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Enriched Categories, Internal Categories and Change of Base

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Chapter 0

Introduction

As soon as we move into the world of enriched, internal or fibered categories we are challenged to consider the way in which categorical properties of those structures transform as we pass from one category of discourse, or *base*, to another.

For example, in topos theory one is keen to describe what happens to the (co)completeness or exactness properties of categories within (or indeed over) a topos $\mathcal{E}$ when they are converted into categories within another topos $\mathcal{F}$ by an application of the direct image of a geometric morphism $f\colon \mathcal{E} \longrightarrow \mathcal{F}$ (say). In homotopy theory, one might seek to study the relationship that holds between the theories of groupoid and simplicially enriched categories; as induced by the *nerve functor* from groupoids to simplicial sets and its left adjoint the *fundamental groupoid functor*. In algebra, we may usefully consider the translation between category theories enriched in modules over related rings. Finally, one might even ask for an analysis of the passage between the theories of enriched and internal categories relative to the same base.

In this work we develop an abstract theory of *change of base* which is adequate to capture all of these examples. We concentrate, in particular, on ensuring that this is strong enough to allow us to prove some very precise results about the way that (co)limits, Kan-extensions and exactness properties of category theories transform under change of base. We go on to apply this framework to study the relationship between certain enriched and internal category theories over the same base.

As an application, we consider (and prove) a conjecture due to Bob Paré [38] regarding a certain well behaved class of limits in 2-category theory. We start by reviewing this more concrete problem, whose solution originally motivated the development of the change of base theory developed herein.

0.1 Limits in 2-Categories

The theory of 2-categorical completeness presents us with subtleties which simply do not arise in its unenriched counterpart. Indeed, many familiar limit notions – such as pullbacks and equalisers – exhibit behaviours which, when viewed from the 2-categorical perspective, can reasonably be described as being somewhat pathological. On studying the archetypal 2-category $\underline{\underline{\mathrm{Cat}}}$ (of all small categories) in greater detail, for example, we find that many of these pathologies arise simply because some limit constructions require us to postulate the strict identity of pairs of objects in the categories whose limit is being taken. Conversely, those 2-categorical limits that are better behaved in $\underline{\underline{\mathrm{Cat}}}$ only require the, quintessentially categorical, condition that certain objects become related by a given, possibly invertible, arrow. However, these observations are far from formally identifying a well-behaved class of 2-categorical limits, a task that has not proved to be an entirely trivial one.

In this context Bob Paré, in his talk to the Bangor category theory meeting in 1989 [38], raised the following question. Suppose, informally, that we are given "diagrams" D and D′ in a 2-category $\underline{\underline{\mathbf{A}}}$ which have "limits" $\varprojlim \mathrm{D}$ and $\varprojlim \mathrm{D}'$ and that these diagrams are related by a "natural transformation" $\alpha\colon \mathrm{D} \longrightarrow \mathrm{D}'$. Then certainly we know that α induces a map of limits $\varprojlim \alpha\colon \varprojlim \mathrm{D} \longrightarrow \varprojlim \mathrm{D}'$ and, furthermore, we know that this induced map is an isomorphism whenever α is an isomorphism. However, when working in a 2-category it is equivalence, not isomorphism, of objects which matters to us, and so we might naturally be led to follow Paré by asking what happens if α is only a *point-wise equivalence*, that is to say comprises components which are equivalences. In general there is no reason to believe that under such conditions $\varprojlim \mathrm{D}'$ should be equivalent to $\varprojlim \mathrm{D}$; however one might hope that for certain "reasonable" 2-categorical limits this would be the case.

More specifically, the notion of 2-dimensional limit used by Paré when asking his original question involved diagrams parameterised by *double categories*. In that context, he says that a double category $\mathbb{D}$ parameterises a *persistent* limit if for any pair of diagrams $\mathrm{D}, \mathrm{D}'\colon \mathbb{D} \longrightarrow \underline{\underline{\mathbf{A}}}$, the map of their *double limits* induced by a *horizontal natural transformation* $\alpha\colon \mathrm{D} \longrightarrow \mathrm{D}'$, all of whose components are equivalences, is itself an equivalence. Paré provided a simple characterisation of such double categories $\mathbb{D}$, a new proof of which is given in section 2.6 herein.

Various other classes of well behaved 2-categorical limits exist in the literature. In particular, in [7] Bird, Kelly, Power and Street introduced the notion of *flexible* limit and they show that these are precisely the ones that can be constructed using products, inserters, equifiers and splitting of idempotents. Importantly this class of 2-categorical limits does not include equalisers and pullbacks, but it does include all *pseudo-limits* and all *(op)lax-limits*. Indeed the flexible limits turn out to be a particularly interesting and useful class since, for example, they are the limits inherited by the 2-category T–Alg of algebras and pseudo-algebra maps for a (nice)

2-monad T.

In fact, in section 2.7 we will show that Paré's class of persistent limits is *closed*,[1] in the sense that "any limit which can be constructed out of persistent limits is persistent". Furthermore, we shall even show that the classes of persistent limits and flexible limits are identical, in a suitable sense which we shall make precise later.

Now, in order to study the relationship between persistent and flexible limits we must first frame them within the same abstract context. However, on the one hand persistent limits have been defined and studied within the theory of double categories, that is to say inside the 2-category $\mathrm{Cat}(\underline{\mathrm{CAT}})$ of *internal categories* in the category $\underline{\mathrm{CAT}}$ of (large) categories and functors between them. So in discussing limits indexed by double categories, we are really regarding 2-categories as certain internal categories in $\underline{\mathrm{CAT}}$ and defining *double limits* as conical limits in that internal context. On the other hand, 2-categories can also be regarded as $\underline{\mathrm{CAT}}$-*enriched categories*, for which we usually define limit notions that are *weighted* by enriched profunctors (cf. Kelly [30]). Furthermore, and somewhat inconveniently for our purposes here, it is this language of enriched categories that has traditionally been used to frame the definitions of flexible limit and closed classes of limits.

A central part of our programme, then, is to establish the relationship between internal categories possessing conically defined limits and enriched categories possessing weighted limits. However in carrying this out we meet an immediate technical obstacle: profunctors in $\mathrm{Cat}(\underline{\mathrm{CAT}})$ do not compose! The problem here is that profunctorial composition is defined using certain coequalisers which must be stable under pullback, and this is not generally true of coequalisers in $\underline{\mathrm{CAT}}$. One solution to this problem, which we adopt here, is to work not in $\mathrm{Cat}(\underline{\mathrm{CAT}})$ but in the larger category $\mathrm{Cat}(\underline{\mathrm{SS}})$, where $\underline{\mathrm{SS}}$ is the category of simplicial sets. Here composition of profunctors is very well behaved, and so we may obtain the results we want as a matter of elementary profunctorial calculation. The bulk of chapter 2, then, is devoted to establishing the relationship between double limits and 2-categorical limits by studying them both in the common context of $\mathrm{Cat}(\underline{\mathrm{SS}})$.

However this strategy involves us in a new task, that of establishing a more formal relationship between the theories of $\underline{\mathrm{CAT}}$-enriched categories, on the one hand, and categories internal to $\underline{\mathrm{SS}}$, on the other. Specifically, our approach will be to provide an abstract account of *change of base* which is general enough to relate these two category theories, via the reflective inclusion $\underline{\mathrm{CAT}} \hookrightarrow \underline{\mathrm{SS}}$, while providing structure sufficient to allow us to translate (co)completeness properties of $\underline{\mathrm{CAT}}$-categories into corresponding properties of categories in $\underline{\mathrm{SS}}$. With this in mind, chapter 1 presents a general theory of change of base for category theories as codified into structures called *equipments*. These provide an abstract framework which combines the calculi of functors and profunctors of a given category theory

[1] in the time since this work was originally written, the terminology *closed* for this concept has been supplanted by the more precise adjective *saturated*.

into a single axiomatised structure, in a way which applies to enriched and internal theories alike. In this context we may describe change of base structures between two category theories as *bicategorical adjunctions* (or *biadjunctions* for short) between their equipments. These share many of the formal properties of geometric morphisms in topos theory, and indeed they may be seen as an indirect generalisation of such things via the work of Carboni, Kelly and Wood [12].

It turns out that the greatest technical challenge of this work has been the development of a fully justified theory of change of base at the level of generality discussed above. This must both be general enough for wide applicability while being specific enough to allow us to prove strong results about how (co)limit notions *within* our category theories transform under change of base. In fact, along the way to this goal we will need to prove very many 2- and 3-categorical results of a foundational character. So finally this thesis contributes to *2-dimensional category theory* on a rather general level.

A more detailed, section by section, summary of the contents of this thesis follows.

0.2 Chapter 1: Change of Base for Abstract Category Theories.

Section 1.1: We begin by discussing the notion of *local adjunction* between bicategories. In theorem 1.1.6, we introduce (a generalised form of) their characterisation in terms of a unit and counit. We then go on, in proposition 1.1.9 and its corollary, to discuss questions of the preservation of Kan extensions and liftings by local adjunctions. This section is little more than a compendium of results and methods which we will be using throughout the remainder of this chapter.

Section 1.2: Here we review the notion of *equipment*, which generalises and abstracts the calculi of functors and profunctors associated with enriched and internal category theories. In summary, an equipment consists of two bicategories $\underline{\underline{\mathcal{M}}}$ and $\underline{\underline{\mathcal{K}}}$ which share the same class of 0-cells and whose 1-cells are respectively thought of as the *profunctors* and *functors* of an abstract category theory. These are related by structures which carry functors $f \in \underline{\underline{\mathcal{K}}}$ to an adjoint pair of profunctors $f_* \dashv f^* \in \underline{\underline{\mathcal{M}}}$, which we think of as being the left and right representables associated with f. Our interest in equipments here is much the same as that which led Wood to introduce them in [56] and [57], viz., they provide just enough abstract bicategorical structure for one to develop within them a complete theory of weighted limits and colimits.

Our goal now is to reduce the question of change of base to that of constructing a biadjoint pair of maps between equipments, and for this we first need to define maps of equipments, along with their attendant transformations and modifications. Maps of equipments do not necessarily preserve the composition of profunctors; instead they are designed to be well behaved with respect to certain *squares* of functors and profunctors in our equipment. These squares are introduced in definition 1.2.4, and their properties are discussed briefly there. However, a complete development of the theory of the structures into which these squares fit is only developed in greater detail in sections 1.3–1.5.

Two important examples of equipments discussed in this section are the *equipment of matrices* derived from a bicategory that possesses stable local coproducts and the *equipment of spans* in a finitely complete category. We also discuss a general *equipment of monads* construction $\mathcal{M}\mathrm{on}(-)$ which yields an equipment whose objects are *monads* within the bicategory of profunctors of an arbitrary equipment satisfying a mild local cocompleteness property. When this construction is applied to an equipment of matrices, this yields the corresponding equipment of enriched categories and profunctors. Furthermore, when it is applied to an equipment of spans this gives us the associated equipment of internal categories.

Section 1.3: As a step towards describing what change of base structures between equipments look like, we start by looking at the general problem of *enrichment over bicategories*. Much in the same way that we introduce category enriched categories (2-categories) in order to abstract the fundamental categorical notions of equivalence and adjunction, our purpose in discussing bicategorical enrichment here is to give an abstract presentation of the corresponding bicategorical notions of *biequivalence* and *biadjunction*. Later on, in section 1.5, we shall construct a number of bicategory enriched categories whose objects are equipments and demonstrate that change of base notions for abstract category theories may be described as biadjoint pair of maps within these structures.

One might hope to apply a fairly conventional approach to establishing a theory of bicategorical enrichment, built upon an appropriate monoidal closed structure on the category of bicategories and homomorphisms. In this case, the theory of biadjointness that we seek forces us to adopt the bicategory $\mathcal{H}om_S(\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{B}}})$, of homomorphisms, *strong* transformations and modifications, as the internal hom between bicategories $\underline{\underline{\mathcal{A}}}$ and $\underline{\underline{\mathcal{B}}}$. Unfortunately, this is neither part of a closed category structure, since there exists no bicategory which can act as its identity object, nor does it possess a corresponding monoidal structure. It does, however, act as the internal hom for a *biclosed multicategory* structure on bicategories, and this is enough to allow us to formulate an appropriate enrichment notion.

At the end of this section we develop a complete, and easily applicable, theory of biadjoint pairs within such bicategory enriched categories. This is closely related to the corresponding presentation of adjunctions in 2-categories in terms of unit, counit and triangle identities. Of course, our definition here coincides with the usual notion of biadjunction (given in terms of equivalences of hom-categories) when we specialise to working within the bicategorically enriched category of bicategories and homomorphisms itself.

Section 1.4: Next we generalise the calculus of *squares* and *cylinders* which should be familiar to the reader from, for instance, Benabou's foundational work on bicategories [3]. By abstracting this calculus we naturally arrive at a theory of *double bicategories*, which may itself be viewed as a hybrid of the theories of double categories and bicategories. Much of this section is given over to constructing various bicategory enriched categories of double bicategories (for example see definition 1.4.7) and to formulating a recipe for constructing biadjunctions inside them (cf. proposition 1.4.8).

Of course, double bicategories may prove to be interesting structures in their own right. However, our primary interest in them here is a consequence of the fact that definition 1.2.4 provides us with a way to build double bicategories from equipments. Now we may define bicategory enriched categories of equipments by "pulling back" the enriched structures that we have constructed between the associated double bicategories introduced in definition 1.2.4.

Section 1.5: Having constructed bicategory enriched categories of equipments indirectly via associated double bicategories, we now provide more concrete and practical descriptions of the 1-cells of these structures, calling them *equipment maps*. These come in a variety of flavours, depending upon how strongly the compositional structure of the "profunctors" (sometimes called *proarrows*) of the domain equipment is preserved. At the weakest level, where raw equipment maps live, we only insist that certain very specific composites of profunctors should be preserved. Stronger gadgets called *equipment morphisms* and *equipment homomorphisms* are defined in a way which makes them into morphisms or homomorphisms (respectively) of bicategories of profunctors.

In this section, we also tease out the structure possessed by the *transformations* and *modifications* which mediate between equipment maps and check that everything here behaves well with respect to the process of passing to dual equipments. Finally we spell out the recipe for constructing biadjoints in this particular situation.

Applying these results, we obtain change of base biadjunctions for equipments of matrices (example 1.5.16) and spans (example 1.5.17). As an aside, we note that these examples generalise earlier results of Carboni, Kelly and Wood [12] on change of base for poset enriched categories of relations.

Section 1.6: Our next step is to obtain change of base results for enriched categories and internal categories by applying the equipment of monads construction $\mathcal{M}\mathrm{on}(-)$ to the equipments of matrices and spans. We do so by defining, in proposition 1.6.5, a suitable enriched functor extending $\mathcal{M}\mathrm{on}(-)$, which we then apply, in examples 1.6.6 and 1.6.7, to give us the desired change of base results. We also show that representable profunctors are preserved by any morphism of equipments obtained by an application of the enriched functor $\mathcal{M}\mathrm{on}(-)$. This allows us to extend 1-categorical actions on categories of functors to 2-categorical ones and to demonstrate, in section 1.7, certain results describing the way in which limits within enriched and internal categories transform under change of base. This analysis provides us with the foundation upon which we shall build the work of chapter 2.

We might mention in passing that there exists a very strong analogy between the change of base notions we develop here and the theory of geometric morphisms in (elementary) topos theory. Specifically, we demonstrate that the change of base structures between category theories may be expressed as a biadjoint pair of equipment maps. Furthermore, we also demonstrate that the left biadjoints of these pairs satisfy the "left exactness" property that they act homomorphically on bicategories of profunctors. That this homomorphism property deserves to be regarded as a form of left exactness is most easily seen in the context of equipments of spans, where it reduces immediately to the preservation of the pullbacks used to compose spans. Indeed, as discussed by Carboni, Kelly and Wood in [12], when we specialise the change of base notions discussed here to the very special case of locally ordered categories of relations in a topos it actually reduces to a theory which is equivalent to the classical theory of geometric morphisms.

Section 1.7: Finally, we would like to use our biadjunctions between equipments to obtain local adjunctions between the corresponding bicategories of profunctors. Having done that, we then apply corollary 1.1.10 to deduce results describing the extent to which colimit cylinders are preserved by change of base. Theorem 1.7.1 presents a useful result of this form, with a yet stronger result being obtained in lemma 1.7.7, which demonstrates that certain "inclusions" of category theories both preserve and create colimit cylinders. It is this result that we apply in chapter 2 to support our representation of enriched colimits as internal ones. Now our careful analysis of the duality construction for equipments, as summarised in corollary 1.5.11, ensures that all of the results we have derived for colimit cylinders also hold for limit cylinders.

0.2.1 Why Generalise to Bicategorical Enrichment?

In the narrative of sections 1.1 to 1.6, most of our examples could have been expressed within a suitable 2-category, constructed in much the same way as the more general bicategorically enriched structures discussed in this thesis. In essence, the structure of our maps of equipments is built upon the structure of their actions on categories of functors and in these examples this is essentially 2-categorical in nature.

However, we have two reasons for not restricting ourselves to a 2-categorical description of these phenomena. The first is that it turns out that the bicategorically enriched theory is not substantially more difficult to develop than the 2-categorical one. The second is that this generalisation, and the full strength of using biadjoints to represent change of base notions, is needed when we come to discussing the natural generalisations of the notion of "sheaf" which arise in enriched category theory.

As discussed at the end of section 1.7, it is natural to follow the lead of Betti [4] and Walters [55] in regarding sheaves on a *site* as being *Cauchy complete* categories enriched in a bicategory derived from that site. Abstracting this idea, we can define a generalised site to be a bicategory $\underline{\underline{\mathcal{B}}}$ over which to enrich, along with a closed class of *absolute* weights; that is, of weights for absolute colimits. Now the equipment of sheaves over such a site has the usual bicategory of $\underline{\underline{\mathcal{B}}}$-enriched categories and $\underline{\underline{\mathcal{B}}}$-profunctors between them as its bicategory of proarrows and has as its arrows a certain sub-bicategory of adjoint profunctors in there. So here we have an example where the arrows of an equipment actually form a genuine bicategory, rather than a category or a 2-category. Furthermore, our generalised sites are related by *co-continuous homomorphisms* which generalise the continuous maps of classical sites and which are subject to a generalisation of the classical *comparison lemma* for sheaves (theorem 1.7.13). This theory may, for example, be used to study change of base processes for stacks over toposes. Now while it is easy to replace our bicategories with biequivalent 2-categories (see appendix A) it is by no means quite so easy to see how one might replace the biadjunctions constructed in our comparison lemma with genuine adjunctions.

0.3 Chapter 2: Double Limits.

Section 2.1: The first section of this chapter simply lays a little groundwork, by collecting together a number of well known technical results. As a general context for this work we introduce a *Gabriel theory* $\mathbb{J} = (\underline{\mathcal{C}}, J)$ and let $\underline{\mathcal{A}}$ denote the category of $\mathbb{J}$-models in $\underline{\text{Set}}$, which we regard as being a monoidal category under cartesian product. Of course, we know that $\underline{\mathcal{A}}$ is a locally presentable category and that it is a reflective full subcategory of $\underline{\widetilde{\mathcal{C}}}$, the category of $\underline{\text{Set}}$-valued presheaves on $\underline{\mathcal{C}}$. Against this backdrop, one of our primary goals will be to study how that adjunction governs the relationship between the theory of $\underline{\mathcal{A}}$-*enriched categories* and that of *categories internal to* $\underline{\widetilde{\mathcal{C}}}$. However, to obtain a well behaved such theory we will also find it necessary to assume an extra, rather mild, technical condition which ensures that the inclusion of $\underline{\mathcal{A}}$ in $\underline{\widetilde{\mathcal{C}}}$ preserves all (small) coproducts.

A canonical example of this setup is provided by the Gabriel theory (Δ, J) whose models are (small) categories. This will allow us to study the $\underline{\text{Cat}}$-enriched theory of 2-categories by representing these as categories internal to the topos $\underline{\text{SS}}$ of simplicial sets.

Section 2.2: In order to represent $\underline{\mathcal{A}}$-enriched categories as categories internal to $\underline{\widetilde{\mathcal{C}}}$, we start by constructing a change of base inclusion of the equipment of $\underline{\mathcal{A}}$-enriched categories into that of $\underline{\widetilde{\mathcal{C}}}$-enriched ones, in the manner discussed in example 1.6.6.

We then observe that the category of (small) sets $\underline{\text{Set}}$ may be identified with the full subcategory of $\widetilde{\underline{\mathcal{C}}}$ determined by the "discrete $\underline{\mathcal{C}}$-sets" (constant presheaves). Furthermore, given an (X, Y)-indexed matrix with entries valued in $\widetilde{\underline{\mathcal{C}}}$ we may construct a span in $\widetilde{\underline{\mathcal{C}}}$ between the discrete $\underline{\mathcal{C}}$-sets on X and Y by taking the coproduct of its entries. This construction provides us with an equipment homomorphism from the equipment of $\widetilde{\underline{\mathcal{C}}}$-matrices into the corresponding equipment of spans in $\widetilde{\underline{\mathcal{C}}}$. Applying the functor $\mathcal{M}\mathrm{on}(-)$ we obtain an equipment homomorphism from the equipment of $\widetilde{\underline{\mathcal{C}}}$-enriched categories to the equipment of internal categories in $\widetilde{\underline{\mathcal{C}}}$, and this in turn possesses a left (bi)adjoint which is obtained by the adjoint lifting argument of proposition 1.6.13. Thus change of base from $\widetilde{\underline{\mathcal{C}}}$-enriched categories to internal categories in $\widetilde{\underline{\mathcal{C}}}$ is also an inclusion of equipments.

Now composing together the inclusions of equipments of the last two paragraphs, we obtain the sought for representation of $\underline{\mathcal{A}}$-enriched categories. This brings with it all of the colimit cylinder creation properties we might hope for, which tell us that the theory of $\underline{\mathcal{A}}$-colimits may be represented faithfully within the theory of colimits in categories internal to $\widetilde{\underline{\mathcal{C}}}$.

Sections 2.3: Here we begin by recalling that the Grothendieck construction for presheaves may be carried out in the internal setting. It turns out that it is precisely this construction that allows us to reduce all weighted colimits in internal category theory to conical ones. Indeed, there is a sense in which one might say that the very lack of such a construction in the general enriched context is precisely the factor which forces us to resort to the more involved weighted theory in the first place.

So by passing along the inclusion of equipments derived in section 2.2 and then applying the Grothendieck construction in $\widetilde{\underline{\mathcal{C}}}$ we find ourselves able to reduce all weighted colimits in $\underline{\mathcal{A}}$-enriched category theory to conical ones in the theory internal to $\widetilde{\underline{\mathcal{C}}}$.

Section 2.4: Returning now to study the specific case where our Gabriel theory $\mathbb{J}$ is taken to be the theory of categories, we show that that the notion of conical (co)limit defined in the previous section, in terms of Kan extensions of profunctors, coincides with Paré's double (co)limit notion. This is, of course, simply a matter of unwinding the definitions.

Section 2.5: Having discussed individual colimits at some length, we move on to deal with classes of colimits. We learn, from the work of Albert and Kelly [1], that a class of weights for colimits is said to be *closed* precisely when any weight for a colimit which may be constructed from that class is itself a member of that class. They characterise classes of weights which have this property in terms of the closure properties of certain enriched subcategories of the category $\mathcal{P}(\underline{\mathbf{A}})$ of all weights on each (small) $\underline{\mathcal{A}}$-enriched category. Consequently, if we wish to export their characterisation to our internal setting in $\underline{\mathcal{A}}$ (or more accurately $\widetilde{\underline{\mathcal{C}}}$) then we must first understand how to use the Grothendieck construction to represent the enriched category $\mathcal{P}(\underline{\mathbf{A}})$ in terms of internal discrete fibrations over the internalisation $\mathrm{I}_\star\underline{\mathbf{A}}$

of $\underline{\mathbf{A}}$.

To do this we start by enriching $\mathrm{Cat}(\underline{\mathcal{A}})$, the category of internal categories and functors in $\underline{\mathcal{A}}$, with the structure of an $\underline{\mathcal{A}}$-enriched category (proposition 2.5.4). Then we show that this gives rise to an $\underline{\mathcal{A}}$-enriched structure on each slice of $\mathrm{Cat}(\underline{\mathcal{A}})$ under which it is (small) $\underline{\mathcal{A}}$-complete and $\underline{\mathcal{A}}$-cocomplete (proposition 2.5.6). Furthermore we show that the Grothendieck construction actually lifts to an $\underline{\mathcal{A}}$-enriched and $\underline{\mathcal{A}}$-fully faithful embedding $\mathbb{G}_{\underline{\mathcal{A}}}\colon \mathcal{P}(\underline{\mathbf{A}}) \longrightarrow \mathrm{Cat}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$ whose image is the full $\underline{\mathcal{A}}$-subcategory of discrete fibrations (theorem 2.5.7). Finally we find that $\mathbb{G}_{\underline{\mathcal{A}}}$ preserves all (small) $\underline{\mathcal{A}}$-colimits, so that the left adjoint possessed by its underlying ordinary functor may be lifted to an $\underline{\mathcal{A}}$-enriched one (corollary 2.5.9).

We next consider the following situation. Suppose that we were given a (possibly large) set $\mathcal{X}$ of categories internal to $\underline{\mathcal{A}}$ whose purpose is to parameterise some class of $\underline{\mathcal{A}}$-colimits. Then we may derive a corresponding class of $\underline{\mathcal{A}}$-enriched weights $\mathcal{X}(\text{-})$ describing the same class of $\underline{\mathcal{A}}$-colimits. Given such a class of weights, the work of Albert and Kelly [1] tells us that a weight on a small $\underline{\mathcal{A}}$-category $\underline{\mathbf{A}}$ parameterises an $\underline{\mathcal{A}}$-colimit which is constructible from the colimits provided by the class $\mathcal{X}$ if and only if it is a member of the category $\mathcal{X}^*(\underline{\mathbf{A}})$ obtained by closing the category of representable weights on $\underline{\mathbf{A}}$ under $\mathcal{X}$-colimits within the category of all weights $\mathcal{P}(\underline{\mathbf{A}})$. However, for our purposes here it is more convenient to ask whether it is possible to characterise this closure process directly in terms of the original class $\mathcal{X}$ of categories in $\mathrm{Cat}(\underline{\mathcal{A}})$.

Theorem 2.5.12 provides the bridge which allows us to do just this. It shows that the $\underline{\mathcal{A}}$-enriched adjunction $\mathbf{L}_{\underline{\mathcal{A}}} \dashv \mathbb{G}_{\underline{\mathcal{A}}}$ exhibiting $\mathcal{P}(\underline{\mathbf{A}})$ as reflective in $\mathrm{Cat}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$ restricts to give an $\underline{\mathcal{A}}$-enriched adjunction exhibiting $\mathcal{X}^*(\underline{\mathbf{A}})$ as reflective in a slice category $\mathcal{X}^\#/_{\mathrm{I}_\star\underline{\mathbf{A}}}$. Here $\mathcal{X}^\#$ denotes the closure under $\mathcal{X}$-colimits in $\mathrm{Cat}(\underline{\mathcal{A}})$ of the full $\underline{\mathcal{A}}$-subcategory $\mathcal{T}(\underline{\mathcal{A}})$ of those internal categories possessing terminal objects (in a global sense). In particular, this result tells us that the closure of the class of weights $\mathcal{X}(\text{-})$ may be given by the formula:

$$\mathcal{X}^*(\underline{\mathbf{A}}) = \left\{ X \in \mathcal{P}(\underline{\mathbf{A}}) \,\middle|\, \mathbb{G}_{\underline{\mathcal{A}}}(X) \in \mathcal{X}^\# \right\}$$

Section 2.6: Here we recall Paré's *persistent limit* notion [38] and prove a characterisation result for this class of double limits. Specifically we show that a double category $\mathbb{D}$ parameterises a persistent limit if and only if each of the connected components of its category of horizontal arrows posses a natural weak initial object (theorem 2.6.6). This result is independent of the preceding general theory and was first established by Paré in loc. cit., although the precise form of the persistency notion given there was not quite correct as stated.

Section 2.7: In this final section, we turn our attention to the class of *flexible 2-limits* as introduced by Bird, Kelly, Power and Street in [7]. This class is known to be the closed class of limits generated by *products*, *inserters*, *equifiers* and *splittings of idempotents*. We show how to present these basic kinds of 2-limit as double limits and follow Paré by demonstrating that every flexible limit is persistent. However, we are now in a position to go further than he did and apply the work of sections 2.2 through 2.5 to reverse this implication and demonstrate that *all* persistent limits are actually flexible (theorem 2.7.1). So as our dénouement we find that these two closed classes of 2-dimensional limits are identical, as originally conjectured by Paré in his Bangor talk [38].

0.4 Appendix: Pasting in Bicategories

In this technical annex, we generalise John Power's work [40] on 2-categorical pasting schemes to show that this notion of pasting can also be made to make unambiguous sense as a description of generalised composition within bicategories. Furthermore, we show that such pasting composites are preserved by homomorphisms between bicategories. Our main tool here is the fact that any bicategory is biequivalent to a 2-category.

0.5 Epilogue

In the time since this thesis was first prepared for examination, the field of higher category theory has thrived. It is now one whose tentacles have propagated into applications so diverse that one hesitates to list them here, simply for fear of missing out a favourite. So it is amusing to recall that in the late 1980's even the majority of category theorists would have said, in a moment of candour, that 2-categories were at best a necessary evil and at worst a complication to be avoided at all costs. Of course, at that time we already knew a very great deal about the (pseudo-)algebra of such structures, thanks to the diligent efforts of a small group of proponents collected around trail-blazers like Max Kelly, Ross Street and Jean Benabou (to name but a very few). However, the influence of these techniques was yet to be fully felt in the broader community and they were certainly far from being accepted as the fundamental part of the category theory toolbox that they are today.

Given this background, I feel that I should pay tribute to the foresight that Martin Hyland showed in recognising in me what I recall him describing as "a definite tendency towards exotic Australian category theory". He had been early to recognise the growing importance and utility of the higher category theory then emerging from the Sydney Category Seminar, and from the very first week of my PhD studies he strongly encouraged me to pursue this interest. I recall being a little less convinced of this choice myself, especially given that my contemporaries were engaged in apparently deeper and more worthwhile pursuits in topos theory and theoretical computer science. However, this unease began to evaporate when I attended the International Category Theory Meeting in 1989, which was held at the

University College of North Wales in Bangor, where I had the good fortune to be present at two talks, on quite distinct topics, which inspired me to pursue the work described here. The first of these was Bob Paré's beautiful exposition of his work on 2-categorical limits from a double categorical perspective [38]. The second I recall as a double act, with Max Kelly and Richard Wood providing a condensed introduction to their radical analysis of change of base for locally ordered categories [12]. Any remaining qualms were completely dispelled for me later that year when I finally got hold of a copy of Makkai and Paré's book on accessible categories [34]. That work makes such a good case for the utility of genuinely 2-categorical techniques in describing and solving problems of broad interest that it finally convinced me to take an Antipodean course, in the first instance mathematically and then later as a naturalised Australian.

In retrospect, one might describe this work as an early *tricategorical* contribution to the meta-theory of abstract category theories. While it does not rely directly upon Gordon, Power and Street's seminal account [21], since that work postdates this by about 2 years, it certainly draws its inspiration from similar sources. Both owe a great debt to John Gray's work on formal category theory [23], which is remarkable for taking such a fundamentally tricategorical approach so early in the development of 2-category theory. The work here also derives great inspiration from Ross Street's labours in that tradition [47], [49], [50], [53] and [54] and most particularly from his account of the theory of fibrations in bicategories [48]. The lesson I learnt from the last mentioned work was that by approaching bicategorical constructions at an appropriate level of abstraction we reduce them to arguments which are only marginally more complicated than corresponding 1- or 2-categorical ones. In contemporary terms, one might say that the complexity of their theory is tamed by working at the level of the (semi-strict) tricategory in which they live, by encoding any new structures we define at that level (wherever possible) and by making judicious use of some basic and well known bicategorical coherence results.

In the past two decades a number of authors have formulated change of base theories which are closely related to the one presented here. Indeed I am partially responsible for the first of these, which may be found in [10]. This follows earlier work of Carboni, Kelly and Wood [12] by expressing equipments, à la Wood [56], as two-sided fibrations over their categories of arrows (functors) and building a 2-category of such structures, wherein commonly occurring change of base processes may be expressed as adjoint pairs of 1-cells. Notably, this theory does not ask that the proarrows (profunctors, or as Max Kelly had it "the Greeks") of these equipments should themselves compose, instead it simply asks that arrows (or in Max's terminology "the Romans") should act on proarrows on both sides. In the theory presented in [10] we also start by ignoring general composites of proarrows, making the mild assumption that they exist without insisting that the basic structure preserving homomorphisms between proarrow equipments should preserve them. Later we enrich these basic maps by asking that they respect composites of proarrows in an (op)lax- or pseudo- sense, and generalise Kelly's doctrinal adjunction results [29] to induce such structures across biadjunctions.

A more recent variation on this theme is provided by Shulman [45] which takes an unapologetically double categorical approach to this theory from the very start. His *framed bicategories* are double categories that are pseudo-associative in the horizontal (profunctorial) direction, in the tradition of Grandis and Paré [22], and which carry a certain kind of *connection* structure [8], expressed as a fibrational property, reflecting vertical (functorial) information into the horizontal. He then builds a family of interrelated 2-categories, whose 1-cells preserve vertical composites "on the nose" and act in an (op)lax- or pseudo-functorial manner on horizontal composites. Here again he is able to express many change of base processes as adjoint pairs in these 2-categories. One might also make similar comments about Cruttwell's thesis [14], which again uses this variety of double category to achieve a similar end.

The work presented here also takes a largely double categorical approach to the change of base question. The *double bicategories* introduced in section 1.4 for this purpose generalise the double categories used by Shulman, Grandis and Paré, Garner [19], [20] and others by allowing composition to be pseudo-associative in both directions. Furthermore, it also recognises, and takes fundamental advantage of, the connection structure that lives on the double bicategory constructed from any equipment. In section 1.5 these become our primary tool in translating the homomorphisms, transformations and modifications of double bicategories, as defined in section 1.3, into corresponding structures between equipments. However, I should admit that I never really viewed the theory of double bicategories as an end in itself, as suggested by my comments about these structures on page (8) in the original introduction above. Instead I had already derived a homomorphism notion for equipments through other, more prosaic means, and I introduced double bicategories as a calculus of cylinders [3] designed to be sufficient only to explain my homomorphism notion and to assemble such things into a bicategorically enriched ensemble. Subsequently, however, these structures have enjoyed a certain life of their own, most notably in the work of Jeffrey Morton [36], [37]. Not only does he provide an account of such structures which is both broader and more detailed than the one given here, but he also builds a double bicategory of *cobordisms with corners* and applies that to the problem of formalising certain extended topological quantum field theories.

In essence all of these works take as their starting points the same pair of absolutely central observations, which in their most elemental form date back to the work of Carboni, Kelly and Wood [12] on poset enriched categories. Firstly, they observe that any change of base theory cannot simply be formulated in terms of the (op)lax structures which operate at the level of the proarrows of our equipments. Instead these must be supported by stronger (pseudo-)functorial structure at the level of the arrows in those equipments. Indeed, it is this fact which largely accounts for the lack of success of the local adjunction notions which had been introduced by Betti and Power [6] and Jay [25] to describe change of base at the purely profunctorial level. Secondly, they demonstrate that the preservation of proarrow composition is best regarded as a secondary consideration. Instead they start from the premiss that the much more important property is the preservation of certain 2-cell *squares* whose horizontal faces are arrows and whose vertical faces are general proarrows. It is this last fact that leads to the ubiquity of certain double category notions in the works discussed above.

It is worth noting, however, that all of the authors cited above provide a purely categorical account of the arrow level structure in equipments. This in turn implies that the collectives they build for their equipments are all 2-categories. The customary reason given for making this simplification is that it results in a theory which is more easily developed because it does not require any, possibly unfamiliar, tricategorical machinery. While there is some veracity to this view, it is also arguable that the presentation given here, while it does involve some aspects of the theory of bicategories which may be unfamiliar, is not in any real sense more complicated to develop or motivate. Its central constructions really are a matter of traditional enriched category theory, albeit over the category of bicategories, and as such they map directly onto their 2-categorical counterparts discussed above. The great benefit of this more general approach is that our tricategories of equipments allow us to actualise important examples which are not available in the 2-categorical theories simply because they give rise to biadjoint, rather than strictly adjoint, pairs. For example, this extra expressiveness allows us to give an account of the cartesian bicategories [11] discussed in example 1.5.19 and to express the comparison lemma considerations discussed at the end of section 1.7.

As a postscript, I would like to mention some very recent work of Jonas Frey [17] on the process of building toposes from triposes [24]. This work studies the extent to which the tripos to topos construction bears a universal characterisation as some kind of left adjoint to the forgetful functor from the 2-category of toposes and left exact functors to a certain, naturally occurring, 2-category of triposes. While results of this kind have been suggested or established by other authors, such as Pitts [39] and Rosolini and Maietti [44], their work implicitly relies upon restricting attention to those topos and tripos morphisms which are *regular*, in the sense that they also preserve existential quantification and disjunction. This latter condition is too restrictive if, for example, we wish to use this universal characterisation to refine our understanding of geometric morphisms between toposes by making concrete calculations with adjunctions in the 2-category of triposes. Put simply, the direct images of geometric morphisms are not in general regular, so one cannot capture them as adjunctions in the 2-category of toposes and regular left exact functors.

The considerations of the last paragraph lead us to seek an extension of this universal characterisation which relates the full 2-categories of triposes and toposes whose 1-cells are not necessarily regular. However, in Frey's own words *"the abandonment of regularity leads to complications which require more sophisticated 2-dimensional techniques"* since, as he demonstrates, the most natural such extension is only functorial in the oplax sense. This observation leads him to regard these 2-categories as weak equipments (in the sense of the comments following definition 1.2.1), in which the proarrows are all left exact morphisms between toposes or triposes and the arrows are the regular morphisms amongst those, and to build a bicategory enriched category of such weak equipments in which the homorphisms are those oplax-functors that act pseudo-functorially on arrows. This construction is a direct analogue of the one used here to derive $\underline{\underline{\underline{\mathcal{E}coMor}}}$ from $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SC}$ in section 1.5 and it gives rise to a structure within which the full tripos to topos construction becomes left biadjoint to the underlying tripos 2-functor.

Observe that Frey's analysis provides us with another important example for which we require the full strength of the tricategorical framework developed here. Of course, were we content to take a much stricter approach to the categorical algebra inherent in Frey's work then it is conceivable that we could devise a purely 2-categorical account of his central result. Unfortunately, such contrivances lead us away from the concrete examples he has in mind by, for example, mandating a preferred choice of (some of) the categorical structure of our triposes and toposes and the "on the nose" preservation of those choices by the morphisms of such. So our relatively inexpensive decision to embrace the tricategorical has paid off, providing us with a biadjoint characterisation of the tripos to topos construction which is both natural and intuitively appealing. However, for me the true appeal of Frey's paper is that it shows how one might apply the theory developed here to some of the very same deeper problems that my compatriots were engaged in when I commenced this work.

In preparing this reprint, I have tried to remain as true as possible to my original text in both style and content. When I first started working on it I had hoped to convert my LaTeX source into a more modern form and to replace my late-80's style diagrams with sparkling new 21st century ones (or even better with string diagrams). However, even with the kind help of Micah McCurdy, I was never able to find the time to complete a full conversion of this form and so I have reluctantly returned to my old sources. Consequently, I would like to beg the reader's indulgence and to apologise for any frustration that my slightly clunky old typesetting may engender. The text itself is much as it was when I first wrote it. In places I have corrected minor misconceptions and mistakes that crept into my original account, but I have largely avoided the urge to reinterpret and rewrite passages to "improve" their presentation in light of subsequent developments. I have, however, taken the single liberty of rephrasing the introduction above so that, while its content and structure are the same as before, I hope it now reads more like a coherent passage and less like the scribblings of a student desperate to submit his work. Furthermore, the reader will find that I have added a few bits and pieces here and there, which are flagged in footnotes, including a new doctrinal adjunction result, proposition 1.4.13, which was originally suggested by Jonas Frey, and the cartesian bicategory example discussed in 1.5.19.

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Finally I should like to express my deepest thanks to my parents, who were tireless in ensuring that I was provided with a firm foundation upon which to build my future. This thesis is dedicated to my wife Sally and my daughter Charlotte; who are the very centre of my life and without whose support this work would never have been written.

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My thanks also go to all of the members of the Australian (née Sydney) Category Seminar past and present. It is my weekly interaction with this excellent group of mathematicians which has fuelled and maintained my enthusiasm for my discipline over the past two decades, and which promises to continue to do so for the next two. Almost all of the research I have engaged in ultimately traces its inspiration back to Ross Street, whose friendship and encouragement drew me back into mathematics a decade ago and for which I am eternally grateful. I would also like to thank Richard Garner and Emily Riehl, whose lively personalities and quick mathematical wits have brought a new momentum to my research life, Micah McCurdy, who has thrived despite my best efforts to thwart him as his supervisor, and Jonas Frey, whose questions and suggestions have encouraged me to view this work in a new light. I would, however, like to single out Steve Lack for particular thanks; without his firm encouragement and support I would never have plucked up the intellectual courage to prepare this thesis for wider circulation after so many years.

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I would like to re-dedicate this work to Sally and Charlotte and to add an extra dedication to my second daughter Florence, who is just about to launch herself into her own scientific career. Their love and support continues to sustain me just as it did 20 years ago and they remain the very centre of my existence.

[1]This document was typeset using version 3 (1990) of Paul Taylor's diagram macro package.

Chapter 1

Change of Base for Abstract Category Theories.

In this chapter we set out to provide abstract structures with which to talk about questions of "change of base" in enriched (or other) category theories. An archetype for the sort of question we are interested in might be:

Question 1.0.1 *Suppose that* $\underline{\underline{\mathcal{B}}}$, $\underline{\underline{\mathcal{C}}}$ *are distributive bicategories with small sets of objects and*

$$\mathrm{F}\colon \underline{\underline{\mathcal{C}}} \longrightarrow \underline{\underline{\mathcal{B}}}$$

is a well behaved homomorphism of bicategories then what sort of structures encapsulate the behaviour of the actions it has on the bicategories of $\underline{\underline{\mathcal{B}}}$*- and* $\underline{\underline{\mathcal{C}}}$*-enriched categories and functors or profunctors? Does the structure we have chosen allow us to deduce anything about, for instance, the stability of the cocompleteness properties of a category under change of base?*

To give a comprehensive answer to this sort of question it is necessary to encapsulate together actions on the bicategories of functors and profunctors, relating them via the left and right representable profunctors associated with each functor. In this context we will see that change of base bears a striking similarity to the notion of geometric morphism in elementary topos theory.

1.1 Local Adjunctions

In this section we introduce the notion of *Local Adjunction*, which will turn out to constitute the action of change of base on bicategories of profunctors. Various structures have been introduced under this name, notably in Betti and Power [6] and Jay [25], but here when we refer to local adjunctions we will always mean the former notion or strengthenings thereof.

Firstly we re-iterate the main definition of [6], which requires a little familiarity with the theory of bicategories as adumbrated in Benabou [3] and Street [48]. Let $\underline{\underline{\mathcal{B}}}$, $\underline{\underline{\mathcal{C}}}$ be bicategories related by a morphism $\mathrm{G}\colon\underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$ and a comorphism $\mathrm{F}\colon\underline{\underline{\mathcal{C}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ from which we define two comorphisms

$$\mathrm{F}_{\#}, \mathrm{G}^{\#}\colon\underline{\underline{\mathcal{C}}}^{\mathrm{op}} \longrightarrow \mathcal{B}icat(\underline{\underline{\mathcal{B}}}, \underline{\mathrm{Cat}})$$

where $\mathrm{F}_{\#}c = \underline{\underline{\mathcal{B}}}(\mathrm{F}c, \text{-})$ and $\mathrm{G}^{\#}c = \underline{\underline{\mathcal{C}}}(c, \mathrm{C}\text{-})$. These definitions may remain a bit unclear without reviewing a little notation. $\mathcal{B}icat(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{C}}})$ denotes the bicategory of Morphisms, Transformations and Modifications between $\underline{\underline{\mathcal{B}}}$ and $\underline{\underline{\mathcal{C}}}$, as described in [48]. An important (locally full) sub-bicategory of $\mathcal{B}icat(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{C}}})$ is $\mathcal{H}om_S(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{C}}})$ consisting of Homomorphisms, Strong Transformations and Modifications. Notice that we use the subscript S to remind us that we are interested only in strong transformations. Notations for the various duals of a bicategory $\underline{\underline{\mathcal{B}}}$ are $\underline{\underline{\mathcal{B}}}^{\mathrm{op}}$ obtained by reversing 1-cells, $\underline{\underline{\mathcal{B}}}^{\mathrm{co}}$ constructed by reversing 2-cells and $\underline{\underline{\mathcal{B}}}^{\mathrm{coop}}$ which we leave up to the imagination of the reader. Related to $\mathcal{B}icat(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{C}}})$ is the bicategory $\mathcal{B}icat^{\mathrm{op}}(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{C}}})$ of Morphisms, Optransformations (these are transformations in which the 2-cellular structure has the opposite orientation) and Modifications, this is (canonically) strictly isomorphic to the dual $(\mathcal{B}icat(\underline{\underline{\mathcal{B}}}^{\mathrm{op}}, \underline{\underline{\mathcal{C}}}^{\mathrm{op}}))^{\mathrm{op}}$.

The definition is:

Definition 1.1.1 (Betti & Power) *We say that* F *is locally left adjoint to* G *mediated by a transformation* $\psi\colon \mathrm{F}_{\#} \longrightarrow \mathrm{G}^{\#}$ *(in symbols* $\mathrm{F} \dashv\!\!|_{\psi} \mathrm{G}$*) iff each 1-cellular component* $\psi_{cb}\colon\underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \longrightarrow \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b)$ *has a left adjoint* φ_{cb}*.*

It it worth pointing out a dual, if $\mathrm{F} \dashv\!\!|_{\psi} \mathrm{G}$ we have duals $\mathrm{G}^{\mathrm{coop}}\colon\underline{\underline{\mathcal{C}}}^{\mathrm{coop}} \longrightarrow \underline{\underline{\mathcal{B}}}^{\mathrm{coop}}$ a comorphism and $\mathrm{F}^{\mathrm{coop}}\colon\underline{\underline{\mathcal{B}}}^{\mathrm{coop}} \longrightarrow \underline{\underline{\mathcal{C}}}^{\mathrm{coop}}$ a morphism. By taking mates of the structure 2-cells of ψ under the various adjunctions $\varphi_{ab} \dashv \psi_{ab}$ we get 2-cells giving the functors $\varphi^{\mathrm{co}}_{ab}\colon\underline{\underline{\mathcal{C}}}^{\mathrm{coop}}(\mathrm{G}b, a) \longrightarrow \underline{\underline{\mathcal{B}}}^{\mathrm{coop}}(b, \mathrm{F}a)$ the structure of a transformation

$$\begin{tikzcd}(\underline{\underline{\mathcal{B}}}^{\mathrm{coop}})^{\mathrm{op}} \arrow[r, bend left, "(\mathrm{G}^{\mathrm{coop}})_{\#}"] \arrow[r, bend right, "(\mathrm{F}^{\mathrm{coop}})^{\#}"'] \arrow[r, phantom, "\Downarrow \varphi^{\mathrm{co}}"] & \mathcal{B}icat(\underline{\underline{\mathcal{C}}}^{\mathrm{coop}}, \underline{\mathrm{Cat}})\end{tikzcd}$$

which clearly mediates a local adjunction $\mathrm{G}^{\mathrm{coop}} \dashv\!\!|_{\varphi^{\mathrm{co}}} \mathrm{F}^{\mathrm{coop}}$.

As Betti and Power point out this notion has many satisfying properties, amongst which are the duality above and the fact that we may compose two (compatible) local adjoints to get a third one. They also demonstrate that a locally cocontinuous homomorphism $\mathrm{H}\colon\underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$ between distributive bicategories gives rise to a locally adjoint pair

$$\begin{tikzcd}\underline{\underline{\mathcal{C}}}\text{–Prof} \arrow[r, bend right, "\mathrm{H}_*"'] \arrow[r, phantom, "\perp_{\psi}"] & \underline{\underline{\mathcal{B}}}\text{–Prof} \arrow[l, bend right, "\mathrm{H}^*"']\end{tikzcd}$$

when $\underline{\underline{\mathcal{B}}}$ is small. Notice that if $\underline{\underline{\mathcal{B}}}$ is small and locally small cocomplete then it must be locally posetal and so this result does not cover the majority of cases in which we will be interested. Later on we prove a more general result, as part of the abstract framework we will be building, then the smallness condition we impose on $\underline{\underline{\mathcal{B}}}$ is simply that each of its "homsets" has a small set of generators.

In [6] the authors provide a method for constructing local adjunctions by a one-sided universal property, this however is inherently non-symmetrical, unlike for instance the one-sided description of adjoints in traditional category theory, and does not suffice to construct all local adjoints. It does however point out the importance of considering local adjoints equipped with some form "unit" or "counit", and even the idea of "triangle identities" for local adjunctions. Since *all* of the local adjoints that we construct in latter sections will be defined in terms of this sort of machinery, we take a little time to develop it here.

The choice of mediating transformation ψ, even for a fixed pair of (co)mor-phisms F and G, is only (relatively) loosely constrained by the structure of those (co)morphisms. In many of the naturally occurring examples, particularly in change of base questions, ψ and φ are obtained from families of unit and counit 1-cells

$$c \xrightarrow{\Psi_c} \mathrm{GF}c \text{ one for each 0-cell } c \in \underline{\underline{\mathcal{C}}}$$
$$\mathrm{FG}b \xrightarrow{\Phi_b} b \quad \text{one for each 0-cell } b \in \underline{\underline{\mathcal{B}}}$$

by setting

$$\begin{aligned} \psi_{cb} = \quad & \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \xrightarrow{\mathrm{G}} \underline{\underline{\mathcal{C}}}(\mathrm{GF}c, \mathrm{G}b) \xrightarrow{- \otimes \Psi_c} \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b) \\ \varphi_{cb} = \quad & \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b) \xrightarrow{\mathrm{F}} \underline{\underline{\mathcal{B}}}(\mathrm{F}c, \mathrm{FG}b) \xrightarrow{\Phi_b \otimes -} \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \end{aligned} \tag{1.1}$$

in fact this is exactly the sort of approach that is taken by Jay (in [25]) as part of the definition of local adjunction. On its own this would not be enough structure to allow us to prove the sorts of theorem we will examine later on since, not only should each φ_{cb} be left adjoint to ψ_{cb}, but these should support complimentary transformation structures (in the sense of the duality result above) derived by some extra structure on the units and counits.

Betti and Power point out in remark 4.6 of [6] that if F and G are bicategory homomorphisms then the composites FG and GF are well defined and we can take the Ψ_c and Φ_b to be the 1-cellular components of optransformations:

$$\begin{aligned} \mathrm{I}_{\underline{\underline{\mathcal{C}}}} & \xrightarrow{\Psi} \mathrm{GF} \\ \mathrm{FG} & \xrightarrow{\Phi} \mathrm{I}_{\underline{\underline{\mathcal{B}}}} \end{aligned} \tag{1.2}$$

From this information we may derive the transformations ψ and φ^{co} and show that adjunctions $\psi_{cb} \dashv \varphi_{cb}$ satisfying the various compatibility conditions with respect to the 2-cellular structure of ψ and φ^{co} correspond to having modifications

$$\mathrm{I_G} \xrightarrow{\alpha} \mathrm{G}\Phi \otimes \Psi\mathrm{G}$$
$$\Phi\mathrm{F} \otimes \mathrm{F}\Psi \xrightarrow{\beta} \mathrm{I_F}$$

satisfying two identities corresponding to the classical triangle identities.

What the authors of [6] do not mention is that this description may be extended to include cases in which F and G are not homomorphisms. The important point is that we are not really interested in forming the composites GF and FG as morphisms (or comorphisms) but rather we are concerned with defining the optransformations of (1.2) without having to explicitly compose F and G as follows:

Definition 1.1.2 *If* $\mathrm{G}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$ *is a morphism and* $\mathrm{F}\colon \underline{\underline{\mathcal{C}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ *a comorphism then a (generalised) optransformation* $\Psi\colon \mathrm{I}_{\underline{\underline{\mathcal{C}}}} \longrightarrow \mathrm{GF}$ *is given by the following data:*

$$\textit{1-cells} \quad (c \xrightarrow{\Psi_c} \mathrm{G}(\mathrm{F}c)) \in \underline{\underline{\mathcal{C}}},\ \textit{one for each 0-cell } c \in \underline{\underline{\mathcal{C}}}.$$

and

$$\textit{2-cells} \quad \begin{tikzcd} c \arrow[r, "\Psi_c"] \arrow[d, "p"'] \arrow[dr, phantom, "\Uparrow \Psi_p"] & \mathrm{G}(\mathrm{F}c) \arrow[d, "\mathrm{G}(\mathrm{F}p)"] \\ c' \arrow[r, "\Psi_{c'}"'] & \mathrm{G}(\mathrm{F}c') \end{tikzcd} \quad \textit{in } \underline{\underline{\mathcal{C}}},\ \textit{one for each 1-cell } (c \xrightarrow{p} c') \in \underline{\underline{\mathcal{C}}}.$$

subject to the conditions:

$$\begin{tikzcd} c \arrow[rr, "\Psi_c"] \arrow[d, bend right, "p"'] \arrow[d, bend left, "q"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[drr, phantom, "\Uparrow \Psi_q"] & & \mathrm{G}(\mathrm{F}c) \arrow[d, "\mathrm{G}(\mathrm{F}q)"] \\ c' \arrow[rr, "\Psi_{c'}"'] & & \mathrm{G}(\mathrm{F}c') \end{tikzcd} \quad = \quad \begin{tikzcd} c \arrow[rr, "\Psi_c"] \arrow[d, "p"'] \arrow[drr, phantom, "\Psi_p \Uparrow"] & & \mathrm{G}(\mathrm{F}c) \arrow[d, bend right, "\mathrm{G}(\mathrm{F}p)"'] \arrow[d, bend left, "\mathrm{G}(\mathrm{F}q)"] \arrow[d, phantom, "\overset{\mathrm{G}(\mathrm{F}\alpha)}{\Rightarrow}"] \\ c' \arrow[rr, "\Psi_{c'}"'] & & \mathrm{G}(\mathrm{F}c') \end{tikzcd} \tag{1.3}$$

for each 2-cell $(\alpha\colon p \longrightarrow q) \in \underline{\underline{\mathcal{C}}}$.

$$
\begin{tikzcd}
 & c \arrow[r, "\Psi_c"] \arrow[dl, "p"'] \arrow[d, phantom, "\Uparrow \Psi_p"] & \mathrm{G}(\mathrm{F}c) \arrow[dl, "\mathrm{G}(\mathrm{F}p)"] \arrow[dd, "\mathrm{G}(\mathrm{F}p'\otimes\mathrm{F}p)"] \arrow[d, phantom, "\overset{\mathrm{can}}{\Rightarrow}"] \\
c' \arrow[r, "\Psi_{c'}"] \arrow[dr, "p'"'] & \mathrm{G}(\mathrm{F}c') \arrow[dr, "\mathrm{G}(\mathrm{F}p')"] \arrow[d, phantom, "\Uparrow \Psi_{p'}"] & \\
 & c'' \arrow[r, "\Psi_{c''}"'] & \mathrm{G}(\mathrm{F}c'')
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
c \arrow[r, "\Psi_c"] \arrow[d, "p'\otimes p"'] \arrow[dr, phantom, "\Psi_{p'\otimes p} \Uparrow"] & \mathrm{G}(\mathrm{F}c) \arrow[d, shift right=6, "\mathrm{G}(\mathrm{F}(p'\otimes p))"'] \arrow[d, shift left=6, "\mathrm{G}(\mathrm{F}p'\otimes\mathrm{F}p)"] \arrow[d, phantom, "\overset{\mathrm{G}(\mathrm{can})}{\Rightarrow}"] \\
c'' \arrow[r, "\Psi_{c''}"'] & \mathrm{G}(\mathrm{F}c'')
\end{tikzcd}
\tag{1.4}
$$

for each compatible pair of 1-cells $p, p' \in \underline{\underline{\mathcal{C}}}$*; and*

$$
\begin{tikzcd}
c \arrow[r, "\Psi_c"] \arrow[d, "\mathrm{i}_c"'] \arrow[dr, phantom, "\Psi_{\mathrm{i}_c} \Uparrow"] & \mathrm{G}(\mathrm{F}c) \arrow[d, shift right=6, "\mathrm{G}(\mathrm{Fi}_c)"'] \arrow[d, shift left=6, "\mathrm{G}(\mathrm{i}_{\mathrm{F}c})"] \arrow[d, phantom, "\overset{\mathrm{G}(\mathrm{can})}{\Rightarrow}"] \\
c \arrow[r, "\Psi_c"'] & \mathrm{G}(\mathrm{F}c)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
c \arrow[r, "\Psi_c"] \arrow[d, "\mathrm{i}_c"'] \arrow[dr, phantom, "\overset{\mathrm{can}}{\cong}"] & \mathrm{G}(\mathrm{F}c) \arrow[d, shift right=6, "\mathrm{i}_{\mathrm{G}(\mathrm{F}c)}"'] \arrow[d, shift left=6, "\mathrm{G}(\mathrm{i}_{\mathrm{F}c})"] \arrow[d, phantom, "\overset{\mathrm{can}}{\Rightarrow}"] \\
c \arrow[r, "\Psi_c"'] & \mathrm{G}(\mathrm{F}c)
\end{tikzcd}
\tag{1.5}
$$

for each 0-cell $c \in \underline{\underline{\mathcal{C}}}$.

