced the "abstract" with "generalized".

long rather arduous construction takes place in the category of set-valued functors.

In a paper written just before this one (see bibliography) [11] I show that the category of small-categories-and-natural-equivalence-classes-of-functors is not concrete. I also give an unenlightening proof that the category of groups-and-conjugacy-classes-of-homomorphism *is* concrete, a fact rather easily seen from the sufficiency of the generalized-regular-subobject condition. Also, the characterization therein of those categories $\mathcal{C}$ for which the category of "petty" functors from $\mathcal{C}$ is concrete becomes much easier. The Eckmann-Hilton analogue of homotopy in abelian categories usually yields non-concrete categories, as do the notions of homotopy on chain complexes.

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Addendum: p-height done right

I'm going to construct a leaner G_α (it appears as a subgroup of the previously defined group). Let W_α be the set of finite ascending words of ordinals *strictly less* than α.

The generators-and-relations construction does not easily yield the necessary proofs, so I'm going to use a different approach, one very familiar to logicians and computer scientists. Let $\mathbb{S}_\alpha$ be the set of words on W_α, which instead of being called words of words will be called "sentences". By a "rewrite rule" is meant an ordered pair of sentences denoted $S_1 \Rightarrow S_2$. By an application of such a rule we mean the result of starting with a sentence S, finding a subsentence equal to S_1 and replacing that subsentence with S_2. We'll stipulate a set of rewrite rules on $\mathbb{S}_\alpha$ and observe that 1) the rules are "strongly normalizing", that is, there is a chain condition on applications of the rewrite rules, or put another was, starting with any sentence we must in a finite number of steps reach a "terminal" sentence, one on which no rewrite rules apply; 2) the terminal sentence reached is independent of the order in which the rewrite

[11] There's another result from this paper that should be mentioned here. The Whitehead Theorem says that the (ordinary spherical) homotopy functors jointly reflect isomorphisms in $\mathcal{H}$ and that seems to be saying that $\mathcal{H}$ though not concrete is not as unconcrete as it could be. Not so. In the mentioned paper I observed that every locally small category has a canonical isomorphism-reflecting set-valued functor. It's not hard to construct. First define yet another equivalence relation on maps targeted at A, to wit, $X \to A \equiv X' \to A$ if for all $A \to Y$ it is the case that $X \to A \to Y$ is an isomorphism iff $X' \to A \to Y$ is an isomorphism. Define $F(A)$ to be the family of equivalence classes and note that F—if we ignore for the moment the possibility that it is too big—is easily seen to be a covariant functor. If there are maps into A that are not split monos we use $\star$ to denote their (common) equivalence class. Define $F_\star(A)$ to be the same as $F(A)$ if there are such maps into A and $F_\star(A) = F(A) \cup \{\star\}$ if not. Clearly $F_\star$ remains a covariant functor. It isn't hard to see that it reflects both right and left invertability and that's a stronger property than merely reflecting isomorphisms. To finish, construct a partial map from the set, $End(A)$, of endomorphisms of A to $F(A)$ that hits every element except $\star$. Given $e \in End(A)$ such that e is a split-idempotent, that is, such that there exist maps $A \to X$ and $X \to A$ with $A \to X \to A = e$ and $X \to A \to X = 1$ then the equivalence class of $X \to A$ is independent of choice of splitting and we use that fact to construct a function as advertised from the set of split-idempotents on A to the non-trivial elements of $F_\star(A)$.

rules are applied.

G_α is then defined as the set of terminal sentences. The binary operation applied to elements S and S' of G_α is the result of starting with the catenation SS' and then normalizing. The result is clearly a monoid; the associativity is an immediate consequence of the associativity of catenation and the uniqueness of the terminal sentence reached and the empty sentence is clearly a unit.

I do the case $p = 2$ and mention how to do the general case. (Note that any prime would suffice for the non-concreteness of homotopy.)

We stipulate two kinds of rules:

<u>The Order Rules</u>: $vu \Rightarrow uv$ if u is a shorter word than v; or if they are the same length and u lexicographically precedes v.

<u>The Shortening Rules</u>: $uu \Rightarrow u'$ if $u = \beta u'$; and $ee \Rightarrow 0$ where e is the empty word and 0 is the empty sentence.

The terminal sentences are then easily seen to be just the strictly ascending sequences of words (with the ordering on words obvious from the order rule). That the rules are strongly normalizing is clear.

The uniqueness is a clear consequence of the "confluence" property:

Suppose that S_1 is a sentence and that S_2 and S_3 are each the result of a single application of a rewrite rule to S_1. Then, using the rules one may reach a common sentence S_4 from each of S_2 and S_3.