Since we have expressed these conditions in terms of pasting diagrams it is worth pointing out that appendix A contains a development of the theory of these as extended to bicategories. In accordance with comments made there we will generally write iterated composites of the 1-cells of a bicategory without explicit bracketing, unless that might help with the exposition. Similarly we follow the usual convention of only introducing identity 1-cells into a composite if they are necessary as the domain or codomain of a 2-cell. We should point out that at some places in our work with bicategories we will assume familiarity with the conventions and results presented in that appendix, particularly with respect to the notion of applying a homomorphism to a pasting cell.

We will usually denote the tensorial horizontal composition of a bicategory $\underline{\underline{\mathcal{B}}}$ by $\otimes$ with identity i_b on each 0-cell $b \in \underline{\underline{\mathcal{B}}}$, and use $\bullet$ for vertical composition of 2-cells. The canonical 2-cells that form part of the structure of morphisms, comorphisms etc. as well as the associativity and identity isomorphisms of bicategories (when we display them explicitly) will generally all carry the name "can". We rely on the context of a canonical 2-cell, in terms of its domain and codomain, to relate exactly which one it is. For instance in (1.4) we have two instances of "can" in contexts

$$\begin{aligned} \mathrm{can}\colon \mathrm{G}(\mathrm{F}p') \otimes \mathrm{G}(\mathrm{F}p) &\Rightarrow \mathrm{G}(\mathrm{F}p' \otimes \mathrm{F}p) \\ \mathrm{G}(\mathrm{can})\colon \mathrm{G}(\mathrm{F}(p' \otimes p)) &\Rightarrow \mathrm{G}(\mathrm{F}p' \otimes \mathrm{F}p) \end{aligned}$$

from which we may infer that they are the compositional comparison maps of the morphism G (instantiated at $\mathrm{F}p' \otimes \mathrm{F}p$) and comorphism F (instantiated at $p' \otimes p$) respectively. The interpretation of the conditions of (1.3)–(1.5) should now be clear.

Returning to definition 1.1.2, if either of F or G is a homomorphism then the composite GF may be formed as a morphism or comorphism (respectively) and our definition becomes that of a traditional optransformation from the identity homomorphism $\mathrm{I}_{\underline{\underline{\mathcal{C}}}}$ on $\underline{\underline{\mathcal{C}}}$ to this composite. This justifies the use "optransformation" for the structure presented in definition 1.1.2. By taking the $(-)^{\mathrm{coop}}$ dual of everything in definition 1.1.2 we obtain the concept of a (generalised) optransformation $\Phi\colon \mathrm{FG} \longrightarrow \mathrm{I}_{\underline{\underline{\mathcal{B}}}}$ which we leave to the reader to spell out.

In their discussion of local adjoints induced by one-sided universal properties, in section 4 of [6], its authors start with a morphism G and provide a method of constructing local left adjoints to it. It turns out that the comorphism structure on one of these, F say, is chosen precisely to ensure that the 1-cells $\Psi_c\colon c \longrightarrow \mathrm{G}(\mathrm{F}c)$, involved in the construction process, lift to an optransformation $\Psi\colon \mathrm{I}_{\underline{\underline{\mathcal{C}}}} \longrightarrow \mathrm{GF}$ of definition 1.1.2.

As promised transformations $\psi\colon \mathrm{F}_{\#} \longrightarrow \mathrm{G}^{\#}$ arise from the kind of optransformation we have defined, notice that the following lemma simply makes more explicit a part of the construction in proposition 4.2 of [6]:

Lemma 1.1.3 *If* $\mathrm{G}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$ *is a morphism,* $\mathrm{F}\colon \underline{\underline{\mathcal{C}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ *a comorphism and* $\Psi\colon \mathrm{I}_{\underline{\underline{\mathcal{C}}}} \longrightarrow \mathrm{GF}$ *an optransformation then the functors*

$$\psi_{cb} = \quad \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \xrightarrow{\ \mathrm{G}\ } \underline{\underline{\mathcal{C}}}(\mathrm{GF}c, \mathrm{G}b) \xrightarrow{\ -\otimes\Psi_c\ } \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b)$$

may be given the structure of a transformation:

$$\begin{tikzcd} \underline{\underline{\mathcal{C}}}^{\mathrm{op}} \arrow[rr, bend left, "\mathrm{F}_{\#}"] \arrow[rr, bend right, "\mathrm{G}^{\#}"'] \arrow[rr, phantom, "\Downarrow\psi"] & & \mathcal{B}icat(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathrm{Cat}}}) \end{tikzcd} \tag{1.6}$$

Proof. Remark 3.2 of [6] sets out the 2-cellular structure lifting the collection of functors ψ_{cb} to a transformation ψ, this consists of:

(a) for each 0-cell $c \in \underline{\underline{\mathcal{C}}}$ and 1-cell $(p\colon b \longrightarrow b') \in \underline{\underline{\mathcal{B}}}$ a 2-cell

$$\begin{tikzcd}
\underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \arrow[r, "\psi_{cb}"] \arrow[d, "\underline{\underline{\mathcal{B}}}(\mathrm{F}c,p)"'] \arrow[dr, phantom, "\Downarrow \psi_{cp}"] & \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b) \arrow[d, "\underline{\underline{\mathcal{C}}}(c,\mathrm{G}p)"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c, b') \arrow[r, "\psi_{cb'}"'] & \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b')
\end{tikzcd}$$

in $\underline{\mathrm{Cat}}$, subject to the coherence conditions making ψ_{c-} into a transformation $\underline{\underline{\mathcal{B}}}(\mathrm{F}c, -) \longrightarrow \underline{\underline{\mathcal{C}}}(c, \mathrm{G}-)$ for each 0-cell $c \in \underline{\underline{\mathcal{C}}}$.

In this case we let ψ_{cp} be given by the pasting

$$\begin{tikzcd}
\underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \arrow[r, "\mathrm{G}"] \arrow[d, "p \otimes -"'] \arrow[dr, phantom, "\mathrm{can} \Downarrow"] & \underline{\underline{\mathcal{C}}}(\mathrm{GF}c, \mathrm{G}b) \arrow[r, "- \otimes \Psi_c"] \arrow[d, "\mathrm{G}p \otimes -"] \arrow[dr, phantom, "\cong"] & \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b) \arrow[d, "\mathrm{G}p \otimes -"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c, b') \arrow[r, "\mathrm{G}"'] & \underline{\underline{\mathcal{C}}}(\mathrm{GF}c, \mathrm{G}b') \arrow[r, "- \otimes \Psi_c"'] & \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b')
\end{tikzcd}$$

where the 2-cell "can" is the compositional comparison of the morphism G and the isomorphism in the right hand square is the associativity of $\underline{\underline{\mathcal{C}}}$. In other words ψ_{cp} is the natural transformation with component at $r \in \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b)$ given by the composite:

$$\mathrm{G}p \otimes (\mathrm{G}r \otimes \Psi_c) \xrightarrow[\cong]{\mathrm{assoc}} (\mathrm{G}p \otimes \mathrm{G}r) \otimes \Psi_c \xrightarrow{\mathrm{can} \otimes \Psi_c} \mathrm{G}(p \otimes r) \otimes \Psi_c$$

Notice that the definition of the ψ_{cp}s does not involve the 2-cellular structure of Ψ in any way and so checking that they satisfy the conditions necessary for them to be the 2-cells of a transformation $\psi_{c-}\colon \underline{\underline{\mathcal{B}}}(\mathrm{F}c, -) \longrightarrow \underline{\underline{\mathcal{C}}}(c, \mathrm{G}-)$ is easy, directly from the coherence properties of the morphism G.

(b) for each 0-cell $b \in \underline{\underline{\mathcal{B}}}$ and 1-cell $(q\colon c' \longrightarrow c) \in \underline{\underline{\mathcal{C}}}$ a 2-cell

$$\begin{tikzcd}
\underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \arrow[r, "\psi_{cb}"] \arrow[d, "\underline{\underline{\mathcal{B}}}(\mathrm{F}q,b)"'] \arrow[dr, phantom, "\Downarrow \psi_{qb}"] & \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b) \arrow[d, "\underline{\underline{\mathcal{C}}}(q,\mathrm{G}b)"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c', b) \arrow[r, "\psi_{c'b}"'] & \underline{\underline{\mathcal{C}}}(c', \mathrm{G}b)
\end{tikzcd}$$

in $\underline{\underline{\mathrm{Cat}}}$, subject to the coherence conditions making $\psi_{\text{-}b}$ into a transformation $\underline{\underline{\mathcal{B}}}(\mathrm{F}\text{–}, b) \longrightarrow \underline{\underline{\mathcal{C}}}(\text{–}, \mathrm{G}b)$ for each 0-cell $b \in \underline{\underline{\mathcal{B}}}$.

In this case we let ψ_{qb} be given by the pasting

$$\begin{tikzcd}
\underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \arrow[r, "\mathrm{G}"] \arrow[d, "- \otimes \mathrm{F}q"'] \arrow[dr, phantom, "\mathrm{can} \Downarrow"] & \underline{\underline{\mathcal{C}}}(\mathrm{GF}c, \mathrm{G}b) \arrow[r, "- \otimes \Psi_c"] \arrow[d, "- \otimes \mathrm{GF}q"'] \arrow[dr, phantom, "- \otimes \Psi_q \Downarrow"] & \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b) \arrow[d, "- \otimes q"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c', b) \arrow[r, "\mathrm{G}"'] & \underline{\underline{\mathcal{C}}}(\mathrm{GF}c', \mathrm{G}b) \arrow[r, "- \otimes \Psi_{c'}"'] & \underline{\underline{\mathcal{C}}}(c', \mathrm{G}b)
\end{tikzcd}$$

where the 2-cell "can" is the compositional comparison of the morphism G so ψ_{qb} is the natural transformation with component at $r \in \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b)$ given by the composite:

$$\mathrm{G}r \otimes \Psi_c \otimes q \xrightarrow{\mathrm{G}r \otimes \Psi_q} \mathrm{G}r \otimes \mathrm{GF}q \otimes \Psi_{c'} \xrightarrow{\mathrm{can} \otimes \Psi_{c'}} \mathrm{G}(r \otimes \mathrm{F}q) \otimes \Psi_{c'}$$

These do involve the 2-cellular structure of the optransformation Ψ and as a result the conditions that are necessary for them to constitute the 2-cells of a transformation $\psi_{\text{-}b} \colon \underline{\underline{\mathcal{B}}}(\mathrm{F}\text{–}, b) \longrightarrow \underline{\underline{\mathcal{C}}}(\text{–}, \mathrm{G}b)$ each follow from the corresponding condition on Ψ and the coherence properties of the morphism G.

(c) finally this data must satisfy the cubical identity

$$\begin{tikzcd}
\underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \arrow[r, "\psi_{cb}"] \arrow[d, "\underline{\underline{\mathcal{B}}}(\mathrm{F}q,b)"'] \arrow[dr, phantom, "\psi_{qb} \Downarrow"] & \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b) \arrow[d, "\underline{\underline{\mathcal{C}}}(q,\mathrm{G}b)"] \arrow[dr, "\underline{\underline{\mathcal{C}}}(c,\mathrm{G}p)"] & \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c', b) \arrow[r, "\psi_{c'b}"] \arrow[dr, "\underline{\underline{\mathcal{B}}}(\mathrm{F}c',p)"'] \arrow[r, phantom, shift right=6, "\psi_{c'p} \Downarrow"] & \underline{\underline{\mathcal{C}}}(c', \mathrm{G}b) \arrow[r, phantom, "\cong"] \arrow[dr, "\underline{\underline{\mathcal{C}}}(c',\mathrm{G}p)"] & \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b') \arrow[d, "\underline{\underline{\mathcal{C}}}(q,\mathrm{G}b')"] \\
& \underline{\underline{\mathcal{B}}}(\mathrm{F}c', b') \arrow[r, "\psi_{c'b'}"'] & \underline{\underline{\mathcal{C}}}(c', \mathrm{G}b')
\end{tikzcd}
=
\begin{tikzcd}
\underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \arrow[r, "\psi_{cb}"] \arrow[d, "\underline{\underline{\mathcal{B}}}(\mathrm{F}q,b)"'] \arrow[dr, "\underline{\underline{\mathcal{B}}}(\mathrm{F}c,p)"] & \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b) \arrow[dr, "\underline{\underline{\mathcal{C}}}(c,\mathrm{G}p)"] \arrow[d, phantom, "\Downarrow \psi_{cp}"] & \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c', b) \arrow[r, phantom, "\cong"] \arrow[dr, "\underline{\underline{\mathcal{B}}}(\mathrm{F}c',p)"'] & \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b') \arrow[r, "\psi_{cb'}"] \arrow[d, "\underline{\underline{\mathcal{C}}}(\mathrm{F}q,b')"'] \arrow[dr, phantom, "\Downarrow \psi_{qb'}"] & \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b') \arrow[d, "\underline{\underline{\mathcal{C}}}(q,\mathrm{G}b')"] \\
& \underline{\underline{\mathcal{B}}}(\mathrm{F}c', b') \arrow[r, "\psi_{c'b'}"'] & \underline{\underline{\mathcal{C}}}(c', \mathrm{G}b')
\end{tikzcd}$$

for each pair of 1-cells $(p \colon b \longrightarrow b') \in \underline{\underline{\mathcal{B}}}$ and $(q \colon c' \longrightarrow c) \in \underline{\underline{\mathcal{C}}}$. From the definitions of the 2-cells in this diagram as given in parts (a) and (b), establishing this condition is an easy diagram chase, involving only the coherence properties of the morphism G. $\square$

Notice that it is not true in general that all transformations $\psi\colon \mathrm{F}_{\#} \longrightarrow \mathrm{G}^{\#}$ arise from optransformations by the construction of the last lemma. A case in which they always do is that in which G is a homomorphism and ψ_{cp} is an isomorphism for each 0-cell $c \in \underline{\underline{\mathcal{C}}}$ and 1-cell $(p\colon b \longrightarrow b') \in \underline{\underline{\mathcal{B}}}$, implying that $\mathrm{G}^{\#}$ and ψ restrict to

$$\begin{tikzcd} \underline{\underline{\mathcal{C}}}^{\mathrm{op}} \arrow[r, bend left, "\mathrm{F}_{\#}"] \arrow[r, bend right, "\mathrm{G}^{\#}"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathcal{H}om_S(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathrm{Cat}}}) \end{tikzcd}$$

and so we may construct an optransformation by applying the bicategorical Yoneda lemma in the form:

Lemma 1.1.4 (Street [48]) *The functors*

$$\mathcal{H}om_S(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathrm{Cat}}})(\underline{\underline{\mathcal{B}}}(b,\text{-}), \mathrm{H}) \xrightarrow{\mathrm{ev}_{\mathrm{i}_b}} \mathrm{H}(b)\ ,$$

denoting evaluation at the identity on the 0-cell $b \in \underline{\underline{\mathcal{B}}}$, are the components of a strong transformation

$$\begin{tikzcd} \underline{\underline{\mathcal{B}}} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathcal{H}om_S(\mathcal{H}om_S(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathrm{Cat}}}), \underline{\underline{\mathrm{Cat}}}) \end{tikzcd}$$

and each one is an equivalence. □

It follows that we have equivalences

$$\mathcal{H}om_S(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathrm{Cat}}})(\underline{\underline{\mathcal{B}}}(\mathrm{F}c', \text{-}), \underline{\underline{\mathcal{C}}}(c', \mathrm{G}\text{-})) \xrightarrow[\sim]{\mathrm{ev}_{\mathrm{i}_{\mathrm{F}c'}}} \underline{\underline{\mathcal{C}}}(c, \mathrm{GF}c') \tag{1.7}$$

which we exploit to obtain 1-cells $\Psi_c\colon c \longrightarrow \mathrm{GF}c$ and 2-cells $\Psi_q\colon \Psi_c \otimes q \Rightarrow \mathrm{GF}q \otimes \Psi_{c'}$ from the corresponding cells of ψ. We verify that these cells satisfy each optransformation condition from the corresponding condition on ψ using the "naturality", in c and c', of the functors in (1.7). It is now a simple matter to demonstrate that applying lemma 1.1.3 to this optransformation yields a transformation which is isomorphic to ψ in $\mathcal{B}icat(\underline{\underline{\mathcal{C}}}^{\mathrm{op}}, \mathcal{H}om_S(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathrm{Cat}}}))(\mathrm{F}_{\#}, \mathrm{G}^{\#})$.

Returning the main thrust of the argument, lemma 1.1.3 governs the way that the transformation $\psi\colon \mathrm{F}_{\#} \longrightarrow \mathrm{G}^{\#}$ in the definition of a local adjoint may be derived from a "unit". Taking its $(\text{-})^{\mathrm{coop}}$ dual provides us with the construction of the transformation $\varphi^{\mathrm{co}}\colon (\mathrm{G}^{\mathrm{coop}})_{\#} \longrightarrow (\mathrm{F}^{\mathrm{coop}})^{\#}$ from a "counit", but it is useful to re-express this dual as we do in the following lemma. Here we exploit the fact that $\mathrm{G}^{\#}$ and $\mathrm{F}_{\#}$ may also be considered to be comorphisms into $\mathcal{B}icat^{\mathrm{op}}(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathrm{Cat}}})$ and then transformations φ^{co} correspond to optransformations $\varphi\colon \mathrm{G}^{\#} \longrightarrow \mathrm{F}_{\#}$:

Lemma 1.1.3$^{\mathrm{coop}}$ *If* $\mathrm{G}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$ *is a morphism,* $\mathrm{F}\colon \underline{\underline{\mathcal{C}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ *a comorphism and* $\Phi\colon \mathrm{FG} \longrightarrow \mathrm{I}_{\underline{\underline{\mathcal{B}}}}$ *an optransformation then the functors*

$$\varphi_{cb} = \quad \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b) \xrightarrow{\mathrm{F}} \underline{\underline{\mathcal{B}}}(\mathrm{F}c, \mathrm{FG}b) \xrightarrow{\Phi_b \otimes \text{-}} \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b)$$

may be given the structure of an optransformation:

$$\underline{\underline{\mathcal{C}}}^{\mathrm{op}} \underset{\mathrm{F}_{\#}}{\overset{\mathrm{G}^{\#}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \varphi}}}} \mathcal{B}icat^{\mathrm{op}}(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathrm{Cat}}}) \tag{1.8}$$

Proof. The 2-cellular structure of the optransformation φ is given by the pastings:

$$\varphi_{cp} = \begin{tikzcd} \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b) \arrow[r, "\mathrm{F}"] \arrow[d, "\mathrm{G}p \otimes -"'] \arrow[dr, phantom, "\mathrm{can} \Uparrow"] & \underline{\underline{\mathcal{B}}}(\mathrm{F}c, \mathrm{FG}b) \arrow[r, "\Phi_b \otimes -"] \arrow[d, "\mathrm{FG}p \otimes -"'] \arrow[dr, phantom, "\Uparrow \Phi_p \otimes -"] & \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \arrow[d, "p \otimes -"] \\ \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b') \arrow[r, "\mathrm{F}"'] & \underline{\underline{\mathcal{B}}}(\mathrm{F}c, \mathrm{FG}b') \arrow[r, "\Phi_{b'} \otimes -"'] & \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b') \end{tikzcd}$$

$$\varphi_{qb} = \begin{tikzcd} \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b) \arrow[r, "\mathrm{F}"] \arrow[d, "- \otimes q"'] \arrow[dr, phantom, "\mathrm{can} \Uparrow"] & \underline{\underline{\mathcal{B}}}(\mathrm{F}c, \mathrm{FG}b) \arrow[r, "\Phi_b \otimes -"] \arrow[d, "- \otimes \mathrm{F}q"] \arrow[dr, phantom, "\cong"] & \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \arrow[d, "- \otimes \mathrm{F}q"] \\ \underline{\underline{\mathcal{C}}}(c', \mathrm{G}b) \arrow[r, "\mathrm{F}"'] & \underline{\underline{\mathcal{B}}}(\mathrm{F}c', \mathrm{FG}b) \arrow[r, "\Phi_b \otimes -"'] & \underline{\underline{\mathcal{B}}}(\mathrm{F}c', b) \end{tikzcd}$$

□

For the remainder of the section assume that we have been given unit and counit optransformations $\Psi\colon \mathrm{I}_{\underline{\underline{\mathcal{C}}}} \longrightarrow \mathrm{GF}$ and $\Phi\colon \mathrm{FG} \longrightarrow \mathrm{I}_{\underline{\underline{\mathcal{B}}}}$ respectively and that ψ and φ are defined from these as in the last lemma and its dual. We need to derive conditions which ensure that $\varphi_{cb} \dashv \psi_{cb}$ for each pair of 0-cells $b \in \underline{\underline{\mathcal{B}}}, c \in \underline{\underline{\mathcal{C}}}$ with the proviso that φ_{cp}, ψ_{cp} and φ_{qb}, ψ_{qb} are pairs of mates under these adjunctions. To this end we prove:

Lemma 1.1.5 *Families of 2-cells*

$$\{\kappa_{cb}\colon \mathrm{i}_{\underline{\underline{\mathcal{C}}}(c,\mathrm{G}b)} \longrightarrow \psi_{cb} \circ \varphi_{cb} | c \in \underline{\underline{\mathcal{C}}}, b \in \underline{\underline{\mathcal{B}}}\} \tag{1.9}$$

satisfying the conditions

$$
\begin{tikzcd}
& \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b) \arrow[dl, "\varphi_{cb}"'] \arrow[d, "\mathrm{i}"] \arrow[dl, phantom, shift left=4ex, "\overset{\kappa_{cb}}{\Leftarrow}"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c,b) \arrow[r, "\psi_{cb}"'] \arrow[d, "p\otimes -"'] \arrow[dr, phantom, "\Downarrow\psi_{cp}"] & \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b) \arrow[d, "\mathrm{G}p\otimes -"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c,b') \arrow[r, "\psi_{cb'}"'] & \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b')
\end{tikzcd}
=
\begin{tikzcd}
\underline{\underline{\mathcal{B}}}(\mathrm{F}c,b) \arrow[d, "p\otimes -"'] \arrow[dr, phantom, "\Uparrow\varphi_{cp}"] & \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b) \arrow[l, "\varphi_{cb}"'] \arrow[d, "\mathrm{G}p\otimes -"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c,b') \arrow[dr, "\psi_{cb'}"'] & \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b') \arrow[l, "\varphi_{cb'}"'] \arrow[d, "\mathrm{i}"] \arrow[d, phantom, shift right=4ex, "\underset{\kappa_{cb'}}{\Leftarrow}"] \\
& \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b')
\end{tikzcd}
\tag{1.10}
$$

for each 0-cell $c \in \underline{\underline{\mathcal{C}}}$ *and 1-cell* $(p\colon b \longrightarrow b') \in \underline{\underline{\mathcal{B}}}$ *and*

$$
\begin{tikzcd}
& \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b) \arrow[dl, "\varphi_{cb}"'] \arrow[d, "\mathrm{i}"] \arrow[dl, phantom, shift left=4ex, "\overset{\kappa_{cb}}{\Leftarrow}"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c,b) \arrow[r, "\psi_{cb}"'] \arrow[d, "-\otimes \mathrm{F}q"'] \arrow[dr, phantom, "\Downarrow\psi_{qb}"] & \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b) \arrow[d, "-\otimes q"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c',b) \arrow[r, "\psi_{c'b}"'] & \underline{\underline{\mathcal{C}}}(c',\mathrm{G}b)
\end{tikzcd}
=
\begin{tikzcd}
\underline{\underline{\mathcal{B}}}(\mathrm{F}c,b) \arrow[d, "-\otimes \mathrm{F}q"'] \arrow[dr, phantom, "\Uparrow\varphi_{qb}"] & \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b) \arrow[l, "\varphi_{cb}"'] \arrow[d, "-\otimes q"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c',b) \arrow[dr, "\psi_{cb'}"'] & \underline{\underline{\mathcal{C}}}(c',\mathrm{G}b) \arrow[l, "\varphi_{c'b}"'] \arrow[d, "\mathrm{i}"] \arrow[d, phantom, shift right=4ex, "\underset{\kappa_{cb'}}{\Leftarrow}"] \\
& \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b')
\end{tikzcd}
\tag{1.11}
$$

for each 0-cell $b \in \underline{\underline{\mathcal{B}}}$ *and 1-cell* $(q\colon c' \longrightarrow c) \in \underline{\underline{\mathcal{C}}}$ *are in bijective correspondence with families of 2-cells*

$$
\begin{tikzcd}
\mathrm{G}b \arrow[r, "\Psi_{\mathrm{G}b}"] \arrow[dr, "\mathrm{i}_{\mathrm{G}b}"'] & \mathrm{GFG}b \arrow[d, "\mathrm{G}\Phi_b"] \arrow[dl, phantom, shift left=3ex, "\Uparrow\alpha_b"] \\
& \mathrm{G}b
\end{tikzcd}
\tag{1.12}
$$

in $\underline{\underline{\mathcal{C}}}$, one for each 0-cell $b \in \underline{\underline{\mathcal{B}}}$, satisfying the identity

$$\begin{tikzcd}
 & \mathrm{G}\Phi_{b'} \otimes \Psi_{\mathrm{G}b'} \otimes \mathrm{G}p \arrow[r, "\mathrm{G}\Phi_{b'} \otimes \Psi_{\mathrm{G}p}"] & \mathrm{G}\Phi_{b'} \otimes \mathrm{GFG}p \otimes \Psi_{\mathrm{G}b} \arrow[dr, "\mathrm{can} \otimes \Psi_{\mathrm{G}b}"] & \\
\mathrm{G}p \arrow[ur, "\alpha_{b'} \otimes \mathrm{G}p"] \arrow[dr, "\mathrm{G}p \otimes \alpha_b"'] & & \parallel & \mathrm{G}(\Phi_{b'} \otimes \mathrm{FG}p) \otimes \Psi_{\mathrm{G}b} \arrow[dl, "\mathrm{G}\Phi_p \otimes \Psi_{\mathrm{G}b}"] \\
 & \mathrm{G}p \otimes \mathrm{G}\Phi_b \otimes \Psi_{\mathrm{G}b} \arrow[r, "\mathrm{can} \otimes \Psi_{\mathrm{G}b}"'] & \mathrm{G}(p \otimes \Phi_b) \otimes \Psi_{\mathrm{G}b} &
\end{tikzcd} \tag{1.13}$$

for each 1-cell $(p\colon b \longrightarrow b') \in \underline{\underline{\mathcal{B}}}$.

Proof. It will help to rewrite conditions (1.10) and (1.11) by substituting the definitions of ψ and φ given in the lemmas above and expressing the results "pointwise", in other words as equations on the components of the various natural transformations at each $r \in \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b)$:

$$\begin{tikzcd}
\mathrm{G}p \otimes r \arrow[r, "\kappa^{\mathrm{G}p\otimes r}_{cb'}"] \arrow[d, "\mathrm{G}p \otimes \kappa^r_{cb}"] & \mathrm{G}(\Phi_{b'} \otimes \mathrm{F}(\mathrm{G}p \otimes r)) \otimes \Psi_c \arrow[r, "\mathrm{G}(\Phi_{b'} \otimes \mathrm{can}) \otimes \Psi_c"] & \mathrm{G}(\Phi_{b'} \otimes \mathrm{FG}p \otimes \mathrm{F}r) \otimes \Psi_c \arrow[d, "\mathrm{G}(\Phi_p \otimes \mathrm{F}r) \otimes \Psi_c"'] \\
\mathrm{G}p \otimes \mathrm{G}(\Phi_b \otimes \mathrm{F}r) \otimes \Psi_c \arrow[rr, "\mathrm{can} \otimes \Psi_c"'] & & \mathrm{G}(p \otimes \Phi_b \otimes \mathrm{F}r) \otimes \Psi_c
\end{tikzcd} \tag*{$(1.10)_{pr}$}$$

for each 1-cell $(p\colon b \longrightarrow b') \in \underline{\underline{\mathcal{B}}}$ and

$$\begin{tikzcd}
r \otimes q \arrow[r, "\kappa^r_{cb} \otimes q"] \arrow[d, "\kappa^{r\otimes q}_{c'b}"'] & \mathrm{G}(\Phi_b \otimes \mathrm{F}r) \otimes \Psi_c \otimes q \arrow[r, "\mathrm{G}(\Phi_b \otimes \mathrm{F}r) \otimes \Psi_q"] & \mathrm{G}(\Phi_b \otimes \mathrm{F}r) \otimes \mathrm{GF}q \otimes \Psi_{c'} \arrow[d, "\mathrm{can} \otimes \Psi_{c'}"] \\
\mathrm{G}(\Phi_b \otimes \mathrm{F}(r \otimes q)) \otimes \Psi_{c'} \arrow[rr, "\mathrm{G}(\Phi_b \otimes \mathrm{can}) \otimes \Psi_{c'}"'] & & \mathrm{G}(\Phi_b \otimes \mathrm{F}r \otimes \mathrm{F}q) \otimes \Psi_{c'}
\end{tikzcd} \tag*{$(1.11)_{rq}$}$$

for each 1-cell $(q\colon c' \longrightarrow c) \in \underline{\underline{\mathcal{C}}}1$, where κ^r_{cb} is the component of the natural transformation κ_{cb} at r.

We first show that families of 2-cells of the kind portrayed in (1.9) and satisfying condition (1.11) correspond to families like those shown in (1.12) with no conditions to satisfy:

$\overset{\theta}{\Longrightarrow}$: for a family $\underset{\sim}{\kappa} = \{\kappa_{cb}\colon \mathrm{i}_{\underline{\underline{\mathcal{C}}}(c,\mathrm{G}b)} \longrightarrow \psi_{cb} \circ \varphi_{cb} | c \in \underline{\underline{\mathcal{C}}}, b \in \underline{\underline{\mathcal{B}}}\}$ satisfying (1.11) define $\underset{\sim}{\alpha} = \theta(\underset{\sim}{\kappa})$ with components α_b given by the composite

$$\mathrm{i}_{\mathrm{G}b} \xrightarrow{\kappa^{\mathrm{i}_{\mathrm{G}b}}_{\mathrm{G}b,b}} \mathrm{G}(\Phi_b \otimes \mathrm{Fi}_{\mathrm{G}b}) \otimes \Psi_{\mathrm{G}b} \xrightarrow{\mathrm{G}(\Phi_b \otimes \mathrm{can}) \otimes \Psi_{\mathrm{G}b}} \mathrm{G}(\Phi_b \otimes \mathrm{i}_{\mathrm{FG}b}) \otimes \Psi_{\mathrm{G}b}$$
$$\xrightarrow[\cong]{\mathrm{G}(\mathrm{can}) \otimes \Psi_{\mathrm{G}b}} \mathrm{G}\Phi_b \otimes \Psi_{\mathrm{G}b}$$

for each 0-cell $b \in \underline{\underline{\mathcal{B}}}$.

$\overset{\phi}{\Longleftarrow}$: for a family $\underset{\sim}{\alpha} = \{\alpha_b\colon \mathrm{i}_{\mathrm{G}b} \longrightarrow \mathrm{G}\Phi_b \otimes \Psi_{\mathrm{G}b} | b \in \underline{\underline{\mathcal{B}}}\}$ define the 2-cell κ^r_{cb} to be the composite

$$r \xrightarrow{\alpha_b \otimes r} \mathrm{G}\Phi_b \otimes \Psi_{\mathrm{G}b} \otimes r \xrightarrow{\mathrm{G}\Phi_b \otimes \Psi_r} \mathrm{G}\Phi_b \otimes \mathrm{GF}r \otimes \Psi_c$$
$$\xrightarrow{\mathrm{can} \otimes \Psi_c} \mathrm{G}(\Phi_b \otimes \mathrm{F}r) \otimes \Psi_c$$

for each pair of 0-cells $b \in \underline{\underline{\mathcal{B}}}$, $c \in \underline{\underline{\mathcal{C}}}$ and 1-cell $r \in \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b)$.

Combining the interchange rule, naturality of the canonical 2-cells associated with the morphism G and condition (1.3) in the definition of the optransformation Ψ in a simple diagram chase we demonstrate that these constitute the components of natural transformations:

$$\mathrm{i}_{\underline{\underline{\mathcal{C}}}(c,\mathrm{G}b)} \xrightarrow{\kappa_{cb}} \psi_{cb} \circ \varphi_{cb}$$

Furthermore another chase, this time involving condition (1.4) on Ψ and the properties of G, shows that they also satisfy the condition scheme $(1.11)_{rq}$.

By simple diagram chases we show that $\phi\theta\underset{\sim}{\alpha} = \underset{\sim}{\alpha}$ and $\theta\phi\underset{\sim}{\kappa} = \underset{\sim}{\kappa}$ from condition (1.5) for the optransformation Ψ and condition scheme $(1.11)_{rq}$ for $\underset{\sim}{\kappa}$ respectively.

All that remains is to prove that ϕ carries families satisfying condition (1.12) to families satisfying scheme $(1.10)_{pr}$, and vice versa for θ. To do this suppose that $\underset{\sim}{\alpha}$ and $\underset{\sim}{\kappa}$ correspond under this bijection and consider the map σ^{pr}_{cb}, defined to be the composite:

$$\mathrm{G}(p \otimes \Phi_b) \otimes \Psi_{\mathrm{G}b} \otimes r \xrightarrow{\mathrm{G}(p \otimes \Phi_b) \otimes \Psi_r} \mathrm{G}(p \otimes \Phi_b) \otimes \mathrm{GF}r \otimes \Psi_c$$
$$\xrightarrow{\mathrm{can} \otimes \Psi_c} \mathrm{G}(p \otimes \Phi_b \otimes \mathrm{F}r) \otimes \Psi_c$$

It is a matter of easy diagram chases to show that the map obtained by applying the functor $- \otimes r$ to the upper (lower) leg of diagram (1.13) and then composing with σ^{pr}_{cb} is equal to the upper (lower) leg of diagram $(1.10)_{pr}$. Notice that condition (1.5) on the optransformation Ψ implies that if $c = \mathrm{G}b$ and $r = \mathrm{i}_{\mathrm{G}b}$ then composing σ^{pr}_{cb} with

$$\mathrm{G}(p \otimes \Phi_b \otimes \mathrm{Fi}_{\mathrm{G}b}) \otimes \Psi_{\mathrm{G}b} \xrightarrow{\mathrm{G}(p \otimes \Phi_b \otimes \mathrm{can}) \otimes \Psi_{\mathrm{G}b}} \mathrm{G}(p \otimes \Phi_b \otimes \mathrm{i}_{\mathrm{FG}b}) \otimes \Psi_{\mathrm{G}b}$$

gives the canonical isomorphism:

$$\begin{aligned} \mathrm{G}(p \otimes \Phi_b) \otimes \Psi_{\mathrm{G}b} \otimes \mathrm{i}_{\mathrm{G}b} &\xrightarrow[\cong]{\mathrm{can}} \mathrm{G}(p \otimes \Phi_b) \otimes \Psi_{\mathrm{G}b} \\ &\xrightarrow[\cong]{\mathrm{G}(\mathrm{can}) \otimes \Psi_{\mathrm{G}b}} \mathrm{G}(p \otimes \Phi_b \otimes \mathrm{i}_{\mathrm{FG}b}) \otimes \Psi_{\mathrm{G}b} \end{aligned}$$

It now becomes clear that the condition on $\underset{\sim}{\alpha}$ holds iff the one on $\underset{\sim}{\kappa}$ does. $\square$

Again it is worth mentioning the dual lemma 1.1.5$^{\mathrm{coop}}$ relating families of natural transformations

$$\varphi_{cb} \otimes \psi_{cb} \xrightarrow{\tau_{cb}} \mathrm{i}_{\underline{\underline{\mathcal{B}}}(\mathrm{F}c,b)}\ ,$$

obeying similar compatibility conditions with respect to the 2-cells of ψ and φ, and families of 2-cells

$$\begin{tikzcd} \mathrm{F}c \arrow[d, "\mathrm{F}\Psi_c"'] \arrow[dr, "\mathrm{i}_{\mathrm{F}c}"] & \\ \mathrm{FGF}c \arrow[r, "\Phi_{\mathrm{F}c}"'] \arrow[ur, phantom, "\beta_c \Uparrow"] & \mathrm{F}c \end{tikzcd}$$

satisfying the dual condition (1.13)$^{\mathrm{coop}}$. Notice that if G and F are homomorphisms then we may form (honest) optransformations

$$\mathrm{GFG} \xrightarrow{\mathrm{G}\Phi} \mathrm{G}$$
$$\mathrm{F} \xrightarrow{\mathrm{F}\Psi_c} \mathrm{FGF}$$

then (1.13) and (1.13)$^{\mathrm{coop}}$ are exactly the conditions that the 2-cells α_b and β_c must satisfy to be the components of modifications

$$\mathrm{I}_{\mathrm{G}} \xrightarrow{\alpha} \mathrm{G}\Phi \circ \Psi\mathrm{G}$$
$$\Phi\mathrm{F} \circ \mathrm{F}\Psi \xrightarrow{\beta} \mathrm{I}_{\mathrm{F}}$$

In interpreting lemma 1.1.5, and its dual, we think of κ_{cb} and τ_{cb} as the unit and counit of the (hoped for) adjunction $\varphi_{cb} \dashv \psi_{cb}$, then conditions (1.10) and (1.11) will ensure that the structure of φ as an optransformation is that derived from ψ by taking the mates of its structure 2-cells. All that remains is to formulate a condition which makes sure that these proposed units and counits do in fact satisfy the triangle identity. This motivates:

Theorem 1.1.6 (Triangle Identities for Local Adjunctions) *The following are equivalent:*

(i) F *is a local left adjoint of* G *as mediated by a unit* $\Psi\colon \mathrm{I}_{\underline{\underline{\mathcal{C}}}} \longrightarrow \mathrm{GF}$ *and counit* $\Phi\colon \mathrm{FG} \longrightarrow \mathrm{I}_{\underline{\underline{\mathcal{B}}}}$*, in symbols* $\mathrm{F} \dashv\!\!|_{\Psi,\Phi} \mathrm{G}$*. In other words* Ψ *and* Φ *are (generalised) optransformations, such that the transformation* $\psi\colon \mathrm{F}_{\#} \longrightarrow \mathrm{G}^{\#}$ *and optransformation* $\varphi\colon \mathrm{G}^{\#} \longrightarrow \mathrm{F}_{\#}$ *derived from them are "pointwise adjoint" (i.e.* $\varphi_{cb} \dashv \psi_{cb}$ *for each pair of 0-cells) and "compatible" (i.e. the structure 2-cells of* φ *are the mates of those of* ψ *under these adjunctions).*

(ii) there exist families of 2-cells

$$\begin{tikzcd} \mathrm{G}b \arrow[r, "\Psi_{\mathrm{G}b}"] \arrow[dr, "\mathrm{i}_{\mathrm{G}b}"'] & \mathrm{GFG}b \arrow[d, "\mathrm{G}\Phi_b"] \arrow[dl, phantom, "\Uparrow \alpha_b"] \\ & \mathrm{G}b \end{tikzcd}$$

one for each 0-cell $b \in \underline{\underline{\mathcal{B}}}$ *and*

$$\begin{tikzcd} \mathrm{F}c \arrow[d, "\mathrm{F}\Psi_c"'] \arrow[dr, "\mathrm{i}_{\mathrm{F}c}"] & \\ \mathrm{FGF}c \arrow[r, "\Phi_{\mathrm{F}c}"'] \arrow[ur, phantom, "\beta_c \Uparrow"] & \mathrm{F}c \end{tikzcd}$$

one for each 0-cell $c \in \underline{\underline{\mathcal{C}}}$*, which satisfy conditions (1.13) and (1.13)*$^{\mathrm{coop}}$ *respectively. Furthermore they also satisfy the identity*

$$\begin{tikzcd} \mathrm{G}\Phi_{\mathrm{F}c} \otimes \Psi_{\mathrm{GF}c} \otimes \Psi_c \arrow[r, "\mathrm{G}\Phi_{\mathrm{F}c} \otimes \Psi_{\Psi_c}"] & \mathrm{G}\Phi_{\mathrm{F}c} \otimes \mathrm{GF}\Psi_c \otimes \Psi_c \arrow[r, "\mathrm{can} \otimes \Psi_c"] \arrow[d, phantom, "="] & \mathrm{G}(\Phi_{\mathrm{F}c} \otimes \mathrm{F}\Psi_c) \otimes \Psi_c \arrow[d, "\mathrm{G}\beta_c \otimes \Psi_c"] \\ \mathrm{i}_{\mathrm{GF}c} \otimes \Psi_c \arrow[u, "\alpha_{\mathrm{F}c} \otimes \Phi_c"] \arrow[rr, "\mathrm{can} \otimes \Psi_c"'] & {} & \mathrm{Gi}_{\mathrm{F}c} \otimes \Psi_c \end{tikzcd} \tag{1.14}$$

for each 0-cell $c \in \underline{\underline{\mathcal{C}}}$*, and its dual (1.14)*$^{\mathrm{coop}}$ *for each 0-cell* $b \in \underline{\underline{\mathcal{B}}}$*, which between them relate the two families.*

Proof. It is clear from lemmas 1.1.3, 1.1.3$^{\mathrm{coop}}$, 1.1.5 and 1.1.5$^{\mathrm{coop}}$ that all we need to prove is that condition (1.14) holds iff the triangle identity

$$
\begin{tikzcd}
\psi_{cb} \arrow[r, "\kappa_{cb}\psi_{cb}"] \arrow[dr, "\mathrm{i}_{\psi_{cb}}"'] & \psi_{cb}\varphi_{cb}\psi_{cb} \arrow[d, "\psi_{cb}\tau_{cb}"] \\
& \psi_{cb}
\end{tikzcd}
\tag{1.15}
$$

holds for all 0-cells $c \in \underline{\underline{\mathcal{C}}}$ and $b \in \underline{\underline{\mathcal{B}}}$. The equivalence of condition $(1.14)^{\mathrm{coop}}$ with the other triangle identity would then follow by duality.

First substitute the expressions for ψ_{cb}, φ_{cb}, κ_{cb} and τ_{cb} in terms of Ψ, Φ, α and β into (1.15) and consider the components of the resulting diagram of natural transformations as evaluated at $t \in \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b)$. It is a matter of an easy diagram chase involving rule (1.4) for the optransformation Ψ and condition (1.13) on the α_bs to show that this reduces to the diagram

$$
\begin{tikzcd}
& \mathrm{G}t \otimes \mathrm{G}\Phi_{\mathrm{F}c} \otimes \mathrm{GF}\Psi_c \otimes \Psi_c \arrow[dr, "\mathrm{can} \otimes \Psi_c"] & \\
\mathrm{G}t \otimes \mathrm{G}\Phi_{\mathrm{F}c} \otimes \Psi_{\mathrm{GF}c} \otimes \Psi_c \arrow[ur, "\mathrm{G}t \otimes \mathrm{G}\Phi_{\mathrm{F}c} \otimes \Psi_{\Psi_c}"] & & \mathrm{G}(t \otimes \Phi_{\mathrm{F}c} \otimes \mathrm{F}\Psi_c) \otimes \Psi_c \arrow[d, "\mathrm{G}(t \otimes \beta_c) \otimes \Psi_c"] \\
\mathrm{G}t \otimes \Psi_c \arrow[u, "\mathrm{G}t \otimes \alpha_{\mathrm{F}c} \otimes \Psi_c"] \arrow[rr, "\mathrm{i}"'] & & \mathrm{G}t \otimes \Psi_c
\end{tikzcd}
\tag{1.16}
$$

and so the triangle identity (1.15) holds iff diagram (1.16) commutes for all 1-cells $(t\colon \mathrm{F}c \longrightarrow b) \in \underline{\underline{\mathcal{B}}}$.

Once expressed in this form it is clear, by easy diagram chases involving the coherence properties of the canonical 2-cells associated with the morphism G, that we may obtain diagram (1.16) by applying the functor $\mathrm{G}t\otimes\text{–}$ to (1.14) and composing with:

$$
\mathrm{G}t \otimes \mathrm{Gi}_{\mathrm{F}c} \otimes \Psi_c \xrightarrow{\mathrm{can} \otimes \Psi_c} \mathrm{G}(t \otimes \mathrm{i}_{\mathrm{F}c}) \otimes \Psi_c \xrightarrow[\simeq]{\mathrm{G}(\mathrm{can}) \otimes \Psi_c} \mathrm{G}t \otimes \Psi_c
$$

Conversely we get diagram (1.14) by setting $b = \mathrm{F}c$ and $t = \mathrm{i}_{\mathrm{F}c}$ in (1.16) and then composing with:

$$
\mathrm{i}_{\mathrm{GF}c} \otimes \Psi_c \xrightarrow{\mathrm{can} \otimes \Psi_c} \mathrm{Gi}_{\mathrm{F}c} \otimes \Psi_c
$$

It follows therefore that, for any given pair of 0-cells $c \in \underline{\underline{\mathcal{C}}}$ and $b \in \underline{\underline{\mathcal{B}}}$, the diagram (1.14) commutes iff (1.16) commutes for all $t \in \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b)$ iff the triangle identity in (1.15) holds. $\square$

Example 1.1.7 (Biadjunctions) By definition a homomorphism $\mathrm{G}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$ has a *left biadjoint* if for each 0-cell $c \in \underline{\underline{\mathcal{C}}}$ there exists a 0-cell $\mathrm{F}c \in \underline{\underline{\mathcal{B}}}$ and a 1-cell $\Psi_c\colon c \longrightarrow \mathrm{GF}c$ such that the functor

$$\psi_{cb} = \quad \underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \xrightarrow{\mathrm{G}} \underline{\underline{\mathcal{C}}}(\mathrm{GF}c, \mathrm{G}b) \xrightarrow{-\otimes \Psi_c} \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b)$$

is an equivalence for each pair of 0-cells $b \in \underline{\underline{\mathcal{B}}}$, $c \in \underline{\underline{\mathcal{C}}}$.

Given a 1-cell $(q\colon c' \longrightarrow c) \in \underline{\underline{\mathcal{C}}}$ we define $\mathrm{F}q\colon \mathrm{F}c' \longrightarrow \mathrm{F}c$ to be a 1-cell equipped with some isomorphism

$$\Psi_c \otimes q \underset{\Psi_q}{\xrightarrow{\cong}} \mathrm{GF}q \otimes \Psi_{c'}$$

which exists and is unique up to isomorphism by the universal property of biadjoints given above. Now define the action of F on a 2-cell $\gamma\colon q \longrightarrow r$ of $\underline{\underline{\mathcal{C}}}$ to be the unique 2-cell $\mathrm{F}\gamma\colon \mathrm{F}q \longrightarrow \mathrm{F}r$ such that:

$$\Psi_r \bullet (\Psi_c \otimes \gamma) = (\mathrm{GF}\gamma \otimes \Psi_{c'}) \bullet \Psi_q$$

This action is clearly functorial by the uniqueness clause of its definition. The identity and compositional comparison isomorphisms making F into a homomorphism are the uniquely existing ones chosen precisely to ensure that the 1-cells Ψ_c and 2-cells Ψ_q become the components of a strong transformation $\Psi\colon \mathrm{I} \longrightarrow \mathrm{GF}$. This sort of construction is illustrated in proposition 4.2 of [6] and will feature in latter sections so we do not dwell on it here.

Of course we may pick an equivalence inverse φ_{cb} for each ψ_{cb} along with isomorphisms $\kappa_{cb}\colon \mathrm{i} \xrightarrow{\cong} \psi_{cb} \circ \varphi_{cb}$ and $\tau_{cb}\colon \varphi_{cb} \circ \psi_{cb} \xrightarrow{\cong} \mathrm{i}$ satisfying the triangle identities and displaying this as an adjoint equivalence. Applying lemma 1.1.3 to Ψ we give the ψ_{cb} the structure of a strong transformation $\psi\colon \mathrm{F}_\# \longrightarrow \mathrm{G}^\#$ and taking mates of its structure 2-cells under the adjoint equivalences above we lift the φ_{cb} to a strong transformation $\varphi\colon \mathrm{G}^\# \longrightarrow \mathrm{F}_\#$. By the $(-)^{\mathrm{coop}}$ dual of the comment after lemma 1.1.3 we know that since F is a homomorphism and φ is a strong transformation then there exists a strong transformation $\Phi\colon \mathrm{FG} \longrightarrow \mathrm{I}_{\underline{\underline{\mathcal{B}}}}$ which induces φ as in lemma $(1.1.3)^{\mathrm{coop}}$. This is enough to demonstrate that F is a local left adjoint of G mediated by unit Ψ and counit Φ.

Notice that when we express this local adjunction in terms of the triangle identities given in theorem 1.1.6 the 2-cells α_b ($b \in \underline{\underline{\mathcal{B}}}$) and β_c ($c \in \underline{\underline{\mathcal{C}}}$) are all isomorphisms. In fact, reversing the process above, we see that the notion of triangle identities for biadjoints consists of homomorphisms F and G, unit Ψ and counit Φ which are *strong* transformations and isomorphisms

$$
\begin{tikzcd}
\mathrm{G} \arrow[r, "\Psi\mathrm{G}"] \arrow[dr, "\mathrm{i}_{\mathrm{G}}"'] & \mathrm{GFG} \arrow[d, "\mathrm{G}\Phi"] \arrow[dl, phantom, "\overset{\cong}{\alpha}"] \\
 & \mathrm{G}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{F} \arrow[d, "\mathrm{F}\Psi"'] \arrow[dr, "\mathrm{i}_{\mathrm{F}}"] & \\
\mathrm{FGF} \arrow[r, "\Phi\mathrm{F}"'] \arrow[ur, phantom, "\underset{\cong}{\beta}"] & \mathrm{F}
\end{tikzcd}
$$

in $\mathcal{H}om_S(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{C}}})(\mathrm{G}, \mathrm{G})$ and $\mathcal{H}om_S(\underline{\underline{\mathcal{C}}}, \underline{\underline{\mathcal{B}}})(\mathrm{F}, \mathrm{F})$ respectively. The extra conditions on this as a local adjunction, (1.14) and its dual, simply ensure that the equivalences between $\underline{\underline{\mathcal{B}}}(\mathrm{F}c, b)$ and $\underline{\underline{\mathcal{C}}}(c, \mathrm{G}b)$ derived from this information are adjoint ones. Its is clear though that for any biadjoint we may always choose the isomorphisms α and β such that they satisfy these conditions, and we will often assume that this has been done, for precisely the sorts of reason that we often assume that all equivalences are presented as adjoint ones. We will return to biadjoints and their triangle identities as a foundation for change of base in later sections.

Example 1.1.8 *The local adjunctions in [6] are mediated by a unit and counit in this way, but we leave detailed verification of that fact until later.*

Local adjunctions mediated by a unit and counit suffer from a number of drawbacks:

(i) There is no convenient way, in any great generality, of expressing them in terms of a one-sided universal property. The one given in definition 4.1 of [6] certainly produces local adjunctions with a unit, but there seems to be no simple condition that we may use to ensure that we also get a counit.

(ii) They do not compose. Although a composite of two such locally adjoint pairs will always be a local adjunction it will not necessarily be mediated by a unit and counit. This stems from the fact that notion of applying a morphism to an optransformation has no real meaning in general.

Their real utility will become apparent as we examine change of base in more detail. For the moment we present a "Day type" result which will become important later on when we consider colimits in enriched categories.

Proposition 1.1.9 *If* $\mathrm{F} \dashv\!|_{\psi} \mathrm{G} \colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$ *is a local adjunction and*

$$
\begin{tikzcd}
\mathrm{F}c' \arrow[rr, "\mathrm{F}q"] \arrow[dr, "p"'] & \arrow[d, phantom, "\overset{\gamma}{\Leftarrow}"] & \mathrm{F}c \arrow[dl, "(p \Leftarrow \mathrm{F}q)"] \\
 & b &
\end{tikzcd}
\tag{1.17}
$$

is a right Kan extension diagram in $\underline{\underline{\mathcal{B}}}$ then the diagram

$$\begin{tikzcd} c' \arrow[rr, "q"] \arrow[dr, "\psi_{c'b}(p)"'] & \arrow[d, phantom, "\overset{\check{\gamma}}{\Leftarrow}"] & c \arrow[dl, "\psi_{cb}(p \Leftarrow \mathrm{F}q)"] \\ & \mathrm{G}b & \end{tikzcd} \tag{1.18}$$

where $\check{\gamma}$ is the composite

$$\psi_{cb}(p \Leftarrow \mathrm{F}q) \otimes q \xrightarrow{\psi_{qb}^{(p \Leftarrow \mathrm{F}q)}} \psi_{c'b}((p \Leftarrow \mathrm{F}q) \otimes \mathrm{F}q) \xrightarrow{\psi_{c'b}(\gamma)} \psi_{c'b}(p)$$

is also a right extension diagram in $\underline{\underline{\mathcal{C}}}$, so long as the natural transformation φ_{qb} is an isomorphism.

Proof. We have the following series of bijections, which are natural in
$r \in \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b)$:

$$\begin{array}{cl} r \longrightarrow \psi_{cb}(p \Leftarrow \mathrm{F}q) & \\ \hline & \text{since } \varphi_{cb} \dashv \psi_{cb} \\ \varphi_{cb}(r) \longrightarrow p \Leftarrow \mathrm{F}q & \\ \hline & \text{since (1.17) is a Kan extn.} \\ \varphi_{c'b}(r \otimes q) \xrightarrow[\cong]{\varphi_{qb}^{r}} \varphi_{cb}(r) \otimes \mathrm{F}q \longrightarrow p & \\ \hline & \text{since } \varphi_{c'b} \dashv \psi_{c'b} \\ r \otimes q \longrightarrow \psi_{c'b}(p) & \end{array} \tag{1.19}$$

This certainly identifies $\psi_{cb}(p \Leftarrow \mathrm{F}q)$ as the right Kan extension of $\psi_{c'b}(p)$ along q, all that remains is to show that $\check{\gamma}$ is indeed to counit displaying that Kan extension. To do this we simply follow what happens to the identity map

$$\psi_{cb}(p \Leftarrow \mathrm{F}q) \xrightarrow{\mathrm{i}_{\psi_{cb}(p \Leftarrow \mathrm{F}q)}} \psi_{cb}(p \Leftarrow \mathrm{F}q)$$

under these bijections as follows:

$$\begin{array}{c} \psi_{cb}(p \Leftarrow \mathrm{F}q) \xrightarrow{\mathrm{i}_{\psi_{cb}(p \Leftarrow \mathrm{F}q)}} \psi_{cb}(p \Leftarrow \mathrm{F}q) \\ \hline \varphi_{cb}\psi_{cb}(p \Leftarrow \mathrm{F}q) \xrightarrow{\tau_{cb}^{(p \Leftarrow \mathrm{F}q)}} p \Leftarrow \mathrm{F}q \\ \hline \varphi_{cb}\psi_{cb}(p \Leftarrow \mathrm{F}q) \otimes \mathrm{F}q \xrightarrow{\tau_{cb}^{(p \Leftarrow \mathrm{F}q)} \otimes \mathrm{F}q} (p \Leftarrow \mathrm{F}q) \otimes \mathrm{F}q \xrightarrow{\gamma} p \end{array} \tag{1.20}$$

We know that the structure 2-cells of ψ and φ are mates under the adjunctions $\varphi_{cb} \dashv \psi_{cb}$ and so we have the identity:

$$
\begin{tikzcd}
\underline{\underline{\mathcal{B}}}(\mathrm{F}c,b) \arrow[d, "\mathrm{i}"'] \arrow[dr, "\psi_{cb}"] \arrow[d, phantom, shift left=4ex, "\overset{\tau_{cb}}{\Leftarrow}"] & \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c,b) \arrow[d, "-\otimes \mathrm{F}q"'] & \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b) \arrow[l, "\varphi_{cb}"] \arrow[d, "-\otimes q"] \arrow[dl, phantom, "\Uparrow \varphi_{qb}"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c',b) & \underline{\underline{\mathcal{C}}}(c',\mathrm{G}b) \arrow[l, "\varphi_{c'b}"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\underline{\underline{\mathcal{B}}}(\mathrm{F}c,b) \arrow[r, "\psi_{cb}"] \arrow[d, "-\otimes \mathrm{F}q"'] \arrow[dr, phantom, "\Downarrow \psi_{qb}"] & \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b) \arrow[d, "-\otimes q"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c',b) \arrow[r, "\psi_{c'b}"] \arrow[d, "\mathrm{i}"'] \arrow[d, phantom, shift left=4ex, "\underset{\tau_{cb'}}{\Leftarrow}"] & \underline{\underline{\mathcal{C}}}(c',\mathrm{G}b) \arrow[dl, "\varphi_{cb'}"] \\
\underline{\underline{\mathcal{B}}}(\mathrm{F}c,b') &
\end{tikzcd}
\tag{1.21}
$$

On composing the map at the bottom of (1.20) with $\varphi_{qb}^{(p \Leftarrow \mathrm{F}q)}$ we get γ composed with the component of the natural transformation on the left and side of the above equality at $(p \Leftarrow \mathrm{F}q)$. Applying this identity and the naturality of $\varphi_{c'b}$ we get the composite

$$
\begin{aligned}
\varphi_{c'b}(\psi_{cb}(p \Leftarrow \mathrm{F}q) \otimes q) &\xrightarrow{\varphi_{c'b}(\psi_{qb})} \varphi_{c'b}\psi_{c'b}((p \Leftarrow \mathrm{F}q) \otimes \mathrm{F}q) \\
&\xrightarrow{\varphi_{c'b}\psi_{c'b}(\gamma)} \varphi_{c'b}\psi_{c'b}(p) \\
&\xrightarrow{\tau_{c'b}^{p}} p
\end{aligned}
$$

which corresponds under the last bijection in our sequence to $\check{\gamma}$. $\square$

A useful and immediate corollary of this proposition is:

Corollary 1.1.10 *If* $\mathrm{F} \dashv\!\!\vdash_{\Psi,\Phi} \mathrm{G}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$ *is a locally adjoint pair mediated by a unit* Ψ *and counit* Φ *and*

$$
\begin{tikzcd}
\mathrm{F}c' \arrow[rr, "\mathrm{F}q"] \arrow[dr, "p"'] & \arrow[d, phantom, "\overset{\gamma}{\Leftarrow}"] & \mathrm{F}c \arrow[dl, "(p \Leftarrow \mathrm{F}q)"] \\
& b &
\end{tikzcd}
\tag{1.22}
$$

is a right Kan extension diagram in $\underline{\underline{\mathcal{B}}}$ then the pasting of

$$\begin{tikzcd}
c' \arrow[rr, "q"] \arrow[d, "\Psi_{c'}"'] & \arrow[d, phantom, "\overset{\Psi_q}{\Leftarrow}"] & c \arrow[d, "\Psi_c"] \\
\mathrm{GF}c' \arrow[rr, "\mathrm{GF}q"] \arrow[dr, "\mathrm{G}p"'] & \arrow[d, phantom, "\overset{\mathrm{G}\gamma}{\Leftarrow}"] & \mathrm{GF}c \arrow[dl, "\mathrm{G}(p \Leftarrow \mathrm{F}q)"] \\
& \mathrm{G}b &
\end{tikzcd} \tag{1.23}$$

is also a right Kan extension diagram in $\underline{\underline{\mathcal{C}}}$, so long as the comorphism F *"preserves pre-composition with q". This means that for any 1-cell $(r\colon c \longrightarrow \mathrm{G}b) \in \underline{\underline{\mathcal{C}}}$ the compositional comparison*

$$\mathrm{F}(r \otimes q) \xrightarrow{\text{can}} \mathrm{F}r \otimes \mathrm{F}q$$

is an isomorphism. □

Notice that in order to interpret diagram (1.23) correctly we need to examine more closely the context of the 2-cell denoted by $\mathrm{G}\gamma$. This cannot be that obtained by merely applying G to γ, since that has codomain $\mathrm{G}((p \Leftarrow \mathrm{F}q) \otimes \mathrm{F}q)$, the natural (and correct) interpretation is the composite:

$$\mathrm{G}(p \Leftarrow \mathrm{F}q) \otimes \mathrm{GF}q \xrightarrow{\text{can}} \mathrm{G}((p \Leftarrow \mathrm{F}q) \otimes \mathrm{F}q) \xrightarrow{\mathrm{G}\gamma} \mathrm{G}p$$

1.2 Equipments.

In this section we introduce structures suited for an abstract study of category theories in the context of change of base. We will call these *equipments* since they are minor variants of the *proarrow equipments* introduced by Wood in [56] and [57]. Most of the material in this section is folklore, but seems to need a little elaboration in order to fit in with an abstract approach to questions of base change. First the definition:

Definition 1.2.1 *An* equipment $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$ *consists of bicategories* $\underline{\underline{\mathcal{M}}}$, $\underline{\underline{\mathcal{K}}}$ *with the same sets of 0-cells and a homomorphism*

$$\underline{\underline{\mathcal{K}}}^{\mathrm{co}} \xrightarrow{\quad (-)_* \quad} \underline{\underline{\mathcal{M}}}$$

with the properties:

(i) $(-)_*$ *is the identity on 0-cells,*

(ii) if $f\colon a \longrightarrow b$ *is a 1-cell in* $\underline{\underline{\mathcal{K}}}$ *then* f_* *has a right adjoint* f^* *in* $\underline{\underline{\mathcal{M}}}$*,*

An equipment becomes a proarrow equipment, in the sense of Wood, if it also satisfies the condition:

(iii) $(-)_*$ *is locally fully faithful, in other words the functor*

$$\underline{\underline{\mathcal{K}}}^{\mathrm{co}}(a,b) \xrightarrow{\quad (-)_* \quad} \underline{\underline{\mathcal{M}}}(a,b)$$

is fully faithful for each pair of 0-cells $a, b \in \underline{\underline{\mathcal{K}}}$*.*

While it is true that we may always replace an equipment with a proarrow equipment, it is not true that this process always behaves well with respect to the maps between those equipments introduced in latter sections. Often, without loss of generality, we will also assume that the homomorphism $(-)_*$ is *normal*, in other words it preserves identities on the nose rather than just up to isomorphism. So by a slight abuse of notation we may confuse the identities on a 0-cell a in each bicategory $\underline{\underline{\mathcal{K}}}$ and $\underline{\underline{\mathcal{M}}}$ and simply use the symbol i_a for both of them.

Another variant of the equipments theme is that of a *weak* equipment in which we also abandon condition (ii) of the definition. Our principle purpose in introducing these is that much of the theory of change of base we will develop is independent of condition (ii), a fact which holds some importance in later work which we do not develop here. The extra axiom only begins to play a part in our considerations, and in some cases simplify matters, much further on in the narrative.

The intended meaning of the definition becomes a little clearer on pointing out that we think of $\underline{\underline{\mathcal{K}}}$ as a bicategory of categories and *functors* (or sets and functions) and $\underline{\underline{\mathcal{M}}}$ as the corresponding bicategory of categories and *profunctors* (or sets and relations). In this context given a functor $f\colon \mathbb{A} \longrightarrow \mathbb{B} \in \underline{\mathcal{K}}$, the 1-cell $f_* \in \underline{\mathcal{M}}$ becomes the *left* representable profunctor $\mathbb{B}(f(*), -)\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!+\,\, \mathbb{B}$ and f^* the corresponding *right* representable $\mathbb{B}(*, f(-))\colon \mathbb{B} \longrightarrow\!\!\!\!\!\!+\,\, \mathbb{A}$. A few important examples should make things clearer.

Example 1.2.2 (equipment of matrices) Let $\underline{\underline{\mathcal{B}}}$ be a bicategory with:

(i) a small set of 0-cells,

(ii) small local stable coproducts, in other words each "homset" $\underline{\underline{\mathcal{B}}}(b, b')$ has all small coproducts and all functors

$$\underline{\underline{\mathcal{B}}}(b, b') \xrightarrow{p \otimes -} \underline{\underline{\mathcal{B}}}(b, c') \text{ for each 1-cell } p\colon b' \longrightarrow c',$$
$$\underline{\underline{\mathcal{B}}}(b, b') \xrightarrow{- \otimes q} \underline{\underline{\mathcal{B}}}(c, b') \text{ for each 1-cell } q\colon c \longrightarrow b$$

preserve these coproducts.

The bicategory of *matrices* in $\underline{\underline{\mathcal{B}}}$, denoted by $\underline{\underline{\mathcal{B}}}$–Mat, is defined as follows:

0-cells pairs (A, α) where A is a small set and $\alpha\colon A \longrightarrow |\underline{\underline{\mathcal{B}}}|$ is a function into the (small) set of 0-cells of $\underline{\underline{\mathcal{B}}}$. Of course these are simply the 0-cells of the slice category $\underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}$.

1-cells $m\colon (A, \alpha) \longrightarrow\!\!\!\!\!\!\ominus\,\, (B, \beta)$ consist of a family of 1-cells:

$$\{(m_{ab}\colon \alpha(a) \longrightarrow \beta(b)) \in \underline{\underline{\mathcal{B}}} \,|\, a \in A, b \in B\}$$

Vertical composition of 2-cells in $\underline{\underline{\mathcal{B}}}$–Mat is simply pointwise composition of 2-cells in $\underline{\underline{\mathcal{B}}}$.

2-cells $\tau\colon m \Rightarrow n\colon (A, \alpha) \longrightarrow\!\!\!\!\!\!\ominus\,\, (B, \beta)$ consist of a family of 2-cells:

$$\{(\tau_{ab}\colon m_{ab} \Rightarrow n_{ab}) \in \underline{\underline{\mathcal{B}}} \,|\, a \in A, b \in B\}$$

composition this gives $\underline{\underline{\mathcal{B}}}$–Mat its name since it is, in essence, traditional matrix multiplication. Given $m\colon (A, \alpha) \longrightarrow\!\!\!\!\!\!\ominus\,\, (B, \beta)$ and $n\colon (B, \beta) \longrightarrow\!\!\!\!\!\!\ominus\,\, (C, \gamma)$ the composite $n \otimes m$ is the family given by

$$(n \circledast m)_{ac} = \coprod_{b \in B} n_{bc} \otimes m_{ab}$$

with a similar formula for horizontal composition of 2-cells. The associativity isomorphism for this composite is a direct consequence of condition (ii) on $\underline{\underline{\mathcal{B}}}$

because if $l\colon (C,\gamma) \longrightarrow\!\!\!\!\!\!\!\circ\!\!> (D,\delta)$ is another 1-cell we have:

$$\begin{array}{rcl} (l \circledast (n \circledast m))_{ad} & = & \coprod_{c\in C} l_{cd} \otimes \Big(\coprod_{b\in B} n_{bc} \otimes m_{ab}\Big) \\ & \underset{\text{rule (ii)}}{\cong} & \coprod_{c\in C}\coprod_{b\in B} l_{cd} \otimes (n_{bc} \otimes m_{ab}) \\ & \cong & \coprod_{b\in B}\coprod_{c\in C} (l_{cd} \otimes n_{bc}) \otimes m_{ab} \\ & \underset{\text{rule (ii)}}{\cong} & \coprod_{b\in B} \Big(\coprod_{c\in C} l_{cd} \otimes n_{bc}\Big) \otimes m_{ab} \\ & = & ((l \circledast n) \circledast m)_{ad} \end{array}$$

identity on a 0-cell (A,α) is the diagonal matrix

$$\Big(\mathrm{i}_{(A,\alpha)}\Big)_{aa'} = \begin{cases} \mathrm{i}_{\alpha(a)} & \text{if } a = a' \\ 0 & \text{otherwise.} \end{cases}$$

where the 0 symbol is used to denote the initial object in each "homset" of $\underline{\underline{\mathcal{B}}}$. These, being the coproducts of empty diagrams, are guarantied to exist and be stable by condition (ii). To derive the isomorphism displaying this as a right identity consider the composite $m \circledast \mathrm{i}_{(A,\alpha)}$:

$$\begin{array}{rcl} \Big(m \otimes \mathrm{i}_{(A,\alpha)}\Big)_{ab} & \cong & \Big(m_{ab} \otimes \Big(\mathrm{i}_{(a,\alpha)}\Big)_{aa}\Big) \amalg \coprod_{\substack{a'\in a \\ a'\neq a}} \Big(m_{a'b} \otimes \Big(\mathrm{i}_{(a,\alpha)}\Big)_{aa'}\Big) \\ & \cong & \Big(m_{ab} \otimes \mathrm{i}_{\alpha(a)}\Big) \amalg \coprod_{\substack{a'\in a \\ a'\neq a}} (m_{a'b} \otimes 0) \end{array}$$

but $m_{a'b} \otimes 0 \cong 0$, since coproducts are stable, and so the expression at the bottom of this display is canonically isomorphic to m_{ab} as required. Provision of the left identity isomorphism is similar; then verifying the coherence conditions of these and the associativity isomorphisms is easy.

Now define an equipment $(\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (\text{–})_\circ)$, where the homomorphism $(\text{–})_\circ$ maps a function $f\colon (A,\alpha) \longrightarrow (B,\beta)$ in $\underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}$ to the matrix $f_\circ\colon (A,\alpha) \longrightarrow\!\!\!\!\!\!\!\circ\!\!> (B,\beta)$ given by

$$(f_\circ)_{ab} = \begin{cases} \mathrm{i}_{\alpha(a)} & \text{if } f(a) = b \\ 0 & \text{otherwise.} \end{cases}$$

which is well defined since when $f(a) = b$ it follows that $\beta(b) = (\beta \circ f)(a) = \alpha(a)$. It is a matter of routine calculations to check that this defines a normal homomorphism $(\text{–})_\circ$ which acts as the identity on objects.

Of course we have another matrix $f^\circ\colon (B,\beta) \mathrel{\circ\!\!\!\longrightarrow} (A,\alpha)$ associated with f which is simply the matrix transpose of $f_\circ$, and $f_\circ \dashv f^\circ$ in $\underline{\underline{\mathcal{B}}}$–Mat because

$$
\begin{array}{rcl}
(f_\circ \circledast f^\circ)_{bb'} &=& \coprod_{a\in A} (f_\circ)_{ab'} \otimes (f^\circ)_{ba} \;\cong\; \begin{cases} \coprod_{a\in f^{-1}(\{b\})} \mathrm{i}_{\beta(b)} & \text{if } b = b' \\ 0 & \text{otherwise} \end{cases} \\
(f^\circ \circledast f_\circ)_{aa'} &=& \coprod_{b\in B} (f^\circ)_{ba} \otimes (f_\circ)_{a'b} \;\cong\; \begin{cases} \mathrm{i}_{\alpha(a)} & \text{if } f(a) = f(a') \\ 0 & \text{otherwise} \end{cases}
\end{array}
$$

suggesting natural candidates for unit and counit.

Notice that we may express *re-indexing* of matrices in terms of composition with matrices $g_\circ$ and h°. If

$$m\colon (A,\alpha) \mathrel{\circ\!\!\!\longrightarrow} (B,\beta)$$

is a matrix and

$$
\begin{array}{l}
g\colon (A',\alpha') \longrightarrow (A,\alpha), \\
h\colon (B',\beta') \longrightarrow (B,\beta)
\end{array}
$$

are functions in $\underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}$ then:

$$
\begin{array}{rcl}
(m \circledast g_\circ)_{a'b} &\cong& (m_{(ga')b} \otimes \mathrm{i}_{\alpha'(a')}) \amalg \left(\coprod_{\substack{a\in A \\ a\neq g(a')}} m_{ab} \otimes 0 \right) \;\cong\; m_{(ga')b} \\
(h^\circ \circledast m)_{ab'} &\cong& (\mathrm{i}_{\beta'(b')} \otimes m_{a(hb')}) \amalg \left(\coprod_{\substack{b\in B \\ b\neq h(b')}} 0 \otimes m_{ab} \right) \;\cong\; m_{a(hb')}
\end{array}
$$

Clearly equipments of matrices will play a part in the theory of enriched categories, and were introduced in [5] to just that end. The corresponding equipment in the theory of internal categories is:

Example 1.2.3 (equipment of spans) Suppose that $\underline{\mathcal{E}}$ is a category with finite limits then the bicategory of *spans* in $\underline{\mathcal{E}}$, denoted by Span($\underline{\mathcal{C}}$), is defined as follows:

0-cells are simply the objects of $\underline{\mathcal{E}}$,

1-cells $(s_0, S, s_1)\colon A \mathrel{\circ\!\!\!\longrightarrow} B$ are spans

$$
\begin{tikzcd}
& S \arrow[dl, "s_0"'] \arrow[dr, "s_1"] & \\
B & & A
\end{tikzcd}
$$

in $\underline{\mathcal{E}}$.

2-cells $\tau\colon (r_0, R, r_1) \Rightarrow (s_0, S, s_1)\colon A \mathrel{-\!\!\!\circ\!\!\!\rightarrow} B$ are maps $(\tau\colon R \longrightarrow S) \in \underline{\mathcal{E}}$ such that the diagram

$$\begin{tikzcd}
& R \arrow[dl, "r_0"'] \arrow[dr, "r_1"] \arrow[dd, "\tau"'] & \\
B & & A \\
& S \arrow[ul, "s_0"] \arrow[ur, "s_1"'] &
\end{tikzcd}$$

commutes. The vertical composition of 2-cells is achieved by composing the underlying maps in $\underline{\mathcal{E}}$.

composition given spans $(s_0, S, s_1)\colon A \longrightarrow B$ and $(t_0, T, t_1)\colon B \longrightarrow C$ their composite denoted $(t_0, T, t_1) \underset{B}{\times} (s_0, S, s_1)$ is the span $(t_0 \circ \pi_T, T \underset{B}{\times} S, s_1 \circ \pi_S)$ where $T \underset{B}{\times} S$ is the pullback of T and S over B with canonical projections π_T and π_S, pictorially:

$$\begin{tikzcd}
& & T \underset{B}{\times} S \arrow[dl, "\pi_T"'] \arrow[dr, "\pi_S"] \arrow[dd, phantom, "\lrcorner", very near start, rotate=-45] & & \\
& T \arrow[dl, "t_0"'] \arrow[dr, "t_1"] & & S \arrow[dl, "s_0"'] \arrow[dr, "s_1"] & \\
C & & B & & A
\end{tikzcd}$$

The horizontal composite of 2-cells τ and $\tau'\colon (r'_0, R', r'_1) \Rightarrow (s'_0, S', s'_1)$ is the unique map $\tau' \underset{B}{\times} \tau\colon R' \underset{B}{\times} R \longrightarrow S' \underset{B}{\times} S$ induced by the universal property of $S' \underset{B}{\times} S$. We are furnished with associativity isomorphisms as a simple corollary of the composition lemma for pullback squares.

identity on a 0-cell $A \in \mathrm{Span}(\underline{\mathcal{E}})$ is the span:

$$\begin{tikzcd}
& A \arrow[dl, "\mathrm{i}_A"'] \arrow[dr, "\mathrm{i}_A"] & \\
A & & A
\end{tikzcd}$$

The canonical isomorphisms

$$(\mathrm{i}_B, B, \mathrm{i}_B) \underset{B}{\times} (s_0, S, s_1) \cong (s_0, S, s_1) \quad \text{and}$$
$$(s_0, S, s_1) \underset{B}{\times} (\mathrm{i}_A, A, \mathrm{i}_A) \cong (s_0, S, s_1)$$

are a direct consequence of the fact that the pullback functor

$$\underline{\mathcal{E}}/_A \xrightarrow{\ \mathrm{i}_A^*\ } \underline{\mathcal{E}}/_A$$

is canonically (naturally) isomorphic to the identity functor for each object $A \in \underline{\mathcal{E}}$. Verifying the coherence conditions for these and the associativity isomorphisms is an easy exercise.

Now define an equipment $(\mathrm{Span}(\underline{\mathcal{E}}), \underline{\mathcal{E}}, (\text{–})_\circ)$, where $(\text{–})_\circ$ maps a morphism

$$A \xrightarrow{\quad f \quad} B \qquad \in \underline{\mathcal{E}}$$

to the span:

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "\mathrm{i}_A"] & \\ B & & A \end{tikzcd}$$

It is straightforward to show that this is the action of a (normal) homomorphism, again as a direct consequence of the fact that each pullback functor i_B^* is isomorphic to the identity. Finally the right adjoint to $f_\circ$, in $\mathrm{Span}(\underline{\mathcal{E}})$, is its transpose f°:

$$\begin{tikzcd} & A \arrow[dl, "\mathrm{i}_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

This adjunction is easy to demonstrate since

$$\begin{aligned} f^\circ \underset{B}{\times} f_\circ &\cong (k_0, K, k_1) \quad \text{and} \\ f_\circ \underset{A}{\times} f^\circ &\cong (f, A, f) \end{aligned}$$

where

$$\begin{tikzcd} K \arrow[r, shift left, "k_0"] \arrow[r, shift right, "k_1"'] & A \end{tikzcd}$$

is the kernel pair of the morphism f. Then the unit of $f_\circ \dashv f^\circ$ is the "diagonal" map $\Delta\colon A \longrightarrow K$ and its counit is $f\colon A \longrightarrow B$ itself.

Before moving on to the next example, in which we construct equipments of enriched and internal categories from the previous two examples, we should review a few of the notational conventions appropriate to the study of equipments and introduce a "calculus of squares" for them. When working with an equipment $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{–})_*)$ we will tend to use letters

a, b, c, d	to denote its 0-cells,		
f, g, h, k	"	"	the 1-cells of $\underline{\underline{\mathcal{K}}}$ and
p, q, r, s	"	"	the 1-cells of $\underline{\underline{\mathcal{M}}}$.

When no confusion will result we often drop the use of subscripted asterisks and simply rely on context to convey whether we are talking about a 1-cell $f \in \underline{\underline{\mathcal{K}}}$ or the corresponding representable $f_* \in \underline{\underline{\mathcal{M}}}$. In order to aid this rule we write composition in $\underline{\underline{\mathcal{K}}}$ as $g \circ f$ and that in $\underline{\underline{\mathcal{M}}}$ tensorially as $q \otimes p$ ($q \circledast p$ or $q \underset{\bullet}{\times} p$ for matrices and spans). In diagrams we distinguish between the 1-cells in each bicategory by displaying those of $\underline{\underline{\mathcal{K}}}$ as plain arrows $f\colon a \longrightarrow b$ and those of $\underline{\underline{\mathcal{M}}}$ as adorned ones, using $p\colon a \mathrel{\relbar\joinrel\ominus\joinrel\rightarrow} b$ if we are thinking of them as matrices, spans or relations and $p\colon a \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} b$ if they are profunctors.

It is well known that from a homomorphism $\mathrm{H}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$ and any bicategory $\underline{\underline{\mathcal{A}}}$ we may naturally derive a homomorphism,

$$\mathcal{B}icat(\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{B}}}) \xrightarrow{\mathcal{B}icat(\underline{\underline{\mathcal{A}}}, \mathrm{H})} \mathcal{B}icat(\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{C}}})$$

furthermore it is also a matter of folklore that this result no longer applies on replacement of H with a mere morphism $\mathrm{M}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$. On closer analysis it becomes apparent that M fails to act well on transformations. Suppose that $\alpha\colon \mathrm{F} \Rightarrow \mathrm{G}$ is a transformation in $\mathcal{B}icat(\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{B}}})$ then we might reasonably try to define a transformation $\mathrm{M}\alpha\colon \mathrm{MF} \Rightarrow \mathrm{MG}$ with 1-cellular components

$$(\mathrm{M}\alpha)_a = \ \mathrm{M}(\mathrm{F}a) \xrightarrow{\mathrm{M}(\alpha_a)} \mathrm{M}(\mathrm{G}a)$$

but now what do we do to provide 2-cellular components:

$$\begin{tikzcd}
\mathrm{M}(\mathrm{F}a) \arrow[r, "\mathrm{M}(\alpha_a)"] \arrow[d, "\mathrm{M}(\mathrm{F}p)"'] \arrow[dr, phantom, "(\mathrm{M}\alpha)_p \Downarrow"] & \mathrm{M}(\mathrm{G}a) \arrow[d, "\mathrm{M}(\mathrm{G}p)"] \\
\mathrm{M}(\mathrm{F}a') \arrow[r, "\mathrm{M}(\alpha_{a'})"'] & \mathrm{M}(\mathrm{G}a')
\end{tikzcd}$$

Attempting to extend the definition of these which works when M is a homomorphism would fail since it would culminate in trying to compose the incompatible 2-cells:

$$\mathrm{M}(\mathrm{G}p) \otimes \mathrm{M}(\alpha_a) \xrightarrow{\text{can}} \mathrm{M}(\mathrm{G}p \otimes \alpha_a) \xrightarrow{\mathrm{M}(\alpha_p)} \mathrm{M}(\alpha_{a'} \otimes \mathrm{F}p) \xleftarrow{\text{can}} \mathrm{M}(\alpha_{a'}) \otimes \mathrm{M}(\mathrm{F}p)$$

This is no problem for a homomorphism since both of the cells marked "can" would be isomorphisms.

In subsequent sections we will work on reducing change of base to "biadjoint" pairs of maps between equipments, in a way which gives rise both to local adjoints on bicategories of profunctors and true biadjoints on categories of functors (as we might expect). An important part of this program is to unify equipments together into some sort of "bicategorically enriched category", but for this to work we need the sort of property examined in the last paragraph.

Is this program feasible? We know that local adjoints commonly consist of a morphism, comorphism pair one of which, but rarely both, may be a homomorphism (just consider the examples furnished by [6]) and so we cannot rely on them both acting well on the sort of "squares" shown above. The solution that we develop ensures that we only need worry about squares of the form

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\Uparrow \lambda"] & a' \arrow[d, "p'"] \\ \bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \end{tikzcd} \tag{1.24}$$

with $f, \bar{f} \in \underline{\underline{\mathcal{K}}}$ and $p, p' \in \underline{\underline{\mathcal{M}}}$. A map of equipments

$$(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*) \xrightarrow{(\bar{\mathrm{G}}, \mathrm{G})} (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (-)_*)$$

will turn out to consist of separate actions on bicategories of profunctors and functors, which need not even coincide on representables. The important question is now no longer whether or not M is a homomorphism (or even a morphism), but rather do we have the kind of structure relating the actions of functors and profunctors which would allow us to derive a well behaved action on the kind of squares above?

It would therefore seem appropriate to examine the structure that these squares fit into, which might reasonably be dubbed a *Double Bicategory*. The slightly detailed definition of this concept is presented in section 1.4, to which we refer the reader now. We have not given the definition here since it would break up the narrative flow too much, but will use the concepts and notational conventions introduced there from now on. Our principle motivation for crystallising the algebra of squares into a definition is to make explicit exactly what information we will wish to have preserved later on.

Definition 1.2.4 Given a (weak) equipment $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$ we define a double bicategory of squares $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$ with

- bicategories $\underline{\underline{\mathcal{K}}}$, $\underline{\underline{\mathcal{M}}}$ of horizontal and vertical cells respectively,
- the set squares of the type shown in 1.24, more explicitly these are 5-tuples $(f, \bar{f}, p, p', \alpha)$ where $f, \bar{f} \in \underline{\underline{\mathcal{K}}}$ and $p, p' \in \underline{\underline{\mathcal{M}}}$ are 1-cells and $\lambda\colon \bar{f}_* \otimes p \Rightarrow p' \otimes f_*$ is a 2-cell in $\underline{\underline{\mathcal{M}}}$.

  Looking at the diagrams in definition 1.4.1 it is clear that the actions of the 2-cells of $\underline{\underline{\mathcal{K}}}$ and $\underline{\underline{\mathcal{M}}}$ on these squares should be given by the natural pastings, explicitly:

$$\begin{aligned}
\lambda \underset{H}{*} \beta &= \bar{f}_* \otimes q \xrightarrow{\bar{f}_* \otimes \beta} \bar{f}_* \otimes p \xrightarrow{\lambda} p' \otimes f_* \ , \quad \beta\colon q \Rightarrow p \in \underline{\underline{\mathcal{M}}} \\
\beta' \underset{H}{*} \lambda &= \bar{f}_* \otimes p \xrightarrow{\lambda} p' \otimes f_* \xrightarrow{\beta' \otimes f_*} q' \otimes f_* \ , \quad \beta'\colon p' \Rightarrow q' \in \underline{\underline{\mathcal{M}}} \\
\lambda \underset{V}{*} \alpha &= \bar{f}_* \otimes p \xrightarrow{\lambda} p' \otimes f_* \xrightarrow{p' \otimes \alpha_*} q' \otimes g_* \ , \quad \alpha\colon g \Rightarrow f \in \underline{\underline{\mathcal{K}}} \\
\bar{\alpha} \underset{V}{*} \lambda &= \bar{g}_* \otimes p \xrightarrow{\bar{\alpha}_* \otimes p} \bar{f}_* \otimes p \xrightarrow{\lambda} p' \otimes f_* \ , \quad \bar{\alpha}\colon \bar{f} \Rightarrow \bar{g} \in \underline{\underline{\mathcal{K}}}
\end{aligned}$$

Notice that the orientation of the 2-cells in squares and the assumption that $(\text{-})_*$ is a homomorphism from $\underline{\underline{\mathcal{K}}}^{\mathrm{co}}$ (instead of $\underline{\underline{\mathcal{K}}}$) are compatible, in the sense that the actions of the 2-cells of $\underline{\underline{\mathcal{K}}}$ are correctly left or right handed as defined. Checking that all of these are indeed well defined actions and obey the six mutual compatibility conditions of definition 1.4.1 is routine.

- We may picture the horizontal composite of two squares as

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "p"', "\shortmid" marking] \arrow[dr, phantom, "\lambda \Uparrow"] & a' \arrow[r, "f'"] \arrow[d, "p'"', "\shortmid" marking] \arrow[dr, phantom, "\lambda' \Uparrow"] & a'' \arrow[d, "p''", "\shortmid" marking] & a \arrow[r, "f' \circ f"] \arrow[d, "p"', "\shortmid" marking] \arrow[dr, phantom, "\Uparrow (\lambda' \circ \lambda)"] & a'' \arrow[d, "p''", "\shortmid" marking] \\
\bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \arrow[r, "\bar{f}'"'] & \bar{a}'' & \bar{a} \arrow[r, "\bar{f}' \circ \bar{f}"'] & \bar{a}''
\end{tikzcd}$$

(the left square maps to the right, $\longmapsto$)

where the 2-cell $\lambda' \circ \lambda$ is the composite:

$$(\bar{f}' \circ \bar{f})_* \otimes p \xrightarrow[\simeq]{\mathrm{can} \otimes p} \bar{f}'_* \otimes \bar{f}_* \otimes p \xrightarrow{\bar{f}'_* \otimes \lambda} \bar{f}'_* \otimes p' \otimes f_*$$
$$\xrightarrow{\lambda' \otimes f_*} p'' \otimes f'_* \otimes f_* \xrightarrow[\simeq]{p'' \otimes \mathrm{can}} p'' \otimes (f' \circ f)_*$$

Similarly picture vertical composition as

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "p"', "\shortmid" marking] \arrow[dr, phantom, "\lambda \Uparrow"] & a' \arrow[d, "p'", "\shortmid" marking] & & a \arrow[r, "f"] \arrow[dd, "\bar{p} \otimes p"', "\shortmid" marking] \arrow[ddr, phantom, "\Uparrow (\bar{\lambda} \otimes \lambda)"] & a' \arrow[dd, "\bar{p}' \otimes p'", "\shortmid" marking] \\
\bar{a} \arrow[r, "\bar{f}"] \arrow[d, "\bar{p}"', "\shortmid" marking] \arrow[dr, phantom, "\bar{\lambda} \Uparrow"] & \bar{a}' \arrow[d, "\bar{p}'", "\shortmid" marking] \arrow[r, phantom, "\longmapsto"] & {} & & \\
\tilde{a} \arrow[r, "\tilde{f}"'] & \tilde{a}' & & \tilde{a} \arrow[r, "\tilde{f}"'] & \tilde{a}'
\end{tikzcd}$$

where the 2-cell $\bar{\lambda} \otimes \lambda$ is the composite:

$$\tilde{f}_* \otimes \bar{p} \otimes p \xrightarrow{\bar{\lambda} \otimes p} \bar{p}' \otimes \bar{f}_* \otimes p \xrightarrow{\bar{p}' \otimes \lambda} \bar{p}' \otimes p' \otimes f_*$$

Notice that in order to agree with our conventional use of $\circ$ and $\otimes$ for the composition in $\underline{\underline{\mathcal{K}}}$ and $\underline{\underline{\mathcal{M}}}$ we also use these symbols for horizontal and vertical composition of squares. Given 1-cells $f \in \underline{\underline{\mathcal{K}}}$ and $p \in \underline{\underline{\mathcal{M}}}$ the corresponding vertical and horizontal identity squares are

$$\begin{tikzcd} a \arrow[r, "\mathrm{i}_a"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\text{can}}{\cong}"] & a \arrow[d, "p"] \\ \bar{a} \arrow[r, "\mathrm{i}_{\bar{a}}"'] & \bar{a} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "f"] \arrow[d, "\mathrm{i}_a"'] \arrow[dr, phantom, "\overset{\text{can}}{\cong}"] & a' \arrow[d, "\mathrm{i}_{a'}"] \\ a \arrow[r, "f"'] & a' \end{tikzcd}$$

where the maps marked "can" are the composites

$$\mathrm{i}_a \otimes p \xrightarrow[\cong]{\text{can}} p \xleftarrow[\cong]{\text{can}} p \otimes \mathrm{i}_{\bar{a}}$$

and

$$f_* \otimes \mathrm{i}_a \xrightarrow[\cong]{\text{can}} f_* \xleftarrow[\cong]{\text{can}} \mathrm{i}_{a'} \otimes f_*$$

respectively. The verification that the data we have presented satisfies all of the conditions given in definition 1.4.1 is routine (and moderately tedious) and we do not propose to waste time spelling it out here.

Use the notations $\mathrm{Sq}_H(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$ and $\mathrm{Sq}_H(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$ for the bicategories of cylinders Cyl_H and Cyl_V associated with this double bicategory and defined in definition 1.4.1. □

From now on, in agreement with the terminology to be introduced in section 1.4, we will often use the qualifier *depth-wise* rather than *vertical* when talking about composition of 2-cells in bicategory. When working with an equipment $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$ we may now reserve the terms *vertical* and *horizontal* for horizontal composition in $\underline{\underline{\mathcal{M}}}$ and $\underline{\underline{\mathcal{K}}}$ respectively, or the corresponding operations on squares. We exploit an understanding of the double bicategory above in the next example:

Example 1.2.5 (equipment of monads) Let $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ)$ be an equipment in which the bicategory $\underline{\underline{\mathcal{M}}}$ has local stable coequalisers of reflexive pairs. In other words each "homset" $\underline{\underline{\mathcal{M}}}(a, a')$ is a category with coequalisers of reflexive pairs, and the functors $p \otimes -$ and $- \otimes p$ preserve these for each 1-cell $p \in \underline{\underline{\mathcal{M}}}$. Think of this as a bicategory of matrices or spans (and use the appropriate notations) from which we may define an *equipment of monads* $\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ)$ which comprises:

The bicategory of modules $\mathrm{Mon}(\underline{\underline{\mathcal{M}}})$**:**

0-cells monads $(a, \mathbb{A}, \mu_a, \eta_a)$ in $\underline{\underline{\mathcal{M}}}$. These are simply bicategorical morphisms

$$\mathtt{1} \longrightarrow \underline{\underline{\mathcal{M}}}$$

where $\mathtt{1}$ the trivial one object category considered as a bicategory. Explicitly this consists of a 0-cell $a \in \underline{\underline{\mathcal{M}}}$, an endo-1-cell $\mathbb{A}: a \multimap a$ and 2-cells $\mu_a: \mathbb{A} \circledast \mathbb{A} \Rightarrow \mathbb{A}$ and $\eta_a: \mathrm{i}_a \Rightarrow \mathbb{A}$ satisfying the condition that the diagrams

$$\begin{tikzcd} \mathbb{A} \circledast \mathbb{A} \circledast \mathbb{A} \arrow[r, "\mathbb{A} \circledast \mu_a"] \arrow[d, "\mu_a \circledast \mathbb{A}"'] & \mathbb{A} \circledast \mathbb{A} \arrow[d, "\mu_a"] \\ \mathbb{A} \circledast \mathbb{A} \arrow[r, "\mu_a"'] & \mathbb{A} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \arrow[r, "\mathbb{A} \circledast \eta_a"] \arrow[dr, "\mathrm{i}_{\mathbb{A}}"'] & \mathbb{A} \circledast \mathbb{A} \arrow[d, "\mu_a"] & \mathbb{A} \arrow[l, "\eta_a \circledast \mathbb{A}"'] \arrow[dl, "\mathrm{i}_{\mathbb{A}}"] \\ & \mathbb{A} & \end{tikzcd}$$

commute (see [3]).

1-cells which are bimodules $(l_p, p, r_p): (a, \mathbb{A}) \longrightarrow (b, \mathbb{B})$, where $p: a \multimap b$ is a 1-cell and $r_p: p \circledast \mathbb{A} \Rightarrow p$, $l_p: \mathbb{B} \circledast p \Rightarrow p$ are 2-cells which satisfy the usual rules for a left-$\mathbb{B}$ right-$\mathbb{A}$ action, in other words we have identities

$$\begin{tikzcd} p \circledast \mathbb{A} \circledast \mathbb{A} \arrow[r, "p \circledast \mu_a"] \arrow[d, "r_p \circledast \mathbb{A}"'] & p \circledast \mathbb{A} \arrow[d, "r_p"] \\ p \circledast \mathbb{A} \arrow[r, "r_p"'] & p \end{tikzcd} \qquad \begin{tikzcd} p \arrow[r, "p \circledast \eta_a"] \arrow[dr, "\mathrm{i}_p"'] & p \circledast \mathbb{A} \arrow[d, "r_p"] \\ & p \end{tikzcd}$$

with two similar identities for l_p, and a mutual compatibility condition:

$$\begin{tikzcd} \mathbb{B} \circledast p \circledast \mathbb{A} \arrow[r, "\mathbb{B} \circledast r_p"] \arrow[d, "l_p \circledast \mathbb{A}"'] & \mathbb{B} \circledast p \arrow[d, "l_p"] \\ p \circledast \mathbb{A} \arrow[r, "r_p"'] & p \end{tikzcd}$$

2-cells $\alpha: (l_p, p, r_p) \Rightarrow (l_q, q, r_q)$ are 2-cells $\alpha: p \Rightarrow q$ in $\underline{\underline{\mathcal{M}}}$ which are equivariant for the given actions of $\mathbb{A}$ and $\mathbb{B}$ on p and q, which we may express in terms of commutative squares:

$$\begin{tikzcd} p \circledast \mathbb{A} \arrow[r, "r_p"] \arrow[d, "\alpha \circledast \mathbb{A}"'] & p \arrow[d, "\alpha"] \\ q \circledast \mathbb{A} \arrow[r, "r_q"'] & q \end{tikzcd} \qquad \begin{tikzcd} \mathbb{B} \circledast p \arrow[r, "l_p"] \arrow[d, "\mathbb{B} \circledast \alpha"'] & p \arrow[d, "\alpha"] \\ \mathbb{B} \circledast q \arrow[r, "l_q"'] & q \end{tikzcd}$$

Of course right-$\mathbb{A}$ left-$\mathbb{B}$ bimodules are simply the algebras of a monad on $\underline{\underline{\mathcal{M}}}(a,b)$ with functor part $\mathbb{B} \circledast - \circledast \mathbb{A}$, for which algebra homomorphisms are exactly equivariant maps.

composition is the usual tensor product of modules. Given bimodules (l_p, p, r_p) and $(l_q, q, r_q) \colon (b, \mathbb{B}) \longrightarrow (c, \mathbb{C})$ their tensor product $(l_q, q, r_q) \otimes (l_p, p, r_p)$ has underlying 1-cell given by the coequaliser

$$\begin{tikzcd} q \circledast \mathbb{B} \circledast p \arrow[r, shift left, "r_q \circledast p"] \arrow[r, shift right, "q \circledast l_p"'] & q \circledast p \arrow[r, "c_{q\otimes p}"] & q \otimes p \end{tikzcd}$$

which exists, by assumption, in $\underline{\underline{\mathcal{M}}}(a,c)$ since the pair concerned has a reflector $(q \circledast \eta_b \circledast p) \colon q \circledast p \longrightarrow q \circledast \mathbb{B} \circledast p$. The functors $- \circledast \mathbb{A}$ and $\mathbb{C} \circledast -$ preserve this coequaliser and the diagrams

$$\begin{tikzcd} q \circledast \mathbb{B} \circledast p \circledast \mathbb{A} \arrow[r, shift left, "r_q \circledast p \circledast \mathbb{A}"] \arrow[r, shift right, "q \circledast l_p \circledast \mathbb{A}"'] \arrow[d, "q \circledast \mathbb{B} \circledast r_p"'] & q \circledast p \circledast \mathbb{A} \arrow[d, "q \circledast r_p"] \\ q \circledast \mathbb{B} \circledast p \arrow[r, shift left, "r_q \circledast p"] \arrow[r, shift right, "q \circledast l_p"'] & q \circledast p \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \circledast q \circledast \mathbb{B} \circledast p \arrow[r, shift left, "\mathbb{C} \circledast r_q \circledast p"] \arrow[r, shift right, "\mathbb{C} \circledast q \circledast l_p"'] \arrow[d, "l_q \circledast \mathbb{B} \circledast p"'] & \mathbb{C} \circledast q \circledast p \arrow[d, "l_q \circledast p"] \\ q \circledast \mathbb{B} \circledast p \arrow[r, shift left, "r_q \circledast p"] \arrow[r, shift right, "q \circledast l_p"'] & q \circledast p \end{tikzcd}$$

commute serially, inducing a natural right-$\mathbb{A}$ left-$\mathbb{C}$ action on $q \otimes p$. For more detail on, for instance, the provision of associativity isomorphisms etc. see [28].

identity on a monad $(a, \mathbb{A}, \mu_a, \eta_a)$ is the bimodule $(\mu_a, \mathbb{A}, \mu_a)$.

The bicategory $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (-)_\circ)$**:**

The definition of $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (-)_\circ)$ is best explained in terms of the work of section 1.4 on double bicategories, formally it is $\mathcal{B}icat_H(\mathtt{1}, \mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (-)_\circ))$ the construction of which is described in observation 1.4.2. We give a brief description:

0-cells the monads in $\underline{\underline{\mathcal{M}}}$,

1-cells $(f,\lambda)\colon(a,\mathbb{A})\longrightarrow(b,\mathbb{B})$ consist of a 1-cell $(f\colon a\longrightarrow b)\in\underline{\underline{\mathcal{K}}}$ and a 2-cell $\lambda\colon f_\circ\circledast\mathbb{A}\Rightarrow\mathbb{B}\circledast f_\circ$ such that $(f_\circ,\lambda)$ is a monad *opfunctor* in $\underline{\underline{\mathcal{M}}}$ (in the sense of [46]). When interpreting monads as morphisms, with domain $\mathtt{1}$, monad opfunctors correspond to optransformations of morphisms.

2-cells $\alpha\colon(f,\lambda)\Rightarrow(\dot f,\dot\lambda)$ consist of a 2-cell $\alpha\colon f\Rightarrow\dot f$ in $\underline{\underline{\mathcal{K}}}$ such that $\alpha_\circ$ is a monad opfunctor transformation (or modification!) in $\underline{\underline{\mathcal{M}}}$ from $(\dot f_\circ,\dot\lambda)$ to $(f_\circ,\lambda)$. There is no mistake here, the homomorphism $(-)_*$ reverses the orientation of 2-cells.

The homomorphism $(-)_*\colon\mathrm{Mnd}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(-)_\circ)^{\mathrm{co}}\longrightarrow\mathrm{Mon}(\underline{\underline{\mathcal{M}}})$**:**

Given a 1-cell $(f,\lambda)\colon(a,\mathbb{A})\longrightarrow(b,\mathbb{B})\in\mathrm{Mnd}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(-)_\circ)$ define a bimodule $(f,\lambda)_*$ with underlying 1-cell $\mathbb{B}\circledast f_\circ\colon a\longrightarrow\!\!\!\!\!\!\!\circ\;\; b$ in $\underline{\underline{\mathcal{M}}}$ and actions given by the composites:

$$\begin{aligned}
r_{(f,\lambda)_*}&=\mathbb{B}\circledast f_\circ\circledast\mathbb{A}\xrightarrow{\mathbb{B}\circledast\lambda}\mathbb{B}\circledast\mathbb{B}\circledast f_\circ\xrightarrow{\mu_b\circledast f_\circ}\mathbb{B}\circledast f_\circ\\
l_{(f,\lambda)_*}&=\mathbb{B}\circledast\mathbb{B}\circledast f_\circ\xrightarrow{\mu_b\circledast f_\circ}\mathbb{B}\circledast f_\circ
\end{aligned}$$

That $r_{(f,\lambda)_*}$ is a well defined right action of $(a,\mathbb{A})$ follows directly from the conditions on $(f_\circ,\lambda)$ as a monad opfunctor. With no further effort it is clear that $l_{(f,\lambda)_*}$ is a left action and that these two actions commute. Now suppose that $\alpha\colon(f,\lambda)\Rightarrow(\dot f,\dot\lambda)$ is a 2-cell in $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(-)_\circ)$ then since $\alpha_\circ\colon\dot f_\circ\Rightarrow f_\circ$ is a monad opfunctor transformation we know that $\lambda\bullet(\alpha_\circ\circledast\mathbb{A})=(\mathbb{B}\circledast\alpha_\circ)\bullet\dot\lambda$ and so $\mathbb{B}\circledast\alpha_\circ\colon\mathbb{B}\circledast\dot f_\circ\Rightarrow\mathbb{B}\circledast f_\circ$ is an equivariant map. It follows therefore that we have a naturally defined functor

$$\mathrm{Mnd}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(-)_\circ)^{\mathrm{co}}((a,\mathbb{A}),(b,\mathbb{B}))\xrightarrow{(-)_*}\mathrm{Mon}(\underline{\underline{\mathcal{M}}})((a,\mathbb{A}),(b,\mathbb{B}))\tag{1.25}$$

for each pair of monads in $\underline{\underline{\mathcal{M}}}$. We postpone the proof that these functors are the "homset" actions of a naturally defined homomorphism until after the next lemma.

The derivation of a right representable $(f,\lambda)^*$, which will turn out to be right adjoint to $(f,\lambda)_*$ in $\mathrm{Mon}(\underline{\underline{\mathcal{M}}})$, is only slightly more involved. First take the mate $\lambda^\circ\colon\mathbb{A}\circledast f^\circ\Rightarrow f^\circ\circledast\mathbb{B}$ of the 2-cell λ under the adjunction $f_\circ\dashv f^\circ$, then it is a matter of the standard properties of mates (cf. [32]) to show that the pair (f°,λ°) is a monad *functor* from $(b,\mathbb{B})$ to $(a,\mathbb{A})$ (or in other words a transformation of the corresponding bicategorical morphisms). All that remains is to define $(f,\lambda)^*$ from this monad functor dually to the definition of $(f,\lambda)_*$, so we have underlying 1-cell $f^\circ\circledast\mathbb{B}\colon b\longrightarrow\!\!\!\!\!\!\!\circ\;\; a$ in $\underline{\underline{\mathcal{M}}}$ with actions:

$$\begin{aligned}
l_{(f,\lambda)^*}&=\mathbb{A}\circledast f^\circ\circledast\mathbb{B}\xrightarrow{\lambda^\circ\circledast\mathbb{B}}f^\circ\circledast\mathbb{B}\circledast\mathbb{B}\xrightarrow{f^\circ\circledast\mu_b}f^\circ\circledast\mathbb{B}\\
r_{(f,\lambda)^*}&=f^\circ\circledast\mathbb{B}\circledast\mathbb{B}\xrightarrow{f^\circ\circledast\mu_b}f^\circ\circledast\mathbb{B}
\end{aligned}$$

Again the fact that $l_{(f,\lambda)^*}$ is a well defined left action hinges on the conditions that (f°, λ°) satisfies as a monad functor, and we postpone the proof that $(f,\lambda)_* \dashv (f,\lambda)^*$ until after the subsequent lemma.

For a little more insight into this sort of construction see [9]. $\square$

As well as being crucial to the completion of the previous example, the following lemma will become more important once we have unified equipments together into a bicategorically enriched category and wish to show that the $\mathcal{M}$on() construction becomes functorial with respect that sort of structure:

Lemma 1.2.6 *Consider the constructions of observation 1.2.5. The following hold:*

(i) Given 1-cells

$$(a,\mathbb{A}) \xrightarrow{(f,\lambda)} (a',\mathbb{A}') \qquad\qquad (a',\mathbb{A}') \xrightarrow{p'} (\bar{a}',\overline{\mathbb{A}}')$$

in $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}},\underline{\mathcal{K}},(\text{-})_\circ)$ *and* $\mathrm{Mon}(\underline{\underline{\mathcal{M}}})$ *respectively, then the bimodule* $p' \otimes (f,\lambda)_*$ *is isomorphic to the profunctor with underlying 1-cell* $p' \circledast f_\circ \colon a \longrightarrow \bar{a}'$ *and actions:*

$$r_{p'\circledast f_\circ} = p' \circledast f_\circ \circledast \mathbb{A} \xrightarrow{p' \circledast \lambda} p' \circledast \mathbb{A}' \circledast f_\circ \xrightarrow{r_{p'} \circledast f_\circ} p' \circledast f_\circ$$

$$l_{p'\circledast f_\circ} = \mathbb{B} \circledast p' \circledast f_\circ \xrightarrow{l_{p'} \circledast f_\circ} p' \circledast f_\circ$$

This isomorphism is natural in both p' *and* (f,λ).

(ii) Given 1-cells

$$(a,\mathbb{A}) \xrightarrow{p} (\bar{a},\overline{\mathbb{A}}) \qquad\qquad (a,\mathbb{A}) \xrightarrow{q} (\bar{a}',\overline{\mathbb{A}}')$$

in $\mathrm{Mon}(\underline{\underline{\mathcal{M}}})$ *and*

$$(\bar{a},\overline{\mathbb{A}}) \xrightarrow{(\bar{f},\bar{\lambda})} (\bar{a}',\overline{\mathbb{A}}')$$

in $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}},\underline{\mathcal{K}},(\text{-})_\circ)$ *then equivariant maps*

$$(\bar{f},\bar{\lambda})_* \otimes p \xrightarrow{\bar{\gamma}} q$$

are in bijective correspondence (naturally in p,q and $(\bar{f},\bar{\lambda})$*) with 2-cells*

$$\bar{f}_\circ \circledast p \xrightarrow{\gamma} q$$

in $\underline{\underline{\mathcal{M}}}$*, subject to the condition that the diagrams*

$$\begin{tikzcd}
\bar{f}_\circ \circledast p \circledast \mathbb{A} \arrow[r, "\bar{f}_\circ \circledast r_p"] \arrow[d, "\gamma \circledast \mathbb{A}"'] \arrow[dr, phantom, "\text{(a)}"] & \bar{f}_\circ \circledast p \arrow[d, "\gamma"] \\
q \circledast \mathbb{A} \arrow[r, "r_q"'] & q
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{f}_\circ \circledast \overline{\mathbb{A}} \circledast p \arrow[rr, "\bar{\lambda} \circledast p"] \arrow[d, "\bar{f}_\circ \circledast l_p"'] & \text{(b)} & \overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p \arrow[d, "\overline{\mathbb{A}}' \circledast \gamma"] \\
\bar{f}_\circ \circledast p \arrow[r, "\gamma"'] & q & \overline{\mathbb{A}}' \circledast q \arrow[l, "l_q"]
\end{tikzcd}
\tag{1.26}$$

commute.

(iii) There are dual results for the right representables $(f,\lambda)^*$*.*

Proof.

(i) It is a routine verification, directly from the conditions on $r_{p'}$ as a right action, to check that the maps

$$\begin{tikzcd} p' \circledast \mathbb{A}' \circledast \mathbb{A}' \circledast f_\circ \arrow[r, shift left=2, "r_{p'}\circledast\mathbb{A}'\circledast f_\circ"] \arrow[r, shift right=2, "p'\circledast\mu_{a'}\circledast f_\circ"'] & p' \circledast \mathbb{A}' \circledast f_\circ \arrow[l, shift right=6, "p'\circledast\mathbb{A}'\circledast\eta_{a'}\circledast f_\circ"'] \arrow[r, shift right, "r_{p'}\circledast f_0"'] & p' \circledast f_\circ \arrow[l, shift right, "p'\circledast\eta_{a'}\circledast r_\circ"'] \end{tikzcd} \tag{1.27}$$

fit together into a diagram

$$\begin{tikzcd} p' \circledast \mathbb{A}' \circledast f_\circ \arrow[r, "p'\circledast\mathbb{A}'\circledast\eta_{a'}\circledast f_\circ"] \arrow[d, "r_{p'}\circledast f_0"'] & p' \circledast \mathbb{A}' \circledast \mathbb{A}' \circledast f_\circ \arrow[r, "p'\circledast\mu_{a'}\circledast f_\circ"] \arrow[d, "r_{p'}\circledast\mathbb{A}'\circledast f_\circ"'] & p' \circledast \mathbb{A}' \circledast f_\circ \arrow[d, "r_{p'}\circledast f_0"] \\ p' \circledast f_\circ \arrow[r, "p'\circledast\eta_{a'}\circledast f_\circ"'] & p' \circledast \mathbb{A}' \circledast f_\circ \arrow[r, "r_{p'}\circledast f_\circ"'] & p' \circledast f_\circ \end{tikzcd}$$

in which both squares commute and both horizontal composites are equal to identities. In other words (1.27) is a reflexive coequaliser diagram, and is in fact the coequaliser we use to construct the underlying 1-cell of $p' \otimes (f,\lambda)_*$. We check that the maps $r_{p\circledast f_\circ}$ and $l_{p\circledast f_\circ}$ are the actions given in the definition of $p' \otimes (f,\lambda)_*$ by two easy diagram chases, one involving the fact that the actions on p' commute, which we leave to the reader.

(ii) Equivariant maps $\bar{\gamma}\colon (\bar{f},\bar{\lambda})_* \otimes p \longrightarrow q$ correspond,via composition with the canonical quotient map $(f,\lambda)_* \circledast p \longrightarrow (f,\lambda)_* \otimes p$, to maps

$$\overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p \xrightarrow{\tilde{\gamma}} q$$

with the properties represented in the commutative diagrams

$$\begin{tikzcd} \overline{\mathbb{A}}' \circledast \overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p \arrow[r, "\overline{\mathbb{A}}'\circledast\tilde{\gamma}"] \arrow[d, "\mu_{\bar{a}'} \circledast \bar{f}_\circ \circledast p"'] \arrow[dr, phantom, "\text{(a)}"] & \overline{\mathbb{A}}' \circledast q \arrow[d, "l_q"] \\ \overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p \arrow[r, "\tilde{\gamma}"'] & q \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p \circledast \mathbb{A} \arrow[r, "\overline{\mathbb{A}}'\circledast\bar{f}_\circ\circledast r_p"] \arrow[d, "\tilde{\gamma} \circledast \mathbb{A}"'] \arrow[dr, phantom, "\text{(b)}"] & \overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p \arrow[d, "\tilde{\gamma}"] \\ q \circledast \mathbb{A} \arrow[r, "r_q"'] & q \end{tikzcd} \tag{1.28}$$

(corresponding to the equivariance of $\bar{\gamma}$), and

$$\begin{array}{ccccc} \overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast \overline{\mathbb{A}} \circledast p & & \xrightarrow{\overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast l_p} & & \overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p \\ {\scriptstyle \overline{\mathbb{A}}' \circledast \bar{\lambda} \circledast p}\downarrow & & & & \downarrow{\scriptstyle \tilde{\gamma}} \\ \overline{\mathbb{A}}' \circledast \overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p & \xrightarrow[\mu_{\bar{a}'} \circledast \bar{f}_\circ \circledast p]{} & \overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p & \xrightarrow[\tilde{\gamma}]{} & q \end{array} \tag{1.29}$$

(representing $\tilde{\gamma}$'s invariance under "middle action" by $\overline{\mathbb{A}}$).

In turn we have a bijection between maps $\tilde{\gamma}$ satisfying condition 1.28(a) alone and 2-cells $\gamma\colon f_\circ \circledast p \Rightarrow q$ in $\underline{\underline{\mathcal{M}}}$ with no conditions to satisfy. This is given by:

$\boldsymbol{\theta}\colon \vec{\boldsymbol{\gamma}} \mapsto \boldsymbol{\gamma}$**;** given $\tilde{\gamma}$ define $\theta(\tilde{\gamma})$ to be the composite:

$$\bar{f}_\circ \circledast p \xrightarrow{\eta_{\bar{a}'} \circledast \bar{f}_\circ \circledast p} \overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p \xrightarrow{\tilde{\gamma}} q$$

$\boldsymbol{\phi}\colon \boldsymbol{\gamma} \mapsto \vec{\boldsymbol{\gamma}}$**;** given γ define $\phi(\gamma)$ to be the composite:

$$\overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p \xrightarrow{\overline{\mathbb{A}}' \circledast \gamma} \overline{\mathbb{A}}' \circledast q \xrightarrow{l_q} q$$

This satisfies condition 1.28(a) by a simple diagram chase starting from the fact that l_q is a left action map and therefore satisfies $l_q \bullet (\overline{\mathbb{A}}' \circledast l_q) = l_q \bullet (\mu_{\bar{a}'} \circledast q)$.

Straightforward diagram chases demonstrate that $\theta\phi =$ "identity", by starting from the identity rule for the left action l_q, and $\phi\theta =$ "identity", from condition 1.28(a). All that remains is the routine task of showing that under this bijection conditions 1.28(b) and 1.29 correspond to 1.26(a) and 1.26(b) respectively, which we leave up to the reader.

(iii) The dual result mentioned gives an isomorphism between $(\bar{f}, \bar{\lambda})^* \otimes p'$ and $\bar{f}^\circ \circledast p'$, equipped with the canonical actions, and a bijection between equivariant maps out of $p \otimes (f, \lambda)^*$ and maps $p \circledast f^\circ$ satisfying dual conditions to 1.26 and involving λ°, the mate of λ under the adjunction $f_\circ \dashv f^\circ$. $\square$

Corollary 1.2.7 *The functors of (1.25) are the homset actions of a homomorphism* $(\text{-})_*\colon \mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (\text{-})_\circ) \longrightarrow \mathrm{Mon}(\underline{\underline{\mathcal{M}}})$ *with the property that for each 1-cell* (f, λ) *in* $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (\text{-})_\circ)$ *we have* $(f, \lambda)_* \dashv (f, \lambda)^*$*. This completes the verification that, under the conditions of example 1.2.5), we get an equipment:*

$$\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (\text{-})_\circ) = (\mathrm{Mon}(\underline{\underline{\mathcal{M}}}), \mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (\text{-})_\circ), (\text{-})_*)$$

Proof. It is now easy to demonstrate that $(\text{-})_*$ is a homomorphism. Firstly direct verification reveals that $(\mathrm{i}_{(a,\mathbb{A})})_* \cong \mathrm{i}_{(a,\mathbb{A})}$. For 1-cells $(f,\lambda)\colon (a,\mathbb{A}) \longrightarrow (a',\mathbb{A}')$ and $(f',\lambda')\colon (a',\mathbb{A}') \longrightarrow (a'',\mathbb{A}'')$ in $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}})$ lemma 1.2.6(i) shows that the composite $(f',\lambda')_* \otimes (f,\lambda)_*$ has underlying 1-cell $(f',\lambda')_* \circledast f_\circ \cong \mathbb{A}'' \circledast f'_\circ \circledast f_\circ$ with actions which make the isomorphism $\mathbb{A}'' \circledast f'_\circ \circledast f_\circ \cong \mathbb{A}'' \circledast (f' \circ f)_\circ$ into an equivariant isomorphism $(f',\lambda')_* \otimes (f,\lambda)_* \cong ((f',\lambda') \circ (f,\lambda))_*$.