The verification of the confluence property is clear if the "domains" of the applications needed to arrive at S_2 and S_3 are disjoint. In the case at hand, different rules have different domains and all domains are of length two, hence we need consider only the cases where S_1 is a three-word sentence. A little case analysis reduces the problem to four patterns, to wit, S_1 is uuu, vvu, vuu or wvu where $u < v < w$. Each of these three cases is easily dispatched by a follow-your-nose application of the rewrite rules (as must be the case if confluence holds).

As already noted, the set of terminal sentences is clearly a monoid. As such it is generated by one-word sentences and the order rules say that one-word sentences commute with each other, hence the monoid is commutative. Switching to additive notation, we know that each one-word sentence is a torsion element: $2^{n+1}u = 0$ where n is the length of u. A commutative monoid in which the generators are all torsion is, of course, a torsion monoid. And any torsion monoid is a group.

We can now easily verify inductively that $I_\beta G_\alpha$, for $\beta \leq \alpha$, consists of all terminal sentences in which all ordinals are at least β (and, of course, still less than α). Hence $I_\alpha G_\alpha$ has only one non-trivial element (the one-word sentence whose one word is the empty word) and $I_{\alpha+1} G_\alpha$ has none.

(For $p > 2$ keep the order rules but change the shortening rules so that they apply to iterated strings of length p. The terminal sentences are characterized as those of the form $u_1 u_2 \cdots u_n$ where $u_i \leq u_{i+1}$ and in which no word appears more than $p - 1$ times.)

This construction of G_α produces a group isomorphic to the subgroup of the one produced by the 1970 construction, to wit, the subgroup generated by those words that end with α. Note that there are natural inclusions $G_\alpha \subset G_\beta$ for $\alpha \leq \beta$ and that these inclusions preserve the distinguished element (the one named by the empty word).

Addendum: topology done fast

Given X and a subcomplex $X' \subset X$ denote the cokernel of the inclusion map $X' \to X$ as defined in the category of connected pointed CW-complexes, $\mathcal{T}$, as X/X'. It is usually described as the result of "smashing X' to a point".

The homotopy extension theorem tells us that $X \to X/X'$ remains a weak-cokernel in the homotopy category $\mathcal{H}$ because if $f : X \to Y$ is such that $f|X'$ is null-homotopic then the homotopy extension theorem says that f is homotopic to a map $g : X \to Y$ such that $g|X'$ is constant. (If the failure of uniqueness is not evident, consider the case where X is a closed n-ball and X' is its boundary. Then X/X' is an n-sphere.)

All maps in $\mathcal{T}$ have weak-cokernels—not just inclusions of subcomplexes—indeed, canonically so. Give $X \to Y$ we can replace Y with the MAPPING CYLINDER of f, to wit, the pushout in $\mathcal{T}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i\downarrow & & \downarrow \\ X\times I & \to & \mathrm{Cy}(f) \end{array}$$

where I is the unit interval and $i : X \to X{\times}I$ sends x to $\langle x, 1\rangle$. Then $\mathrm{Cy}(f)$ is homotopically equivalent with Y and we can use that fact to construct the MAPPING CONE, C_f, as $\mathrm{Cy}(f)/(X\times\{0\})$.[12]

It is easy to check that $f' : Y \to C_f$ is a weak-cokernel of f. In the case that $X \to Y$ is already the inclusion of a subcomplex this construction obviously results in a different space but one that can be seen to be homotopically equivalent. Note that Y automatically appears as a subcomplex of C_f and we can use the first construction for the weak-cokernel of f'. But now a startling thing becomes evident. By smashing Y to a point we have effectively removed all traces of Y from C_f/Y. Not even f has a trace. This last weak-cokernel is none other than the SUSPENSION of X, denoted ΣX, the result of smashing each of the two ends of the cylinder, $X\times I$, each to a point.).[13]

Thus we obtain a sequence of three maps where each is the weak-cokernel of the previous:

$$X \to Y \to C_f \to \Sigma X.$$

If we apply the same argument to $Y \to C_f$ we obtain

$$Y \to C_f \to \Sigma X \to \Sigma Y$$

where the last map turns out to be Σf "turned upside down". Fortunately, for purposes of the following assertion we can ignore the phrase in quotes. We obtain an infinite sequence of weak-cokernels:

$$X \xrightarrow{f} Y \xrightarrow{f'} C_f \xrightarrow{f''} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f'} \Sigma C_f$$

$$\xrightarrow{\Sigma f''} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \xrightarrow{\Sigma^2 f'} \Sigma^2 Cf \xrightarrow{\Sigma^2 f''} \cdots$$

This may be formalized as a functor from the category whose objects are maps in $\mathcal{T}$ to the category of sequences:

$$\mathcal{T}^{\to} \to \mathcal{T}^{\to\to\cdots}.$$

We need the theorem that a homology functor, H, when applied to

$$X \xrightarrow{f} Y \xrightarrow{f'} C_f$$

[12]For strict homotopy we should use the "reduced cylinder" obtained by smashing the line $\{\star\}\times I$ to a point and then use it to obtain the "reduced mapping cone". But, 1) the same homotopy extension theorem tells us that smashing a contractable subcomplex of a CW-complex to a point doesn't change its homotopy type and 2) for purposes of this paper we could move the discussion to the realm of simply connected spaces where strict and free homotopy are the same.