Suppose that $(f,\lambda)\colon (a,\mathbb{A}) \longrightarrow (a',\mathbb{A}')$ is a 1-cell in $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ)$ then we have the following sequence of natural bijections

$$
\begin{array}{lcl}
 & (f,\lambda)_* \otimes p \longrightarrow q & \text{equivariant} \\
\text{lemma 1.2.6(ii)} & \overline{} & \\
 & f_\circ \circledast p \longrightarrow q & \text{sats. (1.26)(a),(b)} \\
f_\circ \dashv f^\circ & \overline{} & \\
 & p \longrightarrow f^\circ \circledast q & \text{sats. mates of (1.26)(a),(b)}
\end{array}
$$

but we know that $(f,\lambda)^* \otimes q \cong f^\circ \circledast q$, by lemma 1.2.6(iii), and under this isomorphism the conditions on the last map in the display above simply state that it is an equivariant map $p \longrightarrow (f,\lambda)^* \otimes q$. It is a matter of direct calculation to check that this correspondence satisfies the following rule

$$
\frac{(f,\lambda)_* \otimes p \xrightarrow{\ \gamma\ } q}{p \xrightarrow{\ \hat{\gamma}\ } (f,\lambda)^* \otimes q}
\quad\Longrightarrow\quad
\frac{(f,\lambda)_* \otimes p \otimes r \xrightarrow{\ \gamma \otimes r\ } q \otimes r}{p \otimes r \xrightarrow{\ \hat{\gamma} \otimes r\ } (f,\lambda)^* \otimes q \otimes r}
$$

which is equivalent to saying that the 2-cell

$$
\begin{tikzcd}
 & (a'\mathbb{A}) \arrow[dl, "(f,\lambda)^*"'] \arrow[dr, "\mathrm{i}_{(a',\mathbb{A}')}"] \arrow[d, phantom, "\overset{\Rightarrow}{\epsilon}"] & \\
(a,\mathbb{A}) \arrow[rr, "(f,\lambda)_*"'] & {} & (a'\mathbb{A}')
\end{tikzcd}
$$

corresponding to the identity on $(f,\lambda)^*$, has the absolute right lifting property as in Street and Walters [54]. Proposition 2 of that paper then demonstrates that $(f,\lambda)_* \dashv (f,\lambda)^*$. $\square$

We introduced examples 1.2.2 and 1.2.3 as halfway houses towards equipments of enriched and internal categories respectively; the next two examples reveal that this may be achieved by applying the equipment of monads construction. An approach like this is immediately open to the accusation of being rather a roundabout way of defining these equipments, but is motivated by more than a desire to simply unify these two principal examples. This will become more apparent when we come to construct change of base structures between equipments of categories. While these are often quite complicated the corresponding structures between equipments of spans and matrices are far easier to define and study; the functoriality of $\mathcal{M}\mathrm{on}(\text{-})$, which we shall establish, then allows those results to be lifted to the more complex arena of category theory itself:

Example 1.2.8 (equipments of enriched categories) Return to example 1.2.2 and suppose that the bicategory $\underline{\underline{\mathcal{B}}}$ also has stable local coequalisers of reflexive pairs, then it is quite clear that the bicategory of matrices $\underline{\underline{\mathcal{B}}}$–Mat does as well. It follows that we may apply the equipment of monads construction to $(\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (-)_\circ)$ so we shall look at the resulting equipment in more detail.

In fact the work has already been done for us; our equipment of matrices is precisely the structure described in section 1 of [5]; from which its authors derive the (traditional) 2-category of $\underline{\underline{\mathcal{B}}}$-enriched categories $\underline{\underline{\mathcal{B}}}$–Cat, cf. section 2 of that paper. Their construction clearly demonstrates that the 1-skeleton $\underline{\underline{\mathcal{B}}}\text{–Cat}_1$, or $|\underline{\underline{\mathcal{B}}}\text{–Cat}|$ in their notation, is identical to our category $\mathrm{Mnd}(\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (-)_\circ)$.

Section 3 of [5] identifies Mon($\underline{\underline{\mathcal{B}}}$–Mat) as the (traditional) bicategory of $\underline{\underline{\mathcal{B}}}$-profunctors $\underline{\underline{\mathcal{B}}}$–Prof; now it is a matter of a straightforward calculation to show that the homomorphism $(-)_*$ is the usual one taking a $\underline{\underline{\mathcal{B}}}$-functor to its associated left representable profunctor. With this we have identified $\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (-)_\circ)$ as the equipment in which to study $\underline{\underline{\mathcal{B}}}$-enriched category theory. Notice that theorem 8 of loc. cit. shows that if $\underline{\underline{\mathcal{B}}}$ has all right extensions and liftings and all small local limits then so do $\underline{\underline{\mathcal{B}}}$–Mat and Prof($\underline{\underline{\mathcal{B}}}$), we return to this a little later on.

To sum this all up:

$$(\underline{\underline{\mathcal{B}}}\text{–Prof}, \underline{\underline{\mathcal{B}}}\text{–Cat}_1, (-)_*) \overset{\text{def}}{=} \mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (-)_\circ)$$

It is worth pointing out that the argument given for lemma 1.2.6(ii) shows that natural transformations $(f, \lambda) \Rightarrow (f', \lambda')$, as defined in section 2 of [5], correspond to equivariant maps $(f', \lambda')_* \Rightarrow (f, \lambda)_*$. □

Example 1.2.9 (equipments of internal categories) Let $\underline{\mathcal{E}}$ be a locally small category with finite limits and coequalisers of reflexive pairs which are stable under pullback; then we may apply the monads construction to the equipment of spans in $\underline{\mathcal{A}}$, as defined in example 1.2.3.

In identifying the equipment $\mathcal{M}\mathrm{on}(\mathrm{Span}(\underline{\mathcal{E}}), \underline{\mathcal{E}}, (-)_\circ)$ notice that a monad $\mathbb{A}$ in Span($\underline{\mathcal{E}}$) is no more than an internal category in $\underline{\mathcal{E}}$. For a 1-cell $(f, \lambda)\colon \mathbb{A} \longrightarrow \mathbb{A}'$ in Mnd(Span($\underline{\mathcal{E}}$), $\underline{\mathcal{E}}$) the 2-cell $\lambda\colon f_\circ \underset{\bullet}{\times} \mathbb{A} \Rightarrow \mathbb{A}' \underset{\bullet}{\times} f_\circ$ corresponds (under the adjunction $f_\circ \dashv f^\circ$) to another $\hat{\lambda}\colon f_\circ \underset{\bullet}{\times} \mathbb{A} \underset{\bullet}{\times} f^\circ \Rightarrow \mathbb{A}'$ or in other words a map $\hat{\lambda}\colon \mathbb{A}_1 \longrightarrow \mathbb{A}'_1$ making

$$\begin{tikzcd}
\mathbb{A}_0 \arrow[d, "f"'] & \mathbb{A}_1 \arrow[l, "d_0"'] \arrow[r, "d_1"] \arrow[d, "\hat{\lambda}"'] & \mathbb{A}_0 \arrow[d, "f"] \\
\mathbb{A}'_0 & \mathbb{A}'_1 \arrow[l, "d_0"'] \arrow[r, "d_1"] & \mathbb{A}'_0
\end{tikzcd}$$

commute, the top and bottom lines of which are simply the spans underlying the monads $\mathbb{A}$ and $\mathbb{A}'$. The conditions on λ as the 2-cellular part of a monad opfunctor simply translate to saying that the maps f and $\hat{\lambda}$ are the actions on objects and morphisms of an internal functor, it follows that $\mathrm{Mnd}(\mathrm{Span}(\underline{\mathcal{E}}),\underline{\mathcal{E}})$ is the usual category $\mathrm{Cat}(\underline{\mathcal{E}})_1$ of categories and functors internal to $\underline{\mathcal{E}}$.

An examination of $\mathrm{Mon}(\mathrm{Span}(\underline{\mathcal{E}}))$ reveals that it is simply $\mathrm{Prof}(\underline{\mathcal{E}})$, the bicategory of categories and profunctors internal to $\underline{\mathcal{E}}$, as described in chapter 2 of [28], so thi completes our identification of

$$\mathcal{M}\mathrm{on}(\mathrm{Span}(\underline{\mathcal{E}}),\underline{\mathcal{E}},(\text{-})_\circ) = (\mathrm{Prof}(\underline{\mathcal{E}}),\mathrm{Cat}(\underline{\mathcal{E}})_1,(\text{-})_*)$$

as the equipment in which to study the internal category theory of $\underline{\mathcal{E}}$. Again internal natural transformations $(f,\lambda)\Rightarrow(f',\lambda')$ in the traditional sense (cf [28]) correspond to equivariant maps $(f',\lambda')_*\Rightarrow(f,\lambda)_*$.

Notice that the presentation of these matters in [28] demonstrates that if $\underline{\mathcal{E}}$ is locally cartesian closed then $\mathrm{Prof}(\underline{\mathcal{E}})$ has all right extensions and liftings. Later on we will need to make explicit calculations with these so we'll describe how to construct them, leaving the necessary verifications up to the reader. Given a pair of 1-cells

$$\begin{tikzcd} \mathbb{A} \arrow[r, "X", "|" marking] \arrow[dr, "Y"', "|" marking] & \mathbb{B} \\ & \mathbb{C} \end{tikzcd}$$

in $\mathrm{Prof}(\underline{\mathcal{E}})$ the underlying "set" of the right extension of Y along X is given symbolically by:

$$\left| Y \underset{\mathbb{B}}{\Leftarrow} X \right| = \left\{ (c,f,b) \;\middle|\; \begin{array}{l} c\in\mathbb{C}_0, b\in\mathbb{B}_0, f\colon {}_bX \longrightarrow {}_cY, \text{ s.t.} \\ (\forall x\in {}_bX)(y_1(f(x)) = x_1(x)) \text{ and} \\ (\forall x\in {}_bX)(\forall\alpha\in\mathbb{A}_1)((d_0(\alpha)=x_1(x)) \\ \qquad\qquad\qquad \Rightarrow (f(x\cdot\alpha) = f(x)\cdot\alpha)) \end{array} \right\} \tag{1.30}$$

Here the identity of the maps x_0, x_1, y_0 and y_1 is established by noting that the spans which underlie our profunctors are:

$$\mathbb{B}_0 \xleftarrow{x_0} X \xrightarrow{x_1} \mathbb{A}_0$$
$$\mathbb{C}_0 \xleftarrow{y_0} Y \xrightarrow{y_1} \mathbb{A}_0$$

The notation $f\colon {}_bX \longrightarrow {}_cY$ is interpreted as saying "f is a function from the left fibre of X over $b\in\mathbb{B}_0$ to that of Y over $c\in\mathbb{C}_0$. To be more precise consider the objects

$$X\times\mathbb{C}_0 \xrightarrow{x_0\times\mathbb{C}_0} \mathbb{B}_0\times\mathbb{C}_0$$
$$\mathbb{B}_0\times Y \xrightarrow{\mathbb{B}_0\times y_0} \mathbb{B}_0\times\mathbb{C}_0$$

in the slice category $\underline{\mathcal{E}}/_{\mathbb{B}_0 \times \mathbb{C}_0}$, this is cartesian closed (since $\underline{\mathcal{E}}$ is locally so) and f is simply an element of $(\mathbb{B}_0 \times Y \to \mathbb{B}_0 \times \mathbb{C}_0)^{(X\times\mathbb{C}_0 \to \mathbb{B}_0\times\mathbb{C}_0)}$ in the fibre over (b, c).

The object $\left|Y \underset{\mathbb{B}}{\Leftarrow} X\right|$ becomes a span with projections

$$\begin{array}{ccccc} \mathbb{C}_0 & \longleftarrow & \left|Y \underset{\mathbb{B}}{\Leftarrow} X\right| & \longrightarrow & \mathbb{B}_0 \\ c & \longleftarrow\!\!\!\dashv & (c, f, b) & \longmapsto & b \end{array}$$

and a profunctor with actions:

$$\begin{array}{ll} (c, f, b) \cdot \beta = (c, f(\beta \cdot -), b') & \text{for } \beta \colon b' \longrightarrow b \in \mathbb{B} \\ \gamma \cdot (c, f, b) = (c', \gamma \cdot f(-), b) & \text{for } \gamma \colon c \longrightarrow c' \in \mathbb{C} \end{array}$$

□

Observation 1.2.10 In both of the last two examples we commented that natural transformations $(f, \lambda) \Rightarrow (f', \lambda')$ correspond to equivariant maps $(f', \lambda')_* \Rightarrow (f, \lambda)_*$ this suggests the following "repletion" operation on equipments. Suppose that $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$ is a (possibly weak) equipment then we may factor the homomorphism $(-)_*$ as

$$\underline{\underline{\mathcal{K}}}^{\mathrm{co}} \xrightarrow{\quad \mathrm{H}_e \quad} \underline{\underline{\mathcal{A}}} \xrightarrow{\quad \mathrm{H}_m \quad} \underline{\underline{\mathcal{M}}}$$

where H_e is essentially surjective on 0- and 1-cells and H_m is locally fully faithful, see appendix A for a description of this factorisation. Now let $\underline{\underline{\mathcal{K}}}_* = \underline{\underline{\mathcal{A}}}^{\mathrm{co}}$ and we get an equipment

$$(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)_{\mathrm{rep}} \stackrel{\mathrm{def}}{=} (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}_*, (-)_*)$$

where the homomorphism $(-)_* \colon \underline{\underline{\mathcal{K}}}_*^{\mathrm{co}} \longrightarrow \underline{\underline{\mathcal{M}}}$ is simply H_m, the important point being that (by construction) the repletion of an equipment is a proarrow equipment in the sense of [56]. Applying this to the equipments in the last two examples gives

$$\begin{aligned} \underline{\underline{\mathcal{B}}}\text{–Equip} &\stackrel{\mathrm{def}}{=} (\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (-)_\circ))_{\mathrm{rep}} \\ &= (\underline{\underline{\mathcal{B}}}\text{–Prof}, \underline{\underline{\mathcal{B}}}\text{–Cat}, (-)_*) \\ \mathrm{Equip}(\underline{\mathcal{E}}) &\stackrel{\mathrm{def}}{=} (\mathcal{M}\mathrm{on}(\mathrm{Span}(\underline{\mathcal{E}}), \underline{\mathcal{E}}, (-)_\circ))_{\mathrm{rep}} \\ &= (\mathrm{Prof}(\underline{\mathcal{E}}), \mathrm{Cat}(\underline{\mathcal{E}}), (-)_*) \end{aligned}$$

where $\underline{\underline{\mathcal{B}}}$–Cat and Cat($\underline{\mathcal{E}}$) are the usual 2-categories of enriched and internal categories, functors and natural transformations. Now use the notations $\underline{\underline{\mathcal{B}}}\text{–Equip}_1$ and $\mathrm{Equip}(\underline{\mathcal{E}})_1$ to denote the non-replete equipments of the last two examples.

Observation 1.2.11 In the introduction of (proarrow) equipments an important consideration was the fact that within them we can express the notion of *weighted* enriched colimit. To see how this works lets take a simple example; that of $\underline{\mathcal{V}}$-enriched category theory, where $\underline{\mathcal{V}}$ is a locally small, small complete and cocomplete, symmetric monoidal closed category. Of course this is precisely the example studied in detail in [30].

First the traditional approach, in [30] an *indexing type*, which is more commonly known now as a *weight*, consists of a small $\underline{\mathcal{V}}$-enriched category $\underline{\mathbf{A}}$ and a $\underline{\mathcal{V}}$-functor

$$\underline{\mathbf{A}}^{\mathrm{op}} \xrightarrow{\quad \mathrm{W} \quad} \underline{\mathcal{V}}$$

where $\underline{\mathcal{V}}$ is enriched over itself because it is a closed category. Recall also that we may form an enriched functor category $[\underline{\mathbf{A}}^{\mathrm{op}}, \underline{\mathcal{V}}]$ with homsets given by the end

$$[\underline{\mathbf{A}}^{\mathrm{op}}, \underline{\mathcal{V}}](\mathrm{F}, \mathrm{F}') \cong \int_{A \in \underline{\mathbf{A}}} \underline{\mathcal{V}}(\mathrm{F}(a), \mathrm{F}'(a)) \tag{1.31}$$

which exists in $\underline{\mathcal{V}}$ since $\underline{\mathbf{A}}$ is small and $\underline{\mathcal{V}}$ possesses all small limits.

Now we are in a position to define the notion of weighted $\underline{\mathcal{V}}$-colimit. Say that a *diagram* (enriched functor) $\mathrm{G}\colon \underline{\mathbf{A}} \longrightarrow \underline{\mathbf{B}}$ in a $\underline{\mathcal{V}}$-category $\underline{\mathbf{B}}$ has a colimit *weighted* by W iff there exists an object $\mathrm{colim}\,(\mathrm{W}, \mathrm{G}) \in \underline{\mathbf{B}}$ with family of isomorphisms

$$\underline{\mathbf{B}}(\mathrm{colim}\,(\mathrm{W}, \mathrm{G}), B) \cong [\underline{\mathbf{A}}^{\mathrm{op}}, \underline{\mathcal{V}}](W, \underline{\mathbf{B}}(\mathrm{G}(\text{–}), B)) \tag{1.32}$$

$\underline{\mathcal{V}}$-natural in the object $B \in \underline{\mathbf{B}}$.

To see how we might express this in the equipment $\underline{\mathcal{V}}$–Equip notice first that the weight W is no more or less than a profunctor:

$$\underline{\mathbf{A}} \xrightarrow{\quad \mathrm{W} \quad} \underline{\mathbf{1}}$$

Furthermore for any other profunctor $\mathrm{V}\colon \underline{\mathbf{A}} \to \underline{\mathbf{B}}$ the right Kan extension

$$\begin{tikzcd} \underline{\mathbf{A}} \arrow[rr, "\mathrm{W}"] \arrow[dr, "\mathrm{V}"'] & \arrow[d, phantom, "\Leftarrow"] & \underline{\mathbf{1}} \arrow[dl, "\mathrm{V} \Leftarrow \mathrm{W}"] \\ & \underline{\mathbf{B}} & \end{tikzcd}$$

in $\underline{\mathcal{V}}$–Prof is given by the formula:

$$\mathrm{V} \Leftarrow \mathrm{W}(\cdot, B) \cong \int_{A \in \underline{\mathbf{A}}} \underline{\mathcal{V}}(\mathrm{W}(A, \cdot), \mathrm{V}(A, B))$$

and finally notice that objects of $\underline{\mathbf{B}}$ correspond to $\underline{\mathcal{V}}$-functors $\underline{\mathbf{1}} \longrightarrow \underline{\mathbf{B}}$ and for a $\underline{\mathcal{V}}$-functor $\mathrm{G}\colon \underline{\mathbf{A}} \longrightarrow \underline{\mathbf{B}}$ the left representable G_* is given by:

$$\mathrm{G}_*(A, B) \cong \underline{\mathbf{B}}(\mathrm{G}(A), B)$$

Compare this with our definition of the homsets of $[\underline{\mathbf{A}}^{\mathrm{op}}, \underline{\mathcal{V}}]$ in (1.31) and that of the enriched colimit in (1.32). It is now clear that $\mathrm{colim}\,(\mathrm{W}, \mathrm{G})\colon \underline{\mathbf{1}} \longrightarrow \underline{\mathbf{B}}$, if it exists, is the (essentially unique) $\underline{\mathcal{V}}$-functor with the property that

$$\mathrm{colim}\,(\mathrm{W}, \mathrm{G})_* \cong \mathrm{G}_* \Leftarrow \mathrm{W}$$

in $\underline{\mathcal{V}}$–Prof. This motivates the following definition in any equipment:

Definition 1.2.12 (weighted colimits and cylinders) Suppose $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$ is an equipment and that we are given a pair of 1-cells

$$\begin{tikzcd} a \arrow[r, "p", "\shortmid"{marking}] \arrow[dr, "f"'] & b \\ & c \end{tikzcd}$$

with $p \in \underline{\underline{\mathcal{M}}}$ and $f \in \underline{\underline{\mathcal{K}}}$. We say that a 1-cell $\mathrm{colim}\,(p, f): b \longrightarrow c \in \underline{\underline{\mathcal{K}}}$ is a *colimit* of f *weighted* by p if and only if there is a right Kan extension diagram

$$\begin{tikzcd} a \arrow[rr, "p", "\shortmid"{marking}] \arrow[dr, "f_*"', "\shortmid"{marking}] & \arrow[d, phantom, "\overset{\gamma}{\Leftarrow}"] & b \arrow[dl, "\mathrm{colim}\,(p,f)_*", "\shortmid"{marking}] \\ & c & \end{tikzcd}$$

in $\underline{\underline{\mathcal{M}}}$. We say that this diagram is a *colimit cylinder* displaying $\mathrm{colim}\,(p, f)$. $\square$

The dual notion of *weighted limit* involves right liftings and the right representables f^*, but a better way of expressing them is as weighted colimit in some dual of the equipment we started with:

Definition 1.2.13 (Dual Equipments) If $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$ is an equipment, we may form a homomorphism

$$\underline{\underline{\mathcal{K}}} \xrightarrow{(-)^*} \underline{\underline{\mathcal{M}}}^{\mathrm{op}}$$

which is the identity on 0-cells, takes a 1-cell $f: a \longrightarrow a' \in \underline{\underline{\mathcal{K}}}$ to f^*, the right adjoint to f_* in $\underline{\underline{\mathcal{M}}}$, and a 2-cell $\alpha: f \Rightarrow \dot{f}$ to the mate $\alpha^*: f^* \Rightarrow \dot{f}^*$ of $\alpha_*: \dot{f}_* \Rightarrow f_*$ under the adjunctions $f_* \dashv f^*$ and $\dot{f}_* \dashv \dot{f}^*$. The canonical 2-cells making these actions into a homomorphism are again mates of those associated with $(-)_*$, under the various adjunctions $f_* \dashv f^*$. Having defined this homomorphism we get an equipment

$$(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)^{\mathrm{op}} \stackrel{\mathrm{def}}{=} (\underline{\underline{\mathcal{M}}}^{\mathrm{op}}, \underline{\underline{\mathcal{K}}}^{\mathrm{co}}, (-)^*)$$

which satisfies rule (ii) of definition 1.2.1 by dint of the fact that an adjunction $f_* \dashv f^*$ in $\underline{\underline{\mathcal{M}}}$ may be re-interpreted as $f^* \dashv f_*$ in $\underline{\underline{\mathcal{M}}}^{\mathrm{op}}$. As is easily checked the process of taking the dual of an equipment is involutive, in other words $((\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)^{\mathrm{op}})^{\mathrm{op}} = (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$.

It is important to notice that any limit cylinder in $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$ corresponds to a colimit cylinder in $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)^{\mathrm{op}}$. This allows us to state any theorem concerning cylinders only in its colimit form, using $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)^{\mathrm{op}}$ to derive results about limit cylinders. For more information on dual equipments see section 1.5. $\square$

1.3 Bicategory Enriched Categories.

In order to interpret change of base as a biadjoint pair of maps between equipments we must first introduce some kind of structure in which the notion of a biadjoint may be interpreted. Think of this in much the same way as you would the introduction of 2-categories in which to interpret the notion of adjunction in terms of unit and counit. In particular we will seek to provide natural notions of homomorphism, transformation and modification of equipments and unify these together into a "bicategorically enriched category".

It is worth pointing out that by this phrase we do not mean gadgets which might correctly be called weak or pseudo 3-categories, the as yet undefined[1] 3-dimensional analogue of the bicategory notion, involving an outer level composition which is only associative up to coherent equivalences (see [26] and [51]). In fact in the structures we will be considering this composition is strictly associative, essentially due to the fact that composition of bicategorical morphisms is so. In this sense these are really an abstraction on the level of 2-categories rather than bicategories[2] being principally concerned with structure preserving maps between algebraic objects rather than relations or spans between them. In fact we will be considering categories which are bicategory enriched in a slightly modified but still essentially traditional sense.

Let $\underline{\mathcal{H}om}_0$ denote the category of (small) bicategories and homomorphisms between them. Traditionally if we wished to consider categories enriched in $\underline{\mathcal{H}om}_0$ we would firstly need to endow it with some kind of monoidal (or closed) category structure. A first candidate for this might be product of bicategories, but this will not do since we would also like to have a natural bijection

$$\frac{\underline{\underline{\mathcal{A}}} \otimes \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}}{\underline{\underline{\mathcal{A}}} \longrightarrow \mathcal{H}om_S(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{A}}})}$$

so as to ensure that when we enrich $\underline{\mathcal{H}om}_0$ over itself its "homsets" consist of homomorphisms, strong transformations and modifications, as we might nïavely expect.

In [23] Gray examines an entirely analogous problem, that of providing $\underline{\text{2-Cat}}$, the category of 2-category and 2-functors, with tensors $\otimes_p$ (or $\otimes_l$) which provide natural bijections

$$\frac{\mathbb{A} \otimes_p \mathbb{B} \longrightarrow \mathbb{C}}{\mathbb{A} \longrightarrow \text{PSEUDO}(\mathbb{B}, \mathbb{C})} \qquad \frac{\mathbb{A} \otimes_l \mathbb{B} \longrightarrow \mathbb{C}}{\mathbb{A} \longrightarrow \text{LAX}(\mathbb{B}, \mathbb{C})}$$

[1]the structures envisaged when I made this comment are now called *tricategories* and were introduced and studied by Gordon, Power and Street [21] some three years after this thesis was complete.

[2]however by the coherence result presented in [21] the structures introduced here are in a sense no less general than tricategories.

where PSEUDO($\mathbb{B}, \mathbb{C}$) is the 2-category of 2-functors, *pseudo* transformations and modifications and LAX($\mathbb{B}, \mathbb{C}$) that of 2-functors, *lax* transformations and modifications. He is only able to do so by imposing a strictness condition on himself from the outset which we do not, in essence he considers only *strict* homomorphisms (or 2-functors in his context), i.e. those with structural isomorphisms which are in fact identities. A little later on we will give an example which demonstrates that we may not drop this restriction and still expect to define such tensors.

The solution to our problem may also be found in [23], rather than relying on a tensor product, why not return to the concept of "multi-linear map", pictured

$$[\underline{\underline{\mathcal{A}}}_1 \cdots \underline{\underline{\mathcal{A}}}_n] \longrightarrow \underline{\underline{\mathcal{B}}}$$

which we define to ensure suitably natural bijections:

$$\frac{[\underline{\underline{\mathcal{A}}}_1 \cdots \underline{\underline{\mathcal{A}}}_n] \longrightarrow \underline{\underline{\mathcal{B}}}}{[\underline{\underline{\mathcal{A}}}_1 \cdots \underline{\underline{\mathcal{A}}}_{n-1}] \longrightarrow \mathcal{H}om_S(\underline{\underline{\mathcal{A}}}_n, \underline{\underline{\mathcal{B}}})} \ ?$$

Enrichment over such a calculus of multi-linear maps has a natural definition, closely related to enrichment over a closed category, but notice that $\underline{\mathcal{H}om}_0$ is not one of these, lacking as it does a bicategory $\underline{\underline{\mathcal{I}}}$ such that homomorphisms $\underline{\underline{\mathcal{I}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ are in natural bijective correspondence with the 0-cells of $\underline{\underline{\mathcal{B}}}$. If such a bicategory were to exist then Yoneda's lemma would imply that it had a 0-cell $a \in \underline{\underline{\mathcal{I}}}$ such that each map

$$\begin{array}{ccc} \underline{\mathcal{H}om}_0(\underline{\underline{\mathcal{I}}}, \underline{\underline{\mathcal{B}}}) & \xrightarrow{\mathrm{ev}_a} & \underline{\underline{\mathcal{B}}}_0 \\ \mathrm{H}\colon \underline{\underline{\mathcal{I}}} \to \underline{\underline{\mathcal{B}}} & \longmapsto & \mathrm{H}(a) \end{array} \tag{1.33}$$

is a bijection. Now pick any bicategory $\underline{\underline{\mathcal{B}}}$ possessing a 0-cell $b \in \underline{\underline{\mathcal{B}}}$ on which there is an endo-1-cell $p\colon b \longrightarrow b$ isomorphic, but not equal, to i_b. We have at least one homomorphism $\mathrm{H}\colon \underline{\underline{\mathcal{I}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ with $\mathrm{H}(a) = b$ since ev_a is surjective, from which we may construct two distinct homomorphisms H_0 and H_1, with $\mathrm{H}_0(\mathrm{i}_a) = \mathrm{i}_b$ and $\mathrm{H}_1(\mathrm{i}_a) = p$. Therefore $\mathrm{H}_0 \neq \mathrm{H}_1$ and $\mathrm{H}_0(a) = \mathrm{H}_1(a) = b$ contradicting the injectivity of ev_a.

Of course were we willing to insist that all homomorphisms should be *normal*, that is to say preserve identities "on the nose", there is no block to making $\underline{\mathcal{H}om}_0$ into a closed category. We could then exploit the usual definition of enrichment over such a structure in the development of the theory presented in the next few sections. Although using only normal homomorphisms is no real restriction, since most that we meet are naturally normal and anyway all homomorphisms may be canonically replaced with a normal one, we still opt to approach our work from the point of view of multi-linear maps. This choice is basically motivated by necessity; in order to get a practical grip on the enriched categories of interest here we will always need to resort to considering (explicitly or implicitly) the multi-linear maps we describe later.

The properties of multi-linear maps are abstracted to form what Lambek, in [33], calls a *multicategory*; we review his definition:

Definition 1.3.1 (multigraph) Given a set M_0 let M_0^* denote the free monoid generated by M_0, which has elements which are finite, possibly empty, sequences of the elements of M_0. We write sequences in square brackets $[A_1, \cdots, A_n]$ reserving $[\cdot]$ to denote the empty sequence, and sometimes we may drop the brackets around a sequence of length 1. We use vector notation $\vec{A}$ to denote arbitrary sequences and define $\mathrm{lh}(\vec{A})$ to be the sequence length function.

A *multigraph* consists of sets M_0 and M_1, of *objects* and *multi-maps* respectively, and functions

$$M_1 \xrightarrow{s} M_0^*$$

$$M_1 \xrightarrow{t} M_0$$

denoting *source* and *target*. As usual we write a multi-map f with source $\vec{B}$ and target A as

$$\vec{B} \xrightarrow{f} A$$

and say that it is an *n-map* if $\mathrm{lh}(\vec{B}) = n$.

Definition 1.3.2 (multicategory) A *multicategory* $\mathbb{M}$ consists of a multigraph (M_0, M_1, s, t) with

(a) For each object $A \in M_0$ an identity multi-map:

$$A \xrightarrow{\mathrm{i}_A} A$$

(b) A partially defined operation which assigns to each pair of multi-maps

$$\vec{C} \xrightarrow{g} B \qquad [\vec{B}_0, B, \vec{B}_1] \xrightarrow{f} A$$

a "composite":

$$[\vec{B}_0, \vec{C}, \vec{B}_1] \xrightarrow{f\langle \vec{B}_0, g, \vec{B}_1\rangle} A$$

We might think of this as the process of substituting the map g into the map f.

This data is subject to the conditions:

(i) $f\langle \vec{B}_0, \mathrm{i}_B, \vec{B}_1\rangle = f$;

(ii) $\mathrm{i}_A\langle f\rangle = f$'

(iii) given

$$[\vec{C}_0, A, \vec{C}_1, B, \vec{C}_2] \xrightarrow{\quad h \quad} C$$
$$\vec{A} \xrightarrow{\quad g \quad} A$$
$$\vec{B} \xrightarrow{\quad k \quad} B$$

then

$$(h\langle \vec{C}_0, g, \vec{C}_1, B, \vec{C}_2\rangle)\langle \vec{C}_0, \vec{A}, \vec{C}_1, k, \vec{C}_2\rangle = (h\langle \vec{C}_0, A, \vec{C}_1, k, \vec{C}_2\rangle)\langle \vec{C}_0, g, \vec{C}_1, \vec{B}, \vec{C}_2\rangle$$

which we may shorten to $h\langle g\rangle\langle k\rangle = h\langle k\rangle\langle g\rangle$, so long as no confusion then arises;

(iv) given

$$[\vec{B}_0, B, \vec{B}_1] \xrightarrow{\quad h \quad} A$$
$$[\vec{C}_0, C, \vec{C}_1] \xrightarrow{\quad g \quad} B$$
$$\vec{D} \xrightarrow{\quad k \quad} C$$

then $h\langle \vec{B}_0, (g\langle \vec{C}_0, k, \vec{C}_1\rangle), \vec{B}_1\rangle = (h\langle \vec{B}_0, g, \vec{B}_1\rangle)\langle \vec{C}_0, k, \vec{C}_1\rangle$ which we shorten to $h\langle g\langle k\rangle\rangle = h\langle g\rangle\langle k\rangle$.

We adopt the notation $\mathrm{ob}(\mathbb{M})$ and $\mathrm{map}(\mathbb{M})$ for the sets of objects and multi-maps of $\mathbb{M}$. $\square$

Of course multicategories are not of primary interest here and so we refer the reader to [33] for details of notation etcetera. In particular we will be using the notions of *left* and *right closed* multicategory, hoping that our description of the cases of interest here contain enough information for the reader to gather a flavour of the general theory. In recalling the definition above we have two particular multicategories in mind, both of which have objects which are (small) bicategories, for which we use the names $\underline{\mathcal{H}om}$ and $\underline{\mathcal{H}om}_S$:

Definition 1.3.3 (the multicategory $\underline{\mathcal{H}om}$) Has as its objects small bicategories (relative to some fixed set theoretic universe) and multi-maps, called n-homomorphisms

$$[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_n] \xrightarrow{\quad \mathrm{F} \quad} \underline{\underline{\mathcal{A}}}$$

given by the following information

(i) F maps each n-tuple $(b_1, \cdots, b_n)$ of 0-cells, with $b_i \in \underline{\underline{\mathcal{B}}}_i$ for each i, to a 0-cell $\mathrm{F}(b_1, \cdots, b_n) \in \underline{\underline{\mathcal{A}}}$, a 0-homomorphism $[\cdot] \longrightarrow \underline{\underline{\mathcal{A}}}$ simply corresponds to a 0-cell of $\underline{\underline{\mathcal{A}}}$.

(ii) for each fixed $1 \leq i \leq n$ and $(n-1)$-tuple of 0-cells $b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_n$ the function

$$\text{0–cell}(\underline{\underline{\mathcal{B}}}_i) \xrightarrow{\mathrm{F}(b_1,\cdots,b_{i-1},-,b_{i+1},\cdots,b_n)} \text{0–cell}(\underline{\underline{\mathcal{A}}})$$

comes enriched with the structure of a bicategorical homomorphism from $\underline{\underline{\mathcal{B}}}_i$ to $\underline{\underline{\mathcal{A}}}$. In future we are unlikely to explicitly quote all of the variables in the domain of F, in which case we will assume that the structure, definition or result under consideration is understood as being applied for each fixed set of 0-cells filling up the remaining positions in that domain. When using this convention we may subscript the cyphers – and $*$, for instance $\mathrm{F}(-_i)$ denotes any of the homomorphisms $\mathrm{F}(b_1, \cdots, b_{i-1}, -, b_{i+1}, \cdots, b_n)\colon \underline{\underline{\mathcal{B}}}_i \longrightarrow \underline{\underline{\mathcal{A}}}$.

(iii) These homomorphisms are related as follows, fix numbers $1 \leq i < j \leq n$ and then for each pair of 1-cells

$$b_i \xrightarrow{f_i} b_i' \qquad \in \underline{\underline{\mathcal{B}}}_i$$
$$b_j \xrightarrow{f_j} b_j' \qquad \in \underline{\underline{\mathcal{B}}}_j$$

we have a 2-cell

$$\begin{tikzcd}
\mathrm{F}(b_i,b_j) \arrow[r, "\mathrm{F}(f_i,b_j)"] \arrow[d, "\mathrm{F}(b_i,f_j)"'] \arrow[dr, phantom, "\Downarrow \mathrm{F}(f_i,f_j)"] & \mathrm{F}(b_i',b_j) \arrow[d, "\mathrm{F}(b_i',f_j)"] \\
\mathrm{F}(b_i,b_j') \arrow[r, "\mathrm{F}(f_i,b_j')"'] & \mathrm{F}(b_i',b_j')
\end{tikzcd}$$

or to be pedantic one of these for each vector $(b_k)_{k\neq i,j}$ of 0-cells. These must collectively satisfy the usual conditions making them into the structure 2-cells of

(a) a transformation $\mathrm{F}(f_i, -_j)\colon \mathrm{F}(b_i, -_j) \longrightarrow \mathrm{F}(b_i', -_j)$ in the horizontal direction and

(b) an optransformation $\mathrm{F}(-_i, f_j)\colon \mathrm{F}(-_i, b_j) \longrightarrow \mathrm{F}(-_i, b_j')$ in the vertical direction.

(iv) These squares are subject to the condition that given $j < k \leq n$ and a third 1-cell

$$b_k \xrightarrow{f_k} b_k' \qquad \in \underline{\underline{\mathcal{B}}}_k$$

the following cubical pasting equality holds:

$$
\begin{tikzcd}
 & F(b_i,b_j,b_k) \arrow[r] \arrow[dl] \arrow[d] \arrow[dr, phantom, "\Downarrow F(f_i,f_j,b_k)"] & F(b'_i,b_j,b_k) \arrow[d] \\
F(b_i,b_j,b'_k) \arrow[d] \arrow[r, phantom, "\overset{\Leftarrow}{F(b_i,f_j,f_k)}"] & F(b_i,b'_j,b_k) \arrow[r] \arrow[dl] \arrow[dr, phantom, "\overset{F(f_i,b'_j,f_k)}{\Leftarrow}"] & F(b'_i,b'_j,b_k) \arrow[dl] \\
F(b_i,b'_j,b'_k) \arrow[r] & F(b'_i,b'_j,b'_k) &
\end{tikzcd}
=
\begin{tikzcd}
 & F(b_i,b_j,b_k) \arrow[r] \arrow[dl] \arrow[dr, phantom, "\overset{\Leftarrow}{F(f_i,b_j,f_k)}"] & F(b'_i,b_j,b_k) \arrow[dl] \arrow[d] \\
F(b_i,b_j,b'_k) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow F(f_i,f_j,b'_k)"] & F(b'_i,b_j,b'_k) \arrow[d] \arrow[r, phantom, "\overset{\Leftarrow}{F(b'_i,f_j,f_k)}"] & F(b'_i,b'_j,b_k) \arrow[dl] \\
F(b_i,b'_j,b'_k) \arrow[r] & F(b'_i,b'_j,b'_k) &
\end{tikzcd}
$$

Notice the strong similarity between this definition and Remark 3.2 of [6] which we exploited in lemma 1.1.3. To complete the definition of $\underline{\mathcal{Hom}}$ we set about giving a notion of substitution for multi-homomorphisms, given two such

$$[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_n] \xrightarrow{\quad F \quad} \underline{\underline{\mathcal{A}}}$$
$$[\underline{\underline{\mathcal{C}}}_1, \cdots, \underline{\underline{\mathcal{C}}}_m] \xrightarrow{\quad G \quad} \underline{\underline{\mathcal{B}}}_i$$

we may define an $(n+m-1)$-homomorphism $F\langle G\rangle$ with

$$F\langle G\rangle(\vec{b}_0, \vec{c}, \vec{b}_1) = F(\vec{b}_0, G(\vec{c}), \vec{b}_1)$$

and homomorphism structures

$$
\begin{aligned}
F\langle G\rangle(\vec{b}_0, -, \vec{b}_1, \vec{c}, \vec{b}_2) &= F(\vec{b}_0, -, \vec{b}_1, G(\vec{c}), \vec{b}_2) \\
F\langle G\rangle(\vec{b}_0, \vec{c}_0, -, \vec{c}_1, \vec{b}_1) &= F(\vec{b}_0, *, \vec{b}_1) \circ G(\vec{c}_0, -, \vec{c}_1) \\
F\langle G\rangle(\vec{b}_0, \vec{c}, \vec{b}_1, -, \vec{b}_2) &= F(\vec{b}_0, G(\vec{c}), \vec{b}_1, -, \vec{b}_2)
\end{aligned}
$$

where $\circ$ denotes the usual composition of homomorphisms. Let $f_k \in \underline{\underline{\mathcal{B}}}_k$, $f_l \in \underline{\underline{\mathcal{B}}}_l$, $g_r \in \underline{\underline{\mathcal{C}}}_r$ and $g_s \in \underline{\underline{\mathcal{C}}}_s$ denote arbitrary 1-cells of distinct bicategories in the domain of the $(n+m-1)$-homomorphism we are constructing, now define the 2-cells which unify the homomorphisms above by:

$$\begin{aligned}
\mathrm{F}\langle\mathrm{G}\rangle(f_k, f_l) &= \mathrm{F}(f_k, f_l)\\
\mathrm{F}\langle\mathrm{G}\rangle(f_k, g_r) &= \mathrm{F}(f_k, \mathrm{G}(g_r))\\
\mathrm{F}\langle\mathrm{G}\rangle(g_r, f_l) &= \mathrm{F}(\mathrm{G}(g_r), f_l)
\end{aligned}$$

The only case which remains is $\mathrm{F}\langle\mathrm{G}\rangle(g_r, g_s)$, the natural definition of which is forced by its intended context, meaning the 2-cell obtained by applying the homomorphism $\mathrm{F}(\vec{b}_0, -_i, \vec{b}_1)$ to the pasting cell

$$\begin{tikzcd}
\mathrm{G}(c_r, c_s) \arrow[r, "\mathrm{G}(g_r, c_s)"] \arrow[d, "\mathrm{G}(c_r, g_s)"'] \arrow[dr, phantom, "\Downarrow \mathrm{G}(g_r, g_s)"] & \mathrm{G}(c'_r, c_s) \arrow[d, "\mathrm{G}(c'_r, g_s)"] \\
\mathrm{G}(c_r, c'_s) \arrow[r, "\mathrm{G}(g_r, c'_s)"'] & \mathrm{G}(c'_r, c'_s)
\end{tikzcd}$$

as described in definition A.0.8 of appendix A, in other words the composite:

$$\begin{aligned}
\mathrm{F}(\mathrm{G}(c'_r, g_s)) \otimes \mathrm{F}(\mathrm{G}(g_r, c_s)) &\xrightarrow[\cong]{\text{can}} \mathrm{F}(\mathrm{G}(c'_r, g_s) \otimes \mathrm{G}(g_r, c_s))\\
&\xrightarrow{\mathrm{F}(\mathrm{G}(g_r, g_s))} \mathrm{F}(\mathrm{G}(g_r, c'_s) \otimes \mathrm{G}(c_r, g_s))\\
&\xrightarrow[\cong]{\text{can}} \mathrm{F}(\mathrm{G}(g_r, c'_s)) \otimes \mathrm{F}(\mathrm{G}(c_r, g_s))
\end{aligned} \tag{1.34}$$

The proof that these data do indeed satisfy the conditions required of a multi-homomorphism is a matter of routine case checking, and is closely related to, but no more general than, (the proof of) the fact that a homomorphism $\mathrm{H}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$ naturally gives rise to homomorphisms:

$$\mathcal{B}icat(\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{B}}}) \xrightarrow{\mathcal{B}icat(\underline{\underline{\mathcal{A}}}, \mathrm{H})} \mathcal{B}icat(\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{C}}})$$

In fact most work goes into verifying that the formulae above give a well defined result when substituting a multi-homomorphism into a plain (1-) homomorphism, since this solely concerns the more complex 2-cells of composite 1.34. Of course, with a little effort, we may prove this result by direct calculation and a few tedious diagram chases, but by far the easiest and most intuitive approach is to express everything in terms of pasting diagrams and then apply the clause of lemma A.0.12 concerning the preservation of pasting composites by homomorphisms. We commend a closer look at these verifications to the reader as a way of building an intuition for the interactions between pasting and homomorphisms, but do not propose to expand on them here due to restrictions on space.

Simple case checking also demonstrates that this notion of substitution, along with the usual identity homomorphism $\mathrm{I}_{\underline{\underline{\mathcal{A}}}}$ on each bicategory $\underline{\underline{\mathcal{A}}}$, satisfy all of the conditions required of a multicategory. Again we leave detailed verification up to the reader. $\square$

To provide a motivation for the notion of n-homomorphism we first define, for each pair of bicategories $\underline{\underline{\mathcal{A}}}$ and $\underline{\underline{\mathcal{B}}}$, the bicategory $\mathcal{H}om(\underline{\underline{\mathcal{B}}},\underline{\underline{\mathcal{A}}})$ which is the full sub-bicategory of $\mathcal{B}icat(\underline{\underline{\mathcal{B}}},\underline{\underline{\mathcal{A}}})$ on those morphisms which are homomorphisms, and $\mathcal{H}om^{\mathrm{op}}(\underline{\underline{\mathcal{B}}},\underline{\underline{\mathcal{A}}})$ to be the corresponding full sub-bicategory of $\mathcal{B}icat^{\mathrm{op}}(\underline{\underline{\mathcal{B}}},\underline{\underline{\mathcal{A}}})$. We may also define a canonical evaluation 2-homomorphism

$$[\mathcal{H}om(\underline{\underline{\mathcal{B}}},\underline{\underline{\mathcal{A}}}),\underline{\underline{\mathcal{B}}}] \xrightarrow{\mathrm{ev_r}} \underline{\underline{\mathcal{A}}}$$

given by $\mathrm{ev_r}(\mathrm{H},b) = \mathrm{H}(b)$, $\mathrm{ev_r}(\mathrm{H},\text{-}) = \mathrm{H}(\text{-})$ and

$$\begin{array}{rcl} \mathcal{H}om(\underline{\underline{\mathcal{B}}},\underline{\underline{\mathcal{A}}}) & \xrightarrow{\mathrm{ev_r}(\text{-},b)} & \underline{\underline{\mathcal{A}}} \qquad \text{g.b.} \\ \mathrm{H} & \longmapsto & \mathrm{H}(b) \\ \Psi\colon \mathrm{H} \to \mathrm{H}' & \longmapsto & \Psi_b\colon \mathrm{H}(b) \to \mathrm{H}'(b) \\ \alpha\colon \Psi \Rightarrow \dot{\Psi} & \longmapsto & \alpha_b\colon \Psi_b \Rightarrow \dot{\Psi}_b \end{array}$$

which is a strict homomorphism (for each 0-cell $b \in \underline{\underline{\mathcal{B}}}$) because composition of transformations and modifications is performed "pointwise" in $\underline{\underline{\mathcal{A}}}$. Lastly the 2-cells unifying these are provided by the structure 2-cells of each transformation:

$$\mathrm{ev_r}(\Psi,f) = \begin{tikzcd} \mathrm{H}(b) \arrow[r, "\Psi_b"] \arrow[d, "\mathrm{H}(f)"'] \arrow[dr, phantom, "\Psi_f \Downarrow"] & \mathrm{H}'(b) \arrow[d, "\mathrm{H}'(f)"] \\ \mathrm{H}(\bar{b}) \arrow[r, "\Psi_{\bar{b}}"'] & \mathrm{H}'(\bar{b}) \end{tikzcd}$$

Notice that we also have a dually defined 2-homomorphism:

$$[\underline{\underline{\mathcal{B}}},\mathcal{H}om^{\mathrm{op}}(\underline{\underline{\mathcal{B}}},\underline{\underline{\mathcal{A}}})] \xrightarrow{\mathrm{ev_l}} \underline{\underline{\mathcal{A}}}$$

It should come as no surprise that these evaluations play the same rôle in the theory of multicategories as do the traditional evaluation maps of monoidal closed categories:

Lemma 1.3.4 *Substitution into the evaluation 2-homomorphism*

$$[\mathcal{H}om(\underline{\underline{\mathcal{B}}}_{n+1},\underline{\underline{\mathcal{A}}}),\underline{\underline{\mathcal{B}}}_{n+1}] \xrightarrow{\mathrm{ev_r}} \underline{\underline{\mathcal{A}}}$$

sets up a bijection:

$$\frac{[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_n] \xrightarrow{\quad\tilde{\mathrm{F}}\quad} \mathcal{H}om(\underline{\underline{\mathcal{B}}}_{n+1}, \underline{\underline{\mathcal{A}}})}{[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_{n+1}] \xrightarrow{\quad\mathrm{F}\quad} \underline{\underline{\mathcal{A}}}}$$

Dually substitution into

$$[\underline{\underline{\mathcal{B}}}_1, \mathcal{H}om^{\mathrm{op}}(\underline{\underline{\mathcal{B}}}_1, \underline{\underline{\mathcal{A}}})] \xrightarrow{\quad\mathrm{ev}_1\quad} \underline{\underline{\mathcal{A}}}$$

gives a bijection:

$$\frac{[\underline{\underline{\mathcal{B}}}_2, \cdots, \underline{\underline{\mathcal{B}}}_{n+1}] \xrightarrow{\quad\hat{\mathrm{F}}\quad} \mathcal{H}om^{\mathrm{op}}(\underline{\underline{\mathcal{B}}}_1, \underline{\underline{\mathcal{A}}})}{[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_{n+1}] \xrightarrow{\quad\mathrm{F}\quad} \underline{\underline{\mathcal{A}}}}$$

In other words $\underline{\mathcal{H}om}$ is what Lambek calls a biclosed *multicategory.*

Proof. We establish the first bijection, the second is quite clearly dual to that. Consider cases:

$\boldsymbol{n=0}$: By definition a 0-homomorphism $\tilde{\mathrm{F}}\colon [\cdot] \longrightarrow \mathcal{H}om(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{A}}})$ simply picks out a 0-cell of its domain, or in other words a homomorphism $\mathrm{F}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{A}}}$. Examining the definitions of ev_{r} and the substitution operation we see that $\mathrm{F} = \mathrm{ev}_{\mathrm{r}}\langle \tilde{\mathrm{F}}, \underline{\underline{\mathcal{B}}}\rangle$.

$\boldsymbol{n=1}$: given a 2-homomorphism

$$[\underline{\underline{\mathcal{B}}}_1, \underline{\underline{\mathcal{B}}}_2] \xrightarrow{\quad\mathrm{F}\quad} \underline{\underline{\mathcal{A}}}$$

we may define a (1-)homomorphism

$$\underline{\underline{\mathcal{B}}}_1 \xrightarrow{\quad\tilde{\mathrm{F}}\quad} \mathcal{H}om(\underline{\underline{\mathcal{B}}}_2, \underline{\underline{\mathcal{A}}})$$

where $\tilde{\mathrm{F}}(b_1)$ is the homomorphism $\mathrm{F}(b_1, -)$, for each 0-cell b_1 in $\underline{\underline{\mathcal{B}}}_1$, and $\tilde{\mathrm{F}}(f_1)$ is the transformation $\mathrm{F}(f_1, -)\colon \mathrm{F}(b_1, -) \longrightarrow \mathrm{F}(b'_1, -)$, for each 1-cell $f_1\colon b_1 \longrightarrow b'_1$ in $\underline{\underline{\mathcal{B}}}_1$, both of which are guarantied by the definition of 2-homomorphism.

In order to derive the action of $\tilde{\mathrm{F}}$ on 2-cells, and provide it with structural isomorphisms, we consider condition (b) on the 2-cells $\mathrm{F}(f_i, f_j)$. By plotting out the pasting equalities which ensure that $\mathrm{F}(-, f_2)\colon \mathrm{F}(-, b_2) \longrightarrow \mathrm{F}(-, b'_2)$ is an optransformation for each 1-cell $f_2 \in \underline{\underline{\mathcal{B}}}_2$, we see that this condition is equivalent to:

- given a 2-cell $\alpha_1\colon f_1 \Rightarrow \dot{f}_1$ in $\underline{\underline{\mathcal{B}}}_1$ the 2-cells
$$\mathrm{F}(f_1, b_2) \xrightarrow{\mathrm{F}(\alpha_1, b_2)} \mathrm{F}(\dot{f}_1, b_2)$$
one for each 0-cell $b_2 \in \underline{\underline{\mathcal{B}}}_2$, satisfy the conditions required of a modification from $\tilde{\mathrm{F}}(f_1)$ to $\tilde{\mathrm{F}}(\dot{f}_1)$.
- given a pair of 1-cells $f_1\colon b_1 \longrightarrow b_1'$ and $f_1'\colon b_1' \longrightarrow b_1''$ in $\underline{\underline{\mathcal{B}}}_1$ the canonical 2-cell isomorphisms
$$\mathrm{F}(f_1', b_2) \otimes \mathrm{F}(f_1, b_2) \xrightarrow[\cong]{\text{can}} \mathrm{F}(f_1' \otimes f_1, b_2)$$
one for each 0-cell $b_2 \in \underline{\underline{\mathcal{B}}}_2$, form an (isomorphic) modification from $\tilde{\mathrm{F}}(f_1') \otimes \tilde{\mathrm{F}}(f_1)$ to $\tilde{\mathrm{F}}(f_1' \otimes f_1)$.
- given a 0-cell $b_1 \in \underline{\underline{\mathcal{B}}}_1$ the canonical 2-cells
$$\mathrm{i}_{\mathrm{F}(b_1, b_2)} \xrightarrow[\cong]{\text{can}} \mathrm{F}(\mathrm{i}_{b_1}, b_2)$$
one for easy 0-cell $b_2 \in \underline{\underline{\mathcal{B}}}_2$, form an (isomorphic) modification from $\mathrm{i}_{\tilde{\mathrm{F}}(b_1)}$ to $\tilde{\mathrm{F}}(\mathrm{i}_{b_1})$.

These modifications satisfy all of the coherence conditions required of a homomorphism, simply since they do so "pointwise", and therefore complete the definition of $\tilde{\mathrm{F}}$. Returning to our examinations of substitution and ev_r it is clear that $\tilde{\mathrm{F}}$ is the unique homomorphism such that $\mathrm{F} = \mathrm{ev}_\mathrm{r}\langle \tilde{\mathrm{F}}, \underline{\underline{\mathcal{B}}}_2\rangle$.

$\boldsymbol{n > 1}$**:** Given a $(n+1)$-homomorphism
$$[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_{n+1}] \xrightarrow{\mathrm{F}} \underline{\underline{\mathcal{A}}}$$
we construct an n-homomorphism
$$[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_n] \xrightarrow{\tilde{\mathrm{F}}} \underline{\underline{\mathcal{A}}}$$
by letting $\tilde{\mathrm{F}}(b_1, \cdots, b_n) = \mathrm{F}(b_1, \cdots, b_n, \text{–})$ and defining the homomorphism structures in each variable in exactly the same way as in case $n = 1$. All that remains is to provide unifying modifications $\tilde{\mathrm{F}}(f_i, f_j)$ for $1 \leq i < j \leq n$ and each pair of 2-cells $f_i \in \underline{\underline{\mathcal{B}}}_i$ and $f_j \in \underline{\underline{\mathcal{B}}}_j$. But the cubical condition (iv) on F clearly implies that the collection of 2-cells
$$\mathrm{F}(b_i', f_j, b_k) \otimes \mathrm{F}(f_i, b_j, b_k) \xrightarrow{\mathrm{F}(f_i, f_j, b_k)} \mathrm{F}(f_i, b_j', b_k) \otimes \mathrm{F}(b_i, f_j, b_k)$$
where b_k ranges over the 0-cells of $\underline{\underline{\mathcal{B}}}_k$, form a modification from $\tilde{\mathrm{F}}(b_i', f_j) \otimes \tilde{\mathrm{F}}(f_i, b_j)$ to $\tilde{\mathrm{F}}(f_i, b_j') \otimes \tilde{\mathrm{F}}(b_i, f_j)$. These modifications satisfy the conditions required of an n -homomorphism "pointwise", providing us with $\tilde{\mathrm{F}}$ which is again the unique multi -homomorphism such that $\mathrm{F} = \mathrm{ev}_\mathrm{r}\langle \tilde{\mathrm{F}}, \underline{\underline{\mathcal{B}}}_2\rangle$. □

It would now seem to be little more than a formality to define the concept of bicategorical enrichment, by first generalising the notion of enrichment from monoidal to multicategories and then applying it to $\underline{\mathcal{H}om}$. This is not however really the multicategory of interest when studying biadjoints, although we will see later that it is appropriate when used in the interpretation of the triangle identities for local adjunctions between homomorphisms. In essence enrichment over $\underline{\mathcal{H}om}$ gives categories with "homsets" which look like bicategories of homomorphisms, transformations and modifications, but when describing biadjoints in terms of triangle identities it is important that unit and counit should both be *strong* transformations.

As indicated in the last paragraph, for some purposes we might wish to restrict attention to *strong* multi-homomorphisms, that is those

$$[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_n] \xrightarrow{\mathrm{F}} \underline{\underline{\mathcal{A}}}$$

in $\underline{\mathcal{H}om}$ such that for each pair integers $1 \leq i < j \leq n$ and 1-cells $f_i \in \underline{\underline{\mathcal{B}}}_1$, $f_j \in \underline{\underline{\mathcal{B}}}_j$ the 2-cells $\mathrm{F}(f_i, f_j)$ are isomorphisms. Quite clearly this subset is closed in $\underline{\mathcal{H}om}$ under substitution (and identities) and so constitutes a sub-multicategory $\underline{\mathcal{H}om}_S$.

Notice that $\underline{\mathcal{H}om}_S$ is *symmetric* in the sense that S_n, the group of permutations of $\{1, \cdots, n\}$, admits an obvious left action on the set of strong n-homomorphisms under which a permutation $\sigma \in S_n$ acts on F to permute its variables and give another strong n-homomorphism:

$$[\underline{\underline{\mathcal{B}}}_{\sigma(1)}, \cdots, \underline{\underline{\mathcal{B}}}_{\sigma(n)}] \xrightarrow{\sigma \cdot \mathrm{F}} \underline{\underline{\mathcal{A}}}$$

The property that completes the notion of symmetry is the compatibility of these actions with respect to substitution, a concept which we leave up to the imagination of the reader. Notice that $\underline{\mathcal{H}om}$ itself is certainly not symmetric in this way, the orientations of the 2-cells $\mathrm{F}(f_i, f_j)$ are determined by the order of the variables in the domain of F and on permuting these there is little chance that this orientation rule will be preserved. The symmetry properties of strong multi-homomorphisms stem from the fact that after any permutation we may correct any badly oriented 2-cells by replacing them with their inverses.

Symmetry also finds expression in the restriction of lemma 1.3.4 to $\underline{\mathcal{H}om}_S$. Recall that $\mathcal{H}om_S(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{A}}})$ denotes the bicategory of homomorphisms, strong transformations and modifications, which we may consider to be a (locally full) sub-bicategory of both $\mathcal{H}om(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{A}}})$ and $\mathcal{H}om^{\mathrm{op}}(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{A}}})$. The evaluation 2-homomorphisms of the lemma then restrict to strong 2-homomorphisms

$$[\mathcal{H}om_S(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{A}}}), \underline{\underline{\mathcal{B}}}] \xrightarrow{\mathrm{ev_r}} \underline{\underline{\mathcal{A}}} \qquad [\underline{\underline{\mathcal{B}}}, \mathcal{H}om_S(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{A}}})] \xrightarrow{\mathrm{ev_l}} \underline{\underline{\mathcal{A}}}$$

and substitution into these gives bijections

$$\frac{[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_n] \xrightarrow{\tilde{\mathrm{F}}} \mathcal{H}om_S(\underline{\underline{\mathcal{B}}}_{n+1}, \underline{\underline{\mathcal{A}}})}{[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_{n+1}] \xrightarrow{\mathrm{F}} \underline{\underline{\mathcal{A}}}} \qquad \frac{[\underline{\underline{\mathcal{B}}}_2, \cdots, \underline{\underline{\mathcal{B}}}_{n+1}] \xrightarrow{\hat{\mathrm{F}}} \mathcal{H}om_S(\underline{\underline{\mathcal{B}}}_1, \underline{\underline{\mathcal{A}}})}{[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_{n+1}] \xrightarrow{\mathrm{F}} \underline{\underline{\mathcal{A}}}}$$

of strong multi-homomorphisms.

Now we give the definition of multicategory enrichment:

Definition 1.3.5 If $\mathbb{M}$ is a multicategory then a $\mathbb{M}$-enriched category consists of a set $\mathrm{ob}(\mathbb{A})$ of objects and for each pair $a, a' \in \mathrm{ob}(\mathbb{A})$ an object $\mathbb{A}(a,a') \in \mathbb{M}$ equipped with multi-homomorphisms

$$\begin{array}{rll} [\mathbb{A}(a',a''), \mathbb{A}(a,a')] \xrightarrow{\circ} \mathbb{A}(a,a'') & \text{for each triple } a, a', a'' \in \mathrm{ob}(\mathbb{A}) \\ [\cdot] \xrightarrow{\quad \mathrm{i}_a \quad} \mathbb{A}(a,a) & \text{for each } a \in \mathrm{ob}(\mathbb{A}) \end{array}$$

composition and *identity* which satisfy:

$$\begin{array}{rcll} \circ\langle \mathbb{A}(a'',a'''), \circ\rangle & = & \circ\langle \circ, \mathbb{A}(a,a')\rangle & : [\mathbb{A}(a'',a'''), \mathbb{A}(a',a''), \mathbb{A}(a,a')] \longrightarrow \mathbb{A}(a''',a) \\ \circ\langle \mathbb{A}(a,a'), \mathrm{i}_a\rangle & = & \mathrm{i}_{\mathbb{A}(a,a')} & : \mathbb{A}(a,a') \longrightarrow \mathbb{A}(a,a') \\ \circ\langle \mathrm{i}_{a'}, \mathbb{A}(a,a')\rangle & = & \mathrm{i}_{\mathbb{A}(a,a')} & : \mathbb{A}(a,a') \longrightarrow \mathbb{A}(a,a') \end{array}$$

□

A natural example is that of the enrichment of a *right closed* multicategory $\mathbb{M}$ over itself. Such a multicategory has, by definition, an operation which associates with any pair of objects $A, B \in \mathrm{ob}(\mathbb{M})$ a third one $(B \Leftarrow A)$ and a 2-map

$$[B \Leftarrow A, A] \xrightarrow{\mathrm{ev_r}} B$$

substitution into which gives the kind of bijections we saw in lemma 1.3.4. When this condition holds an essentially standard proof (cf. [30] section 1.6) provides us with an $\mathbb{M}$-enriched category with set of objects $\mathrm{ob}(\mathbb{M})$ itself and "homsets" $\mathbb{M}(A,B) = (B \Leftarrow A)$. The composition and identities making these into an $\mathbb{M}$-category are given by the correspondences:

$$\frac{[C \Leftarrow B, B \Leftarrow A, A] \xrightarrow{\mathrm{ev_r}\langle C \Leftarrow B, \mathrm{ev_r}\rangle} C}{[C \Leftarrow B, B \Leftarrow A] \xrightarrow{\qquad \circ \qquad} C \Leftarrow A} \qquad \frac{A \xrightarrow{\mathrm{i}_A} A}{[\cdot] \xrightarrow{\mathrm{i}_A} A \Leftarrow A}$$

Lemma 1.3.4 provides us with a second bijection, and we say as a result that $\underline{\mathcal{H}om}$ is also *left* closed. This we deal with this by defining a dual multicategory $\mathbb{M}^{\mathrm{op}}$ which has the same objects and multi-maps as $\mathbb{M}$ but in which the domain of each multi-map is formed by reversing the order of the objects in its domain sequence in $\mathbb{M}$, then $\mathbb{M}$ is left closed iff $\mathbb{M}^{\mathrm{op}}$ is right closed giving rise to another enrichment, this time over $\mathbb{M}^{\mathrm{op}}$. Of course if $\mathbb{M}$ is biclosed then it admits both of these enrichments, furthermore symmetry would ensure an isomorphism with its dual $\mathbb{M}^{\mathrm{op}}$, implying that left and right internal homs coincide thus demonstrating that the two possible enrichments do as well.

To fix our nomenclature we say that a category is "bicategory enriched" if it is enriched over $\underline{\mathcal{H}om}$ and "*strongly* bicategory enriched" if its composition is a strong

2-homomorphism, which is to say that it's enriched over $\underline{\mathcal{H}om}_S$. We may also use the term "bicategory op-enriched" to denote $\underline{\mathcal{H}om}^{\mathrm{op}}$-categories, although our principle interest is in enrichment of the strong variety. Associated with each of these we have an enriched category of bicategories, for which we use the notations $\underline{\underline{\underline{\mathcal{H}om}}}$, $\underline{\underline{\underline{\mathcal{H}om}}}^{\mathrm{op}}$ and $\underline{\underline{\underline{Hom}}}_S$ where we use three underlines to emphasise their 3-dimensional nature. It seems appropriate to give a more detailed description of bicategorical enrichment:

Description 1.3.6 A bicategory enriched category $\underline{\underline{\underline{\mathcal{A}}}}$ consists of

- a set of objects $\mathrm{ob}(\underline{\underline{\underline{\mathcal{A}}}})$,
- for each pair of objects $A, A' \in \mathrm{ob}(\underline{\underline{\underline{\mathcal{A}}}})$ a bicategory $\underline{\underline{\underline{\mathcal{A}}}}(A, A')$,
- for each pair of 0-cells $\mathrm{F} \in \underline{\underline{\underline{\mathcal{A}}}}(A, A')$ and $\mathrm{G} \in \underline{\underline{\underline{\mathcal{A}}}}(A', A'')$ a composite 0-cell $\mathrm{G} \circ \mathrm{F} \in \underline{\underline{\underline{\mathcal{A}}}}(A, A'')$ and enrichments of this operation to homomorphisms

$$\underline{\underline{\underline{\mathcal{A}}}}(A, A') \xrightarrow{\mathrm{G} \circ -} \underline{\underline{\underline{\mathcal{A}}}}(A', A'')$$
$$\underline{\underline{\underline{\mathcal{A}}}}(A', A'') \xrightarrow{- \circ \mathrm{F}} \underline{\underline{\underline{\mathcal{A}}}}(A, A'')$$

  These should collectively satisfy the associativity rules

$$\begin{array}{rcll} (\mathrm{H} \circ \mathrm{G}) \circ - & = & \mathrm{H} \circ (\mathrm{G} \circ -) & : \underline{\underline{\underline{\mathcal{A}}}}(A, A') \longrightarrow \underline{\underline{\underline{\mathcal{A}}}}(A, A''') \\ - \circ (\mathrm{G} \circ \mathrm{F}) & = & (- \circ \mathrm{G}) \circ \mathrm{F} & : \underline{\underline{\underline{\mathcal{A}}}}(A'', A''') \longrightarrow \underline{\underline{\underline{\mathcal{A}}}}(A, A''') \\ (\mathrm{G} \circ -) \circ \mathrm{F} & = & \mathrm{G} \circ (- \circ \mathrm{F}) & : \underline{\underline{\underline{\mathcal{A}}}}(A', A'') \longrightarrow \underline{\underline{\underline{\mathcal{A}}}}(A, A''') \end{array}$$

  where $\mathrm{H} \in \underline{\underline{\underline{\mathcal{A}}}}(A'', A''')$ is a third, arbitrary, 0-cell.
- for each object A an identity 0-cell $\mathrm{I}_A \in \underline{\underline{\underline{\mathcal{A}}}}(A, A)$ such that the homomorphisms

$$\underline{\underline{\underline{\mathcal{A}}}}(A, B) \xrightarrow{\mathrm{I}_B \circ -} \underline{\underline{\underline{\mathcal{A}}}}(A, B)$$
$$\underline{\underline{\underline{\mathcal{A}}}}(A, B) \xrightarrow{- \circ \mathrm{I}_A} \underline{\underline{\underline{\mathcal{A}}}}(A, B)$$

  are both equal to the identity on $\underline{\underline{\underline{\mathcal{A}}}}(A, B)$ for all objects $A, B \in \mathrm{ob}(\underline{\underline{\underline{\mathcal{A}}}})$
- for each pair of 1-cells

$$\mathrm{F} \xrightarrow{\Phi} \overline{\mathrm{F}} \quad \in \underline{\underline{\underline{\mathcal{A}}}}(A, A')$$
$$\mathrm{G} \xrightarrow{\Psi} \overline{\mathrm{G}} \quad \in \underline{\underline{\underline{\mathcal{A}}}}(A', A'')$$

  a 2-cell

$$\begin{tikzcd} \mathrm{G} \circ \mathrm{F} \arrow[r, "\Psi \circ \mathrm{F}"] \arrow[d, "\mathrm{G} \circ \Phi"'] \arrow[dr, phantom, "\Downarrow \Psi \circ \Phi"] & \overline{\mathrm{G}} \circ \mathrm{F} \arrow[d, "\overline{\mathrm{G}} \circ \Phi"] \\ \mathrm{G} \circ \overline{\mathrm{F}} \arrow[r, "\Psi \circ \overline{\mathrm{F}}"'] & \overline{\mathrm{G}} \circ \overline{\mathrm{F}} \end{tikzcd} \qquad (1.35)$$

in the bicategory $\underline{\underline{\mathcal{A}}}(A, A'')$. These must collectively satisfy the rules which ensure that they form the structure 2-cells of:

(a) a transformation $\Psi \circ - : (\mathrm{G} \circ -) \longrightarrow (\overline{\mathrm{G}} \circ -)$,
(b) an optransformation $- \circ \Phi : (- \circ \mathrm{F}) \longrightarrow (- \circ \overline{\mathrm{F}})$.

- finally these 2-cells must also satisfy an associativity condition, so suppose that

$$\mathrm{H} \xrightarrow{\quad \Omega \quad} \overline{\mathrm{H}} \ \in \underline{\underline{\mathcal{A}}}(A'', A''')$$

is another 1-cell then we have 2-cells

$$\begin{tikzcd}
\mathrm{H}\circ(\mathrm{G}\circ\mathrm{F}) \arrow[r, "\mathrm{H}\circ(\Psi\circ\mathrm{F})"] \arrow[d, "\mathrm{H}\circ(\mathrm{G}\circ\Phi)"'] \arrow[dr, phantom, "\Downarrow \mathrm{H}\circ(\Psi\circ\Phi)"] & \mathrm{H}\circ(\overline{\mathrm{G}}\circ\mathrm{F}) \arrow[d, "\mathrm{H}\circ(\overline{\mathrm{G}}\circ\Phi)"] \\
\mathrm{H}\circ(\mathrm{G}\circ\overline{\mathrm{F}}) \arrow[r, "\mathrm{H}\circ(\Psi\circ\overline{\mathrm{F}})"'] & \mathrm{H}\circ(\overline{\mathrm{G}}\circ\overline{\mathrm{F}})
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{H}\circ\mathrm{G})\circ\mathrm{F} \arrow[r, "(\Omega\circ\mathrm{G})\circ\mathrm{F}"] \arrow[d, "(\mathrm{H}\circ\Psi)\circ\mathrm{F}"'] \arrow[dr, phantom, "\Downarrow (\Omega\circ\Psi)\circ\mathrm{F}"] & (\overline{\mathrm{H}}\circ\mathrm{G})\circ\mathrm{F} \arrow[d, "(\overline{\mathrm{H}}\circ\Psi)\circ\mathrm{F}"] \\
(\mathrm{H}\circ\overline{\mathrm{G}})\circ\mathrm{F} \arrow[r, "(\Omega\circ\overline{\mathrm{G}})\circ\mathrm{F}"'] & (\overline{\mathrm{H}}\circ\overline{\mathrm{G}})\circ\mathrm{F}
\end{tikzcd}$$

which have the natural definitions demanded by their context. In other words they are the results of applying the homomorphisms $\mathrm{H} \circ -$ and $- \circ \mathrm{F}$ to the square pasting cells $\Psi \circ \Phi$ and $\Omega \circ \Psi$ (respectively) in the way described in definition A.0.8 of appendix A. This notation allows us to express the remaining three associativity conditions as:

$$\begin{aligned}
\Omega \circ (\mathrm{G} \circ \Phi) &= (\Omega \circ \mathrm{G}) \circ \Phi \\
\mathrm{H} \circ (\Psi \circ \Phi) &= (\mathrm{H} \circ \Psi) \circ \Phi \\
(\Omega \circ \Psi) \circ \mathrm{F} &= \Omega \circ (\Psi \circ \mathrm{F})
\end{aligned}$$

Each of these postulate an equality between 2-cells which the earlier associativity conditions already ensure to have the same domain and codomain.

The descriptions of strong and op-enrichment only differ from this in requiring that the 2-cells $\Psi \circ \Phi$ should be isomorphisms in the first case or have the opposite orientation in the second. We will meet a number of related examples of strongly bicategory enriched categories specifically designed to cope with change of base questions in terms of equipments and double bicategories in the next section.

In future we follow the naming conventions of bicategory theory and refer to the 0,1 and 2-cells of the "homsets" of a bicategory enriched category as its homomorphisms, transformations (with the modifier "strong" or "op-" where appropriate) and modifications. This will serve to emphasise the pivotal relationship between calculations inside these categories and those with which we are already well accustomed in bicategory theory itself. This resemblance is heightened by dropping the use of $\circ$ and writing these composites by superposition alone, thus clarifying the relationship between the notions of composition and application of homomorphisms etcetera. $\square$

We also expand on the notion of a bicategorically enriched functor:

Description 1.3.7 A bicategorically enriched functor $(-)_\diamond \colon \underline{\underline{\underline{\mathcal{A}}}} \longrightarrow \underline{\underline{\underline{\mathcal{B}}}}$ consists of a map taking each object $A \in \underline{\underline{\underline{\mathcal{A}}}}$ to some object $A_\diamond \in \underline{\underline{\underline{\mathcal{B}}}}$ and for each pair of objects $A, A' \in \underline{\underline{\underline{\mathcal{A}}}}$ a homomorphism of bicategories $(-)_\diamond \colon \underline{\underline{\underline{\mathcal{A}}}}(A, A') \longrightarrow \underline{\underline{\underline{\mathcal{B}}}}(A_\diamond, A'_\diamond)$ which satisfy:

- for each homomorphism $\mathrm{F} \colon \overline{A} \longrightarrow A$ in $\underline{\underline{\underline{\mathcal{A}}}}$ the equality
$$(-)_\diamond \circ \mathrm{F}_\diamond = (- \circ \mathrm{F})_\diamond \colon \underline{\underline{\underline{\mathcal{A}}}}(A, A') \longrightarrow \underline{\underline{\underline{\mathcal{B}}}}(\overline{A}_\diamond, A'_\diamond)$$
of homomorphisms (of bicategories) holds.
- for each homomorphism $\mathrm{F}' \colon A' \longrightarrow \overline{A}'$ in $\underline{\underline{\underline{\mathcal{A}}}}$ the equality
$$\mathrm{F}'_\diamond \circ (-)_\diamond = (\mathrm{F}' \circ -)_\diamond \colon \underline{\underline{\underline{\mathcal{A}}}}(A, A') \longrightarrow \underline{\underline{\underline{\mathcal{B}}}}(A_\diamond, \overline{A}'_\diamond)$$
of homomorphisms (of bicategories) holds.
- for each pair of transformations
$$\mathrm{F} \xrightarrow{\quad\Phi\quad} \overline{\mathrm{F}} \quad \in \underline{\underline{\underline{\mathcal{A}}}}(A, A')$$
$$\mathrm{G} \xrightarrow{\quad\Psi\quad} \overline{\mathrm{G}} \quad \in \underline{\underline{\underline{\mathcal{A}}}}(A', A'')$$
the equality of modifications
$$(\Psi \circ \Phi)_\diamond = \Psi_\diamond \circ \Phi_\diamond \;\; \colon (\overline{\mathrm{G}}_\diamond \circ \Phi_\diamond) \otimes (\Phi_\diamond \circ \mathrm{F}) \longrightarrow (\Psi_\diamond \circ \overline{\mathrm{F}}_\diamond) \otimes (\mathrm{G}_\diamond \circ \Phi_\diamond)$$
holds. In this expression $(\Psi \circ \Phi)_\diamond$ is the modification obtained by applying the homomorphism $(-)_\diamond$ to the square pasting cell (1.35) (as described in appendix A) whereas $\Psi_\diamond \circ \Phi_\diamond$ is the canonical modification associated with the transformations $\Psi_\diamond$ and $\Phi_\diamond$ in $\underline{\underline{\underline{\mathcal{B}}}}$. The domain and codomain of these already match up by the previous two conditions on $(-)_\diamond$.

An obvious example of such a functor is the representable $\underline{\underline{\underline{\mathcal{A}}}}(A, -)$ associated with an object $A \in \underline{\underline{\underline{\mathcal{A}}}}$, which maps an object $A' \in \underline{\underline{\underline{\mathcal{A}}}}$ to the bicategory $\underline{\underline{\underline{\mathcal{A}}}}(A, A') \in \underline{\underline{\underline{\mathcal{H}om}}}$ and has action on the "homset" $\underline{\underline{\underline{\mathcal{A}}}}(A', A'')$ derived from the composition of $\underline{\underline{\underline{\mathcal{A}}}}$ via the correspondence:

$$\frac{[\underline{\underline{\underline{\mathcal{A}}}}(A', A''), \underline{\underline{\underline{\mathcal{A}}}}(A, A')] \xrightarrow{\quad\circ\quad} \underline{\underline{\underline{\mathcal{A}}}}(A, A'')}{\underline{\underline{\underline{\mathcal{A}}}}(A', A'') \xrightarrow{\quad\underline{\underline{\underline{\mathcal{A}}}}(A, -)\quad} \mathcal{H}om(\underline{\underline{\underline{\mathcal{A}}}}(A, A'), \underline{\underline{\underline{\mathcal{A}}}}(A, A''))}$$

Clearly if $\underline{\underline{\underline{\mathcal{A}}}}$ is strongly enriched then each representable $\underline{\underline{\underline{\mathcal{A}}}}(A, -)$ restricts to an enriched functor $\underline{\underline{\underline{\mathcal{A}}}} \longrightarrow \underline{\underline{\underline{\mathcal{H}om}}}_S$. $\square$

Having introduced the material of this section as a context in which to interpret and generalise the notion of biadjunction we should really take this opportunity to elaborate upon that idea. Recall that example 1.1.7 of section 1.1 provided us with a triangle identity description of biadjunctions, which may naturally be cast into the context of the strongly bicategory enriched $\underline{\underline{\underline{\mathcal{H}om}}}_S$. Expressing this in an arbitrary $\underline{\mathcal{H}om}_S$-category we get:

Definition 1.3.8 A *biadjoint pair* $(\mathrm{G} \dashv_b \mathrm{F}, \Psi, \Phi, \alpha, \beta)\colon A \longrightarrow B$ in a strongly bicategory enriched category $\underline{\underline{\underline{\mathcal{A}}}}$ is given by the following data:

- a pair of homomorphisms $\mathrm{G} \in \underline{\underline{\underline{\mathcal{A}}}}(A, B)$ and $\mathrm{F} \in \underline{\underline{\underline{\mathcal{A}}}}(B, A)$,
- two strong transformations

  $$\mathrm{I}_B \xrightarrow{\quad\Psi\quad} \mathrm{G} \circ \mathrm{F}$$

  $$\mathrm{F} \circ \mathrm{G} \xrightarrow{\quad\Phi\quad} \mathrm{I}_A$$

  called the unit and counit respectively.
- triangle identities which consist of two isomorphic modifications

  $$\begin{tikzcd} \mathrm{G} \arrow[r, "\Psi\mathrm{G}"] \arrow[dr, "\mathrm{i_G}"'] & \mathrm{GFG} \arrow[d, "\mathrm{G}\Phi"] \arrow[dl, phantom, "\underset{\alpha}{\cong}"] \\ & \mathrm{G} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{F} \arrow[d, "\mathrm{F}\Psi"'] \arrow[dr, "\mathrm{i_F}"] & \\ \mathrm{FGF} \arrow[r, "\Phi\mathrm{F}"'] \arrow[ur, phantom, "\overset{\beta}{\cong}"] & \mathrm{F} \end{tikzcd} \qquad (1.36)$$

  in $\underline{\underline{\underline{\mathcal{A}}}}(A, B)(\mathrm{G}, \mathrm{G})$ and $\underline{\underline{\underline{\mathcal{A}}}}(B, A)(\mathrm{F}, \mathrm{F})$ respectively.

This is certainly enough to specify a biadjunction but, as we pointed out in example 1.1.7, it is often useful to assume two extra coherence conditions on α and β which ensure that the equivalences $\underline{\underline{\mathcal{B}}}(\mathrm{F}c, b) \simeq \underline{\underline{\mathcal{C}}}(c, \mathrm{G}b)$, as derived from them, are in fact *adjoint* equivalences. Therefore we say that a biadjoint pair is *locally adjoint* if the diagrams

$$\begin{tikzcd} \mathrm{G\Phi F} \otimes \Psi\mathrm{GF} \otimes \Psi \arrow[r, "\cong", "\mathrm{G\Phi F} \otimes (\Psi\Psi)"'] & \mathrm{G\Phi F} \otimes \mathrm{GF}\Psi \otimes \Psi \arrow[r, "\cong", "\mathrm{can} \otimes \Psi"'] & \mathrm{G}(\Phi\mathrm{F} \otimes \mathrm{F}\Psi) \otimes \Psi \arrow[dd, "\mathrm{G}\beta \otimes \Psi", "\wr"'] \\ (\mathrm{G}\Phi \otimes \Psi\mathrm{G})\mathrm{F} \otimes \Psi \arrow[u, "\mathrm{can} \otimes \Psi", "\wr"'] & (\mathrm{a}) & \\ \mathrm{i_G F} \otimes \Psi \arrow[u, "\alpha\mathrm{F} \otimes \Psi", "\wr"'] \arrow[r, "\mathrm{can} \otimes \Psi", "\simeq"'] & \mathrm{i_{GF}} \otimes \Psi \arrow[r, "\mathrm{can} \otimes \Psi", "\simeq"'] & \mathrm{Gi_F} \otimes \Psi \end{tikzcd}$$

$$
\begin{tikzcd}
\Phi\otimes \mathrm{FG}\Phi\otimes \mathrm{F}\Psi\mathrm{G} \arrow[r, "\cong", "(\Phi\Phi)\otimes \mathrm{F}\Psi\mathrm{G}"'] & \Phi\otimes \Phi\mathrm{FG}\otimes \mathrm{F}\Psi\mathrm{G} \arrow[r, "\cong", "\Phi\otimes \mathrm{can}"'] & \Phi\otimes(\Phi\mathrm{F}\otimes \mathrm{F}\Psi)\mathrm{G} \arrow[dd, "\wr\ \Phi\otimes\beta\mathrm{G}"] \\
\Phi\otimes \mathrm{F}(\mathrm{G}\Phi\otimes\Psi\mathrm{G}) \arrow[u, "\Phi\otimes \mathrm{can}\ \wr"] & \text{(b)} & \\
\Phi\otimes \mathrm{Fi_G} \arrow[u, "\Phi\otimes \mathrm{F}\alpha\ \wr"] \arrow[r, "\Phi\otimes \mathrm{can}", "\cong"'] & \Phi\otimes \mathrm{i_{FG}} \arrow[r, "\Phi\otimes \mathrm{can}", "\cong"'] & \Phi\otimes \mathrm{i_F G}
\end{tikzcd}
$$

are commutative. The genesis of these conditions is clear on comparing them with display (1.14) of section 1.1. Re-expressing these in terms of pasting diagrams (as described in appendix A) they simply say that the pastings of diagrams

$$
\begin{tikzcd}
 & \mathrm{I}_B \arrow[dl, "\Psi"'] \arrow[dr, "\Psi"] \arrow[d, phantom, "\overset{\Psi\Psi}{\cong}"] & \\
\mathrm{GF} \arrow[dr, "\Psi\mathrm{GF}"] \arrow[ddr, bend right, "\mathrm{i_{GF}}"'] \arrow[ddr, phantom, "\overset{\alpha\mathrm{F}}{\cong}"] & {} & \mathrm{GF} \arrow[dl, "\mathrm{GF}\Psi"'] \arrow[ddl, bend left, "\mathrm{i_{GF}}"] \arrow[ddl, phantom, "\overset{\mathrm{G}\beta}{\cong}"'] \\
 & \mathrm{GFGF} \arrow[d, "\mathrm{G}\Phi\mathrm{F}"] & \\
 & \mathrm{GF} & \text{(a)}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{FG} \arrow[d, "\mathrm{F}\Psi\mathrm{G}"] \arrow[ddl, bend right, "\mathrm{i_{FG}}"'] \arrow[ddr, bend left, "\mathrm{i_{FG}}"] & \\
{} \arrow[r, phantom, "\overset{\beta\mathrm{G}}{\cong}"] & \mathrm{FGFG} \arrow[dl, "\Phi\mathrm{FG}"'] \arrow[dr, "\mathrm{FG}\Phi"] \arrow[r, phantom, "\overset{\mathrm{F}\alpha}{\cong}"] \arrow[dd, phantom, "\underset{\Phi\Phi}{\cong}"] & {} \\
\mathrm{FG} \arrow[dr, "\Phi"'] & & \mathrm{FG} \arrow[dl, "\Phi"] \\
 & \mathrm{I}_A & \text{(b)}
\end{tikzcd}
\tag{1.37}
$$

are equal to the identity 2-cells on Ψ and Φ respectively. □

This definition is justified and supported by the following lemma:

Lemma 1.3.9 *Biadjoints in strongly bicategory enriched categories enjoy the following properties:*

(i) *We may compose biadjoints in* $\underline{\underline{\mathcal{A}}}$ *in such a way that the composite of two locally adjoint biadjoints is also locally adjoint.*

(ii) *If* $(-)_\diamond\colon \underline{\underline{\mathcal{A}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ *is a bicategory enriched functor then it maps a biadjoint* $\mathrm{F} \dashv_b \mathrm{G}$ *in* $\underline{\underline{\mathcal{A}}}$ *to a biadjoint* $\mathrm{F}_\diamond \dashv_b \mathrm{G}_\diamond$ *in* $\underline{\underline{\mathcal{B}}}$*. This mapping preserves the local adjointness property and is well behaved with respect to composites of biadjoints.*

(iii) *If* $\mathrm{G}\colon B \longrightarrow C$ *and* $\mathrm{F}\colon C \longrightarrow B$ *are homomorphisms and* $\Psi\colon \mathrm{I}_C \longrightarrow \mathrm{GF}$ *a transformation in* $\underline{\underline{\mathcal{A}}}$ *then tfae:*

(a) *For each object $A \in \underline{\underline{\mathcal{A}}}$ the representable $\underline{\underline{\mathcal{A}}}(A,-)\colon \underline{\underline{\mathcal{A}}} \longrightarrow \underline{\underline{\mathcal{H}om}}_S$ maps Ψ to a transformation which constitutes the unit of a traditional biadjoint $\underline{\underline{\mathcal{A}}}(A,\mathrm{F}) \dashv_b \underline{\underline{\mathcal{A}}}(A,\mathrm{G})\colon \underline{\underline{\mathcal{A}}}(A,B) \longrightarrow \underline{\underline{\mathcal{A}}}(A,C)$.*

(b) *There exists a transformation $\Phi\colon \mathrm{FG} \longrightarrow \mathrm{I}_A$ and isomorphic modifications α and β as in (1.36) displaying a biadjoint $\mathrm{F} \dashv_b \mathrm{G}$ in $\underline{\underline{\mathcal{A}}}$.*

(c) *We may find Φ, α and β as in (b) so that they satisfy the conditions required of a locally adjoint biadjunction.*

(iv) *Biadjoints are essentially unique, in the sense that if $\mathrm{G}\colon B \longrightarrow C \in \underline{\underline{\mathcal{A}}}$ is a homomorphism and $\mathrm{F},\mathrm{F}'\colon C \longrightarrow B$ are both left (respectively right) biadjoints to G then there exists a canonical equivalence between F and F' in the bicategory $\underline{\underline{\mathcal{A}}}(C,B)$.*[3]

Proof.

(i) Suppose that

$$\begin{gathered}(\mathrm{G},\mathrm{F},\Psi,\Phi,\alpha,\beta)\colon A \longrightarrow A' \\ (\mathrm{G}',\mathrm{F}',\Psi',\Phi',\alpha',\beta')\colon A' \longrightarrow A''\end{gathered}$$

are a pair of biadjunctions in $\underline{\underline{\mathcal{A}}}$, we demonstrate that $\mathrm{F}\circ\mathrm{F}' \dashv_b \mathrm{G}'\circ\mathrm{G}$ by providing unit and counit given by the composites

$$\begin{gathered}\mathrm{I}_{A''} \xrightarrow{\quad\Psi'\quad} \mathrm{G}'\mathrm{F}' \xrightarrow{\quad\mathrm{G}'\Psi\mathrm{F}'\quad} \mathrm{G}'\mathrm{G}\mathrm{F}\mathrm{F}' \\ \mathrm{F}\mathrm{F}'\mathrm{G}'\mathrm{G} \xrightarrow{\quad\mathrm{F}\Phi'\mathrm{G}\quad} \mathrm{FG} \xrightarrow{\quad\Phi\quad} \mathrm{I}_A\end{gathered}$$

and triangle isomorphisms given by the pastings

$$\begin{tikzcd}
\mathrm{FF'} \arrow[d, "\mathrm{FF'\Psi'}"'] \arrow[dr, "\mathrm{i_{FF'}}"] & & \\
\mathrm{FF'G'F'} \arrow[r, "\mathrm{F\Phi'F'}"'] \arrow[d, "\mathrm{FF'G'\Psi F'}"'] \arrow[ur, phantom, "\overset{\cong}{\mathrm{F}\beta'}"] \arrow[dr, phantom, "\overset{\cong}{(\mathrm{F\Phi'})(\mathrm{\Psi F'})}"] & \mathrm{FF'} \arrow[d, "\mathrm{F\Psi F'}"] \arrow[dr, "\mathrm{i_{FF'}}"] & \\
\mathrm{FF'G'GFF'} \arrow[r, "\mathrm{F\Phi'GFF'}"'] & \mathrm{FGFF'} \arrow[r, "\mathrm{\Phi FF'}"'] \arrow[ur, phantom, "\overset{\cong}{\beta\mathrm{F'}}"] & \mathrm{FF'}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{G'G} \arrow[r, "\mathrm{\Psi'G'G}"] \arrow[dr, "\mathrm{i_{G'G}}"'] & \mathrm{G'F'G'G} \arrow[r, "\mathrm{G'\Psi F'G'G}"] \arrow[d, "\mathrm{G'\Phi'G}"] \arrow[dl, phantom, "\overset{\alpha'\mathrm{G}}{\cong}"] \arrow[dr, phantom, "\overset{(\mathrm{G'\Psi})(\mathrm{\Phi'G})}{\cong}"] & \mathrm{G'GFF'G'G} \arrow[d, "\mathrm{G'GF\Phi'G}"] \\
& \mathrm{G'G} \arrow[r, "\mathrm{G'\Psi G}"] \arrow[dr, "\mathrm{i_{G'G}}"'] & \mathrm{G'GFG} \arrow[d, "\mathrm{G'G\Phi}"] \arrow[dl, phantom, "\overset{\mathrm{G'}\alpha}{\cong}"] \\
& & \mathrm{G'G}
\end{tikzcd}
\tag{1.38}$$

[3]this clause was not included in the original 1992 version of this work and I would like to thank Jonas Frey for suggesting its inclusion here.

or to be more precise these pastings composed with the canonical isomorphisms:

$$\begin{array}{rcl} \mathrm{FF'G'\Psi F'} \otimes \mathrm{FF'\Psi'} & \overset{\text{can}}{\cong} & \mathrm{FF'(G'\Psi F'} \otimes \Psi') \\ \Phi\mathrm{FF'} \otimes \mathrm{F\Phi'GFF'} & \overset{\text{can}}{\cong} & (\Phi \otimes \mathrm{F\Phi'G})\mathrm{FF'} \\ \mathrm{G'\Psi F'G'G} \otimes \Psi'\mathrm{G'G} & \overset{\text{can}}{\cong} & (\mathrm{G'\Psi F'} \otimes \Psi')\mathrm{G'G} \\ \mathrm{G'G}\Phi \otimes \mathrm{G'GF\Phi'G} & \overset{\text{can}}{\cong} & \mathrm{G'G}(\Phi \otimes \mathrm{F\Phi'G}) \end{array} \tag{1.39}$$

The correct interpretation of each 2-cell in these pastings is easily derived from its context as described in appendix A. We postpone checking that composition of biadjunctions behaves well with respect to the local adjointness property until the end of this proof.

(ii) This result is straightforward given the work and conventions of appendix A, though more involved otherwise. Suppose that $(\mathrm{G}, \mathrm{F}, \Psi, \Phi, \alpha, \beta)\colon A \longrightarrow B$ is a biadjoint in $\underline{\underline{\mathcal{A}}}$ then simply applying the homomorphisms which constitute the actions of $(\text{-})_\diamond$ on the "homsets" $\underline{\underline{\mathcal{A}}}(B, C)$ and $\underline{\underline{\mathcal{A}}}(C, B)$ to the triangles in (1.36) (as in definition A.0.8) gives

$$\begin{tikzcd} \mathrm{G}_\diamond \arrow[r, "\Psi_\diamond\mathrm{G}_\diamond"] \arrow[dr, "\mathrm{i}_{\mathrm{G}_\diamond}"'] & \mathrm{G}_\diamond\mathrm{F}_\diamond\mathrm{G}_\diamond \arrow[d, "\mathrm{G}_\diamond\Phi_\diamond"] \arrow[dl, phantom, "\overset{\cong}{\alpha_\diamond}"] \\ & \mathrm{G} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{F}_\diamond \arrow[d, "\mathrm{F}_\diamond\Psi_\diamond"'] \arrow[dr, "\mathrm{i}_{\mathrm{F}_\diamond}"] \arrow[dr, phantom, "\underset{\cong}{\beta_\diamond}"'] & \\ \mathrm{F}_\diamond\mathrm{G}_\diamond\mathrm{F}_\diamond \arrow[r, "\Phi_\diamond\mathrm{F}_\diamond"'] & \mathrm{F}_\diamond \end{tikzcd} \tag{1.40}$$

the triangle isomorphisms for a biadjunction $\mathrm{F}_\diamond \dashv \mathrm{G}_\diamond$ with unit $\Psi_\diamond$ and counit $\Psi_\diamond$.

Now suppose that α and β satisfy the local adjointness condition and observe that applying $(\text{-})_\diamond$ to the pasting diagrams in (1.37), in the way described in definition A.0.8, simply produces the corresponding diagrams for the biadjunction $\mathrm{F}_\diamond \dashv_b \mathrm{G}_\diamond$. Corollary A.0.12 demonstrates that pastings are preserved by homomorphisms so in particular since the diagrams we start with paste to identities on Ψ and Φ respectively the diagrams they map to under application of $(\text{-})_\diamond$ must paste to the identities on $\Psi_\diamond$ and $\Phi_\diamond$. Finally, by our initial observation, this simply states that the modifications $\alpha_\diamond$ and $\beta_\diamond$ satisfy the local adjointness condition.

(iii) Prove this in the following natural order:

(c)⇒(b) Nothing to prove.

(b)⇒(a) By part (ii) of this lemma applying the representable $\underline{\underline{\mathcal{A}}}(A, \text{-})$ to a biadjunction in $\underline{\underline{\mathcal{A}}}$ gives a biadjoint $\underline{\underline{\mathcal{A}}}(A, \mathrm{F}) \dashv_b \underline{\underline{\mathcal{A}}}(A, \mathrm{G})$ in $\underline{\underline{\mathcal{H}om}}_S$ of the type defined in 1.3.8. All that remains is to show that these biadjoints coincide with the traditional variety, but certainly we already know from example 1.1.7 that every traditional biadjoint gives rise to unit and counit related by triangle isomorphisms.

Conversely suppose we have a biadjunction $(\mathrm{G},\mathrm{F},\Psi,\Phi,\alpha,\beta)$ in $\underline{\underline{\mathcal{H}om}}_S$ then consider the homomorphism $\mathrm{G}\colon\underline{\underline{\mathcal{B}}}\longrightarrow\underline{\underline{\mathcal{C}}}$, for each pair of 0-cells $c\in\underline{\underline{\mathcal{C}}}$ and $b\in\underline{\underline{\mathcal{B}}}$ the functor

$$\psi_{cb} = \quad \underline{\underline{\mathcal{B}}}(\mathrm{F}c,b) \xrightarrow{\ \mathrm{G}\ } \underline{\underline{\mathcal{C}}}(\mathrm{GF}c,\mathrm{G}b) \xrightarrow{-\otimes\Psi_c} \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b) \tag{1.41}$$

is an equivalence with inverse:

$$\phi_{cb} = \quad \underline{\underline{\mathcal{C}}}(c,\mathrm{G}b) \xrightarrow{\ \mathrm{F}\ } \underline{\underline{\mathcal{B}}}(\mathrm{F}c,\mathrm{FG}b) \xrightarrow{\Phi_b\otimes -} \underline{\underline{\mathcal{B}}}(\mathrm{F}c,b)$$

We check the truth of this statement by observing that we have isomorphisms

$$\begin{aligned}
p &\xrightarrow[\cong]{\alpha_b\otimes p} \mathrm{G}\Phi_b\otimes\Psi_{\mathrm{G}b}\otimes p \xrightarrow[\cong]{\mathrm{G}\Phi_b\otimes\Psi_p} \mathrm{G}\Phi_b\otimes\mathrm{GF}(p)\otimes\Psi_c \\
&\qquad\qquad\qquad\qquad\qquad\quad \xrightarrow[\cong]{\mathrm{can}\otimes\Psi_c} \mathrm{G}(\Phi_b\otimes\mathrm{F}(p))\otimes\Psi_c \\
\Phi_b\otimes\mathrm{F}(\mathrm{G}(q)\otimes\Psi_c) &\xrightarrow[\cong]{\Phi_b\otimes\mathrm{can}} \Phi_b\otimes\mathrm{FG}(q)\otimes\mathrm{F}\Psi_c \\
&\xrightarrow[\cong]{\Phi_q\otimes\mathrm{F}\Psi_c} q\otimes\Phi_{\mathrm{F}c}\otimes\mathrm{F}\Psi_c \xrightarrow[\cong]{q\otimes\beta_c} q
\end{aligned} \tag{1.42}$$

which are natural in $p\in\underline{\underline{\mathcal{C}}}(c,\mathrm{G}b)$ and $q\in\underline{\underline{\mathcal{B}}}(\mathrm{F}c,b)$, these are just the components of the natural transformations κ_{cb} and τ_{cb} (respectively) as defined in lemma 1.1.5. Returning to the definition at the beginning of example 1.1.7 this exactly says that the homomorphism G has a left biadjoint.

(a)⇒(c) This implication essentially follows from the bicategorical Yoneda lemma (lemma 1.1.4), however we choose to take a more elementary approach here. The fact that each $\underline{\underline{\underline{\mathcal{A}}}}(A,\Psi)$ is the unit of a biadjoint $\underline{\underline{\underline{\mathcal{A}}}}(A,\mathrm{F})\dashv_b\underline{\underline{\underline{\mathcal{A}}}}(A,\mathrm{G})$ implies that for each pair of homomorphisms $\mathrm{H}\in\underline{\underline{\underline{\mathcal{A}}}}(A,C)$ and $\mathrm{K}\in\underline{\underline{\underline{\mathcal{A}}}}(A,B)$ we have:

- For each transformation $\Delta\colon\mathrm{H}\longrightarrow\mathrm{GK}$ there exists another $\hat{\Delta}\colon\mathrm{FH}\longrightarrow\mathrm{K}$ and an isomorphic modification:

$$\begin{tikzcd}
\mathrm{H} \arrow[r, "\Psi\mathrm{H}"] \arrow[dr, "\Delta"'] & \mathrm{GFH} \arrow[d, "\mathrm{G}\hat{\Delta}"] \arrow[dl, phantom, "\overset{\delta}{\cong}"] \\
& \mathrm{GK}
\end{tikzcd}$$

- Given two such pairs $\langle \hat{\Delta}_i, \delta_i \rangle$ $(i = 0, 1)$, associated with Δ as above, there exists a unique isomorphism $\gamma \colon \Delta_0 \xrightarrow{\cong} \Delta_1$ such that

$$\begin{tikzcd}
\Delta \arrow[r, "\delta_0", "\cong"'] \arrow[dr, "\delta_1"'] & (\mathrm{G}\hat{\Delta}_0) \otimes \Psi \arrow[d, "\wr", "(\mathrm{G}\gamma) \otimes \Psi"'] \\
 & (\mathrm{G}\hat{\Delta}_1) \otimes \Psi
\end{tikzcd}$$

  commutes.

Applying the first of these rules with $A = B$, $\mathrm{H} = \mathrm{G}$, $\mathrm{K} = \mathrm{I}_B$ and $\Delta = \mathrm{i}_\mathrm{G}$ we get a counit $\Phi \colon \mathrm{FG} \longrightarrow \mathrm{I}_B$ and an isomorphic modification $\alpha \colon \mathrm{i}_\mathrm{G} \xrightarrow{\cong} \mathrm{G}\Phi \otimes \Psi\mathrm{G}$. The second rule may be applied to the two diagrams

$$\begin{tikzcd}
 & \mathrm{I}_B \arrow[dl, "\Psi"'] \arrow[dr, "\Psi"] & \\
\mathrm{GF} \arrow[dr, "\Psi\mathrm{GF}"] \arrow[ddr, bend right, "\mathrm{i}_{\mathrm{GF}}"'] & \overset{\Psi\Psi}{\cong} & \mathrm{GF} \arrow[dl, "\mathrm{GF}\Psi"'] \arrow[ddl, bend left, "\mathrm{G}(\Phi\mathrm{F} \otimes \mathrm{F}\Psi)"] \\
\overset{\alpha\mathrm{F}}{\cong} & \mathrm{GFGF} \arrow[d, "\mathrm{G}\Phi\mathrm{F}"] & \overset{\mathrm{can}}{\cong} \\
 & \mathrm{GF} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{I}_B \arrow[d, "\Psi"'] \\
\mathrm{GF} \arrow[d, shift right=2, "\mathrm{i}_{\mathrm{GF}}"'] \arrow[d, shift left=2, "\mathrm{Gi}_\mathrm{F}"] \arrow[d, phantom, "\overset{\mathrm{can}}{\cong}"] \\
\mathrm{GF}
\end{tikzcd}$$

to get a unique isomorphism $\beta \colon \Phi\mathrm{F} \otimes \mathrm{F}\Psi \xrightarrow{\cong} \mathrm{i}_\mathrm{F}$ with the defining property that it and α together satisfy local adjointness condition (1.37)(a).

To prove that condition (1.37)(b) holds notice that (by part (ii) of this lemma) if we apply any representable $\underline{\underline{\underline{\mathcal{A}}}}(A, –)$ to the data we have constructed so far we get a biadjoint $\underline{\underline{\underline{\mathcal{A}}}}(A, \mathrm{F}) \dashv_b \underline{\underline{\underline{\mathcal{A}}}}(A, \mathrm{G})$ in $\underline{\underline{\underline{\mathcal{H}om}}}_S$ satisfying (1.37)(a). For a biadjoint in $\underline{\underline{\underline{\mathcal{H}om}}}_S$ we have already pointed out that (1.37)(a) is exactly condition (1.14) of lemma 1.1.6, so by that lemma we know that the natural transformations κ_{cb} and τ_{cb} of (1.42) above satisfy the triangle identity

$$\begin{tikzcd}
\psi_{cb} \arrow[r, "\kappa_{cb}\psi_{cb}", "\cong"'] \arrow[dr, "\mathrm{i}_{\psi_{cb}}"'] & \psi_{cb}\varphi_{cb}\psi_{cb} \arrow[d, "\wr", "\psi_{cb}\tau_{cb}"'] \\
 & \psi_{cb}
\end{tikzcd}$$

for each pair of 0-cells $b \in \underline{\mathcal{B}}$ and $c \in \underline{\mathcal{C}}$. It is a matter of routine verification to check that if we choose a unit and counit for an equivalence (of categories) in such a way that they satisfy one of the triangle identities then they must also satisfy the other. Therefore making this inference for κ_{cb} and τ_{cb} then applying the "coop" dual of lemma 1.1.6 (in reverse) we show that condition $(1.14)^{\mathrm{coop}}$, or in other words (1.37)(b), holds for our biadjoint as well.

Returning to the biadjoint $\mathrm{F} \dashv_b \mathrm{G}$ in $\underline{\underline{\mathcal{A}}}$, all that remains is to note that the pasting diagram of (1.37)(b) for this is exactly the component at $\mathrm{I}_B \in \underline{\underline{\mathcal{A}}}(B,B)$ of the corresponding pasting for $\underline{\underline{\mathcal{A}}}(B,\mathrm{F}) \dashv \underline{\underline{\mathcal{A}}}(B,\mathrm{G})$. We already know from the last paragraph that this latter diagram pastes to the identity on $\underline{\underline{\mathcal{A}}}(B,\Phi)$, therefore since pasting of modifications in $\underline{\underline{\mathcal{H}om}}_S$ is calculated pointwise we may examine components at I_B to re-assure ourselves that condition (1.37)(b) holds for the data $(\mathrm{G},\mathrm{F},\Psi,\Phi,\alpha,\beta)$ in $\underline{\underline{\mathcal{A}}}$ as defined above.

Notice that by this proof we have immediately that for any biadjunction $(\mathrm{G},\mathrm{F},\Psi,\Phi,\alpha,\beta)$ in $\underline{\underline{\mathcal{A}}}$ there is always a uniquely determined isomorphic modification $\beta'\colon \Phi\mathrm{F} \otimes \mathrm{F}\Psi \overset{\cong}{\Longrightarrow} \mathrm{i}_\mathrm{F}$ such that $(\mathrm{G},\mathrm{F},\Psi,\Phi,\alpha,\beta')$ is a locally adjoint biadjunction. Alternatively, of course, we could choose to replace α instead and still achieve the same result.

(iv) Suppose that $(\mathrm{G},\mathrm{F},\Psi,\Phi,\alpha,\beta)$ and $(\mathrm{G},\mathrm{F}',\Psi',\Phi',\alpha',\beta')$ are two biadjunctions which share the same right biadjoint $\mathrm{G}\colon B \longrightarrow C \in \underline{\underline{\mathcal{A}}}$ then we have two transformations between the left biadjoints F and F′ given by the composites:

$$\mathrm{F} \xrightarrow{\mathrm{F}\Psi'} \mathrm{F}\mathrm{G}\mathrm{F}' \xrightarrow{\Phi\mathrm{F}'} \mathrm{F}'$$
$$\mathrm{F}' \xrightarrow{\mathrm{F}'\Psi} \mathrm{F}'\mathrm{G}\mathrm{F} \xrightarrow{\Phi'\mathrm{F}} \mathrm{F}$$

Now the following composite of isomorphic modifications

$$\begin{aligned}
\Phi'\mathrm{F} \otimes \mathrm{F}'\Psi \otimes \Phi\mathrm{F}' \otimes \mathrm{F}\Psi' &\xrightarrow[\cong]{\Phi'\mathrm{F} \otimes \Phi\mathrm{F}'\Psi \otimes \mathrm{F}\Psi'} \Phi'\mathrm{F} \otimes \Phi\mathrm{F}'\mathrm{G}\mathrm{F} \otimes \mathrm{F}\mathrm{G}\mathrm{F}'\Psi \otimes \mathrm{F}\Psi' \\
&\xrightarrow[\cong]{\mathrm{can}_\mathrm{F}} (\Phi' \otimes \Phi\mathrm{F}'\mathrm{G})\mathrm{F} \otimes \mathrm{F}(\mathrm{G}\mathrm{F}'\Psi \otimes \Psi') \\
&\xrightarrow[\cong]{\Phi\Phi'\mathrm{F} \otimes \mathrm{F}\Psi'\Psi} (\Phi \otimes \mathrm{F}\mathrm{G}\Phi')\mathrm{F} \otimes \mathrm{F}(\Psi'\mathrm{G}\mathrm{F} \otimes \Psi) \\
&\xrightarrow[\cong]{\mathrm{can}_\mathrm{F}} \Phi\mathrm{F} \otimes \mathrm{F}(\mathrm{G}\Phi' \otimes \Psi'\mathrm{G})\mathrm{F} \otimes \mathrm{F}\Psi \\
&\xrightarrow[\cong]{\Phi\mathrm{F} \otimes \mathrm{F}\alpha'\mathrm{F} \otimes \mathrm{F}\Psi} \Phi\mathrm{F} \otimes \mathrm{F}\Psi \overset{\alpha}{\underset{\cong}{\Longrightarrow}} \mathrm{i}_\mathrm{F}
\end{aligned}$$

and a corresponding one obtained by swapping the roles of these two biadjunctions shows that these transformations are mutual equivalence inverses in $\underline{\underline{\mathcal{A}}}(C,B)$ as required.

Finally we return again to the proof that local adjointness of biadjoints is preserved by composition. Since pasting is preserved under the application of homomorphisms we know that the composition of biadjoints, as presented in the proof of (i) above, is preserved by all functors of (strongly) bicategory enriched categories. Following the sort of representability argument that we used to establish condition (1.37)(b) in the previous construction, it should be clear that a biadjoint $(\mathrm{G}, \mathrm{F}, \Psi, \Phi, \alpha, \beta)$ in $\underline{\underline{\mathcal{A}}}$ is locally adjoint iff the application of each representable $\underline{\underline{\mathcal{A}}}(A, -)$ gives a locally adjoint biadjoint in $\underline{\underline{\mathcal{H}om}}_S$. Combining these two results we have reduced the problem to checking the proposition for biadjunctions in $\underline{\underline{\mathcal{H}om}}_S$.

In order to verify the latter result consider two (composable) biadjoints $\mathrm{F} \dashv_b \mathrm{G}\colon \underline{\underline{\mathcal{A}}} \to \underline{\underline{\mathcal{B}}}$ and $\mathrm{F}' \dashv_b \mathrm{G}'\colon \underline{\underline{\mathcal{B}}} \to \underline{\underline{\mathcal{C}}}$ in $\underline{\underline{\mathcal{H}om}}_S$. We know that saying these are locally adjoint simply means that we have adjunctions

$$\kappa_{ba}, \tau_{ba}\colon \psi_{ba} \dashv \varphi_{ba}\colon \underline{\underline{\mathcal{A}}}(\mathrm{F}b, a) \longrightarrow \underline{\underline{\mathcal{B}}}(b, \mathrm{G}a)$$
$$\kappa'_{cb}, \tau'_{cb}\colon \psi'_{cb} \dashv \varphi'_{cb}\colon \underline{\underline{\mathcal{B}}}(\mathrm{F}'c, b) \longrightarrow \underline{\underline{\mathcal{C}}}(c, \mathrm{G}'b)$$

derived, as above, from the given structure of the two biadjoints. Now consider the composite biadjoint $\mathrm{F}\mathrm{F}' \dashv \mathrm{G}'\mathrm{G}$ with the unit, counit and triangle isomorphisms we gave earlier. Simple calculations show that the functors ψ''_{ca} and φ''_{ca} given by this structure in the usual way are (naturally) isomorphic to $(\psi'_{c,\mathrm{G}a} \cdot \psi_{\mathrm{F}'c,a})$ and $(\varphi_{\mathrm{F}'c,a} \cdot \varphi'_{c,\mathrm{G}a})$. Furthermore under these isomorphisms the unit κ''_{ca} and counit τ''_{ca} obtained from the triangle isomorphisms of our composite turn out to be exactly those of the adjunction $(\psi'_{c,\mathrm{G}a} \cdot \psi_{\mathrm{F}'c,a}) \dashv (\varphi_{\mathrm{F}'c,a} \cdot \varphi'_{c,\mathrm{G}a})$. In this way we establish that our composite biadjoint is indeed locally adjoint. $\square$

Before winding up this section we should, as promised, give an example demonstrating that there is no monoidal structure on $\underline{\mathcal{H}om}_0$ over which we may re-interpret the notions of bicategory enrichment presented here in terms of traditional enriched category theory. Such a structure would correspond to the multicategories $\underline{\mathcal{H}om}$ and $\underline{\mathcal{H}om}_S$ having *tensor products*, recall (from [33]) that for a multicategory $\mathbb{M}$ this is defined to mean that given a pair of objects $A, B \in \mathbb{M}$ there exists a third one $A \otimes B$ and a 2-map

$$[A, B] \xrightarrow{m_{AB}} A \otimes B$$

with the universal property that there are bijections

$$\frac{[\vec{A}, A \otimes B, \vec{B}] \longrightarrow C}{[\vec{A}, A, B, \vec{B}] \longrightarrow C}$$

mediated by substitution of m_{AB}.

To show that $\underline{\mathcal{H}om}$ (or by the same argument $\underline{\mathcal{H}om}_S$) does not in general admit such tensor products consider 2-homomorphisms $\mathrm{F}\colon [2, 2] \longrightarrow \underline{\underline{\mathcal{B}}}$, where 2 is the ordinal "2" considered as a bicategory. Such a 2-map is simply determined by a choice of four 0-cells, each one accompanied by an endo-1-cell isomorphic to its identity, related by 1- and 2-cells which we may display as a square:

$$\begin{tikzcd} \mathrm{F}(0,0) \arrow[r, "\mathrm{F}(0,<)"] \arrow[d, "\mathrm{F}(<,0)"'] \arrow[dr, phantom, "\mathrm{F}(<,<)\Downarrow"] & \mathrm{F}(0,1) \arrow[d, "\mathrm{F}(<,1)"] \\ \mathrm{F}(1,0) \arrow[r, "\mathrm{F}(1,<)"'] & \mathrm{F}(1,1) \end{tikzcd}$$

Now suppose that there exists a tensor product $2 \otimes_l 2$ displayed by a 2-homomorphism M, then by definition the 2-homomorphisms F described above are in bijective correspondence with homomorphisms $\hat{\mathrm{F}}\colon 2 \otimes_l 2 \longrightarrow \underline{\underline{\mathcal{B}}}$ such that $\hat{\mathrm{F}}\langle \mathrm{M}\rangle = \mathrm{F}$. There certainly exist such 2-homomorphisms for which no pair of the four chosen 0-cells are identical and since these 2-maps factor through $2 \otimes_l 2$ it follows that this property must also hold true for M. So consider the 1-cell

$$\mathrm{M}(<,1) \otimes \mathrm{M}(0,<)\colon \mathrm{M}(0,0) \longrightarrow \mathrm{M}(1,1)\ .$$

None of the 1-cells specified by the 2-homomorphism M have the same domain *and* codomain therefore it cannot be equal to one of these cells.

What exactly do we impose on a homomorphism $\hat{\mathrm{F}}\colon 2 \otimes_l 2 \longrightarrow \underline{\underline{\mathcal{B}}}$ by saying that it satisfies $\hat{\mathrm{F}}\langle \mathrm{M}\rangle = \mathrm{F}$? Clearly we are only requiring conditions on images of the cells $\mathrm{M}(\bullet,\bullet)$ amongst which $\mathrm{M}(<,1) \otimes \mathrm{M}(0,<)$ is not counted, so its image is only constrained *up to an isomorphism* $\hat{\mathrm{F}}(\mathrm{M}(<,1) \otimes \mathrm{M}(0,<)) \cong \mathrm{F}(<,1) \otimes \mathrm{F}(0,<)$. It should now be apparent that we may construct distinct homomorphisms $\hat{\mathrm{F}}_0, \hat{\mathrm{F}}_1\colon 2 \otimes_l 2 \longrightarrow \underline{\underline{\mathcal{B}}}$ differing on $\mathrm{M}(<,1) \otimes \mathrm{M}(0,<)$ but with $\mathrm{F}_0\langle \mathrm{M}\rangle = \mathrm{F}_1\langle \mathrm{M}\rangle$, contradicting the assumption that M displays $2 \otimes_l 2$ as a tensor.