[13]If we use the reduced cone we get the "reduced suspension" $(X\times I)/\{\langle x, t\rangle | x = \star \text{ or } t = 0 \text{ or } t = 1\}$.

yields an exact sequence of abelian groups

$$H(X)\xrightarrow{H(f)}H(Y)\xrightarrow{H(f')}H(C_f).$$

And that easily says that we get a long exact sequence

$$H(X)\to H(Y)\to H(C_f)\to H(\Sigma X)\to\cdots$$

If one uses the fact that for ordinary homology $H_{n+1}(\Sigma X)\simeq H_n(X)$ we obtain

$$H_{n+1}(X)\to H_{n+1}(Y)\to H_{n+1}(C_f)$$
$$\to H_n(X)\to H_n(Y)\to H_n(C_f)\to\cdots$$

Consider the category, $\mathcal{E}$ of exact sequences of abelian groups of the form

$$0\to\coprod_I\mathbb{Z}\xrightarrow{f}\coprod_J\mathbb{Z}\to G\to 0$$

We obtain a functor $\mathcal{E}\to\mathcal{T}$ by replicating the map f as a map between the bouquets[14] of circles, $\bigvee_I S^1\xrightarrow{\hat f}\bigvee_J S^1$, so that $H_1(\bigvee_I S^1)\xrightarrow{H_1(\hat f)}H_1(\bigvee_J S^1)$ is none other than $\coprod_I\mathbb{Z}\xrightarrow{f}\coprod_J\mathbb{Z}$. Define

$$M(0\to\coprod_I\mathbb{Z}\xrightarrow{f}\coprod_J\mathbb{Z}\to G\to 0)$$

to be the mapping cone $C_{\hat f}$.

Given a map between sequences we can play the same game to obtain a functor $\mathcal{E}\to\mathcal{H}$ and the next step is to note that the values of M depend really only on the right end of the particular sequence in $\mathcal{E}$. We obtain, then, a functor $M:\mathcal{G}\to\mathcal{E}\to\mathcal{H}$. It's called the MOORE SPACE functor. For our purposes the critical property is

$$\mathcal{G}\xrightarrow{M}\mathcal{H}\xrightarrow{H_1}\mathcal{G}$$

is naturally equivalent to the identity functor. And so, consequently, is

$$\mathcal{G}\xrightarrow{M}\mathcal{H}\xrightarrow{\Sigma^n}\mathcal{H}\xrightarrow{H_{n+1}}\mathcal{G}$$

To recapitulate: given a family in $\mathcal{G}$

$$\{\mathbb{Z}_p\to G_\alpha\}_\alpha$$

such that

$$\begin{tikzcd} & G_\beta \arrow[d] \\ \mathbb{Z}_p \arrow[ur] \arrow[r] & G_\alpha \end{tikzcd}$$

does not exist for $\beta>\alpha$ we obtain a family of weak cokernels in $\mathcal{H}$

$$\{\Sigma^n M(\mathbb{Z}_p)\to\Sigma^n M(G_\alpha)\}_\alpha$$

such that

$$\begin{tikzcd} & \Sigma^n M(G_\beta) \arrow[d] \\ \Sigma^n M(\mathbb{Z}_p) \arrow[ur] \arrow[r] & \Sigma^n M(G_\alpha) \end{tikzcd}$$

does not exist for $\beta>\alpha$. And that suffices to show that $\Sigma^n M(\mathbb{Z}_p)$ has more than a set of generalized normal subobjects.

Addendum: the 2-sphere is huge

There's a wonderful simplification (which occured to the writer only after he had communicated what he thought was the final draft): erase the last five subscripts from $\mathbb{Z}$ and set $n=1$.

$\Sigma M(\mathbb{Z})$ is the 2-sphere (reduced suspension or not) and we conclude that it has a proper class of generalized normal subobjects. Using the first addendum's observation that

$$\begin{tikzcd} & \Sigma M(G_\beta) \arrow[d] \\ S^2 \arrow[ur] \arrow[r] & \Sigma M(G_\alpha) \end{tikzcd}$$

does exist for $\beta<\alpha$ we further conclude that a strictly ascending chain of the size of the universe—indeed, the order-type of all the ordinals—appears in the GNS-poset of little old S^2.

$\int$

[14] "Bouquet" is the topologist's word for a coproduct of spheres in $\mathcal{T}$

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