This result only proscribes us from having a suitable monoidal structure on $\underline{\mathcal{H}om}_0$, it does not of course prevent us from extending this category to one with a nicely behaved tensor, and to this end we will see that what $\underline{\mathcal{H}om}_0$ does support is a number of interesting *promonoidal* structures. A (small) promonoidal category $(\mathbb{C}, p, i)$, as defined by Day in [15], consists of a (small) category $\mathbb{C}$ and two functors

$$\mathbb{C}^{\mathrm{op}} \times \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \xrightarrow{p(-,-;-)} \underline{\mathrm{Set}}$$
$$\mathbb{C} \xrightarrow{i(-)} \underline{\mathrm{Set}}$$

supported by "associativity" and "identity" isomorphisms

$$\begin{aligned} \int^{e\in\mathbb{C}} p(e,c;d) \times p(a,b;e) &\cong \int^{e\in\mathbb{C}} p(a,e;d) \times p(b,c;e) \\ \int^{a\in\mathbb{C}} p(a,b;c) \times i(a) &\cong \mathbb{C}(b,c) \\ \int^{b\in\mathbb{C}} p(a,b;c) \times i(b) &\cong \mathbb{C}(a,c) \end{aligned}$$

where the integral signs denote coends as usual, which all exist in $\underline{\mathrm{Set}}$ since $\mathbb{C}$ is small. These isomorphisms must satisfy coherence conditions closely related to MacLane's conditions for monoidal categories. The importance of the promonoidal structures on a (small) category $\mathbb{C}$ lies in their correspondence with monoidal closed structures on the presheaf category $[\mathbb{C}, \underline{\mathrm{Set}}]$. Given a promonoidal category $(\mathbb{C}, p, i)$ we define tensor $F \otimes G$, left internal hom $F \Rightarrow H$ and right internal hom $H \Leftarrow G$ on $[\mathbb{C}, \underline{\mathrm{Set}}]$ by

$$\begin{aligned} (F \otimes G)(c) &= \int^{a,b\in\mathbb{C}} p(a,b;c) \times F(a) \times G(b) \\ (F \Rightarrow H)(b) &= \int_{a,c\in\mathbb{C}} \underline{\mathrm{Set}}(p(a,b;c) \times F(a), H(c)) \\ (H \Leftarrow G)(a) &= \int_{b,c\in\mathbb{C}} \underline{\mathrm{Set}}(p(a,b;c) \times G(b), H(c)) \end{aligned}$$

the identity for this structure is the presheaf $i(-)$. As an indication of how we may derive a promonoidal structure on $\mathbb{C}$ from a monoidal closed one on $[\mathbb{C}, \underline{\mathrm{Set}}]$ notice that, by the Yoneda lemma and the usual results concerning coends, we have the natural isomorphisms:

$$\begin{aligned} [\mathbb{C}, \underline{\mathrm{Set}}](\mathbb{C}(c,-), \mathbb{C}(a,-) \otimes \mathbb{C}(b,-)) &\cong \int^{x,y\in\mathbb{C}} p(x,y;c) \times \mathbb{C}(a,x) \times \mathbb{C}(b,y) \\ &\cong p(a,b;c) \end{aligned} \tag{1.43}$$

Of course promonoidal categories and multicategories are not unrelated, and while it is not necessarily true that all multicategories naturally give rise to promonoidal categories it is certainly true that the examples we have studied do. Suppose that $\mathbb{M}$ is a (small) multicategory and let $\mathbb{M}_0$ denote the category with the same set of objects and morphisms the 1-maps with substitution as composition, furthermore use $\mathbb{M}([A_0, \cdots, A_n], B)$ to denote the set on n-maps with the given domain and codomain. If $\mathbb{M}$ is *closed* then we may define a promonoidal structure on $\mathbb{M}_0$ by setting:

$$\begin{aligned} p(A,B;C) &= \mathbb{M}([A,B],C) \\ i(C) &= \mathbb{M}([\cdot],C) \end{aligned}$$

Since $\mathbb{M}$ is closed we know that there exist natural isomorphisms $p(A,B;C) \cong \mathbb{M}_0(A, C \Leftarrow B) \cong \mathbb{M}_0(B, A \Rightarrow C)$ implying that

$$\begin{aligned} \int^{E\in\mathbb{M}_0} p(E,C;D) \times p(A,B;E) &\cong \int^{E\in\mathbb{M}_0} \mathbb{M}_0(E, D \Leftarrow C) \times p(A,B;E) \\ &\cong p(A,B;D \Leftarrow C) \\ &\cong \mathbb{M}([A,B,C],D) \end{aligned} \tag{1.44}$$

naturally in each variable. By an identical argument we get a second isomorphism

$$\int^{E\in\mathbb{M}_0} p(A,E;D) \times p(B,C;E) \cong \mathbb{M}([A,B,C],D)$$

which we compose with the previous one to obtain an associativity isomorphism. We leave the identity isomorphisms up to the reader, checking the coherence conditions is routine.

An important property of the monoidal closed category $([\mathbb{M}_0, \underline{\mathrm{Set}}], \otimes, \Leftarrow, \Rightarrow)$, as derived from this promonoidal one, is that we may extend and combine the calculations of (1.43) and (1.44) to give isomorphisms:

$$\mathbb{M}\,([A_1, \cdots, A_n], B) \cong [\mathbb{M}_0, \underline{\mathrm{Set}}]\,(\mathbb{M}_0(B, \text{-}), \mathbb{M}_0(A_1, \text{-}) \otimes \cdots \otimes \mathbb{M}_0(A_n, \text{-}))$$

Indeed it is in general true that from any monoidal category $(\mathbb{V}, \otimes)$ we may define a multicategory with the same objects and n-maps $B \to A_1 \otimes \cdots \otimes A_n$, the isomorphisms above identify $\mathbb{M}$ with the full sub-multicategory on the representables of that derived in this way from $([\mathbb{M}_0, \underline{\mathrm{Set}}], \otimes)$. The crucial inference that we may draw is that categories enriched in the opposite monoidal category $([\mathbb{M}_0, \underline{\mathrm{Set}}]^{\mathrm{op}}, \otimes)$ with "homsets" which are representables may be identified with those enriched over the multicategory $\mathbb{M}$. Naturally we apply this to $\underline{\mathcal{H}om}$ and $\underline{\mathcal{H}om}_S$ to reduce the notion of bicategory enrichment to a classical situation

It might seem something of an advance to reduce multicategory enrichment to a traditional context but in fact we get no further. This is principally due to the fact that while $([\mathbb{M}_0, \underline{\mathrm{Set}}], \otimes)$ is closed, and therefore supports the kind of enriched category theory described in [30], it by no means follows that the category we are actually interested in enriching over $([\mathbb{M}_0, \underline{\mathrm{Set}}]^{\mathrm{op}}, \otimes)$ is closed as well. The important point about multi-maps in the cases we are considering is that we have a clear intuition for their manipulation.

1.4 Double Bicategories

In this section we introduce the notion of a double bicategory, which formalises the properties of "squares" and "cylinders" in bicategory theory. An introduction to cylinders and an account of their importance may be found in section 8 of Benabou [3]. Most significantly to our study of change of base we will see that double bicategories may be marshalled together into a number of natural and related (strongly) bicategory enriched categories, from which we may derive similar structures relating equipments via the construction given in definition 1.2.4.

Definition 1.4.1 A *double bicategory* $\underline{\underline{\tilde{\mathcal{D}}}} = (\mathcal{A}, \underline{\underline{\mathcal{H}}}, \underline{\underline{\mathcal{V}}}, \mathcal{S}, \underset{H}{\otimes}, \underset{V}{\otimes}, \mathrm{i}^h, \mathrm{i}^v, \underset{H}{*}, \underset{V}{*})$ consists of

(i) Two bicategories $(\underline{\underline{\mathcal{H}}}, \underset{H}{\otimes}, \mathrm{i}^h)$ and $(\underline{\underline{\mathcal{V}}}, \underset{V}{\otimes}, \mathrm{i}^v)$ which share the same set of 0-cells $\mathcal{A}$. The 1/2-cells of $\underline{\underline{\mathcal{H}}}$ are said to be *horizontal* and those of $\underline{\underline{\mathcal{V}}}$ *vertical*. To avoid confusion later on we refer to vertical composition of 2-cells in these bicategories as *depth-wise composition* and use the symbol $\bullet$.

For notational convenience define categories

$$\begin{aligned} \underline{\mathcal{H}} &= \coprod_{a,a' \in \mathcal{A}} \underline{\underline{\mathcal{H}}}(a, a') \\ \underline{\mathcal{V}} &= \coprod_{a,a' \in \mathcal{A}} \underline{\underline{\mathcal{V}}}(a, a') \end{aligned}$$

using single underlines to distinguish these from the bicategories $\underline{\underline{\mathcal{H}}}$,$\underline{\underline{\mathcal{V}}}$ themselves. We have domain and codomain functions

$$\mathrm{dom}, \mathrm{cod} \colon \underline{\mathcal{H}}_0 \longrightarrow \mathcal{A}$$

$$\mathrm{dom}, \mathrm{cod} \colon \underline{\mathcal{V}}_0 \longrightarrow \mathcal{A}$$

taking an object of $\underline{\mathcal{H}}$ ($\underline{\mathcal{V}}$), which is simply a 1-cell of $\underline{\underline{\mathcal{H}}}$ ($\underline{\underline{\mathcal{V}}}$), to the 0-cell which is its domain (codomain).

(ii) A set $\mathcal{S}$ with domain and codomain functions

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "\langle \mathrm{cod}_H, \mathrm{cod}_V \rangle"'] \arrow[dr, "\langle \mathrm{dom}_H, \mathrm{dom}_V \rangle"] & \\ \underline{\mathcal{V}}_0 \times \underline{\mathcal{H}}_0 & & \underline{\mathcal{V}}_0 \times \underline{\mathcal{H}}_0 \end{tikzcd}$$

which satisfy the conditions

$$\begin{array}{cc} \mathrm{dom}(\mathrm{dom}_H(\lambda)) = \mathrm{dom}(\mathrm{dom}_V(\lambda)) & \mathrm{cod}(\mathrm{dom}_H(\lambda)) = \mathrm{dom}(\mathrm{cod}_V(\lambda)) \\ \mathrm{dom}(\mathrm{cod}_H(\lambda)) = \mathrm{cod}(\mathrm{dom}_V(\lambda)) & \mathrm{cod}(\mathrm{cod}_H(\lambda)) = \mathrm{cod}(\mathrm{cod}_V(\lambda)) \end{array}$$

for each $\lambda \in \mathcal{S}$, and this comes equipped with compatible left and right actions by the category $\underline{\mathcal{V}} \times \underline{\mathcal{H}}$. In other words $\mathcal{S}$ is a profunctor $\underline{\mathcal{V}} \times \underline{\mathcal{H}} \xrightarrow{\ \ +\ \ } \underline{\mathcal{V}} \times \underline{\mathcal{H}}$.

In concrete terms this means that we may visualise each element of $\mathcal{S}$ as a *square*

$$\begin{tikzcd} a \arrow[r, "\mathrm{dom}_V(\lambda)"] \arrow[d, "\mathrm{dom}_H(\lambda)"', "|" marking] \arrow[dr, phantom, "\lambda\lrcorner"] & a' \arrow[d, "\mathrm{cod}_H(\lambda)", "|" marking] \\ \bar{a} \arrow[r, "\mathrm{cod}_V(\lambda)"'] & \bar{a}' \end{tikzcd}$$

where we use adorned arrows to distinguish vertical and horizontal 1-cells. We may also use symbols $\Rightarrow_H$ and $\Rightarrow_V$ to indicate and distinguish horizontal and vertical 2-cells where confusion might otherwise arise.

Giving compatible left and right actions of $\underline{\mathcal{V}} \times \underline{\mathcal{H}}$ amounts to giving separate actions of the 2-cells of $\underline{\underline{\mathcal{V}}}$ and $\underline{\underline{\mathcal{H}}}$, for which we gain a little more intuition by rendering them into pictorial form

$$\begin{tikzcd} a \arrow[d, shift right=3ex, "p"', "|" marking] \arrow[d, shift left=3ex, "q", "|" marking] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] \\ \bar{a} \end{tikzcd} + \begin{tikzcd} a \arrow[r, "g"] \arrow[d, "q"', "|" marking] \arrow[dr, phantom, "\lambda\lrcorner"] & a' \arrow[d, "p'", "|" marking] \\ \bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \end{tikzcd} \longmapsto \begin{tikzcd} a \arrow[r, "g"] \arrow[d, "p"', "|" marking] \arrow[dr, phantom, "\lambda \underset{H}{*} \beta\lrcorner"] & a' \arrow[d, "p'", "|" marking] \\ \bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "q"', "|" marking] \arrow[dr, phantom, "\lambda\lrcorner"] & a' \arrow[d, "p'", "|" marking] \\ \bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \end{tikzcd} + \begin{tikzcd} a' \arrow[d, shift right=3ex, "p'"', "|" marking] \arrow[d, shift left=3ex, "q'", "|" marking] \arrow[d, phantom, "\overset{\beta'}{\Rightarrow}"] \\ \bar{a}' \end{tikzcd} \longmapsto \begin{tikzcd} a \arrow[r, "g"] \arrow[d, "q"', "|" marking] \arrow[dr, phantom, "\beta' \underset{H}{*} \lambda\lrcorner"] & a' \arrow[d, "q'", "|" marking] \\ \bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \end{tikzcd}$$

with similar diagrams for horizontal actions $\lambda \underset{V}{*} \alpha$, $\bar{\alpha} \underset{V}{*} \lambda$ (where $\alpha\colon f \Rightarrow g$ and $\bar{\alpha}\colon \bar{f} \Rightarrow \bar{g}$ are 2-cells in $\underline{\underline{\mathcal{H}}}$). Of course these each satisfy the usual action conditions with respect to identities and depth-wise composition of 2-cells in $\underline{\underline{\mathcal{H}}}$ and $\underline{\underline{\mathcal{V}}}$, but in order to unify them into an endo-profunctor on $\underline{\mathcal{V}} \times \underline{\mathcal{H}}$ they must also satisfy six extra "mutual compatibility" conditions:

$$\begin{array}{cc} (\beta' \underset{H}{*} \lambda) \underset{H}{*} \beta = \beta' \underset{H}{*} (\lambda \underset{H}{*} \beta) & (\beta' \underset{H}{*} \lambda) \underset{V}{*} \alpha = \beta' \underset{H}{*} (\lambda \underset{V}{*} \alpha) \\ (\bar{\alpha} \underset{V}{*} \lambda) \underset{H}{*} \beta = \bar{\alpha} \underset{V}{*} (\lambda \underset{H}{*} \beta) & (\bar{\alpha} \underset{V}{*} \lambda) \underset{V}{*} \alpha = \bar{\alpha} \underset{V}{*} (\lambda \underset{V}{*} \alpha) \\ \bar{\alpha} \underset{V}{*} (\beta' \underset{H}{*} \lambda) = \beta' \underset{H}{*} (\bar{\alpha} \underset{V}{*} \lambda) & (\lambda \underset{V}{*} \alpha) \underset{H}{*} \beta = (\lambda \underset{H}{*} \beta) \underset{V}{*} \alpha \end{array}$$

Once we have this much structure we may define a category $\underline{\underline{\mathrm{Cyl}}}_H(p, p')$, for each pair of vertical 1-cells $p, p' \in \underline{\mathcal{V}}$, with

objects squares $\lambda \in \mathcal{S}$ with $\mathrm{dom}_H(\lambda) = p$ and $\mathrm{cod}_H(\lambda) = p'$,

maps $(\alpha_d, \alpha_c) \colon \lambda \Rightarrow_H \dot{\lambda}$ consist of pairs of 2-cells $\alpha_d \colon \mathrm{dom}_V(\lambda) \Rightarrow \mathrm{dom}_V(\dot{\lambda})$ and $\alpha_c \colon \mathrm{cod}_V(\lambda) \Rightarrow \mathrm{cod}_V(\dot{\lambda})$ in $\underline{\underline{\mathcal{H}}}$ satisfying $\alpha_c \underset{V}{*} \lambda = \dot{\lambda} \underset{V}{*} \alpha_d$, these are called *horizontal cylinders*.

composition is pointwise in $\underline{\underline{\mathcal{H}}}$, in other words if $(\dot{\alpha}_d, \dot{\alpha}_c) \colon \dot{\lambda} \Rightarrow_H \ddot{\lambda}$ is another horizontal cylinder then

$$(\dot{\alpha}_d, \dot{\alpha}_c) \bullet (\alpha_d, \alpha_c) = (\dot{\alpha}_d \bullet \alpha_d, \dot{\alpha}_c \bullet \alpha_c)$$

which is well defined since:

$$\begin{aligned} (\dot{\alpha}_c \bullet \alpha_c) \underset{V}{*} \lambda &= \dot{\alpha}_c \underset{V}{*} (\alpha_c \underset{V}{*} \lambda) = \dot{\alpha}_c \underset{V}{*} (\dot{\lambda} \underset{V}{*} \alpha_d) \\ &= (\dot{\alpha}_c \underset{V}{*} \dot{\lambda}) \underset{V}{*} \alpha_d = (\ddot{\lambda} \underset{V}{*} \dot{\alpha}_d) \underset{V}{*} \alpha_d = \ddot{\lambda} \underset{V}{*} (\dot{\alpha}_d \bullet \alpha_d) \end{aligned}$$

Dually we also have a category $\underline{\underline{\mathrm{Cyl}}}_V(f, \bar{f})$ for each pair of horizontal 1-cells $f, \bar{f} \in \underline{\underline{\mathcal{H}}}$, the maps of which are called *vertical cylinders*.

(iii) Equivariant maps

$$\mathcal{S}_V \otimes \mathcal{S}_V \xrightarrow{\underset{H}{\otimes}} \mathcal{S}_V \qquad \underline{\mathcal{V}}_1 \xrightarrow{\mathrm{i}^h} \mathcal{S}_V$$

$$\mathcal{S}_H \otimes \mathcal{S}_H \xrightarrow{\underset{V}{\otimes}} \mathcal{S}_H \qquad \underline{\mathcal{H}}_1 \xrightarrow{\mathrm{i}^v} \mathcal{S}_H$$

where $\mathcal{S}_V$ ($\mathcal{S}_H$) is $\mathcal{S}$ considered simply as an endo-profunctor on $\underline{\mathcal{V}}$ ($\underline{\mathcal{H}}$), $\otimes$ is used to denote tensor product of profunctors and $\underline{\mathcal{V}}_1$, $\underline{\mathcal{H}}_1$ are imbued with the canonical actions making them into identities, on their respective categories, under $\otimes$.

Unravelling these definitions we get an operation $\lambda' \underset{H}{\otimes} \lambda$, acting on pairs of squares $\lambda', \lambda \in \mathcal{S}$ with $\mathrm{dom}_H(\lambda') = \mathrm{cod}_H(\lambda)$, such that

- $\mathrm{dom}_H(\lambda' \underset{H}{\otimes} \lambda) = \mathrm{dom}_H(\lambda)$ and $\mathrm{cod}_H(\lambda' \underset{H}{\otimes} \lambda) = \mathrm{cod}_H(\lambda')$,
- for vertical 2-cells $\beta, \beta'' \in \underline{\underline{\mathcal{V}}}$ with $\mathrm{cod}(\beta) = \mathrm{dom}_H(\lambda)$ and $\mathrm{cod}_H(\lambda') = \mathrm{dom}(\beta'')$ we have

$$\begin{aligned} (\lambda' \underset{H}{\otimes} \lambda) \underset{H}{*} \beta &= \lambda' \underset{H}{\otimes} (\lambda \underset{H}{*} \beta) \\ \beta'' \underset{H}{*} (\lambda' \underset{H}{\otimes} \lambda) &= (\beta'' \underset{H}{*} \lambda') \underset{H}{\otimes} \lambda\,, \end{aligned}$$

- if $\dot{\lambda}$ is another square and $\beta' \colon \mathrm{cod}_H(\lambda) \Rightarrow_V \mathrm{dom}_H(\dot{\lambda})$ is a vertical 2-cell then:

$$\dot{\lambda} \underset{H}{\otimes} (\beta' \underset{H}{*} \lambda) = (\dot{\lambda} \underset{H}{*} \beta') \underset{H}{\otimes} \lambda$$

Likewise giving an equivariant map i^h, as above, corresponds to providing a square $\mathrm{i}^h_p \in \mathcal{S}$, for each vertical 1-cell $p \in \underline{\underline{\mathcal{V}}}$, with $\mathrm{dom}_H(\mathrm{i}^h_p) = \mathrm{cod}_H(\mathrm{i}^h_p) = p$ such that these collectively satisfy

$$\beta \underset{H}{*} \mathrm{i}^h_p = \mathrm{i}^h_q \underset{H}{*} \beta$$

for each vertical 2-cell $\beta\colon p \Rightarrow_V q$. Explicit descriptions of $\underset{V}{\otimes}$ and i^v are dual to these.

It is immediate that we may define a horizontal composite of vertical cylinders, if $(\beta_d, \beta_c)\colon \lambda \Rightarrow_V \dot{\lambda}$ and $(\beta'_d, \beta'_c)\colon \lambda' \Rightarrow_V \dot{\lambda}'$ are cylinders with $\beta'_d = \beta_c$ then (β_d, β'_c) is a vertical cylinder from $\lambda' \underset{H}{\otimes} \lambda$ to $\dot{\lambda}' \underset{H}{\otimes} \dot{\lambda}$ since:

$$\begin{aligned} \beta'_c \underset{H}{*} (\lambda' \underset{H}{\otimes} \lambda) &= (\beta'_c \underset{H}{*} \lambda') \underset{H}{\otimes} \lambda = (\dot{\lambda}' \underset{H}{*} \beta'_d) \underset{H}{\otimes} \lambda \\ &= \dot{\lambda}' \underset{H}{\otimes} (\beta_c \underset{H}{*} \lambda) = \dot{\lambda}' \underset{H}{\otimes} (\dot{\lambda} \underset{H}{*} \beta_d) = (\dot{\lambda}' \underset{H}{\otimes} \dot{\lambda}) \underset{H}{*} \beta_d \end{aligned}$$

We also get vertical composites of horizontal cylinders dually.

Finally all of the preceding data is subject to the following rules. In each case if we state a rule which only involves horizontal properties of squares then a dual rule must also hold for the corresponding vertical property, thus preserving the symmetry of our definition:

(iv) For compatible squares $\lambda, \lambda' \in \mathcal{S}$:

$$\begin{aligned} \mathrm{dom}_V(\lambda' \underset{H}{\otimes} \lambda) &= \mathrm{dom}_V(\lambda') \underset{H}{\otimes} \mathrm{dom}_V(\lambda) \\ \mathrm{cod}_V(\lambda' \underset{H}{\otimes} \lambda) &= \mathrm{cod}_V(\lambda') \underset{H}{\otimes} \mathrm{cod}_V(\lambda) \end{aligned}$$

In diagrammatic terms this means that horizontal composition of squares looks like:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\lambda"] & a' \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lambda'"] & a'' \arrow[d, "p''"] & \longmapsto & a \arrow[rr, "f' \underset{H}{\otimes} f"] \arrow[d, "p"'] \arrow[drr, phantom, "\lambda' \underset{H}{\otimes} \lambda"] & & a'' \arrow[d, "p''"] \\ \bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \arrow[r, "\bar{f}'"'] & \bar{a}'' & & \bar{a} \arrow[rr, "\bar{f}' \underset{H}{\otimes} \bar{f}"'] & & \bar{a}'' \end{tikzcd}$$

(v) For each vertical 1-cell $p \in \underline{\underline{\mathcal{V}}}$:

$$\begin{aligned} \mathrm{dom}_V(\mathrm{i}^h_p) &= \mathrm{i}^h_{\mathrm{dom}_V(p)} \\ \mathrm{cod}_V(\mathrm{i}^h_p) &= \mathrm{i}^h_{\mathrm{cod}_V(p)} \end{aligned}$$

In diagrammatic terms this means that i^h_p looks like:

$$\begin{tikzcd}
a \arrow[r, "\mathrm{i}^h_a"] \arrow[d, "p"', "|" marking] \arrow[dr, phantom, "\mathrm{i}^h_p \lrcorner"] & a \arrow[d, "p", "|" marking] \\
\bar{a} \arrow[r, "\mathrm{i}^h_{\bar{a}}"'] & \bar{a}
\end{tikzcd}$$

(vi) Horizontal composition preserves horizontal cylinders. In other words suppose that $(\alpha_d, \alpha_c)\colon \lambda \Rightarrow_H \dot{\lambda} \in \underline{\underline{\mathrm{Cyl}}}_H(p, p')$ and $(\alpha'_d, \alpha'_c)\colon \lambda' \Rightarrow_H \dot{\lambda}' \in \underline{\underline{\mathrm{Cyl}}}_H(p', p'')$ are horizontal cylinders, then the pair $(\alpha'_d \underset{H}{\otimes} \alpha_d\,,\, \alpha'_c \underset{H}{\otimes} \alpha_c)$ is a cylinder from $\lambda' \underset{H}{\otimes} \lambda$ to $\dot{\lambda}' \underset{H}{\otimes} \dot{\lambda}$.

We may re-express this in terms of a "middle four interchange" rule, for any (domain/codomain compatible) horizontal 2-cells and squares we have cylinders:

$$\begin{array}{ll}
(\mathrm{i}^h_{\mathrm{dom}_V(\lambda)}, \alpha_c)\colon \lambda \Rightarrow_H \alpha_c \underset{V}{*} \lambda & (\mathrm{i}^h_{\mathrm{dom}_V(\lambda')}, \alpha'_c)\colon \lambda' \Rightarrow_H \alpha'_c \underset{V}{*} \lambda' \\
(\alpha_d, \mathrm{i}^h_{\mathrm{cod}_V(\dot{\lambda})})\colon \dot{\lambda} \Rightarrow_H \dot{\lambda} \underset{V}{*} \alpha_d & (\alpha'_d, \mathrm{i}^h_{\mathrm{dom}_V(\dot{\lambda}')})\colon \dot{\lambda}' \Rightarrow_H \dot{\lambda}' \underset{V}{*} \alpha'_d
\end{array}$$

Now write out what it means to say that horizontal composition of these pairs preserves the cylinder property and we get the equalities

$$\begin{aligned}
(\alpha'_c \underset{H}{\otimes} \alpha_c) \underset{V}{*} (\lambda' \underset{H}{\otimes} \lambda) &= (\alpha'_c \underset{V}{*} \lambda') \underset{H}{\otimes} (\alpha_c \underset{V}{*} \lambda) \\
(\dot{\lambda}' \underset{H}{\otimes} \dot{\lambda}) \underset{V}{*} (\alpha'_d \underset{H}{\otimes} \alpha_d) &= (\dot{\lambda}' \underset{V}{*} \alpha'_d) \underset{H}{\otimes} (\dot{\lambda} \underset{V}{*} \alpha_d)
\end{aligned}$$

from which, in turn, we may clearly re-derive our original condition. This axiom gives us a functor

$$\underline{\underline{\mathrm{Cyl}}}_H(p', p'') \times \underline{\underline{\mathrm{Cyl}}}_H(p, p') \xrightarrow{\underset{H}{\otimes}} \underline{\underline{\mathrm{Cyl}}}_H(p, p'')$$

enriching the horizontal composite of squares.

(vii) given three squares

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "p"', "|" marking] \arrow[dr, phantom, "\lambda \lrcorner"] & a' \arrow[r, "f'"] \arrow[d, "p'"', "|" marking] \arrow[dr, phantom, "\lambda' \lrcorner"] & a'' \arrow[r, "f''"] \arrow[d, "p''"', "|" marking] \arrow[dr, phantom, "\lambda'' \lrcorner"] & a^{(3)} \arrow[d, "p^{(3)}", "|" marking] \\
\bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \arrow[r, "\bar{f}'"'] & \bar{a}'' \arrow[r, "\bar{f}''"'] & \bar{a}^{(3)}
\end{tikzcd}$$

the canonical isomorphisms

$$(f'' \underset{H}{\otimes} f') \underset{H}{\otimes} f \xrightarrow[\cong]{\text{can}} f'' \underset{H}{\otimes} (f' \underset{H}{\otimes} f)$$
$$(\bar{f}'' \underset{H}{\otimes} \bar{f}') \underset{H}{\otimes} \bar{f} \xrightarrow[\cong]{\text{can}} \bar{f}'' \underset{H}{\otimes} (\bar{f}' \underset{H}{\otimes} \bar{f})$$

of $\underline{\underline{\mathcal{H}}}$ form a horizontal cylinder $(\lambda'' \underset{H}{\otimes} \lambda') \underset{H}{\otimes} \lambda \Rightarrow_H \lambda'' \underset{H}{\otimes} (\lambda' \underset{H}{\otimes} \lambda)$ which is necessarily then an isomorphism in $\underline{\underline{\mathrm{Cyl}}}_H(p, p^{(3)})$.

(viii) for any square λ (as in the diagram above) the canonical isomorphisms

$$f \underset{H}{\otimes} \mathrm{i}^h_a \xrightarrow[\cong]{\text{can}} f \qquad \text{and} \qquad \mathrm{i}^h_{a'} \underset{H}{\otimes} f \xrightarrow[\cong]{\text{can}} f$$
$$\bar{f} \underset{H}{\otimes} \mathrm{i}^h_{\bar{a}} \xrightarrow[\cong]{\text{can}} \bar{f} \qquad \qquad \mathrm{i}^h_{\bar{a}'} \underset{H}{\otimes} \bar{f} \xrightarrow[\cong]{\text{can}} \bar{f}$$

of $\underline{\underline{\mathcal{H}}}$ are the components of horizontal cylinders $\lambda \underset{H}{\otimes} \mathrm{i}^h_p \Rightarrow_H \lambda$ and $\mathrm{i}^h_{p'} \underset{H}{\otimes} \lambda \Rightarrow_H \lambda$, which are necessarily then isomorphisms in $\underline{\underline{\mathrm{Cyl}}}_H(p, p')$.

The functors defined at the end of (vi) and the isomorphisms provided by this and the previous axiom render the categories $\underline{\underline{\mathrm{Cyl}}}_H(p, p')$ into the "homsets" of a bicategory of horizontal cylinders $\underline{\underline{\mathrm{Cyl}}}_H$, whose 0-cells are the vertical 1-cells. Since the various composites of cylinders are all defined pointwise in $\underline{\underline{\mathcal{H}}}$ the required coherence conditions on the canonical isomorphisms follow trivially from the corresponding properties in there. Immediately the vertical domain and codomain functions on squares enrich to *strict* homomorphisms:

$$\mathrm{cod}_V, \mathrm{dom}_V \colon \underline{\underline{\mathrm{Cyl}}}_H \longrightarrow \underline{\underline{\mathcal{H}}}$$

Dually we have a bicategory of vertical cylinders $\underline{\underline{\mathrm{Cyl}}}_V$, and strict homomorphisms:

$$\mathrm{cod}_H, \mathrm{dom}_H \colon \underline{\underline{\mathrm{Cyl}}}_V \longrightarrow \underline{\underline{\mathcal{V}}}$$

(ix) given four compatible squares

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\lambda"] & a' \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lambda'"] & a'' \arrow[d, "p''"] \\
\bar{a} \arrow[r, "\bar{f}"] \arrow[d, "\bar{p}"'] \arrow[dr, phantom, "\bar{\lambda}"] & \bar{a}' \arrow[r, "\bar{f}'"] \arrow[d, "\bar{p}'"'] \arrow[dr, phantom, "\bar{\lambda}'"] & \bar{a}'' \arrow[d, "\bar{p}''"] \\
\tilde{a} \arrow[r, "\tilde{f}"'] & \tilde{a}' \arrow[r, "\tilde{f}'"'] & \tilde{a}''
\end{tikzcd}$$

the middle four interchange rule holds, in other words:

$$(\bar{\lambda}' \underset{H}{\otimes} \bar{\lambda}) \underset{V}{\otimes} (\lambda' \underset{H}{\otimes} \lambda) = (\bar{\lambda}' \underset{V}{\otimes} \lambda') \underset{H}{\otimes} (\bar{\lambda} \underset{V}{\otimes} \lambda)$$

As was pointed out in [3] proposition 7.4.1 the theory of bicategories (and *strict* homomorphisms) is essentially algebraic and that the inclusion of this category into the category of bicategories and morphisms preserves limits. In particular we may form the limit of a diagram of strict homomorphisms in the naïve way, that is to say pointwise, and so compose the spans $(\mathrm{cod}_H, \underline{\underline{\mathrm{Cyl}}}_V, \mathrm{dom}_H)$ and $(\mathrm{cod}_V, \underline{\underline{\mathrm{Cyl}}}_H, \mathrm{dom}_V)$ with themselves by pulling back. Now a combination of the comment at the end of (iii), about composition of cylinders, and the middle four interchange rule provides us with strict homomorphisms of spans:

$$(\mathrm{cod}_V, \underline{\underline{\mathrm{Cyl}}}_H, \mathrm{dom}_V) \underset{\underline{\underline{\mathcal{H}}}}{\times} (\mathrm{cod}_V, \underline{\underline{\mathrm{Cyl}}}_H, \mathrm{dom}_V) \xrightarrow{\underset{V}{\otimes}} (\mathrm{cod}_V, \underline{\underline{\mathrm{Cyl}}}_H, \mathrm{dom}_V)$$
$$(\mathrm{cod}_H, \underline{\underline{\mathrm{Cyl}}}_V, \mathrm{dom}_H) \underset{\underline{\underline{\mathcal{V}}}}{\times} (\mathrm{cod}_H, \underline{\underline{\mathrm{Cyl}}}_V, \mathrm{dom}_H) \xrightarrow{\underset{H}{\otimes}} (\mathrm{cod}_H, \underline{\underline{\mathrm{Cyl}}}_V, \mathrm{dom}_H)$$

enriching the corresponding composition of squares.

And finally:

(x) for any 0-cell $a \in \mathcal{A}$ the squares $\mathrm{i}^h_{\mathrm{i}^v_a}$ and $\mathrm{i}^v_{\mathrm{i}^h_a}$ are identical. In other words the horizontal identity square on the identity $\mathrm{i}^v_a \in \underline{\underline{\mathcal{V}}}$ is equal to the vertical identity square on $\mathrm{i}^h_a \in \underline{\underline{\mathcal{H}}}$. Furthermore horizontal (vertical) identity squares are well behaved with respect to vertical (horizontal) composition, i.e.:

$$\begin{aligned} \mathrm{i}^h_q \underset{V}{\otimes} \mathrm{i}^h_p &= \mathrm{i}^h_{q \underset{V}{\otimes} p} \\ \mathrm{i}^v_g \underset{H}{\otimes} \mathrm{i}^v_f &= \mathrm{i}^v_{g \underset{H}{\otimes} f} \end{aligned}$$

This axiom and the conditions imposed on identity squares in rules (iii) and (v) ensure that i^h and i^v enrich to canonical strict homomorphisms

$$\underline{\underline{\mathcal{H}}} \xrightarrow{\quad \mathrm{i}^v \quad} \underline{\underline{\mathrm{Cyl}}}_H$$
$$\underline{\underline{\mathcal{V}}} \xrightarrow{\quad \mathrm{i}^h \quad} \underline{\underline{\mathrm{Cyl}}}_V$$

Notice that, as we pointed out in the course of the definition, double bicategories are inherently symmetrical. To be precise we may swap the rôles of horizontal and vertical cells of $\tilde{\underline{\underline{\mathcal{D}}}}$ and reflect all squares about their leading diagonals to obtain another double bicategory $\tilde{\underline{\underline{\mathcal{D}}}}^{\mathrm{rf}}$. $\square$

We presented an example of this sort of structure in section 1.2, the double bicategory $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ associated with a (weak) equipment. Another canonical example related to that one is $\mathrm{Sq}(\underline{\underline{\mathcal{B}}})$, the double bicategory of squares derived from a single bicategory $\underline{\underline{\mathcal{B}}}$. This has $\underline{\underline{\mathcal{H}}} = \underline{\underline{\mathcal{B}}}^{\mathrm{co}}$, $\underline{\underline{\mathcal{V}}} = \underline{\underline{\mathcal{B}}}$ and squares of the form

$$\begin{tikzcd} a \arrow[r, "q", "\shortmid"{marking}] \arrow[d, "p"', "\shortmid"{marking}] & a' \arrow[d, "p'", "\shortmid"{marking}] \\ \bar{a} \arrow[r, "\bar{q}"', "\shortmid"{marking}] \arrow[ur, phantom, "\lambda \Uparrow"] & \bar{a}' \end{tikzcd}$$

where $p, p', q, \bar{q} \in \underline{\underline{\mathcal{B}}}$ are 1-cells and $(\lambda\colon p' \otimes q \Rightarrow \bar{q} \otimes p) \in \underline{\underline{\mathcal{B}}}$ is a 2-cell. This example is exactly the double bicategory of squares associated with the weak equipment $(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{B}}}^{\mathrm{co}}, \mathrm{I}_{\underline{\underline{\mathcal{B}}}})$. The work (and notation) on cylinders in [3] is clearly related to that presented here via this example. The principle reason for introducing squares and cylinders seems to have been to facilitate the definition of the bicategory $\mathcal{B}icat(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{C}}})$, this is subsumed by the slightly more general version:

Observation 1.4.2 Suppose that $\tilde{\underline{\underline{\mathcal{D}}}} = (\mathcal{A}, \underline{\underline{\mathcal{H}}}, \underline{\underline{\mathcal{V}}}, \mathcal{S})$ is a double bicategory and $\underline{\underline{\mathcal{B}}}$ a bicategory then we may define a natural bicategory $\mathcal{B}icat_H(\underline{\underline{\mathcal{B}}}, \tilde{\underline{\underline{\mathcal{D}}}})$ with:

0-cells bicategory morphisms $\mathrm{M}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{V}}}$,

1-cells bicategory morphisms $\Omega\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathrm{Cyl}_V}}$, which we call *horizontal transformations.* Such cells have domain and codomain given by the composites $(\mathrm{dom}_H \circ \Omega)$ and $(\mathrm{cod}_H \circ \Omega)$ respectively, where $\mathrm{dom}_H, \mathrm{cod}_H\colon \underline{\underline{\mathrm{Cyl}_V}} \longrightarrow \underline{\underline{\mathcal{V}}}$ are the strict homomorphisms defined at the end of definition 1.4.1(viii).

More explicitly a transformation $\Omega\colon \mathrm{M} \longrightarrow \mathrm{M}'$ consists of a family of horizontal 1-cells

$$\mathrm{M}(b) \xrightarrow{\quad\Omega_b\quad} \mathrm{M}'(b)$$

one for each 0-cell $b \in \underline{\underline{\mathcal{B}}}$, and a square

$$\begin{tikzcd} \mathrm{M}b \arrow[r, "\Omega_b"] \arrow[d, "\mathrm{M}p"', "\shortmid"{marking}] & \mathrm{M}'b \arrow[d, "\mathrm{M}'p", "\shortmid"{marking}] \\ \mathrm{M}\bar{b} \arrow[r, "\Omega_{\bar{b}}"'] \arrow[ur, phantom, "\Omega_{\underline{p}} \rfloor"] & \mathrm{M}'\bar{b} \end{tikzcd}$$

for each 1-cell $(p\colon b \longrightarrow \bar{b}) \in \underline{\underline{\mathcal{B}}}$. These must satisfy the rules

- for any 2-cell $\beta\colon p \Rightarrow \dot{p}$ the vertical 2-cells

$$\mathrm{M}p \xrightarrow{\mathrm{M}\beta} \mathrm{M}\dot{p}$$
$$\mathrm{M}'p \xrightarrow{\mathrm{M}'\beta} \mathrm{M}'\dot{p}$$

  are the components of a vertical cylinder $\Omega_p \Rightarrow_V \Omega_{\dot{p}}$.
- for any pair of 1-cells p and $\bar{p}\colon \bar{b} \longrightarrow \tilde{b}$ in $\underline{\underline{\mathcal{B}}}$ the canonical 2-cells

$$\mathrm{M}\bar{p} \underset{V}{\otimes} \mathrm{M}p \xrightarrow{\text{can}} \mathrm{M}(\bar{p} \otimes p)$$
$$\mathrm{M}'\bar{p} \underset{V}{\otimes} \mathrm{M}'p \xrightarrow{\text{can}} \mathrm{M}'(\bar{p} \otimes p)$$

  constitute a cylinder $\Omega_{\bar{p}} \underset{V}{\otimes} \Omega_p \Rightarrow_V \Omega_{\bar{p}\otimes p}$.
- for any 0-cell $b \in \underline{\underline{\mathcal{B}}}$ the canonical identity comparison 2-cells

$$\mathrm{i}^v_{\mathrm{M}b} \xrightarrow{\text{can}} \mathrm{M}(\mathrm{i}_b)$$
$$\mathrm{i}^v_{\mathrm{M}'b} \xrightarrow{\text{can}} \mathrm{M}'(\mathrm{i}_b)$$

  constitute a vertical cylinder $\mathrm{i}^v_{\Omega_b} \Rightarrow_V \Omega_{\mathrm{i}_b}$.

2-cells $m\colon \Omega \Rightarrow \dot{\Omega}$, called *horizontal modifications*, consist of a family of horizontal 2-cells $m_b\colon \Omega_b \Rightarrow_H \dot{\Omega}_b$, one for each 0-cell $b \in \underline{\underline{\mathcal{B}}}$, such that for each 1-cell $(p\colon b \longrightarrow \bar{b}) \in \underline{\underline{\mathcal{B}}}$ the pair of cells $(m_b, m_{\bar{b}})$ is a horizontal cylinder $\Omega_p \Rightarrow_H \dot{\Omega}_p$. As usual depth-wise composition of 2-cells is calculated pointwise in $\underline{\underline{\mathcal{H}}}$.

composition given transformations Ω and Ω' with $(\mathrm{cod}_H \circ \Omega) = (\mathrm{dom}_H \circ \Omega')$, in other words they may be composed, we get a unique morphism

$$\langle \Omega', \Omega \rangle \colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathrm{Cyl}}}_V \underset{\underline{\underline{\mathcal{V}}}}{\times} \underline{\underline{\mathrm{Cyl}}}_V$$

and we obtain a third transformation $\Omega' \underset{H}{\otimes} \Omega$ as the composite

$$\underline{\underline{\mathcal{B}}} \xrightarrow{\langle \Omega', \Omega \rangle} \underline{\underline{\mathrm{Cyl}}}_V \underset{\underline{\underline{\mathcal{V}}}}{\times} \underline{\underline{\mathrm{Cyl}}}_V \xrightarrow{\underset{H}{\otimes}} \underline{\underline{\mathrm{Cyl}}}_V$$

where $\underset{H}{\otimes}$ is the strict homomorphism defined at the end of 1.4.1(ix). Since $\underset{H}{\otimes}$ is in fact a homomorphism of suitable spans it follows that $\mathrm{dom}_H(\Omega' \underset{H}{\otimes} \Omega) = \mathrm{dom}_H \Omega$ and $\mathrm{cod}_H(\Omega' \underset{H}{\otimes} \Omega) = \mathrm{cod}_H \Omega'$. More explicitly the formulae for the structure cells of $\Omega' \underset{H}{\otimes} \Omega$ are

$$\begin{aligned}(\Omega' \otimes \Omega)_b &= \Omega'_b \underset{H}{\otimes} \Omega_b \text{ for 0-cells } b \in \underline{\underline{\mathcal{B}}}\\ (\Omega' \otimes \Omega)_p &= \Omega'_p \underset{H}{\otimes} \Omega_p \text{ for 1-cells } p \in \underline{\underline{\mathcal{B}}}\end{aligned}$$

from which it is clear how to extend this composition to horizontal modifications, simply calculate pointwise in $\underline{\underline{\mathcal{H}}}$. Rule (vii) of definition 1.4.1 provides us with associativity modifications.

identity on the 0-cell M is the composite

$$\underline{\underline{\mathcal{B}}} \xrightarrow{\quad \mathrm{M} \quad} \underline{\underline{\mathcal{V}}} \xrightarrow{\quad \mathrm{i}^h \quad} \underline{\underline{\mathrm{Cyl}_V}}$$

where i^h denotes the strict homomorphism defined at the end of 1.4.1(x), explicitly:

$$\begin{aligned} (\mathrm{i}_{\mathrm{M}})_b &= \mathrm{i}^h_{\mathrm{M}b} \text{ for 0-cells } b \in \underline{\underline{\mathcal{B}}} \\ (\mathrm{i}_{\mathrm{M}})_p &= \mathrm{i}^h_{\mathrm{M}p} \text{ for 0-cells } p \in \underline{\underline{\mathcal{B}}} \end{aligned}$$

The modifications displaying this as an identity, in the bicategorical sense, are provided pointwise by definition 1.4.1(viii).

Some examples of this construction are $\mathcal{B}icat(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{C}}})$ which is just $\mathcal{B}icat_H(\underline{\underline{\mathcal{B}}}, \mathrm{Sq}(\underline{\underline{\mathcal{C}}}))$ and the bicategory $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$ introduced in section 1.2, which is simply $\mathcal{B}icat_H(1, \mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$. Notice that if $\underline{\underline{\mathcal{H}}}$ is a 2-category, rather than just a bicategory, then so is any bicategory $\mathcal{B}icat_H(\underline{\underline{\mathcal{B}}}, \tilde{\underline{\underline{\mathcal{D}}}})$.

We now set about presenting a number of natural (strongly) bicategory enriched categories of double bicategories. In essence we start by presenting notions which only involve the *horizontal* structure of each double bicategory, and then build layers of *vertical* information upon that. On turning back to equipments in later sections we will see that this corresponds to first providing important foundational structure at the level of bicategories of functors and then building the actions upon profunctors on top of this. First the following technical lemmas:

Observation 1.4.3 *The category $\underline{\mathcal{SH}om}$ of (small) bicategories and strict homomorphisms has all small limits which are calculated pointwise, furthermore the functors*

$$\underline{\mathcal{SH}om} \xrightarrow{\mathcal{H}om(\underline{\underline{\mathcal{A}}}, \text{-})} \underline{\mathcal{SH}om}$$
$$\underline{\mathcal{SH}om} \xrightarrow{\mathcal{H}om_S(\underline{\underline{\mathcal{A}}}, \text{-})} \underline{\mathcal{SH}om}$$

preserve these for each bicategory $\underline{\underline{\mathcal{A}}}$.

Proof. Notice that this is essentially just proposition 7.4.1 of [3], $\underline{\mathcal{SH}om}$ is essentially algebraic and therefore possesses all small limits which are calculated pointwise. Describing the construction of these limits explicitly gives an indication of why the functors given should preserve them, let $\mathbb{I}$ be a small category and $\underline{\underline{\mathcal{D}}}\colon \mathbb{I} \longrightarrow \underline{\mathcal{SH}om}$ a diagram, then $\varprojlim_{\mathbb{I}}[\underline{\underline{\mathcal{D}}}(-)]$ has:

0-cells families $\{d_i\}_{i\in\mathbb{I}}$ of 0-cells $d_i \in \underline{\underline{\mathcal{D}}}_i$ satisfying the condition $\underline{\underline{\mathcal{D}}}_\gamma(d_i) = d_{i'}$ for all maps $\gamma\colon i \longrightarrow i'$.

1-cells $\{f_i\}_{i\in\mathbb{I}}\colon \{d_i\}_{i\in\mathbb{I}} \longrightarrow \{d'_i\}_{i\in\mathbb{I}}$ families of 1-cells $f_i \in \underline{\underline{\mathcal{D}}}_i$ satisfying the same condition with respect to the maps of $\mathbb{I}$.

2-cells $\{\alpha_i\}_{i\in\mathbb{I}}\colon \{f_i\}_{i\in\mathbb{I}} \Rightarrow \{f'_i\}_{i\in\mathbb{I}}$ families of 2-cells $\alpha_i \in \underline{\underline{\mathcal{D}}}_i$ again satisfying this naturality condition.

composition All compositions of families are calculated pointwise, at this stage strictness of the transition homomorphisms $\underline{\underline{\mathcal{D}}}_\gamma$ becomes necessary, allowing us to establish that pointwise horizontal composition is well defined. For instance if $\{f_i\}_{i\in\mathbb{I}}$ and $\{g_i\}_{i\in\mathbb{I}}$ are compatible families then the calculation

$$\underline{\underline{\mathcal{D}}}_\gamma(g_i \otimes f_i) = \underline{\underline{\mathcal{D}}}_\gamma(g_i) \otimes \underline{\underline{\mathcal{D}}}_\gamma(f_i) = g_{i'} \otimes f_{i'}$$

establishes that the family $\{g_i \otimes f_i\}_{i\in\mathbb{I}}$ satisfies the required naturality condition, but is only made possible by requiring strictness of $\underline{\underline{\mathcal{D}}}_\gamma$.

identities Also defined pointwise, again strictness of the transition homomorphisms ensure that this is well defined.

The limit cone $\pi_i\colon \varprojlim_{\mathbb{I}}[\underline{\underline{\mathcal{D}}}(-)] \longrightarrow \underline{\underline{\mathcal{D}}}_i$ simply consists of the strict homomorphisms which project families on to each of their components.

Now consider the limit of the diagram $\mathcal{H}om(\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{D}}}(-))$, a family of homomorphisms $\{\mathrm{F}_i \in \mathcal{H}om(\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{D}}}_i)\}_{i\in\mathbb{I}}$ is a 0-cell of $\varprojlim_{\mathbb{I}}[\mathcal{H}om(\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{D}}}(-))]$ iff $\underline{\underline{\mathcal{D}}}_\gamma \circ \mathrm{F}_i = \mathrm{F}_{i'}$ for all maps $\gamma \in \mathbb{I}$. Of course this means that if a is a (0-, 1- or 2-) cell of $\underline{\underline{\mathcal{A}}}$ then the family $\{\mathrm{F}_i(a)\}_{i\in\mathbb{I}}$ satisfies the naturality condition making it a (0-, 1- or 2-) cell of $\varprojlim_{\mathbb{I}}[\underline{\underline{\mathcal{D}}}(-)]$, but this is not enough to characterise an equality of homomorphisms. We must also insist that the families of canonical isomorphisms

$$\mathrm{F}_i(q) \otimes \mathrm{F}_i(p) \xrightarrow[\simeq]{\mathrm{can}_i} \mathrm{F}_i(q \otimes p)$$
$$\mathrm{i}_{\mathrm{F}_i(a)} \xrightarrow[\simeq]{\mathrm{can}_i} \mathrm{F}_i(\mathrm{i}_a)$$

are natural and therefore 2-cells of $\varprojlim_{\mathbb{I}}[\underline{\underline{\mathcal{D}}}(-)]$, here again we require the strictness of transition homomorphisms. The compositions of $\varprojlim_{\mathbb{I}}[\underline{\underline{\mathcal{D}}}(-)]$ are defined pointwise, so we see straightaway that natural families of homomorphisms $\mathrm{F}_i \in \mathcal{H}om(\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{D}}}_i)$ are in bijective correspondence with homomorphisms $\mathrm{F} \in \mathcal{H}om(\underline{\underline{\mathcal{A}}}, \varprojlim_{\mathbb{I}}[\underline{\underline{\mathcal{D}}}(-)])$. It should be no surprise that this bijection is exactly the action of the canonical comparison homomorphism

$$\mathcal{H}om(\underline{\underline{\mathcal{A}}}, \varprojlim_{\mathbb{I}}[\underline{\underline{\mathcal{D}}}(-)]) \longrightarrow \varprojlim_{\mathbb{I}}[\mathcal{H}om(\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{D}}}(-))] \tag{1.45}$$

on 0-cells.

With this we have certainly established that the limit of $\underline{\underline{\mathcal{D}}}(-)$ is preserved by the embedding $\underline{\mathcal{SH}om} \hookrightarrow \underline{\mathcal{H}om}_0$, which might profitably be rephrased by saying that $\underline{\mathcal{H}om}_0$ has limits of all (small) diagrams with transition homomorphisms which are strict, and these are calculated as in $\underline{\mathcal{SH}om}$. A little work is needed to establish our preservation result, but this amounts to no more than noticing that families of transformations $\{\Psi_i \in \mathcal{H}om(\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{D}}}_i)\}_{i\in\mathbb{I}}$ are natural iff the family $\{(\Psi_i)_a\}_{i\in\mathbb{I}}$ is a (1- or 2-) cell of $\varprojlim_{\mathbb{I}}[\underline{\underline{\mathcal{D}}}(-)]$ for all (0- or 1-) cells $a \in \underline{\underline{\mathcal{A}}}$ along with a similar component-wise result for modifications. Using these we finally establish that the comparison homomorphism of (1.45) is also bijective on 1- and 2-cells. The proof for $\mathcal{H}om_S(\underline{\underline{\mathcal{A}}}, -)$ is identical. $\square$

Corollary 1.4.4 *The multicategories $\underline{\mathcal{H}om}$ and $\underline{\mathcal{H}om}_S$ have limits of (small) diagrams of strict homomorphisms calculated as in $\underline{\mathcal{SH}om}$. This means that for any diagram $\underline{\underline{\mathcal{D}}}(-)$ in $\underline{\mathcal{SH}om}$, families of (strong) n-homomorphisms*

$$[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_n] \xrightarrow{\mathrm{F}_i} \underline{\underline{\mathcal{D}}}_i$$

satisfying $\underline{\underline{\mathcal{D}}}_\gamma\langle\mathrm{F}_i\rangle = \mathrm{F}_{i'}$ for all $\gamma \in \mathbb{I}$ (call this an n-cone) factor uniquely though the limit cone $\pi_i\colon \varprojlim_{\mathbb{I}}[\underline{\underline{\mathcal{D}}}(-)] \longrightarrow \underline{\underline{\mathcal{D}}}_i$ as

$$[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_n] \xrightarrow{\mathrm{F}} \varprojlim_{\mathbb{I}}[\underline{\underline{\mathcal{D}}}(-)]$$

with $\pi_i\langle\mathrm{F}\rangle = \mathrm{F}_i$.

Proof. Simple induction on n the order of n-homomorphisms.

$\boldsymbol{n = 0}$: 0-homomorphisms into a bicategory correspond to its 0-cells therefore all the corollary says for this case is that we form the set of 0-cells of a limit in $\underline{\mathcal{SH}om}$ by taking the limit of the composite diagram:

$$\mathbb{I} \xrightarrow{\underline{\underline{\mathcal{D}}}(-)} \underline{\mathcal{SH}om} \xrightarrow{\text{0-cells}} \underline{\mathrm{Set}}$$

$\boldsymbol{n > 0}$: and assuming result is true for m-homomorphisms with $m < n$. Simply use the natural bijective correspondence of lemma 1.3.4 to reduce an n-cone, as in the statement of the lemma, to an $(n-1)$-cone:

$$[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_{n-1}] \xrightarrow{\hat{\mathrm{F}}_i} \mathcal{H}om(\underline{\underline{\mathcal{B}}}_n, \underline{\underline{\mathcal{D}}}_i)$$

The last observation demonstrated that $\mathcal{H}om(\underline{\underline{\mathcal{B}}}_n, -)$ preserves limits in $\underline{\mathcal{SH}om}$ so we may apply the induction hypothesis to get a unique $(n-1)$-homomorphism

$$[\underline{\underline{\mathcal{B}}}_1, \cdots, \underline{\underline{\mathcal{B}}}_{n-1}] \xrightarrow{\hat{\mathrm{F}}} \mathcal{H}om(\underline{\underline{\mathcal{B}}}_n, \varprojlim_{\mathbb{I}}[\underline{\underline{\mathcal{D}}}(-)])$$

which in turn corresponds to a unique n-homomorphism factoring the n-cone we started with. Again the proof for $\underline{\mathcal{H}om}_S$ is identical. $\square$

Definition 1.4.5 Let $\underline{\underline{\underline{\mathcal{A}}}}$ be a (strongly) bicategory enriched category, we say that a homomorphism $\mathrm{H} \in \underline{\underline{\underline{\mathcal{A}}}}(A, B)$ is *strict* iff each homomorphism of bicategories

$$\underline{\underline{\underline{\mathcal{A}}}}(A', A) \xrightarrow{\underline{\underline{\underline{\mathcal{A}}}}(A', \mathrm{H})} \underline{\underline{\underline{\mathcal{A}}}}(A', B) \quad \text{and}$$
$$\underline{\underline{\underline{\mathcal{A}}}}(B, B') \xrightarrow{\underline{\underline{\underline{\mathcal{A}}}}(\mathrm{H}, B')} \underline{\underline{\underline{\mathcal{A}}}}(A, B')$$

is strict in the traditional sense. The strict homomorphisms in $\underline{\underline{\underline{\mathcal{A}}}}$ are well behaved with respect to composition and identities so therefore constitute the maps of an (un -enriched) subcategory $\underline{\underline{\underline{\mathcal{A}}}}_{\mathrm{st}}$ of $\underline{\underline{\underline{\mathcal{A}}}}_0$. Notice that the strict homomorphisms of $\underline{\underline{\underline{\mathcal{H}om}}}$, $\underline{\underline{\underline{\mathcal{H}om}}}^{\mathrm{op}}$ and $\underline{\underline{\underline{\mathcal{H}om}}}_S$ coincide with the traditional notion, eg. $\underline{\underline{\underline{\mathcal{H}om}}}_{\mathrm{st}} = \underline{\mathcal{SH}om}$.

Lemma 1.4.6 *Given any small (un-enriched) category $\mathbb{C}$ we may naturally form a (strongly) bicategory enriched category $\underline{\underline{\underline{\mathcal{A}}}}^{\mathbb{C}}$ the objects of which are (honest) functors $\mathcal{F}\colon \mathbb{C} \longrightarrow \underline{\underline{\underline{\mathcal{A}}}}_{\mathrm{st}}$.*

Proof. This is essentially the same as the usual construction of enriched functor categories by dint of corollary 1.4.4, for comparison see [30] chapter 2. Define the (strongly) bicategory enriched category $\underline{\underline{\underline{\mathcal{A}}}}^{\mathbb{C}}$ with objects functors $\mathcal{F}\colon \mathbb{C} \longrightarrow \underline{\underline{\underline{\mathcal{A}}}}_{\mathrm{st}}$ and homsets given by the end formula:

$$\underline{\underline{\underline{\mathcal{A}}}}^{\mathbb{C}}(\mathcal{F}, \mathcal{G}) = \int_{c \in \mathbb{C}} \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{G}c)$$

This limit only exists in $\underline{\mathcal{H}om}_0$ since we have insisted on restricting our attention to those functors on $\mathbb{C}$ which map only to strict homomorphisms in $\underline{\underline{\underline{\mathcal{A}}}}$. It is formed by taking the limit of a diagram of disjoint components

$$\begin{tikzcd}
 & \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{G}c) \arrow[d, "\underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{G}\gamma)"] \\
\underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c', \mathcal{G}c') \arrow[r, "\underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}\gamma, \mathcal{G}c')"'] & \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{G}c')
\end{tikzcd}$$

one for each map $(\gamma\colon c \longrightarrow c') \in \mathbb{C}$, which we are assured is in $\underline{\mathcal{SH}om}$ by virtue of the fact that each $\mathcal{G}\gamma$ and $\mathcal{F}\gamma$ is a strict homomorphism in $\underline{\underline{\underline{\mathcal{A}}}}$. Applying corollary 1.4.4 we may define a unique 2-homomorphism

$$[\underline{\underline{\underline{\mathcal{A}}}}^{\mathbb{C}}(\mathcal{G}, \mathcal{H}), \underline{\underline{\underline{\mathcal{A}}}}^{\mathbb{C}}(\mathcal{F}, \mathcal{G})] \xrightarrow{\circ} \underline{\underline{\underline{\mathcal{A}}}}^{\mathbb{C}}(\mathcal{F}, \mathcal{H})$$

with the property that

$$\begin{tikzcd}
{[\int_{c \in \mathbb{C}} \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{G}c, \mathcal{H}c), \int_{c \in \mathbb{C}} \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{G}c)]} \arrow[r, "\circ"] \arrow[d, "{[\pi_c, \pi_c]}"'] & \int_{c \in \mathbb{C}} \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{H}c) \arrow[d, "\pi_c"] \\
{[\underline{\underline{\underline{\mathcal{A}}}}(\mathcal{G}c, \mathcal{H}c), \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{G}c)]} \arrow[r, "\circ"] & \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{H}c)
\end{tikzcd}$$

commutes (in other words $\circ\langle\pi_c, \pi_c\rangle = \pi_c\langle\circ\rangle$) for each object $c \in \mathbb{C}$. In a similar fashion we may define a 0-homomorphism

$$[\cdot] \xrightarrow{\ \mathrm{I}_{\mathcal{F}}\ } \underline{\underline{\underline{\mathcal{A}}}}^{\mathbb{C}}(\mathcal{F}, \mathcal{F})$$

by requiring that it be the unique one making

$$\begin{tikzcd} {[\cdot]} \arrow[r, "\mathrm{I}_{\mathcal{F}}"] \arrow[dr, "\mathrm{I}_{\mathcal{F}c}"'] & \int_{c\in\mathbb{C}} \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{F}c) \arrow[d, "\pi_c"] \\ & \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{F}c) \end{tikzcd}$$

commute for each object $c \in \mathbb{C}$. By the usual arguments involving uniqueness properties of universals and the associativity / identity rules of $\underline{\underline{\underline{\mathcal{A}}}}$ we establish that this data satisfies the conditions required of the compositions and identities of a (strongly) bicategory enriched category $\underline{\underline{\underline{\mathcal{A}}}}^{\mathbb{C}}$.

Explicitly the homomorphisms, transformations and modifications of $\underline{\underline{\underline{\mathcal{A}}}}^{\mathbb{C}}$ are families $\{\mathrm{F}_c \in \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{G}c)\}_{c\in\mathbb{C}}$, $\{\Phi_c \in \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{G}c)\}_{c\in\mathbb{C}}$ and $\{\alpha_c \in \underline{\underline{\underline{\mathcal{A}}}}(\mathcal{F}c, \mathcal{G}c)\}_{c\in\mathbb{C}}$ of the corresponding structures in $\underline{\underline{\underline{\mathcal{A}}}}$, satisfying the obvious naturality condition with respect to the maps of $\mathbb{C}$. The various composites making these a (strongly) bicategory enriched category are all defined pointwise from those of $\underline{\underline{\underline{\mathcal{A}}}}$. $\square$

Armed with these lemmas we may now define a fundamental (strongly) bicategory enriched category of (small) double bicategories:

Definition 1.4.7 ($\underline{\underline{\underline{\boldsymbol{\mathcal{H}oriz}}}}$ and $\underline{\underline{\underline{\boldsymbol{\mathcal{H}oriz}}}}_S$) Given a double bicategory $\underline{\underline{\tilde{\mathcal{D}}}}$ we have already constructed the bicategory of horizontal cylinders $\underline{\underline{\mathrm{Cyl}_H}}$ and strict homomorphisms

$$\underline{\underline{\mathrm{Cyl}_H}} \underset{\mathrm{cod}_V}{\overset{\mathrm{dom}_V}{\rightrightarrows}} \underline{\underline{\mathcal{H}}} \tag{1.46}$$

this of course is simply an object of the (strongly) bicategory enriched category $\underline{\underline{\underline{\mathcal{H}om}}}^{\mathbb{P}}$ ($\underline{\underline{\underline{\mathcal{H}om}}}_S^{\mathbb{P}}$), where $\mathbb{P}$ is the category given pictorially by:

$$\mathbb{P} = \boxed{\bullet \rightrightarrows \bullet}$$

Let $\mathcal{DB}icat$ denote the set of (small) double bicategories and

$$\mathcal{DB}icat \xrightarrow{\ \mathcal{H}or\ } \mathrm{ob}(\underline{\underline{\underline{\mathcal{H}om}}}^{\mathbb{P}})$$

the function taking $\underline{\underline{\tilde{\mathcal{D}}}}$ to the diagram of strict homomorphisms in 1.46.

Notice that any enriched functor $\underline{\underline{\mathrm{F}}}\colon \underline{\underline{\mathcal{A}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ between (strongly) bicategory enriched categories may be factored uniquely as $\underline{\underline{\mathrm{F}}}_m \circ \underline{\underline{\mathrm{F}}}_e$, where $\underline{\underline{\mathrm{F}}}_m$ is fully faithful and $\underline{\underline{\mathrm{F}}}_e$ is an isomorphism on objects. Of course we may consider $\mathcal{DB}icat$ to be a discrete (strongly) bicategory enriched category and then $\mathcal{H}or$ becomes an enriched functor into either $\underline{\underline{\underline{\mathcal{H}om}}}^{\mathbb{P}}$ or $\underline{\underline{\underline{\mathcal{H}om}}}^{\mathbb{P}}_S$. The enriched categories $\underline{\underline{\underline{\mathcal{H}oriz}}}$ and $\underline{\underline{\underline{\mathcal{H}oriz}}}_S$ are simply the intermediate categories obtained by applying our factorisation to those versions of $\mathcal{H}or$. In a little more detail:

- $\mathrm{ob}(\underline{\underline{\underline{\mathcal{H}oriz}}}) = \mathcal{DB}icat$,
- $\underline{\underline{\underline{\mathcal{H}oriz}}}(\underline{\underline{\tilde{\mathcal{D}}}}, \underline{\underline{\tilde{\mathcal{D}}}}') = \underline{\underline{\underline{\mathcal{H}om}}}^{\mathbb{P}}(\mathcal{H}or(\underline{\underline{\tilde{\mathcal{D}}}}), \mathcal{H}or(\underline{\underline{\tilde{\mathcal{D}}}}'))$,
- compositions and identities are those of $\underline{\underline{\underline{\mathcal{H}om}}}^{\mathbb{P}}$. $\square$

As usual we really ought to elucidate a little on the structure of the entities which comprise $\underline{\underline{\underline{\mathcal{H}oriz}}}$ ($\underline{\underline{\underline{\mathcal{H}oriz}}}_S$):

homomorphisms: called *horizontal maps*. Given a pair of double bicategories $\underline{\underline{\tilde{\mathcal{D}}}} = (\mathcal{A}, \underline{\underline{\mathcal{H}}}, \underline{\underline{\mathcal{V}}}, \mathcal{S})$ and $\underline{\underline{\tilde{\mathcal{D}}}}' = (\mathcal{A}', \underline{\underline{\mathcal{H}}}', \underline{\underline{\mathcal{V}}}', \mathcal{S}')$ a horizontal map

$$\underline{\underline{\tilde{\mathcal{D}}}} \xrightarrow{\;\tilde{\mathrm{G}} = (\mathrm{G}, \mathrm{G}_H, \mathrm{G}_V, \mathrm{G}_S)\;} \underline{\underline{\tilde{\mathcal{D}}}}'$$

consists of:

(i) A homomorphism $\mathrm{G}_H\colon \underline{\underline{\mathcal{H}}} \longrightarrow \underline{\underline{\mathcal{H}}}'$ with action $\mathrm{G}\colon \mathcal{A} \longrightarrow \mathcal{A}'$ on 0-cells.

(ii) A function G_V acting on vertical 1-cells, mapping $p\colon a \mathrel{\longrightarrow\mkern-14mu+\mkern8mu} \bar{a} \in \underline{\underline{\mathcal{V}}}$ to some $\mathrm{G}_V p\colon \mathrm{G}a \mathrel{\longrightarrow\mkern-14mu+\mkern8mu} \mathrm{G}\bar{a} \in \underline{\underline{\mathcal{V}}}'$.

(iii) A map of squares $\mathrm{G}_S\colon \mathcal{S} \longrightarrow \mathcal{S}'$ given pictorially by:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "p"', "|" marking] \arrow[dr, phantom, "\lambda\lrcorner"] & a' \arrow[d, "p'", "|" marking] \\ \bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \end{tikzcd} \longmapsto \begin{tikzcd} \mathrm{G}a \arrow[r, "\mathrm{G}_H(f)"] \arrow[d, "\mathrm{G}_V(p)"', "|" marking] \arrow[dr, phantom, "\mathrm{G}_S\lambda\lrcorner"] & \mathrm{G}a' \arrow[d, "\mathrm{G}_V(p')", "|" marking] \\ \mathrm{G}\bar{a} \arrow[r, "\mathrm{G}_H(\bar{f})"'] & \mathrm{G}\bar{a}' \end{tikzcd} \tag{1.47}$$

This data must satisfy the following three "cylinder" conditions.

(iv) For any horizontal cylinder $(\alpha, \bar{\alpha})\colon \lambda \Rightarrow_H \dot{\lambda}$ the pair of horizontal 1-cells $(\mathrm{G}_H(\alpha), \mathrm{G}_H(\bar{\alpha}))$ form a cylinder $\mathrm{G}_S(\lambda) \Rightarrow_H \mathrm{G}_S(\dot{\lambda})$. This condition is clearly equivalent to saying that the action of horizontal 2-cells on squares is preserved, in the sense that:

$$\mathrm{G}_S(\dot{\lambda} \underset{V}{*} \alpha) = \mathrm{G}_S(\dot{\lambda}) \underset{V}{*} \mathrm{G}_H(\alpha) \qquad \mathrm{G}_S(\bar{\alpha} \underset{V}{*} \lambda) = \mathrm{G}_H(\bar{\alpha}) \underset{V}{*} \mathrm{G}_S(\lambda)$$

(v) Given a second square

$$\begin{tikzcd} a' \arrow[r, "f'"] \arrow[d, "p'"', "|" marking] \arrow[dr, phantom, "\underline{\lambda'}\rfloor"] & a'' \arrow[d, "p''", "|" marking] \\ \bar{a}' \arrow[r, "\bar{f}'"'] & \bar{a}'' \end{tikzcd}$$

the (isomorphic) canonical horizontal 2-cells

$$\mathrm{G}_H(f') \underset{H}{\otimes} \mathrm{G}_H(f) \Rightarrow_H \mathrm{G}_H(f' \underset{H}{\otimes} f) \qquad \mathrm{G}_H(\bar{f}') \underset{H}{\otimes} \mathrm{G}_H(\bar{f}) \Rightarrow_H \mathrm{G}_H(\bar{f}' \underset{H}{\otimes} \bar{f})$$

are the components of a horizontal cylinder:

$$\mathrm{G}_S(\lambda') \underset{H}{\otimes} \mathrm{G}_S(\lambda) \Rightarrow_H \mathrm{G}_S(\lambda' \underset{H}{\otimes} \lambda)$$

(vi) For each vertical 1-cell $p\colon a \longrightarrow\!\!\!\!\!\!\mid\;\; \bar{a} \in \underline{\underline{\mathcal{V}}}$ the (isomorphic) canonical horizontal 2-cells $\mathrm{i}^h_{\mathrm{G}a} \Rightarrow_H \mathrm{G}_H(\mathrm{i}^h_a)$, $\mathrm{i}^h_{\mathrm{G}\bar{a}} \Rightarrow_H \mathrm{G}_H(\mathrm{i}^h_{\bar{a}})$ are the components of a horizontal cylinder $\mathrm{i}^h_{\mathrm{G}_V p} \Rightarrow_H \mathrm{G}_S(\mathrm{i}^h_p)$.

transformations: $\tilde{\Psi} = (\Psi, \overline{\Psi})\colon \tilde{\mathrm{F}} \longrightarrow \tilde{\mathrm{G}}$ called *horizontal transformations* consist of:

(i) A traditional (strong) transformation $\Psi\colon \mathrm{F}_H \longrightarrow \mathrm{G}_H$.

(ii) For each vertical 1-cell $p\colon a \longrightarrow \bar{a} \in \underline{\underline{\mathcal{V}}}$ a square:

$$\begin{tikzcd} \mathrm{F}a \arrow[r, "\Psi_a"] \arrow[d, "\mathrm{F}_V p"', "|" marking] \arrow[dr, phantom, "\underline{\overline{\Psi}_p}\rfloor"] & \mathrm{G}a \arrow[d, "\mathrm{G}_V p", "|" marking] \\ \mathrm{F}\bar{a} \arrow[r, "\Psi_{\bar{a}}"'] & \mathrm{G}\bar{a} \end{tikzcd}$$

This data must satisfy the cylinder condition:

(iii) Given any square $\lambda \in \mathcal{S}$ as in (1.47) the horizontal structure 2-cells

$$\mathrm{G}_H(f) \underset{H}{\otimes} \Psi_a \xRightarrow{\;\Psi_f\;} \Psi_{a'} \underset{H}{\otimes} \mathrm{F}_H(f)$$

$$\mathrm{G}_H(\bar{f}) \underset{H}{\otimes} \Psi_{\bar{a}} \xRightarrow{\;\Psi_{\bar{f}}\;} \Psi_{\bar{a}'} \underset{H}{\otimes} \mathrm{F}_H(\bar{f})$$

of the transformation Ψ constitute the components of a cylinder:

$$\mathrm{G}_H(\lambda) \underset{H}{\otimes} \overline{\Psi}_p \xRightarrow{\;\overline{\Psi}_\lambda\;} \overline{\Psi}_{p'} \underset{H}{\otimes} \mathrm{F}_H(\lambda)$$

modifications: $\alpha\colon(\Psi,\overline{\Psi})\Rightarrow(\Phi,\overline{\Phi})$ again called *horizontal modifications* comprise:

(i) A (traditional) modification $\alpha\colon\Psi\Rightarrow\Phi$, satisfying the supplementary cylinder condition:

(ii) For each vertical 1-cell $p\colon a \mathrel{\mapstochar\!\!\longrightarrow} \bar{a}$ the horizontal 2-cells $\alpha_a\colon\Psi_a\Rightarrow_H\Phi_a$ and $\alpha_{\bar{a}}\colon\Psi_{\bar{a}}\Rightarrow_H\Phi_{\bar{a}}$ are the components of a cylinder $\alpha_p\colon\Psi_p\Rightarrow_H\Phi_p$.

The following proposition elucidates the structure of biadjoints in $\underline{\underline{\underline{\mathcal{H}oriz}}}_S$ by giving a "one sided" universal property:

Proposition 1.4.8 *A horizontal map of double bicategories* $\tilde{\mathrm{G}}\colon\tilde{\underline{\underline{\mathcal{D}}}}\longrightarrow\tilde{\underline{\underline{\mathcal{D}}}}'$ *has a left biadjoint in* $\underline{\underline{\underline{\mathcal{H}oriz}}}_S$ *iff*

(i) The homomorphism $\mathrm{G}_H\colon\underline{\underline{\mathcal{H}}}\longrightarrow\underline{\underline{\mathcal{H}}}'$ *has a left biadjoint* $\mathrm{F}_H\colon\underline{\underline{\mathcal{H}}}'\longrightarrow\underline{\underline{\mathcal{H}}}$ *(in the traditional sense) with a unit we will call* $\Psi\colon\mathrm{I}_{\underline{\underline{\mathcal{H}}}'}\longrightarrow\mathrm{G}_H\circ\mathrm{F}_H$.

(ii) For all vertical 1-cells $q\colon b\mathrel{\mapstochar\!\!\longrightarrow}\bar{b}$ *in* $\tilde{\underline{\underline{\mathcal{D}}}}'$ *there exists a vertical 1-cell*

$$\mathrm{F}_V(q)\colon\mathrm{F}b\mathrel{\mapstochar\!\!\longrightarrow}\mathrm{F}\bar{b}$$

in $\tilde{\underline{\underline{\mathcal{D}}}}$ *and a square*

$$\begin{tikzcd}
b \arrow[r, "\Psi_b"] \arrow[d, "q"', "|" marking] \arrow[dr, phantom, "\overline{\Psi}_q"] & \mathrm{GF}b \arrow[d, "\mathrm{G}_V\mathrm{F}_V(q)", "|" marking] \\
\bar{b} \arrow[r, "\Psi_{\bar{b}}"'] & \mathrm{GF}\bar{b}
\end{tikzcd}$$

with the property that for each fixed collection of 1-cells

$$\begin{array}{llll}
q\colon b\mathrel{\mapstochar\!\!\longrightarrow}\bar{b} \in \underline{\underline{\mathcal{V}}}' & & p\colon a\mathrel{\mapstochar\!\!\longrightarrow}\bar{a} \in \underline{\underline{\mathcal{V}}} \\
f\colon\mathrm{F}b\longrightarrow a \in \underline{\underline{\mathcal{H}}} & & \bar{f}\colon\mathrm{F}\bar{b}\longrightarrow\bar{a} \in \underline{\underline{\mathcal{H}}}
\end{array}$$

there is a bijection between squares

$$\begin{tikzcd}
b \arrow[r, "\mathrm{G}_H(f)\underset{H}{\otimes}\Psi_b"] \arrow[d, "q"', "|" marking] \arrow[dr, phantom, "\lambda"] & \mathrm{G}a \arrow[d, "\mathrm{G}_V(p)", "|" marking] & \arrow[r, leftrightarrow] & {} & \mathrm{F}b \arrow[r, "f"] \arrow[d, "\mathrm{F}_V(q)"', "|" marking] \arrow[dr, phantom, "\tilde{\lambda}"] & a \arrow[d, "p", "|" marking] \\
\bar{b} \arrow[r, "\mathrm{G}_H(\bar{f})\underset{H}{\otimes}\Psi_b"'] & \mathrm{G}\bar{a} & & & \mathrm{F}\bar{b} \arrow[r, "\bar{f}"'] & \bar{a}
\end{tikzcd} \tag{1.48}$$

satisfying the equation $\lambda=\mathrm{G}_S(\tilde{\lambda})\underset{H}{\otimes}\overline{\Psi}_q$.

Proof.
"⇒" Suppose we are given a horizontal map $\tilde{\mathrm{F}}\colon \underline{\underline{\tilde{\mathcal{D}}}}' \longrightarrow \underline{\underline{\tilde{\mathcal{D}}}}$ which is left biadjoint to $\tilde{\mathrm{G}}$ in $\underline{\underline{\mathcal{H}oriz}}_S$. Examining the construction of this enriched category, which we gave in terms of $\underline{\underline{\mathcal{H}om}}_S^{\mathbb{P}}$, and recalling the properties of biadjunctions discussed in section 1.3 we see that we have biadjoints

$$(\Psi,\Phi,\alpha,\beta):\ \underline{\underline{\mathcal{H}}} \underset{\mathrm{G}_H}{\overset{\mathrm{F}_H}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp_b}}}} \underline{\underline{\mathcal{H}}}'$$

$$(\overline{\Psi},\overline{\Phi},\bar{\alpha},\bar{\beta}):\underline{\underline{\mathrm{Cyl}}}_H \underset{\mathrm{G}_S}{\overset{\mathrm{F}_S}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp_b}}}} \underline{\underline{\mathrm{Cyl}}}'_H$$

in $\underline{\underline{\mathcal{H}om}}_S$, related by conditions which are all of the same form as the following few:

$$\begin{array}{ccc} \mathrm{dom}_V \circ \mathrm{F}_S = \mathrm{F}_H \circ \mathrm{dom}_V & \mathrm{dom}_V \circ \overline{\Psi} = \Psi \circ \mathrm{dom}_V & \mathrm{dom}_V \circ \bar{\alpha} = \alpha \circ \mathrm{dom}_V \\ \mathrm{cod}_V \circ \mathrm{F}_S = \mathrm{F}_H \circ \mathrm{cod}_V & \mathrm{cod}_V \circ \overline{\Phi} = \Phi \circ \mathrm{cod}_V & \mathrm{cod}_V \circ \bar{\beta} = \beta \circ \mathrm{cod}_V \end{array}$$

This certainly provides us with a left biadjoint to G_H, but what about condition (ii) of the lemma? As usual adopt the notation F for the action of F_H on 0-cells and F_V for that of F_S on vertical 1-cells, then the conditions relating the two biadjunctions ensure that given a vertical 1-cell $q \in \underline{\underline{\mathcal{V}}}'$ we may take $\mathrm{F}_V(q)$ and the component of $\overline{\Psi}$ at q as suitable candidates for the 1-cell and square stipulated in our condition. To check the universal property of these first consider a square λ as on the left hand side of (1.48), since G_S has a left biadjoint with unit $\overline{\Psi}$ we certainly know that there exists a square

$$\begin{tikzcd} \mathrm{F}b \arrow[r, "\hat{g}"] \arrow[d, "\mathrm{F}_V(q)"'] \arrow[dr, phantom, "\hat{\lambda}"] & a \arrow[d, "p"] \\ \mathrm{F}\bar{b} \arrow[r, "\ddot{g}"'] & \bar{a} \end{tikzcd} \tag{1.49}$$

and an (isomorphic) horizontal cylinder $(\delta,\bar{\delta})\colon \mathrm{G}_S(\hat{\lambda}) \underset{H}{\otimes} \overline{\Psi}_q \overset{\cong}{\Longrightarrow}_H \lambda$. Consider the components of this cylinder

$$\mathrm{G}_H(\hat{g}) \underset{H}{\otimes} \Psi_b \overset{\delta}{\underset{\cong}{\Longrightarrow}}_H \mathrm{G}_H(f) \underset{H}{\otimes} \Psi_b$$
$$\mathrm{G}_H(\ddot{g}) \underset{H}{\otimes} \Psi_{\bar{b}} \overset{\bar{\delta}}{\underset{\cong}{\Longrightarrow}}_H \mathrm{G}_H(\bar{f}) \underset{H}{\otimes} \Psi_{\bar{b}}$$

we know that the functor $\mathrm{G}_H(-) \underset{H}{\otimes} \Psi_b\colon \underline{\underline{\mathcal{H}}}(\mathrm{F}b,a) \longrightarrow \underline{\underline{\mathcal{H}}}'(b,\mathrm{G}a)$ is fully faithful for all 0-cells $a \in \underline{\underline{\mathcal{H}}}$ and $b \in \underline{\underline{\mathcal{H}}}'$ because $\mathrm{F}_H \dashv_b \mathrm{G}_H$ with unit Ψ, therefore there exist (unique) horizontal 2-cells $\hat{\delta}\colon \hat{g} \overset{\sim}{\Longrightarrow}_H f$ with $\delta = \mathrm{G}_H(\hat{\delta}) \underset{H}{\otimes} \Psi_b$ and $\ddot{\delta}\colon \ddot{g} \overset{\sim}{\Longrightarrow}_H \bar{f}$ with $\bar{\delta} = \mathrm{G}_H(\ddot{\delta}) \underset{H}{\otimes} \Psi_{\bar{b}}$. Now set $\tilde{\lambda} = \ddot{\delta} \underset{V}{*} \hat{\lambda} \underset{V}{*} \hat{\delta}^{-1}$ and we get the calculation

$$\begin{aligned}
\mathrm{G}_S(\tilde{\lambda}) \underset{H}{\otimes} \overline{\Psi}_q &= (\mathrm{G}_H(\ddot{\delta}) \underset{V}{*} \mathrm{G}_S(\hat{\lambda}) \underset{V}{*} \mathrm{G}_H(\hat{\delta}^{-1})) \underset{H}{\otimes} \overline{\Psi}_q \\
&= (\mathrm{G}_H(\ddot{\delta}) \underset{H}{\otimes} \Psi_{\bar{b}}) \underset{V}{*} (\mathrm{G}_S(\hat{\lambda}) \underset{H}{\otimes} \overline{\Psi}_q) \underset{V}{*} (\mathrm{G}_H(\hat{\delta}^{-1}) \underset{H}{\otimes} \Psi_b) \\
&= \bar{\delta} \underset{V}{*} (\mathrm{G}_S(\hat{\lambda}) \underset{H}{\otimes} \overline{\Psi}_q) \underset{V}{*} \delta^{-1} \qquad = \quad \lambda
\end{aligned}$$

where the last equality holds since $(\delta, \bar{\delta})$ is a horizontal cylinder.

Suppose now that we were given two squares $\tilde{\lambda}$ and $\tilde{\lambda}'$, as on the right hand side of (1.48), with $\mathrm{G}_S(\tilde{\lambda}) \otimes \overline{\Psi}_q = \mathrm{G}_S(\tilde{\lambda}') \otimes \overline{\Psi}_q (= \lambda)$ then there exists a horizontal cylinder $(\kappa, \bar{\kappa})\colon \tilde{\lambda} \overset{\sim}{\Longrightarrow}_H \tilde{\lambda}'$ with $\mathrm{G}_S(\kappa, \bar{\kappa}) \underset{H}{\otimes} \overline{\Psi}_q = \mathrm{i}^h_\lambda$, simply because $\mathrm{F}_S \dashv_b \mathrm{G}_S$ with unit $\overline{\Psi}$. Look at the components of this cylinder, κ is a horizontal endo-2-cell on f such that $\mathrm{G}_H(\kappa) \underset{H}{\otimes} \Psi_b = \mathrm{i}^h_\bullet$ therefore, by the faithfulness of $\mathrm{G}_H(\text{–}) \underset{H}{\otimes} \Psi_b$, we have $\kappa = \mathrm{i}^h_f$. Similarly $\bar{\kappa} = \mathrm{i}^h_{\bar{f}}$ and so $\tilde{\lambda}' = \tilde{\lambda}' \underset{V}{*} \kappa = \bar{\kappa} \underset{V}{*} \tilde{\lambda} = \tilde{\lambda}$, where the middle equality is exactly a statement of the fact that $(\kappa, \bar{\kappa})$ is a cylinder. Thus we have established the bijection of (1.48).

“$\Leftarrow$” First notice the following two simple consequences of condition (ii) of the lemma:

(a) For any square

$$\begin{tikzcd}
b \arrow[r, "g"] \arrow[d, "p"', "|" marking] \arrow[dr, phantom, "\lambda\lrcorner"] & \mathrm{G}a \arrow[d, "\mathrm{G}_V(q)", "|" marking] \\
\bar{b} \arrow[r, "\bar{g}"'] & \mathrm{G}\bar{a}
\end{tikzcd}$$

and horizontal 1-cells $\hat{g}$ and $\ddot{g}$ as in (1.49) with given (isomorphic) horizontal 2-cells $\delta\colon \mathrm{G}(\hat{g}) \underset{H}{\otimes} \Psi_b \overset{\sim}{\Longrightarrow}_H g$ and $\bar{\delta}\colon \mathrm{G}(\ddot{g}) \underset{H}{\otimes} \Psi_{\bar{b}} \overset{\sim}{\Longrightarrow}_H \bar{g}$ there exists a unique square $\hat{\lambda}$, as in (1.49), such that $(\delta, \bar{\delta})$ is a cylinder $\mathrm{G}(\hat{\lambda}) \underset{H}{\otimes} \overline{\Psi}_q \overset{\sim}{\Longrightarrow}_H \lambda$. This is simply the square corresponding to $(\bar{\delta})^{-1} \underset{V}{*} \lambda \underset{V}{*} \delta$ under the bijection of (1.48).

(b) For any pair of squares $\tilde{\lambda}$ as in (1.48) and

$$\begin{tikzcd}
\mathrm{F}b \arrow[r, "h"] \arrow[d, "\mathrm{G}_V(q)"', "|" marking] \arrow[dr, phantom, "\tilde{\mu}\lrcorner"] & a \arrow[d, "p", "|" marking] \\
\mathrm{F}\bar{b} \arrow[r, "\bar{h}"'] & \bar{a}
\end{tikzcd}$$

a pair of horizontal 2-cells $\alpha\colon f \Rightarrow_H h$ and $\bar{\alpha}\colon \bar{f} \Rightarrow_H \bar{h}$ constitute a cylinder $\tilde{\lambda} \Rightarrow_H \tilde{\mu}$ iff $(\mathrm{G}_H(\alpha) \underset{H}{\otimes} \Psi_b\,,\, \mathrm{G}_H(\bar{\alpha}) \underset{H}{\otimes} \Psi_{\bar{b}})$ is a cylinder $\mathrm{G}_H(\tilde{\lambda}) \underset{H}{\otimes} \overline{\Psi}_q \Rightarrow_H \mathrm{G}_H(\tilde{\mu}) \underset{H}{\otimes} \overline{\Psi}_q$. The proof of the "if" part is clear, so to get the reverse implication consider the calculation

$$\begin{aligned}
\mathrm{G}_S(\bar{\alpha} \underset{V}{*} \tilde{\lambda}) \underset{H}{\otimes} \overline{\Psi}_q &= (\mathrm{G}_H(\bar{\alpha}) \underset{H}{\otimes} \Psi_b) \underset{V}{*} (\mathrm{G}_S(\tilde{\lambda}) \underset{H}{\otimes} \overline{\Psi}_q) \\
&= (\mathrm{G}_S(\tilde{\mu}) \underset{H}{\otimes} \overline{\Psi}_q) \underset{V}{*} (\mathrm{G}_H(\alpha) \underset{H}{\otimes} \Psi_{\bar{b}}) \quad = \quad \mathrm{G}_S(\tilde{\mu} \underset{V}{*} \alpha) \underset{H}{\otimes} \overline{\Psi}_q
\end{aligned}$$

in which the middle equality simply states that $(\mathrm{G}_H(\alpha) \underset{H}{\otimes} \Psi_b\,,\, \mathrm{G}_H(\bar{\alpha}) \underset{H}{\otimes} \Psi_{\bar{b}})$ is a cylinder, the other two follow from the preservation of $\underset{V}{*}$ by $\tilde{\mathrm{G}}$ and rule (vi) in the definition of a double bicategory. Applying the bijection of (1.48) to this equality we get $\bar{\alpha} \underset{V}{*} \tilde{\lambda} = \tilde{\mu} \underset{V}{*} \alpha$ which simply states that $(\alpha, \bar{\alpha})$ is a cylinder as required.

Using these observations we turn to constructing the postulated left biadjoint to $\tilde{\mathrm{G}}$ in $\underline{\underline{\mathcal{H}oriz}}_S$, first we provide a candidate horizontal map $\tilde{\mathrm{F}}$ and unit $\tilde{\Psi}\colon \mathrm{I}_{\underline{\underline{\tilde{\mathcal{D}}}}'} \longrightarrow \tilde{\mathrm{G}}\tilde{\mathrm{F}}$. Condition (i) states that we already have a homomorphism F_H left biadjoint to G_H in $\underline{\underline{\mathcal{H}om}}_S$ (with unit Ψ), which we set about enriching to a horizontal map of double bicategories by supplementing it with the action on vertical 1-cells F_V as given in condition (ii). We define a map of squares

$$\begin{tikzcd}
b \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\underline{\mu}\rfloor"] & b' \arrow[d, "q'"] & & \mathrm{F}b \arrow[r, "\mathrm{F}_H(g)"] \arrow[d, "\mathrm{F}_V(q)"'] \arrow[dr, phantom, "\mathrm{F}_S(\underline{\mu})\rfloor"] & \mathrm{F}b' \arrow[d, "\mathrm{F}_V(q')"] \\
\bar{b} \arrow[r, "\bar{g}"'] & \bar{b}' \in \underline{\underline{\tilde{\mathcal{D}}}}' \arrow[r, phantom, "\longmapsto"] & & \mathrm{F}(\bar{b}) \arrow[r, "\mathrm{F}_H(\bar{g})"'] & \mathrm{F}(\bar{b}') \in \underline{\underline{\tilde{\mathcal{D}}}}
\end{tikzcd}$$

using observation (a) above by insisting that $\mathrm{F}_S(\mu)$ is the unique square for which the pair $(\Psi_g, \Psi_{\bar{g}})$, comprising 2-cellular components of the (traditional) strong transformation Ψ, is a cylinder $\mathrm{G}_S\mathrm{F}_S(\mu) \underset{H}{\otimes} \overline{\Psi}_q \overset{\sim}{\Longrightarrow}_H \overline{\Psi}_{q'} \underset{H}{\otimes} \mu$.

In order to demonstrate that $(\mathrm{F}, \mathrm{F}_H, \mathrm{F}_V, \mathrm{F}_S)$ defined in this way is a horizontal map we need to demonstrate that it satisfies the three "cylinder conditions" of the explicit description we gave above. For instance the first of these states that it should preserve horizontal cylinders, so let $(\alpha, \bar{\alpha})\colon \mu_1 \Rightarrow_H \mu_2$ be a cylinder in $\underline{\underline{\mathrm{Cyl}}}_H$

and notice that we have commutative diagrams

$$
\begin{tikzcd}
\mathrm{G}_H\mathrm{F}_H(g_1) \underset{H}{\otimes} \Psi_b \arrow[r, "\Psi_{g_1}", "\simeq"'] \arrow[d, "\mathrm{G}_H\mathrm{F}_H(\alpha) \underset{H}{\otimes} \Psi_b"] & \Psi_{b'} \underset{H}{\otimes} g_1 \arrow[d, "\Psi_{b'} \underset{H}{\otimes} \alpha"] \\
\mathrm{G}_H\mathrm{F}_H(g_2) \underset{H}{\otimes} \Psi_b \arrow[r, "\simeq", "\Psi_{g_2}"'] & \Psi_{b'} \underset{H}{\otimes} g_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{G}_H\mathrm{F}_H(\bar{g}_1) \underset{H}{\otimes} \Psi_{\bar{b}} \arrow[r, "\Psi_{\bar{g}_1}", "\simeq"'] \arrow[d, "\mathrm{G}_H\mathrm{F}_H(\bar{\alpha}) \underset{H}{\otimes} \Psi_{\bar{b}}"] & \Psi_{\bar{b}'} \underset{H}{\otimes} \bar{g}_1 \arrow[d, "\Psi_{\bar{b}'} \underset{H}{\otimes} \bar{\alpha}"] \\
\mathrm{G}_H\mathrm{F}_H(\bar{g}_2) \underset{H}{\otimes} \Psi_{\bar{b}} \arrow[r, "\simeq", "\Psi_{\bar{g}_2}"'] & \Psi_{\bar{b}'} \underset{H}{\otimes} \bar{g}_2
\end{tikzcd}
$$

since Ψ is a transformation. Of course we defined $\mathrm{F}_S(\mu_1)$ and $\mathrm{F}_S(\mu_2)$ precisely to ensure that we get the first two cylinders in the following display

$$
\begin{gathered}
(\Psi_{g_1}, \Psi_{\bar{g}_1})\colon \mathrm{G}_H\mathrm{F}_H(\mu_1) \underset{H}{\otimes} \overline{\Psi}_q \overset{\simeq}{\Longrightarrow}_H \Psi_{q'} \underset{H}{\otimes} \mu_1 \\
(\Psi_{g_2}, \Psi_{\bar{g}_2})\colon \mathrm{G}_H\mathrm{F}_H(\mu_2) \underset{H}{\otimes} \overline{\Psi}_q \overset{\simeq}{\Longrightarrow}_H \Psi_{q'} \underset{H}{\otimes} \mu_2 \\
(\Psi_{b'} \underset{H}{\otimes} \alpha, \Psi_{\bar{b}'} \underset{H}{\otimes} \bar{\alpha})\colon \Psi_{q'} \underset{H}{\otimes} \mu_1 \Rightarrow_H \Psi_{q'} \underset{H}{\otimes} \mu_2
\end{gathered}
$$

the last of which is also a cylinder by the assumption that $(\alpha, \bar{\alpha})$ is. It follows that $(\mathrm{G}_S\mathrm{F}_S(\alpha) \underset{H}{\otimes} \Psi_b, \mathrm{G}_S\mathrm{F}_S(\bar{\alpha}) \underset{H}{\otimes} \Psi_{\bar{b}})$, as the component-wise composite of three cylinders, is itself one to which we may apply observation (b) completing the proof that application of $\tilde{\mathrm{F}}$ gives a cylinder $(\mathrm{F}_H(\alpha), \mathrm{F}_H(\bar{\alpha}))\colon \mathrm{F}_S(\mu_1) \Rightarrow_H \mathrm{F}_S(\mu_2)$.

Similar proofs establish the other two cylinder conditions, which are imposed on the canonical isomorphisms of the homomorphism F_H, this time we start with commutative diagrams of the form

$$
\begin{tikzcd}
\mathrm{G}_H(\mathrm{F}_H(g') \underset{H}{\otimes} \mathrm{F}_H(g)) \underset{H}{\otimes} \Psi_b \arrow[rr, "\mathrm{G}_H(\mathrm{can}) \otimes_H \Psi_b", "\simeq"'] & & \mathrm{G}_H(\mathrm{F}_H(g' \underset{H}{\otimes} g)) \underset{H}{\otimes} \Psi_b \arrow[d, "\wr"', "\Psi_{(g' \underset{H}{\otimes} g)}"] \\
\mathrm{G}_H\mathrm{F}_H(g') \underset{H}{\otimes} \mathrm{G}_H\mathrm{F}_H(g) \underset{H}{\otimes} \Psi_b \arrow[u, "\mathrm{can} \underset{H}{\otimes} \Psi_b", "\wr"'] \arrow[r, "\simeq", "\mathrm{G}_H\mathrm{F}_H(g') \underset{H}{\otimes} \Psi_g"'] & \mathrm{G}_H\mathrm{F}_H(g') \underset{H}{\otimes} \Psi_{b'} \underset{H}{\otimes} g \arrow[r, "\simeq", "\Psi_{g'} \underset{H}{\otimes} g"'] & \Psi_{b''} \underset{H}{\otimes} g' \underset{H}{\otimes} g
\end{tikzcd}
$$

for one of the conditions and

$$
\begin{tikzcd}
\mathrm{G}_H(\mathrm{i}^h_{\mathrm{F}b}) \underset{H}{\otimes} \Psi_b \arrow[r, "\mathrm{G}_H(\mathrm{can}) \underset{H}{\otimes} \Psi_b", "\simeq"'] & \mathrm{G}_H\mathrm{F}_H(\mathrm{i}^h_b) \underset{H}{\otimes} \Psi_b \arrow[d, "\wr"', "\Psi_{\mathrm{i}^h_b}"] \\
\mathrm{i}^h_{\mathrm{GF}b} \underset{H}{\otimes} \Psi_b \arrow[u, "\mathrm{can} \otimes \Psi_b", "\wr"'] \arrow[r, "\simeq", "\mathrm{can}"'] & \Psi_b \underset{H}{\otimes} \mathrm{i}^h_b
\end{tikzcd}
$$

for the other, which are again well known coherence properties of a (strong) transformation. This completes the proof that $\tilde{\mathrm{F}}$ is a horizontal map. Of course we selected the defining property of its action F_S on squares exactly to ensure that the squares $\overline{\Psi}_q$ satisfy the cylinder condition allowing them to enrich the unit $\Psi\colon \mathrm{I}_{\underline{\underline{\mathcal{H}}}'} \longrightarrow \mathrm{G}_H\mathrm{G}_F$ to a horizontal transformation $\tilde{\Psi}\colon \mathrm{I}_{\underline{\underline{\tilde{\mathcal{D}}}}'} \longrightarrow \tilde{\mathrm{G}}\tilde{\mathrm{F}}$.

Consulting lemma 1.3.9, and in particular the proof of its clause (iii) (c)⇒(a), it is clear we may check that $\tilde{\Psi}$ is the unit of a biadjoint $\tilde{\mathrm{F}} \dashv_b \tilde{\mathrm{G}}$ in $\underline{\underline{\underline{\mathcal{H}oriz}}}_S$ by establishing two "generalised element" properties. The first of these states that for an arbitrary pair of horizontal maps $\tilde{\mathrm{H}} \in \underline{\underline{\underline{\mathcal{H}oriz}}}_S(\underline{\underline{\tilde{\mathcal{C}}}}, \underline{\underline{\tilde{\mathcal{D}}}}')$ and $\tilde{\mathrm{K}} \in \underline{\underline{\underline{\mathcal{H}oriz}}}_S(\underline{\underline{\tilde{\mathcal{C}}}}, \underline{\underline{\tilde{\mathcal{D}}}})$ and each horizontal transformation $\tilde{\Delta}\colon \tilde{\mathrm{H}} \longrightarrow \tilde{\mathrm{G}}\tilde{\mathrm{F}}$ there exists a horizontal transformation $\tilde{\Gamma}\colon \tilde{\mathrm{F}}\tilde{\mathrm{H}} \longrightarrow \tilde{\mathrm{K}}$ accompanied by an isomorphic horizontal modification:

$$\begin{tikzcd} \tilde{\mathrm{H}} \arrow[r, "\tilde{\Psi}\tilde{\mathrm{H}}"] \arrow[dr, "\tilde{\Delta}"'] \arrow[r, phantom, shift right=5ex, "\overset{\delta}{\cong}"] & \tilde{\mathrm{G}}\tilde{\mathrm{F}}\tilde{\mathrm{H}} \arrow[d, "\tilde{\mathrm{G}}\tilde{\Gamma}"] \\ & \tilde{\mathrm{G}}\tilde{\mathrm{K}} \end{tikzcd} \tag{1.50}$$

We are given that $\mathrm{F}_H \dashv_b \mathrm{G}_H$ with unit Ψ in $\underline{\underline{\underline{\mathcal{H}om}}}_S$, therefore by lemma 1.3.9 we get a transformation $\Gamma\colon \mathrm{F}_H\mathrm{H}_H \longrightarrow \mathrm{K}_H$ and an isomorphic modification:

$$\begin{tikzcd} \mathrm{H}_H \arrow[r, "\Psi\mathrm{H}_H"] \arrow[dr, "\Delta"'] \arrow[r, phantom, shift right=5ex, "\overset{\delta}{\cong}"] & \mathrm{G}_H\mathrm{F}_H\mathrm{H}_H \arrow[d, "\mathrm{G}_H\Gamma"] \\ & \mathrm{G}_H\mathrm{K}_H \end{tikzcd}$$

We enrich Γ to a horizontal transformation by the addition of squares

$$\begin{tikzcd} \mathrm{FH}c \arrow[r, "\Gamma_c"] \arrow[d, "\mathrm{F}_V\mathrm{H}_V(r)"', "\shortmid" marking] \arrow[dr, phantom, "\overline{\Gamma}_r \lrcorner"] & \mathrm{K}c \arrow[d, "\mathrm{K}_V(r)", "\shortmid" marking] \\ \mathrm{FH}\bar{c} \arrow[r, "\Gamma_{\bar{c}}"'] & \mathrm{K}\bar{c} \end{tikzcd}$$

one for each vertical 1-cell $r \in \underline{\underline{\tilde{\mathcal{C}}}}$, defined (using observation (a) above) by the property that it is the unique such square making the pair $(\delta_c, \delta_{\bar{c}})$ into a cylinder $\mathrm{G}_S(\overline{\Gamma}_r) \underset{H}{\otimes} \overline{\Psi}_{\mathrm{H}_V(r)} \overset{\simeq}{\Longrightarrow}_H \overline{\Delta}_r$. Of course we need to check that these satisfy a cylinder condition, so suppose that

$$\begin{tikzcd} c \arrow[r, "h"] \arrow[d, "r"', "|" marking] \arrow[dr, phantom, "\nu\rfloor"] & c' \arrow[d, "r'", "|"' marking] \\ \bar{c} \arrow[r, "\bar{h}"'] & \bar{c}' \end{tikzcd}$$

is a square in $\underline{\underline{\tilde{\mathcal{C}}}}$ then we show that $(\Gamma_h, \Gamma_{\bar{h}})\colon \mathrm{K}_S(\nu) \underset{H}{\otimes} \overline{\Gamma}_r \Rightarrow_H \overline{\Gamma}_{r'} \underset{H}{\otimes} \mathrm{F}_S\mathrm{H}_S(\nu)$ is a cylinder with reference to the commutative diagram

$$\begin{tikzcd} \mathrm{G}_H(\mathrm{K}_H(h) \underset{H}{\otimes} \Gamma_c) \underset{H}{\otimes} \Psi_{\mathrm{H}c} \arrow[rr, "\mathrm{G}_H(\Gamma_h) \underset{H}{\otimes} \Psi_{\mathrm{H}c}", "\simeq"'] \arrow[d, "\mathrm{can} \underset{H}{\otimes} \Psi_{\mathrm{H}c}"', "\wr"] & & \mathrm{G}_H(\Gamma_{c'} \underset{H}{\otimes} \mathrm{F}_H\mathrm{H}_H(h)) \underset{H}{\otimes} \Psi_{\mathrm{H}c} \arrow[d, "\mathrm{can} \underset{H}{\otimes} \Psi_{\mathrm{H}c}", "\wr"'] \\ \mathrm{G}_H\mathrm{K}_H(h) \underset{H}{\otimes} \mathrm{G}_H(\Gamma_c) \underset{H}{\otimes} \Psi_{\mathrm{H}c} \arrow[d, "\mathrm{G}_H\mathrm{K}_H(h) \underset{H}{\otimes} \delta_c"', "\wr"] & & \mathrm{G}_H(\Gamma_{c'}) \underset{H}{\otimes} \mathrm{G}_H\mathrm{F}_H\mathrm{H}_H(h) \underset{H}{\otimes} \Psi_{\mathrm{H}c} \arrow[d, "\mathrm{G}_H(\Gamma_{c'}) \underset{H}{\otimes} \Psi_{\mathrm{H}(h)}", "\wr"'] \\ \mathrm{G}_H\mathrm{K}_H(h) \underset{H}{\otimes} \Delta_c \arrow[r, "\simeq", "\Delta_h"'] & \Delta_{c'} \underset{H}{\otimes} \mathrm{H}_H(h) & \mathrm{G}_H(\Gamma_{c'}) \underset{H}{\otimes} \Psi_{\mathrm{H}c'} \underset{H}{\otimes} \mathrm{H}_H(h) \arrow[l, "\simeq"', "\delta_{c'} \underset{H}{\otimes} \mathrm{H}_H(h)"] \end{tikzcd}$$

alongside a similar one for $\bar{h}$, which are of course instances of the condition that δ satisfies as a modification. Pairs of corresponding maps in these diagrams, except for $(\mathrm{G}_H(\Gamma_h) \underset{H}{\otimes} \Psi_{\mathrm{H}c}, \mathrm{G}_H(\Gamma_{\bar{h}}) \otimes \Psi_{\mathrm{H}\bar{c}})$, are known to be cylinders between suitably defined squares, either because $\tilde{\mathrm{G}}$ is a horizontal map, $\tilde{\Psi}$ and $\tilde{\Delta}$ are horizontal transformations or by the defining property of each $\overline{\Gamma}_r$. Therefore, as a pointwise composite of cylinders, the remaining pair is a cylinder with domain and codomain $\mathrm{G}_S(\mathrm{K}_S(\nu) \underset{H}{\otimes} \overline{\Gamma}_r) \underset{H}{\otimes} \overline{\Psi}_{\mathrm{H}_V(r)}$ and $\mathrm{G}_S(\overline{\Gamma}_{r'} \underset{H}{\otimes} \mathrm{F}_S\mathrm{H}_S(\nu)) \underset{H}{\otimes} \overline{\Psi}_{\mathrm{H}_V(r)}$ respectively, to which we may apply observation (b), completing the proof that $(\Gamma_h, \Gamma_{\bar{h}})$ is itself a cylinder. So $\tilde{\Gamma}$ is a horizontal transformation. Furthermore the defining property of each $\tilde{\Lambda}_r$ was selected so as to ensure that the modification δ would satisfy the cylinder condition making it into a horizontal modification as in (1.50).

Of course we may restate this result by saying that the functor

$$\underline{\underline{\mathcal{H}oriz}}_S(\underline{\underline{\tilde{\mathcal{C}}}}, \underline{\underline{\tilde{\mathcal{D}}}})(\tilde{\mathrm{F}}\tilde{\mathrm{H}}, \tilde{\mathrm{K}}) \xrightarrow{(\tilde{\mathrm{G}}\text{–}) \otimes \tilde{\Psi}\tilde{\mathrm{H}}} \underline{\underline{\mathcal{H}oriz}}_S(\underline{\underline{\tilde{\mathcal{C}}}}, \underline{\underline{\tilde{\mathcal{D}}}}')(\tilde{\mathrm{H}}, \tilde{\mathrm{G}}\tilde{\mathrm{K}})$$

is essentially surjective; the second of the “generalised element” properties we mentioned simply stipulates that it should be fully faithful as well. To establish this suppose that $\gamma\colon \tilde{\mathrm{G}}\tilde{\Gamma} \otimes \tilde{\Psi}\tilde{\mathrm{H}} \Longrightarrow \tilde{\mathrm{G}}\tilde{\Gamma}' \otimes \tilde{\Psi}\tilde{\mathrm{H}}$ is a horizontal modification, then since $\mathrm{F}_H \dashv_b$

G_H we know (by lemma 1.3.9) that there exists a unique modification $\hat{\gamma}\colon \Gamma \Longrightarrow \Gamma'$ in $\underline{\underline{\mathcal{H}om}}_S$ with $\gamma = \mathrm{G}_H\hat{\gamma} \otimes \Psi\mathrm{H}_H$. All that remains is to demonstrate the cylinder property required of $\hat{\gamma}$ as a horizontal modification, which states that for each vertical 1-cell $r\colon c \longrightarrow \bar{c} \in \tilde{\underline{\underline{\mathcal{C}}}}$ the pair $(\hat{\gamma}_c, \hat{\gamma}_{\bar{c}})$ is a cylinder from $\overline{\Gamma}_r$ to $\overline{\Gamma}'_r$. Notice though that γ is a horizontal modification therefore $(\gamma_c, \gamma_{\bar{c}}) = (\mathrm{G}_H(\hat{\gamma}_c) \underset{H}{\otimes} \Psi_{\mathrm{H}c}, \mathrm{G}_H(\hat{\gamma}_{\bar{c}}) \underset{H}{\otimes} \Psi_{\mathrm{H}\bar{c}})$ is a cylinder from $\mathrm{G}_S(\overline{\Gamma}_r) \underset{H}{\otimes} \overline{\Psi}_{\mathrm{H}_V(r)}$ to $\mathrm{G}_S(\overline{\Gamma}'_r) \underset{H}{\otimes} \overline{\Psi}_{\mathrm{H}_V(r)}$; but now applying observation (b) from above we see that $(\hat{\gamma}_c, \hat{\gamma}_{\bar{c}})$ is also a cylinder as required. $\square$

It should be clear that the definition of $\underline{\underline{\mathcal{H}oriz}}_S$ takes little account of the vertical structure of double bicategories, to fix this omission we turn to the process of "layering" vertical information on top of the fundamental horizontal maps. The (strongly) bicategory enriched categories presented in the remainder of this section are all constructed in a similar way, their objects are double bicategories and homomorphisms between them consist of horizontal maps enriched with extra "structure", for instance actions on vertical 2-cells or added canonical isomorphisms. Transformations between these are not specified by adding structure to those of $\underline{\underline{\mathcal{H}oriz}}_S$, but rather by imposing extra conditions stipulating compatibility with the data enriching the underlying horizontal maps of their proposed domains and codomains. So long as these conditions commute with composition of horizontal transformations in the homsets of $\underline{\underline{\mathcal{H}oriz}}_S$ we may go on to define bicategories $\underline{\underline{\mathcal{H}oriz}}'_S(\tilde{\underline{\underline{\mathcal{D}}}}, \tilde{\underline{\underline{\mathcal{D}}}}')$ of enriched maps, compatible transformations and all modifications between them (with no extra conditions to fulfil). Of course these are defined precisely to ensure the existence of strict homomorphisms

$$\underline{\underline{\mathcal{H}oriz}}'_S(\tilde{\underline{\underline{\mathcal{D}}}}, \tilde{\underline{\underline{\mathcal{D}}}}') \xrightarrow{(-)_\diamond} \underline{\underline{\mathcal{H}oriz}}_S(\tilde{\underline{\underline{\mathcal{D}}}}, \tilde{\underline{\underline{\mathcal{D}}}}')$$

which are both locally fully faithful and locally injective on 1-cells. We define a composition of homomorphisms in $\underline{\underline{\mathcal{H}oriz}}'_S$ using that of their underlying horizontal maps alongside a choice of (associative) composition of enriching data. Checking again that this commutes with the supplementary compatibility conditions on transformations we get a strongly bicategory enriched category supplied with a canonical functor:

$$\underline{\underline{\mathcal{H}oriz}}'_S \xrightarrow{(-)_\diamond} \underline{\underline{\mathcal{H}oriz}}_S$$

In presenting our particular cases of this construction we will describe homomorphisms and transformations explicitly, leaving the rest up to the reader. This we do to simplify rather than complicate matters, the natural composition of any enriching material should suggest itself immediately while the verifications mentioned above will turn out to be no more than routine.

As an easy first example we supplement horizontal maps with a functorial action on vertical 2-cells thus obtaining a strongly bicategory enriched category $\underline{\underline{\mathcal{H}oriz}}_{S*}$ which has:

homomorphisms: which comprise a horizontal map

$$(\mathcal{A}, \underline{\underline{\mathcal{H}}}, \underline{\underline{\mathcal{V}}}, \mathcal{S}) \xrightarrow{\tilde{\mathrm{G}}} (\mathcal{A}', \underline{\underline{\mathcal{H}}}', \underline{\underline{\mathcal{V}}}', \mathcal{S}')$$

and for each pair of 0-cells $a, \bar{a} \in \mathcal{A}$ a functor

$$\underline{\underline{\mathcal{V}}}(a, \bar{a}) \xrightarrow{\mathrm{G}_V} \underline{\underline{\mathcal{V}}}'(\mathrm{G}a, \mathrm{G}\bar{a})$$

enriching the action of $\tilde{\mathrm{G}}$ on vertical 1-cells. These must satisfy a "cylinder condition" which requires that for any vertical cylinder $(\beta, \beta'): \lambda \Rightarrow_V \dot{\lambda}$ the pair $(\mathrm{G}_V(\beta), \mathrm{G}_V(\beta'))$ is also a cylinder $\mathrm{G}_S(\lambda) \Rightarrow_V \mathrm{G}_S(\dot{\lambda})$. This is equivalent to saying that the actions of vertical 2-cells on squares are preserved by $\tilde{\mathrm{G}}$, in the sense that:

$$\mathrm{G}_S(\dot{\lambda} \underset{H}{*} \beta) = \mathrm{G}_S(\dot{\lambda}) \underset{H}{*} \mathrm{G}_V(\beta) \qquad \mathrm{G}_S(\beta' \underset{H}{*} \lambda) = \mathrm{G}_V(\beta') \underset{H}{*} \mathrm{G}_S(\lambda)$$

transformations: are horizontal transformations $\tilde{\Psi}: \tilde{\mathrm{F}} \longrightarrow \tilde{\mathrm{G}}$ between underlying horizontal maps satisfying the additional condition that for each vertical 2-cell $\beta: p \Rightarrow_V q \in \underline{\underline{\mathcal{V}}}$ the pair of 2-cells $(\mathrm{F}_V(\beta), \mathrm{G}_V(\beta))$ forms a vertical cylinder $\overline{\Psi}_p \Rightarrow_V \overline{\Psi}_q$.

In fact the following lemma shows that when we restrict attention to double bicategories of the form $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ all horizontal maps admit a unique enrichment to a homomorphism of $\underline{\underline{\mathcal{H}oriz}}_{S*}$ and all transformations are compatible with that enrichment.

Lemma 1.4.9 *Suppose that the double bicategory* $\tilde{\underline{\underline{\mathcal{D}}}}'$ *satisfies a property which states that for each pair of vertical 1-cells* $q, \dot{q}: b \longrightarrow \bar{b}$ *there exists a bijection*

$$\begin{tikzcd} b \arrow[r, "\mathrm{i}^h_b"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner\delta\urcorner"] & b \arrow[d, "\dot{q}"] \\ \bar{b} \arrow[r, "\mathrm{i}^h_{\bar{b}}"'] & \bar{b} \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} b \arrow[d, bend right, "q"'] \arrow[d, bend left, "\dot{q}"] \arrow[d, phantom, "\overset{\delta}{\Rightarrow}"] \\ \bar{b} \end{tikzcd} \tag{1.51}$$

with the defining property $\ulcorner\delta\urcorner = \mathrm{i}^h_{\dot{q}} \underset{H}{*} \delta \; (= \delta \underset{H}{*} \mathrm{i}^h_q)$. *For instance each double bicategory* $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ *has this property. It follows that the canonical strict homomorphism*

$$\underline{\underline{\mathcal{H}oriz}}_{S*}(\tilde{\underline{\underline{\mathcal{D}}}}, \tilde{\underline{\underline{\mathcal{D}}}}') \xrightarrow{(\text{-})_\diamond} \underline{\underline{\mathcal{H}oriz}}_{S}(\tilde{\underline{\underline{\mathcal{D}}}}, \tilde{\underline{\underline{\mathcal{D}}}}')$$

is an isomorphism of bicategories.

Proof. Given a horizontal map $\tilde{\mathrm{G}} \in \underline{\underline{\mathcal{H}oriz}}_S(\underline{\underline{\tilde{\mathcal{D}}}}, \underline{\underline{\tilde{\mathcal{D}}}}')$ we provide an action on vertical 2-cells $\beta\colon p \Rightarrow_V \dot{p}\colon a \to \bar{a} \in \underline{\underline{\mathcal{V}}}$ by letting $\mathrm{G}_V(\beta)\colon \mathrm{G}_V(p) \Rightarrow_V \mathrm{G}_V(\dot{p})$ be the unique vertical 2-cell with the property that the canonical isomorphisms

$$\mathrm{can}\colon \mathrm{i}^h_{\mathrm{G}a} \overset{\simeq}{\Longrightarrow}_H \mathrm{G}_H(\mathrm{i}^h_a) \qquad \overline{\mathrm{can}}\colon \mathrm{i}^h_{\mathrm{G}\bar{a}} \overset{\simeq}{\Longrightarrow}_H \mathrm{G}_H(\mathrm{i}^h_{\bar{a}}) \tag{1.52}$$

constitute a horizontal cylinder $\ulcorner \mathrm{G}_V(\beta) \urcorner \Rightarrow_H \mathrm{G}_S(\ulcorner \beta \urcorner)$. This property may be rephrased by saying that under the bijection of (1.51) $\mathrm{G}_V(\beta)$ corresponds to the square $\overline{\mathrm{can}}^{-1} \underset{H}{*} \mathrm{G}_S(\ulcorner \beta \urcorner) \underset{H}{*} \mathrm{can}$.

Notice that for any horizontal map $\tilde{\mathrm{G}}$ the 2-cells of (1.52) form a horizontal cylinder from $\mathrm{i}^h_{\mathrm{G}_V(p)}$ to $\mathrm{G}_S(\mathrm{i}^h_p)$, therefore if $\tilde{\mathrm{G}}$ underlies some homomorphism of $\underline{\underline{\mathcal{H}oriz}}_{S*}$ these 2-cells also constitute a cylinder:

$$\mathrm{G}_S(\beta \underset{H}{*} \mathrm{i}^h_p) =\!=\!=\!= \mathrm{G}_V(\beta) \underset{H}{*} \mathrm{G}_S(\mathrm{i}^h_p) \xRightarrow[\sim]{(\mathrm{can}, \overline{\mathrm{can}})}_H \mathrm{G}_S(\beta) \underset{H}{*} \mathrm{i}^h_{\mathrm{G}_V(p)}$$

This of course simply states that $\mathrm{G}_V(\beta)$ satisfies the defining property of the previous paragraph therefore if $\underline{\underline{\tilde{\mathcal{D}}}}'$ obeys the condition in the statement of this lemma there can be at most one enrichment of $\tilde{\mathrm{G}}$ to a homomorphism of $\underline{\underline{\mathcal{H}oriz}}_{S*}$.

We proceed by proving that $\mathrm{G}_V(\beta)$ as given above is a well defined action enriching $\tilde{\mathrm{G}}$ to a homomorphism of $\underline{\underline{\mathcal{H}oriz}}_{S*}$. In $\underline{\underline{\tilde{\mathcal{D}}}}'$ we may characterise the left action of vertical 2-cell δ' on a square θ as follows, $\delta' \underset{H}{*} \theta$ is the unique square such that the canonical horizontal 2-cells

$$\mathrm{can}\colon \mathrm{i}^h_{b'} \underset{H}{\otimes} g \overset{\simeq}{\Longrightarrow}_H g \qquad \overline{\mathrm{can}}\colon \mathrm{i}^h_{\bar{b}'} \underset{H}{\otimes} \bar{g} \overset{\simeq}{\Longrightarrow}_H \bar{g}$$

are the components of a cylinder $\ulcorner \delta' \urcorner \underset{H}{\otimes} \theta \overset{\simeq}{\Longrightarrow}_H \delta' \underset{V}{*} \theta$. Suppose that λ is a square of $\underline{\underline{\tilde{\mathcal{D}}}}$, as in (1.47), and $\beta'\colon p \Rightarrow_H \dot{p}'$ a vertical 2-cell then the coherence conditions on G_H as a homomorphism ensure that we have commutative diagrams:

$$\begin{tikzcd}
\mathrm{G}_H(\mathrm{i}^h_a) \underset{H}{\otimes} \mathrm{G}_H(f) \arrow[r, Rightarrow, "\mathrm{can}", "\simeq"'] & \mathrm{G}_H(\mathrm{i}^h_a \underset{H}{\otimes} f) \arrow[d, Rightarrow, "\mathrm{G}_H(\mathrm{can})"', "\wr"] \\
\mathrm{i}^h_{\mathrm{G}a} \underset{H}{\otimes} \mathrm{G}_H(f) \arrow[u, Rightarrow, "\mathrm{can} \underset{H}{\otimes} \mathrm{G}_H(f)", "\wr"'] \arrow[r, "\simeq", "\mathrm{can}"'] & \mathrm{G}_H(f)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{G}_H(\mathrm{i}^h_{\bar{a}}) \underset{H}{\otimes} \mathrm{G}_H(\bar{f}) \arrow[r, Rightarrow, "\overline{\mathrm{can}}", "\simeq"'] & \mathrm{G}_H(\mathrm{i}^h_{\bar{a}} \underset{H}{\otimes} \bar{f}) \arrow[d, Rightarrow, "\mathrm{G}_H(\overline{\mathrm{can}})"', "\wr"] \\
\mathrm{i}^h_{\mathrm{G}\bar{a}} \underset{H}{\otimes} \mathrm{G}_H(\bar{f}) \arrow[u, Rightarrow, "\overline{\mathrm{can}} \underset{H}{\otimes} \mathrm{G}_H(\bar{f})", "\wr"'] \arrow[r, "\simeq", "\overline{\mathrm{can}}"'] & \mathrm{G}_H(\bar{f})
\end{tikzcd}$$

Taking corresponding pairs of 2-cells in the upper legs of these two diagrams we get cylinders

$$\begin{gathered}
(\mathrm{can} \underset{H}{\otimes} \mathrm{G}_H(f), \overline{\mathrm{can}} \underset{H}{\otimes} \mathrm{G}_H(\bar{f}))\colon \ulcorner \mathrm{G}_V(\beta') \urcorner \underset{H}{\otimes} \mathrm{G}_S(\lambda) \overset{\simeq}{\Longrightarrow}_H \mathrm{G}_S(\ulcorner \beta' \urcorner) \underset{H}{\otimes} \mathrm{G}_S(\lambda) \\
(\mathrm{can}, \overline{\mathrm{can}})\colon \mathrm{G}_S(\ulcorner \beta' \urcorner) \underset{H}{\otimes} \mathrm{G}_S(\lambda) \overset{\simeq}{\Longrightarrow}_H \mathrm{G}_S(\ulcorner \beta' \urcorner \underset{H}{\otimes} \lambda) \\
(\mathrm{G}_H(\mathrm{can}), \mathrm{G}_H(\overline{\mathrm{can}}))\colon \mathrm{G}_S(\ulcorner \beta' \urcorner \underset{H}{\otimes} \lambda) \overset{\simeq}{\Longrightarrow}_H \mathrm{G}_S(\beta' \underset{H}{*} \lambda)
\end{gathered}$$

the first by the definition of $\mathrm{G}_V(\beta')$, the second since $\tilde{\mathrm{G}}$ is a horizontal map and the last because as such $\tilde{\mathrm{G}}$ preserves horizontal cylinders. It follows therefore that the pair of 2-cells at the bottom of our squares are the components of a cylinder $\ulcorner \mathrm{G}_V(\beta')\urcorner \underset{H}{\otimes} \mathrm{G}_S(\lambda) \overset{\simeq}{\Longrightarrow}_H \mathrm{G}_S(\beta' \underset{H}{*} \lambda)$; but by our characterisation of the left actions of vertical 2-cells in $\underline{\underline{\tilde{\mathcal{D}}}}'$ this simply establishes that $\mathrm{G}_V(\beta') \underset{H}{*} \mathrm{G}_S(\lambda) = \mathrm{G}_S(\beta' \underset{V}{*} \lambda)$. An identical proof establishes this preservation result for right actions.

Consider composable vertical 2-cells $\beta, \dot{\beta} \in \underline{\underline{\mathcal{V}}}$. We know that $\ulcorner \dot{\beta} \bullet \beta \urcorner = (\dot{\beta} \bullet \beta) \underset{H}{*} \mathrm{i}^h_p = \dot{\beta} \underset{H}{*} (\beta \underset{H}{*} \mathrm{i}^h_p) = \dot{\beta} \underset{H}{*} \ulcorner \beta \urcorner$ and by the same argument $\ulcorner \mathrm{G}_V(\dot{\beta}) \bullet \mathrm{G}_V(\beta)\urcorner = \mathrm{G}_V(\dot{\beta}) \underset{H}{*} \ulcorner \mathrm{G}_V(\beta)\urcorner$. We have already shown that $\mathrm{G}_S(\dot{\beta} \underset{H}{*} \ulcorner\beta\urcorner) = \mathrm{G}_V(\dot{\beta}) \underset{H}{*} \mathrm{G}_S(\ulcorner\beta\urcorner)$ and the defining property of $\mathrm{G}_V(\dot{\beta})$ implies that the 2-cells in (1.52) form a cylinder:

$$\ulcorner \mathrm{G}_V(\dot{\beta}) \bullet \mathrm{G}_V(\beta)\urcorner = \mathrm{G}_V(\dot{\beta}) \underset{H}{*} \ulcorner \mathrm{G}_V(\beta)\urcorner \overset{\simeq}{\Longrightarrow}_H \mathrm{G}_V(\dot{\beta}) \underset{H}{*} \mathrm{G}_S(\ulcorner\beta\urcorner) = \mathrm{G}_S(\ulcorner \dot{\beta} \bullet \beta\urcorner) .$$

This establishes that $\mathrm{G}_V(\dot{\beta}) \bullet \mathrm{G}_V(\beta)$ satisfies the definition of $\mathrm{G}_V(\dot{\beta} \bullet \beta)$, in other words the two are equal and as a result G_V is functorial on vertical 2-cells.

It remains to show that if $\tilde{\Psi}\colon \tilde{\mathrm{F}} \longrightarrow \tilde{\mathrm{G}}$ is a transformation in $\underline{\underline{\mathcal{H}oriz}}_S(\underline{\underline{\tilde{\mathcal{D}}}}, \underline{\underline{\tilde{\mathcal{D}}}}')$ then it satisfies the compatibility condition required of transformations in $\underline{\underline{\mathcal{H}oriz}}_{S*}$ for the unique structures enriching its domain and codomain. Let $\beta\colon p \Rightarrow_V \dot{p}\colon a \longrightarrow\!\!\!\!\!\!+\; \bar{a}$ be any vertical 2-cell in $\underline{\underline{\tilde{\mathcal{D}}}}$. Then because $\Psi\colon \mathrm{F}_H \longrightarrow \mathrm{G}_H$ is a transformation (in $\underline{\underline{\mathcal{H}om}}_S$) ensures we have a commutative diagram

$$\begin{tikzcd}
\mathrm{i}^h_{\mathrm{G}a} \underset{H}{\otimes} \Psi_a \arrow[r, "\mathrm{can}_1", "\simeq"'] \arrow[d, "\wr"', "\mathrm{can} \underset{H}{\otimes} \Psi_a"'] & \Psi_a & \Psi_a \underset{H}{\otimes} \mathrm{i}^h_{\mathrm{F}a} \arrow[l, "\mathrm{can}_2"', "\simeq"] \arrow[d, "\wr", "\Psi_a \underset{H}{\otimes} \mathrm{can}"] \\
\mathrm{G}_H(\mathrm{i}^h_a) \underset{H}{\otimes} \Psi_a \arrow[rr, "\simeq", "\Psi_{\mathrm{i}^h_a}"'] & & \Psi_a \underset{H}{\otimes} \mathrm{F}_H(\mathrm{i}^h_a)
\end{tikzcd}$$

for a and a similar one for $\bar{a}$. We may pair the maps at the left, right and bottom of these diagrams to get horizontal cylinders

$$(\mathrm{can} \underset{H}{\otimes} \Psi_a, \overline{\mathrm{can}} \underset{H}{\otimes} \Psi_{\bar{a}})\colon \ulcorner \mathrm{G}_V(\beta)\urcorner \underset{H}{\otimes} \overline{\Psi}_p \overset{\simeq}{\Longrightarrow}_H \mathrm{G}_S(\ulcorner\beta\urcorner) \underset{H}{\otimes} \overline{\Psi}_p$$
$$(\Psi_{\mathrm{i}^h_a}, \Psi_{\mathrm{i}^h_{\bar{a}}})\colon \mathrm{G}_S(\ulcorner\beta\urcorner) \underset{H}{\otimes} \overline{\Psi}_p \overset{\simeq}{\Longrightarrow}_H \overline{\Psi}_{\dot{p}} \underset{H}{\otimes} \mathrm{F}_S(\ulcorner\beta\urcorner)$$
$$(\Psi_a \underset{H}{\otimes} \mathrm{can}, \Psi_{\bar{a}} \underset{H}{\otimes} \overline{\mathrm{can}})\colon \overline{\Psi}_{\dot{p}} \underset{H}{\otimes} \ulcorner \mathrm{F}_V(\beta)\urcorner \overset{\simeq}{\Longrightarrow}_H \overline{\Psi}_{\dot{p}} \underset{H}{\otimes} \mathrm{F}_S(\ulcorner\beta\urcorner)$$

the first and third by the definition of $\mathrm{G}_V(\beta)$, $\mathrm{F}_V(\beta)$ respectively and the second since $\tilde{\Psi}$ is a horizontal transformation. From this it follows that the pair of 2-cells $(\mathrm{can}_2^{-1} \bullet \mathrm{can}_1, \overline{\mathrm{can}}_2^{-1} \bullet \overline{\mathrm{can}}_1)$ is also a cylinder, a condition which translates to the equality:

$$\overline{\mathrm{can}}_1 \underset{V}{*} (\ulcorner \mathrm{G}_V(\beta)\urcorner \underset{H}{\otimes} \overline{\Psi}_p) \underset{V}{*} \mathrm{can}_1^{-1} = \overline{\mathrm{can}}_2 \underset{V}{*} (\overline{\Psi}_{\dot{p}} \underset{H}{\otimes} \ulcorner \mathrm{F}_V(\beta)\urcorner) \underset{V}{*} \mathrm{can}_2^{-1}$$

Of course we already know that

$$\begin{aligned}
\overline{\mathrm{can}}_1 \underset{V}{*} (\ulcorner \mathrm{G}_V(\beta)\urcorner \underset{H}{\otimes} \overline{\Psi}_p) \underset{V}{*} \mathrm{can}_1^{-1} &= \mathrm{G}_V(\beta) \underset{H}{*} \overline{\Psi}_p \\
\overline{\mathrm{can}}_2 \underset{V}{*} (\overline{\Psi}_{\dot{p}} \underset{H}{\otimes} \ulcorner \mathrm{F}_V(\beta)\urcorner) \underset{V}{*} \mathrm{can}_2^{-1} &= \overline{\Psi}_{\dot{p}} \underset{H}{*} \mathrm{F}_V(\beta)
\end{aligned}$$

therefore we get $\mathrm{G}_V(\beta) \underset{H}{*} \overline{\Psi}_p = \overline{\Psi}_{\dot{p}} \underset{H}{*} \mathrm{F}_V(\beta)$ which is exactly the desired compatibility condition. □

To go a step further we enrich the homomorphisms of $\underline{\underline{\underline{\mathcal{H}oriz}}}_{S*}$ with the structural 2-cells of vertical morphisms so as to obtain a strongly bicategory enriched category $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ with:

homomorphisms: comprising a horizontal map

$$(\mathcal{A}, \underline{\underline{\mathcal{H}}}, \underline{\underline{\mathcal{V}}}, \mathcal{S}) \xrightarrow{\tilde{\mathrm{G}}} (\mathcal{A}', \underline{\underline{\mathcal{H}}}', \underline{\underline{\mathcal{V}}}', \mathcal{S}')$$

and a morphism

$$\underline{\underline{\mathcal{V}}} \xrightarrow{\quad \mathrm{G}_V \quad} \underline{\underline{\mathcal{V}}}'$$

enriching the action of $\tilde{\mathrm{G}}$ on vertical 1-cells. The local action of G_V must satisfy the cylinder condition required of a homomorphism in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{S*}$ and its canonical isomorphisms the following conditions:

(a) Given squares

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\lambda"] & a' \arrow[d, "p'"] \\
\bar{a} \arrow[r, "\bar{f}"'] & \bar{a}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{a} \arrow[r, "\bar{f}"] \arrow[d, "\bar{p}"'] \arrow[dr, phantom, "\bar{\lambda}"] & \bar{a}' \arrow[d, "\bar{p}'"] \\
\tilde{a} \arrow[r, "\tilde{f}"'] & \tilde{a}'
\end{tikzcd}$$

in $\mathcal{S}$, the canonical vertical 2-cells

$$\mathrm{can}\colon \mathrm{G}_V(\bar{p}) \underset{V}{\otimes} \mathrm{G}_V(p) \Rightarrow_V \mathrm{G}_V(\bar{p} \underset{V}{\otimes} p) \quad \mathrm{can}\colon \mathrm{G}_V(\bar{p}') \underset{V}{\otimes} \mathrm{G}_V(p') \Rightarrow_V \mathrm{G}_V(\bar{p}' \underset{V}{\otimes} p')$$

of the morphism G_V are the components of a vertical cylinder:

$$\mathrm{G}_S(\bar{\lambda}) \underset{V}{\otimes} \mathrm{G}_S(\lambda) \Rightarrow_V \mathrm{G}_S(\bar{\lambda} \underset{V}{\otimes} \lambda)$$

(b) For each horizontal 1-cell $f\colon a \longrightarrow a' \in \underline{\underline{\mathcal{H}}}$ the canonical vertical cells

$$\mathrm{can}\colon \mathrm{i}^v_{\mathrm{G}(a)} \Rightarrow_V \mathrm{G}_V(\mathrm{i}^v_a) \quad \mathrm{can}\colon \mathrm{i}^v_{\mathrm{G}(a')} \Rightarrow_V \mathrm{G}_V(\mathrm{i}^v_{a'})$$

are the components of a vertical cylinder $\mathrm{i}^v_{\mathrm{G}_H(f)} \Rightarrow_V \mathrm{G}_S(\mathrm{i}^v_f)$.

transformations: horizontal ones $\tilde{\Psi}\colon \tilde{\mathrm{F}} \longrightarrow \tilde{\mathrm{G}}$ satisfying the cylinder condition required of those in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{S*}$ and:

(i) Given vertical 1-cells $p\colon a \longrightarrow\!\!\!\!\!\!\!\!+\;\; \bar{a}$ and $\bar{p}\colon \bar{a} \longrightarrow\!\!\!\!\!\!\!\!+\;\; \tilde{a}$ in $\underline{\underline{\mathcal{V}}}$ the canonical vertical 2-cells

$$\mathrm{can}\colon \mathrm{G}_V(\bar{p}) \underset{V}{\otimes} \mathrm{G}_V(p) \Rightarrow_V \mathrm{G}_V(\bar{p} \underset{V}{\otimes} p) \quad \mathrm{can}\colon \mathrm{F}_V(\bar{p}) \underset{V}{\otimes} \mathrm{F}_V(p) \Rightarrow_V \mathrm{F}_V(\bar{p} \underset{V}{\otimes} p)$$

are the components of a cylinder $\overline{\Psi}_{\bar{p}} \underset{V}{\otimes} \overline{\Psi}_p \Rightarrow_V \overline{\Psi}_{\bar{p} \underset{V}{\otimes} p}$.

(ii) For each 0-cell $a \in \mathcal{A}$ the canonical 2-cells

$$\mathrm{can}\colon \mathrm{i}^v_{\mathrm{G}(a)} \Rightarrow_V \mathrm{G}_V(\mathrm{i}^v_a) \quad \mathrm{can}\colon \mathrm{i}^v_{\mathrm{F}(a)} \Rightarrow_V \mathrm{F}_V(\mathrm{i}^v_a)$$

are the components of a cylinder $\mathrm{i}^v_{\Psi_a} \Rightarrow_V \overline{\Psi}_{\mathrm{i}^v_a}$.

In a similar fashion we get strongly bicategory enriched categories $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SC}$ and $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH}$ by enriching actions on vertical cells with the structure of comorphisms or homomorphisms respectively and imposing corresponding cylinder conditions on the map transformations between these.

Observation 1.4.10 The various forgetful (enriched) functors between these categories now fit into a commutative diagram:

$$\begin{tikzcd}
& \underline{\underline{\underline{\mathcal{H}oriz}}}_{SM} \arrow[dr] & & \\
\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH} \arrow[ur] \arrow[dr] & & \underline{\underline{\underline{\mathcal{H}oriz}}}_{S*} \arrow[r] & \underline{\underline{\underline{\mathcal{H}oriz}}}_{S} \\
& \underline{\underline{\underline{\mathcal{H}oriz}}}_{SC} \arrow[ur] & &
\end{tikzcd}$$

Here the construction methodology discussed on page 112 onward ensures that each of these forgetful functors has a full and faithful action on modifications and a faithful action on transformations.

Now when carrying a homomorphism $\tilde{\mathrm{F}}$ from $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH}$ to $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ or $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SC}$ we don't actually forget any of its structure. Instead all we do is to forget the property that some pieces of that structure are invertible. Furthermore, suppose that $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$ are homomorphisms in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH}$ and that $\tilde{\Phi}\colon \tilde{\mathrm{F}} \longrightarrow \tilde{\mathrm{G}}$ is a transformation between underlying homomorphisms in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{S*}$. Then it is clear that $\tilde{\Phi}$ satisfies the cylinder conditions which apply to make it a transformation in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH}$ if and only if it satisfies those that apply to transformations in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ or $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SC}$. Consequently we find that the forgetful enriched functors $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH} \longrightarrow \underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ and $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH} \longrightarrow \underline{\underline{\underline{\mathcal{H}oriz}}}_{SC}$ also act in a full manner on transformations and in a faithful manner on homomorphisms.

The most important proposition of this section follows:

Proposition 1.4.11 *Suppose that we are given a biadjunction*

$$(\tilde{\mathrm{F}} \dashv_b \tilde{\mathrm{G}}, \tilde{\Psi}, \tilde{\Phi}, \alpha, \beta) \colon \underline{\underline{\tilde{\mathcal{D}}}} \longrightarrow \underline{\underline{\tilde{\mathcal{D}}}}'$$

in $\underline{\underline{\mathcal{H}oriz}}_{S*}$ *where* $\underline{\underline{\tilde{\mathcal{D}}}}$ *and* $\underline{\underline{\tilde{\mathcal{D}}}}'$ *both satisfy the condition in lemma 1.4.9 then there is a bijection:*

$$\left\{\begin{array}{l}\textit{Enrichments of } \tilde{\mathrm{F}} \textit{ to a homo-}\\ \textit{morphism in } \underline{\underline{\mathcal{H}oriz}}_{SC}.\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{Enrichments of } \tilde{\mathrm{G}} \textit{ to a homo-}\\ \textit{morphism in } \underline{\underline{\mathcal{H}oriz}}_{SM}.\end{array}\right\}$$

Furthermore if we use $\mathrm{can}_{\mathrm{F}}$ *and* $\mathrm{can}_{\mathrm{G}}$ *to denote the various instances of the canonical 2-cells of corresponding enrichments of* $\tilde{\mathrm{F}}$ *and* $\tilde{\mathrm{G}}$ *they satisfy:*

$$\begin{array}{ll} \mathrm{can}_{\mathrm{G}} \underset{H}{*} (\overline{\Psi}_{\bar{q}} \underset{V}{\otimes} \overline{\Psi}_q) = \mathrm{G}_V(\mathrm{can}_{\mathrm{F}}) \underset{H}{*} \overline{\Psi}_{\bar{q} \underset{V}{\otimes} q} & \textit{for 1-cells } q \colon b \to \bar{b}', \bar{q} \colon \bar{b} \to \tilde{b} \textit{ in } \underline{\underline{\mathcal{V}}}'. \\ \mathrm{can}_{\mathrm{G}} \underset{H}{*} \mathrm{i}^v_{\Psi_b} = \mathrm{G}_V(\mathrm{can}_{\mathrm{F}}) \underset{H}{*} \overline{\Psi}_{\mathrm{i}^v_b} & \textit{for each 0-cell } b \in \mathcal{A}'. \end{array} \tag{1.53}$$

and:

$$\begin{array}{ll} (\overline{\Phi}_{\bar{p}} \underset{V}{\otimes} \overline{\Phi}_p) \underset{H}{*} \mathrm{can}_{\mathrm{F}} = \overline{\Phi}_{\bar{p} \underset{V}{\otimes} p} \underset{H}{*} \mathrm{F}_V(\mathrm{can}_{\mathrm{G}}) & \textit{for 1-cells } p \colon a \to \bar{a}, \bar{p} \colon \bar{a} \to \tilde{a} \textit{ in } \underline{\underline{\mathcal{V}}}. \\ \mathrm{i}^v_{\Phi_a} \underset{H}{*} \mathrm{can}_{\mathrm{F}} = \overline{\Phi}_{\mathrm{i}^v_a} \underset{H}{*} \mathrm{F}_V(\mathrm{can}_{\mathrm{G}}) & \textit{for each 0-cell } a \in \mathcal{A}. \end{array} \tag{1.54}$$

Proof. Most importantly it should be noted that on combining the condition of lemma 1.4.9 with the one sided universal property for $\tilde{\mathrm{F}} \dashv_b \tilde{\mathrm{G}}$, as furnished by proposition 1.4.8, we get a bijection

$$\begin{tikzcd} b \arrow[r, "\Psi_b"] \arrow[d, "q"'] \arrow[dr, phantom, "\lambda \lrcorner"] & \mathrm{GF}b \arrow[d, "\mathrm{G}_V(p)"] \\ \bar{b} \arrow[r, "\Psi_{\bar{b}}"'] & \mathrm{GF}\bar{b} \end{tikzcd} \longleftrightarrow \begin{tikzcd} \mathrm{F}b \arrow[d, bend right, "\mathrm{F}_V(q)"'] \arrow[d, bend left, "p"] \arrow[d, phantom, "\overset{\hat{\lambda}}{\Rightarrow}"] \\ \mathrm{F}\bar{b} \end{tikzcd} \tag{1.55}$$

with the defining property $\lambda = \mathrm{G}_V(\hat{\lambda}) \underset{H}{*} \overline{\Psi}_q$. This rule allows us to define unique vertical 2-cells

$$\mathrm{can}_{\mathrm{F}} \colon \mathrm{F}_V(\bar{q} \underset{V}{\otimes} q) \Rightarrow_V \mathrm{F}_V(\bar{q}) \underset{V}{\otimes} \mathrm{F}_V(q) \quad , \quad \mathrm{can}_{\mathrm{F}} \colon \mathrm{F}_V(\mathrm{i}^v_b) \Rightarrow_V \mathrm{i}^v_{\mathrm{F}b}$$

exactly so as to satisfy the equations in (1.53). That these also satisfy the coherence and naturality properties required of the canonical cells of a comorphism F_V follows from the corresponding rules for G_V and the uniqueness properties of bijection (1.55), via a few easy calculations. For instance in order to prove the rule

$$\begin{tikzcd} \mathrm{F}_V(\tilde q \underset{V}{\otimes} \bar q \underset{V}{\otimes} q) \arrow[rr, Rightarrow, "\mathrm{can}_\mathrm{F}"] \arrow[d, Rightarrow, "\mathrm{can}_\mathrm{F}"'] \arrow[drr, phantom, "="] & & \mathrm{F}_V(\tilde q) \underset{V}{\otimes} \mathrm{F}_V(\bar q \underset{V}{\otimes} q) \arrow[d, Rightarrow, "\mathrm{F}_V(\tilde q) \underset{V}{\otimes} \mathrm{can}_\mathrm{F}"] \\ \mathrm{F}_V(\tilde q \underset{V}{\otimes} \bar q) \underset{V}{\otimes} \mathrm{F}_V(q) \arrow[rr, Rightarrow, "\mathrm{can}_\mathrm{F} \underset{V}{\otimes} \mathrm{F}_V(q)"'] & & \mathrm{F}_V(\tilde q) \underset{V}{\otimes} \mathrm{F}_V(\bar q) \underset{V}{\otimes} \mathrm{F}_V(q) \end{tikzcd} \tag{1.56}$$

it is a routine matter to show that

$$\begin{aligned} \mathrm{G}_V((\mathrm{F}_V(\tilde q) \underset{V}{\otimes} \mathrm{can}_\mathrm{F}) \bullet \mathrm{can}_\mathrm{F}) \underset{H}{*} \overline{\Psi}_{\tilde q \underset{V}{\otimes} \bar q \underset{V}{\otimes} q} &= (\mathrm{can}_\mathrm{G} \bullet (\mathrm{G}_V\mathrm{F}_V(\tilde q) \underset{V}{\otimes} \mathrm{can}_\mathrm{G})) \underset{H}{*} (\overline{\Psi}_{\tilde q} \underset{V}{\otimes} \overline{\Psi}_{\bar q} \underset{V}{\otimes} \overline{\Psi}_q) \\ \mathrm{G}_V((\mathrm{can}_\mathrm{F} \underset{V}{\otimes} \mathrm{F}_V(q)) \bullet \mathrm{can}_\mathrm{F}) \underset{H}{*} \overline{\Psi}_{\tilde q \underset{V}{\otimes} \bar q \underset{V}{\otimes} q} &= (\mathrm{can}_\mathrm{G} \bullet (\mathrm{can}_\mathrm{G} \underset{V}{\otimes} \mathrm{G}_V\mathrm{F}_V(q))) \underset{H}{*} (\overline{\Psi}_{\tilde q} \underset{V}{\otimes} \overline{\Psi}_{\bar q} \underset{V}{\otimes} \overline{\Psi}_q) \end{aligned}$$

by using the definition of the 2-cells can_F, then applying the corresponding coherence rule for G_V we see that the expressions on the right hand side are equal. This establishes the equality of the terms on the left and we apply bijection (1.55) to demonstrate that (1.56) holds. Notice that we have followed our usual conventions and eliminated explicit mention of the canonical associativity isomorphisms of the bicategories in question.

We use a similar sort of argument to verify the identities in (1.54). Applying the definition of the 2-cells can_F we get equalities:

$$\begin{aligned} \mathrm{G}_S((\overline{\Phi}_{\bar p} \underset{V}{\otimes} \overline{\Phi}_p) \underset{H}{*} \mathrm{can}_\mathrm{F}) \underset{H}{\otimes} \overline{\Psi}_{\mathrm{G}_V(\bar p) \underset{V}{\otimes} \mathrm{G}_V(p)} &= \mathrm{can}_\mathrm{G} \underset{H}{*} ((\mathrm{G}_S(\overline{\Phi}_{\bar p}) \underset{H}{\otimes} \overline{\Psi}_{\mathrm{G}_V(\bar p)}) \underset{V}{\otimes} (\mathrm{G}_S(\overline{\Phi}_p) \underset{H}{\otimes} \overline{\Psi}_{\mathrm{G}_V(p)})) \\ (\mathrm{G}_S(\overline{\Phi}_{\bar p \underset{V}{\otimes} p}) \underset{H}{\otimes} \overline{\Psi}_{\mathrm{G}_V(\bar p \underset{V}{\otimes} p)}) \underset{H}{*} \mathrm{can}_\mathrm{G} &= \mathrm{G}_S((\overline{\Phi}_{\bar p \underset{V}{\otimes} p}) \underset{H}{*} \mathrm{F}_V(\mathrm{can}_\mathrm{G})) \underset{H}{\otimes} \overline{\Psi}_{\mathrm{G}_V(\bar p) \underset{V}{\otimes} \mathrm{G}_V(p)} \,. \end{aligned} \tag{1.57}$$

Consider now the transformation $\tilde{\mathrm{G}}\tilde\Phi \otimes \tilde\Psi\tilde{\mathrm{G}}$ in $\underline{\underline{\underline{\mathcal{H}oriz}}}_S$, this is isomorphic to the identity $\mathrm{i}_{\tilde{\mathrm{G}}}\colon \tilde{\mathrm{G}} \longrightarrow \tilde{\mathrm{G}}$ by the triangle isomorphism α of the biadjoint $\tilde{\mathrm{F}} \dashv_b \tilde{\mathrm{G}}$. Now $\mathrm{i}_{\tilde{\mathrm{G}}}$ is also the identity transformation on $\tilde{\mathrm{G}}$ as enriched to a homomorphism in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$, and it is easily shown that any transformation isomorphic, in $\underline{\underline{\underline{\mathcal{H}oriz}}}_S$, to one from $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ also satisfies the cylinder conditions required of those in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ therefore $\tilde{\mathrm{G}}\tilde\Phi \otimes \tilde\Psi\tilde{\mathrm{G}}$ is a transformation in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$.

Returning to the equations in (1.57), the equality of their right hand terms simply expresses one of the cylinder conditions required of $\tilde{\mathrm{G}}\tilde\Phi \otimes \tilde\Psi\tilde{\mathrm{G}}$ as a transformation in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$. Finally applying the uniqueness clause of the one sided universal property possessed by $\tilde{\mathrm{F}} \dashv_b \tilde{\mathrm{G}}$ to the resulting equality of left hand terms in (1.57) we get the first equation in (1.54); the second follows by a similar argument.

This completes the construction of the *unique* vertical comorphism structure on $\tilde{\mathrm{F}}$ induced by a given enrichment of $\tilde{\mathrm{G}}$ to $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ and satisfying the conditions given in (1.53) and (1.54). Dually we may construct a unique morphism structure on $\tilde{\mathrm{G}}$ when given a comorphism one on $\tilde{\mathrm{F}}$, with which we have established the bijection required in the statement of the proposition.

Notice that the archetype for this kind of argument was given by Kelly in [29]. □

The last proposition gives us an important way of constructing biadjoints in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ (or dually in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SC}$):

Corollary 1.4.12 *If* $\tilde{\mathrm{F}}\colon \underline{\tilde{\mathcal{D}}}' \longrightarrow \underline{\tilde{\mathcal{D}}}$ *is a homomorphism in* $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH}$ *admitting a right biadjoint in* $\underline{\underline{\underline{\mathcal{H}oriz}}}_{S}$*, and* $\underline{\tilde{\mathcal{D}}},\underline{\tilde{\mathcal{D}}}'$ *both satisfy the condition of lemma 1.4.9; then there is a unique enrichment of* $\tilde{\mathrm{G}}$ *lifting* $\tilde{\mathrm{F}} \dashv_b \tilde{\mathrm{G}}$ *to a biadjoint in* $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$*.*

Proof. We may consider $\tilde{\mathrm{F}}$ to be a homomorphism in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SC}$ (as well as in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$) and so the last proposition provides us with a unique enrichment of $\tilde{\mathrm{G}}$ to a homomorphism in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ with the proviso that the conditions in (1.53) and (1.54) hold. In these conditions all instances of $\mathrm{can}_{\mathrm{F}}$ are isomorphisms, since $\tilde{\mathrm{F}}$ is in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH}$, allowing them to be translated into a form in which they simply state that $\tilde{\Psi}$ and $\tilde{\Phi}$ are transformations in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$; that is once we have re-interpreted $\tilde{\mathrm{F}}$ as a homomorphism in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$. In other words we have (uniquely) lifted $\tilde{\mathrm{F}} \dashv_b \tilde{\mathrm{G}}$ to $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ as required. □

The following result was not included in the original version of this work. While it is not required in the sequel, it is a useful result whose omission was a clear oversight. Where our proposition 1.4.11 may be thought of as a direct analogue of Kelly's theorem 1.2 in [29] we should regard the following proposition as a direct analogue of proposition 1.3 of that same work.

I should like to thank Jonas Frey for pointing out that this result holds and for encouraging me to include it here.

Proposition 1.4.13 *Suppose that we are given a biadjunction*

$$(\tilde{\mathrm{F}} \dashv_b \tilde{\mathrm{G}}, \tilde{\Psi}, \tilde{\Phi}, \alpha, \beta)\colon \underline{\tilde{\mathcal{D}}} \longrightarrow \underline{\tilde{\mathcal{D}}}'$$

in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ *where* $\underline{\tilde{\mathcal{D}}}$ *and* $\underline{\tilde{\mathcal{D}}}'$ *both satisfy the condition in lemma 1.4.9. Now by applying proposition 1.4.11 we find that the homomorphism in* $\underline{\underline{\underline{\mathcal{H}oriz}}}_{S*}$ *which underlies* $\tilde{\mathrm{F}}$ *has a (unique) enrichment to a homomorphism in* $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SC}$ *induced by the structure of* $\tilde{\mathrm{G}}$ *as a homomorphism in* $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$*.*

Let $\mathrm{can}_{\mathrm{F}}$ *and* $\mathrm{can}_{\mathrm{G}}$ *denote the various instances of the canonical vertical 2-cells that enrich* $\tilde{\mathrm{F}}$ *and* $\tilde{\mathrm{G}}$ *with the structure of homomorphisms in* $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ *and let* $\overline{\mathrm{can}}_{\mathrm{F}}$ *denote the various instances of the induced vertical 2-cells which enrich* $\tilde{\mathrm{F}}$ *to a homomorphism in* $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SC}$ *under proposition 1.4.11.*

Then for each pair of 1-cells $q\colon b \to \bar{b}, \bar{q}\colon \bar{b} \to \tilde{b}$ *in* $\underline{\mathcal{V}}'$ *the canonical 2-cells*

$$\mathrm{can}_{\mathrm{F}}\colon \mathrm{F}_V(\bar{q}) \underset{V}{\otimes} \mathrm{F}_V(q) \Rightarrow_V \mathrm{F}_V(\bar{q} \underset{V}{\otimes} q) \;\; \textit{and} \;\; \overline{\mathrm{can}}_{\mathrm{F}}\colon \mathrm{F}_V(\bar{q} \underset{V}{\otimes} q) \Rightarrow_V \mathrm{F}_V(\bar{q}) \underset{V}{\otimes} \mathrm{F}_V(q) \tag{1.58}$$

are mutual inverses and for each 0-cell b *in* $\underline{\mathcal{A}}'$ *the canonical 2-cells*

$$\mathrm{can}_{\mathrm{F}}\colon \mathrm{i}^v_{\mathrm{F}b} \Rightarrow_V \mathrm{F}_V(\mathrm{i}^v_b) \;\; \textit{and} \;\; \overline{\mathrm{can}}_{\mathrm{F}}\colon \mathrm{F}_V(\mathrm{i}^v_b) \Rightarrow_V \mathrm{i}^v_{\mathrm{F}b} \tag{1.59}$$

are also mutual inverses. It follows therefore that $\tilde{\mathrm{F}}$ *is a homomorphism in* $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH}$*.*

Proof. We show that the 2-cells in equation (1.58) are mutually inverse. Observe first that the biadjunction of the statement takes place in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ so we know that the unit $\tilde{\Psi}$ is a horizontal transformation in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$. Consequently, $\tilde{\Psi}$ satisfies the cylinder conditions page 117 which, in particular, provide us with the specific equation:

$$(\mathrm{G}_V(\mathrm{can}_\mathrm{F}) \bullet \mathrm{can}_\mathrm{G}) \underset{H}{*} (\overline{\Psi}_{\bar{q}} \underset{V}{\otimes} \overline{\Psi}_q) = \overline{\Psi}_{\bar{q} \underset{V}{\otimes} q} \tag{1.60}$$

Furthermore, we also have the defining equation for $\overline{\mathrm{can}}_\mathrm{F}$

$$\mathrm{can}_\mathrm{G} \underset{H}{*} (\overline{\Psi}_{\bar{q}} \underset{V}{\otimes} \overline{\Psi}_q) = \mathrm{G}_V(\overline{\mathrm{can}}_\mathrm{F}) \underset{H}{*} \overline{\Psi}_{\bar{q} \underset{V}{\otimes} q} \tag{1.61}$$

from equation (1.53) of the statement of proposition 1.4.11. Combining these two we find that

$$\begin{array}{rcll} \mathrm{G}_V(\mathrm{can}_\mathrm{F} \bullet \overline{\mathrm{can}}_\mathrm{F}) \underset{H}{*} \overline{\Psi}_{\bar{q} \underset{V}{\otimes} q} & = & \mathrm{G}_V(\mathrm{can}_\mathrm{F}) \underset{H}{*} (\mathrm{G}_V(\overline{\mathrm{can}}_\mathrm{F}) \underset{H}{*} \overline{\Psi}_{\bar{q} \underset{V}{\otimes} q}) & \\ & = & \mathrm{G}_V(\mathrm{can}_\mathrm{F}) \underset{H}{*} (\mathrm{can}_\mathrm{G} \underset{H}{*} (\overline{\Psi}_{\bar{q}} \underset{V}{\otimes} \overline{\Psi}_q)) & \text{by (1.61)} \\ & = & (\mathrm{G}_V(\mathrm{can}_\mathrm{F}) \bullet \mathrm{can}_\mathrm{G}) \underset{H}{*} (\overline{\Psi}_{\bar{q}} \underset{V}{\otimes} \overline{\Psi}_q) & \\ & = & \overline{\Psi}_{\bar{q} \underset{V}{\otimes} q} & \text{by (1.60)} \end{array} \tag{1.62}$$

to which we may apply the uniqueness property of the bijection displayed in equation (1.55) to show that can_F is left inverse to $\overline{\mathrm{can}}_\mathrm{F}$.

Now to prove that can_F is also right inverse to $\overline{\mathrm{can}}_\mathrm{F}$ we apply a similar argument to the counit $\tilde{\Phi}$ in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$. Starting with a pair of vertical 1-cells $p \overset{\mathrm{def}}{=} \mathrm{F}_V(q)$ and $\bar{p} \overset{\mathrm{def}}{=} \mathrm{F}_V(\bar{q})$ in $\underline{\underline{\mathcal{V}}}$, we combine the cylinder condition from page 117 for $\tilde{\Phi}$ as a homomorphism in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$

$$\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{V}{\otimes} \overline{\Phi}_{\mathrm{F}_V(q)} = \overline{\Phi}_{\mathrm{F}_V(\bar{q}) \underset{V}{\otimes} \mathrm{F}_V(q)} \underset{H}{*} (\mathrm{F}_V(\mathrm{can}_\mathrm{G}) \bullet \mathrm{can}_\mathrm{F})$$

with the equality from equation (1.54) of the statement of proposition 1.4.11

$$(\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{V}{\otimes} \overline{\Phi}_{\mathrm{F}_V(q)}) \underset{H}{*} \overline{\mathrm{can}}_\mathrm{F} = \overline{\Phi}_{\mathrm{F}_V(\bar{q}) \underset{V}{\otimes} \mathrm{F}_V(q)} \underset{H}{*} \mathrm{F}_V(\mathrm{can}_\mathrm{G})$$

to obtain the equation

$$(\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{V}{\otimes} \overline{\Phi}_{\mathrm{F}_V(q)}) \underset{H}{*} (\overline{\mathrm{can}}_\mathrm{F} \bullet \mathrm{can}_\mathrm{F}) = \overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{V}{\otimes} \overline{\Phi}_{\mathrm{F}_V(q)}$$

by arguing much as before. At this point it is worth saying that our overloading of the notation can for canonical 2-cells involved in the last equation has probably obscured a little more information than we might hope for in this expression. However, a second look reveals that the sub-expression $\overline{\mathrm{can}}_\mathrm{F} \bullet \mathrm{can}_\mathrm{F}$ of the last equation actually refers to the composite:

$$\begin{array}{rcl} \mathrm{F}_V\mathrm{G}_V\mathrm{F}_V(\bar{q}) \underset{V}{\otimes} \mathrm{F}_V\mathrm{G}_V\mathrm{F}_V(q) & \overset{\mathrm{can}_\mathrm{F}}{\Longrightarrow}_V & \mathrm{F}_V(\mathrm{G}_V\mathrm{F}_V(\bar{q}) \underset{V}{\otimes} \mathrm{G}_V\mathrm{F}_V(q)) \\ & \overset{\overline{\mathrm{can}}_\mathrm{F}}{\Longrightarrow}_V & \mathrm{F}_V\mathrm{G}_V\mathrm{F}_V(\bar{q}) \underset{V}{\otimes} \mathrm{F}_V\mathrm{G}_V\mathrm{F}_V(q) \end{array} \tag{1.63}$$

Next we may horizontally right tensor each side of equation (1.62) by the square $\mathrm{F}_S(\overline{\Psi}_{\bar{q}}) \underset{V}{\otimes} \mathrm{F}_S(\overline{\Psi}_q)$ to obtain the equation:

$$\begin{aligned}
&(\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{V}{\otimes} \overline{\Phi}_{\mathrm{F}_V(q)}) \underset{H}{\otimes} (\mathrm{F}_S(\overline{\Psi}_{\bar{q}}) \underset{V}{\otimes} \mathrm{F}_S(\overline{\Psi}_q)) \\
&\qquad = ((\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{V}{\otimes} \overline{\Phi}_{\mathrm{F}_V(q)}) \underset{H}{*} (\overline{\mathrm{can}}_{\mathrm{F}} \bullet \mathrm{can}_{\mathrm{F}})) \underset{H}{\otimes} (\mathrm{F}_S(\overline{\Psi}_{\bar{q}}) \underset{V}{\otimes} \mathrm{F}_S(\overline{\Psi}_q)) \\
&\qquad = (\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{V}{\otimes} \overline{\Phi}_{\mathrm{F}_V(q)}) \underset{H}{\otimes} (\overline{\mathrm{can}}_{\mathrm{F}} \underset{H}{*} (\mathrm{can}_{\mathrm{F}} \underset{H}{*} (\mathrm{F}_S(\overline{\Psi}_{\bar{q}}) \underset{V}{\otimes} \mathrm{F}_S(\overline{\Psi}_q))
\end{aligned} \tag{1.64}$$

However, we know that the maps labelled $\mathrm{can}_{\mathrm{F}}$ are the canonical maps of a $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ structure on $\tilde{\mathrm{F}}$ and that those labelled $\overline{\mathrm{can}}_{\mathrm{F}}$ are the canonical maps of a $\underline{\underline{\underline{\mathcal{H}oriz}}}_{CM}$ structure on $\tilde{\mathrm{F}}$, so the cylinder condition (a) of page 116 provides us with the equations:

$$\begin{aligned}
\mathrm{can}_{\mathrm{F}} \underset{H}{*} (\mathrm{F}_S(\overline{\Psi}_{\bar{q}}) \underset{V}{\otimes} \mathrm{F}_S(\overline{\Psi}_q)) &= \mathrm{F}_S(\overline{\Psi}_{\bar{q}} \underset{V}{\otimes} \overline{\Psi}_q) \underset{H}{*} \mathrm{can}_{\mathrm{F}} \\
\overline{\mathrm{can}}_{\mathrm{F}} \underset{H}{*} \mathrm{F}_S(\overline{\Psi}_{\bar{q}} \underset{V}{\otimes} \overline{\Psi}_q) &= (\mathrm{F}_S(\overline{\Psi}_{\bar{q}}) \underset{V}{\otimes} \mathrm{F}_S(\overline{\Psi}_q)) \underset{H}{*} \overline{\mathrm{can}}_{\mathrm{F}}
\end{aligned} \tag{1.65}$$

These allow us to reduce the expression on the right hand side of equation (1.64) to

$$\begin{aligned}
&(\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{V}{\otimes} \overline{\Phi}_{\mathrm{F}_V(q)}) \underset{H}{\otimes} (((\mathrm{F}_S(\overline{\Psi}_{\bar{q}}) \underset{V}{\otimes} \mathrm{F}_S(\overline{\Psi}_q)) \underset{H}{*} \overline{\mathrm{can}}_{\mathrm{F}}) \underset{H}{*} \mathrm{can}_{\mathrm{F}}) \\
&\qquad = ((\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{V}{\otimes} \overline{\Phi}_{\mathrm{F}_V(q)}) \underset{H}{\otimes} (\mathrm{F}_S(\overline{\Psi}_{\bar{q}}) \underset{V}{\otimes} \mathrm{F}_S(\overline{\Psi}_q))) \underset{H}{*} (\overline{\mathrm{can}}_{\mathrm{F}} \bullet \mathrm{can}_{\mathrm{F}})
\end{aligned} \tag{1.66}$$

where the sub-expression $\overline{\mathrm{can}}_{\mathrm{F}} \bullet \mathrm{can}_{\mathrm{F}}$ now refers to the composite:

$$\mathrm{F}_V(\bar{q}) \underset{V}{\otimes} \mathrm{F}_V(q) \overset{\mathrm{can}_{\mathrm{F}}}{\Longrightarrow}_V \mathrm{F}_V(\bar{q} \underset{V}{\otimes} q) \overset{\overline{\mathrm{can}}_{\mathrm{F}}}{\Longrightarrow}_V \mathrm{F}_V(\bar{q}) \underset{V}{\otimes} \mathrm{F}_V(q) \tag{1.67}$$

Now the sub-expression $(\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{V}{\otimes} \overline{\Phi}_{\mathrm{F}_V(q)}) \underset{H}{\otimes} (\mathrm{F}_S(\overline{\Psi}_{\bar{q}}) \underset{V}{\otimes} \mathrm{F}_S(\overline{\Psi}_q))$ is equal to $(\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{H}{\otimes} \mathrm{F}_S(\overline{\Psi}_{\bar{q}})) \underset{V}{\otimes} (\overline{\Phi}_{\mathrm{F}_V(q)} \underset{H}{\otimes} \mathrm{F}_S(\overline{\Psi}_q))$ my the middle four interchange rule. Furthermore, the triangle isomorphism β of our biadjunction is an isomorphic modification in $\mathcal{H}oriz_{SM}$ from $\tilde{\Phi}\tilde{\mathrm{F}} \otimes \tilde{\mathrm{F}}\tilde{\Phi}$ to $\mathrm{i}_{\tilde{\mathrm{F}}}$. So in particular we know, from modification condition (ii) of page 105, that its pair of components $(\beta_b, \beta_{\bar{b}})$ is a horizontal cylinder from $\overline{\Phi}_{\mathrm{F}_V(q)} \underset{H}{\otimes} \mathrm{F}_S(\overline{\Psi}_q)$ to i^h_q and that its pair of components $(\beta_{\bar{b}}, \beta_{\bar{\bar{b}}})$ is a horizontal cylinder from $\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{H}{\otimes} \mathrm{F}_S(\overline{\Psi}_{\bar{q}})$ to $\mathrm{i}^h_{\bar{q}}$. It follows that $(\beta_b, \beta_{\bar{\bar{b}}})$ is also a horizontal cylinder from $(\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{H}{\otimes} \mathrm{F}_S(\overline{\Psi}_{\bar{q}})) \underset{V}{\otimes} (\overline{\Phi}_{\mathrm{F}_V(q)} \underset{H}{\otimes} \mathrm{F}_S(\overline{\Psi}_q))$ to $\mathrm{i}^h_{\bar{q}} \underset{V}{\otimes} \mathrm{i}^h_q = \mathrm{i}^h_{\bar{q} \underset{V}{\otimes} q}$. So combining this with our earlier middle four observation we find that:

$$\beta_{\bar{\bar{b}}} \underset{V}{*} (\overline{\Phi}_{\mathrm{F}_V(\bar{q})} \underset{V}{\otimes} \overline{\Phi}_{\mathrm{F}_V(q)}) \underset{H}{\otimes} (\mathrm{F}_S(\overline{\Psi}_{\bar{q}}) \underset{V}{\otimes} \mathrm{F}_S(\overline{\Psi}_q)) \underset{V}{*} \beta_b^{-1} = \mathrm{i}^h_{\bar{q} \underset{V}{\otimes} q} \tag{1.68}$$

Finally, by applying $\beta_{\bar{\bar{b}}} \underset{V}{*} -$ and $- \underset{V}{*} \beta_b^{-1}$ to equations (1.64) and (1.66) and reducing using the last equation we obtain

$$\mathrm{i}_{\bar{q} \underset{V}{\otimes} q} = \mathrm{i}_{\bar{q} \underset{V}{\otimes} q} \underset{H}{*} (\overline{\mathrm{can}}_{\mathrm{F}} \bullet \mathrm{can}_{\mathrm{F}}) \tag{1.69}$$

which, when combined with our assumption that the condition of lemma 1.4.9 holds for $\underline{\underline{\tilde{\mathcal{D}}}}'$, implies that $\mathrm{can}_{\mathrm{F}}$ is also right inverse to $\overline{\mathrm{can}}_{\mathrm{F}}$ as required.

A similar argument establishes the corresponding result for the 2-cells in equation (1.59), and that is left to the reader. □

1.5 Bicategory Enriched Categories of Equipments.

Recall from definition 1.2.4 the construction of the double bicategory of squares associated with a (weak) equipment. This gives a map

$$\text{Equipments} \xrightarrow{\text{Sq}} \mathrm{ob}(\underline{\underline{\underline{\mathcal{H}oriz}}}_S)$$

to which we may apply the construction introduced in definition 1.4.7 (with which we defined $\underline{\underline{\underline{\mathcal{H}oriz}}}_S$ itself) to obtain strongly bicategory enriched categories:

$$\begin{array}{rcl}
\underline{\underline{\underline{\mathcal{E}Map}}} & \text{derived from} & \underline{\underline{\underline{\mathcal{H}oriz}}}_{S*} \\
\underline{\underline{\underline{\mathcal{E}Mor}}} & \text{derived from} & \underline{\underline{\underline{\mathcal{H}oriz}}}_{SM} \\
\underline{\underline{\underline{\mathcal{E}coMor}}} & \text{derived from} & \underline{\underline{\underline{\mathcal{H}oriz}}}_{SC} \\
\underline{\underline{\underline{\mathcal{E}Hom}}} & \text{derived from} & \underline{\underline{\underline{\mathcal{H}oriz}}}_{SH}
\end{array}$$

We refer to the homomorphisms in these categories as *maps, morphisms, comorphisms* and *homomorphisms* of equipments respectively. To elucidate this definition consider $\underline{\underline{\underline{\mathcal{E}Map}}}$ in a little more detail:

- $\mathrm{ob}(\underline{\underline{\underline{\mathcal{E}Map}}}) =$ the collection of (small, weak) equipments,
- $\underline{\underline{\underline{\mathcal{E}Map}}}((\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*), (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)) = \underline{\underline{\underline{\mathcal{H}oriz}}}_{S*}(\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*), \mathrm{Sq}(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*))$,
- composition and identities are those of $\underline{\underline{\underline{\mathcal{H}oriz}}}_{S*}$.

Observation 1.4.10 provides us with a commuting diamond of enriched forgetful functors

$$\begin{tikzcd}
 & \underline{\underline{\underline{\mathcal{E}Mor}}} \arrow[dr] & \\
\underline{\underline{\underline{\mathcal{E}Hom}}} \arrow[ur] \arrow[dr] & & \underline{\underline{\underline{\mathcal{E}Map}}} \\
 & \underline{\underline{\underline{\mathcal{E}coMor}}} \arrow[ur] &
\end{tikzcd}$$

which all act in a full and faithful manner on modifications and in a faithful manner on transformations. That observation also tells us that the forgetful functors from $\underline{\underline{\underline{\mathcal{E}Hom}}}$ to $\underline{\underline{\underline{\mathcal{E}Mor}}}$ and $\underline{\underline{\underline{\mathcal{E}coMor}}}$ also act in a full manner on transformations and in a faithful manner on homomorphisms.

Notice that since each $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ satisfies the condition of lemma 1.4.9 we could equally well have used $\underline{\underline{\underline{\mathcal{H}oriz}}}_S$ to define $\underline{\underline{\underline{\mathcal{E}Map}}}$. It follows therefore that biadjoints in $\underline{\underline{\underline{\mathcal{E}Map}}}$ have, and may be constructed using, the one sided universal property of proposition 1.4.8; furthermore proposition 1.4.11 and corollary 1.4.12

hold, allowing us to relate enrichments to $\underline{\underline{\underline{\mathcal{E}Mor}}}$ and $\underline{\underline{\underline{\mathcal{E}coMor}}}$ via biadjoints in $\underline{\underline{\underline{\mathcal{E}Map}}}$.

From now onwards, in order to agree with the notation we introduced for equipments, we will use $\circ$ and $\otimes$ to denote horizontal and vertical composition (respectively) in the double bicategory $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$. For some purposes it is useful to translate the definitions given above into a form more suited to the study of equipments:

Proposition 1.5.1 *$\underline{\underline{\underline{\mathcal{E}Map}}}$ has homomorphisms, transformations and modifications which admit more concrete descriptions as follows:*

Definition 1.5.2 (Equipment Maps) *the homomorphisms of $\underline{\underline{\underline{\mathcal{E}Map}}}$, such a map $\ddot{\mathrm{G}}\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$ consists of:*

(a) A homomorphism $\mathrm{G}\colon \underline{\underline{\mathcal{K}}} \longrightarrow \underline{\underline{\mathcal{L}}}$.

(b) For each pair of 0-cells $a, \bar{a} \in \underline{\underline{\mathcal{K}}}$ a functor:

$$\underline{\underline{\mathcal{M}}}(a, \bar{a}) \xrightarrow{\overline{\mathrm{G}}_{a\bar{a}}} \underline{\underline{\mathcal{N}}}(\mathrm{G}(a), \mathrm{G}(\bar{a}))$$

(c) For each pair $f\colon a \longrightarrow a' \in \underline{\underline{\mathcal{K}}}$, $p'\colon a' \longrightarrow\!\!\!\!\!\!+\;\; \bar{a}' \in \underline{\underline{\mathcal{M}}}$ a 2-cell:

$$\overline{\mathrm{G}}(p' \otimes f_*) \overset{\rho_{p'f}}{\Longrightarrow} \overline{\mathrm{G}}(p') \otimes (\mathrm{G}f)_* \text{ in } \underline{\underline{\mathcal{N}}}(\mathrm{G}(a), \mathrm{G}(\bar{a}'))$$

These must be natural in both $p' \in \underline{\underline{\mathcal{M}}}(a', \bar{a}')$ and $f \in \underline{\underline{\mathcal{K}}}(a, a')$. Dually for each pair $p\colon a \longrightarrow\!\!\!\!\!\!+\;\; \bar{a} \in \underline{\underline{\mathcal{M}}}$, $\bar{f}\colon \bar{a} \longrightarrow \bar{a}' \in \underline{\underline{\mathcal{K}}}$ a 2-cell:

$$(\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(p) \overset{\mu_{\bar{f}p}}{\Longrightarrow} \overline{\mathrm{G}}(\bar{f}_* \otimes p) \text{ in } \underline{\underline{\mathcal{N}}}(\mathrm{G}(a), \mathrm{G}(\bar{a}'))$$

Again these must be natural in both $p \in \underline{\underline{\mathcal{M}}}(a, \bar{a})$ and $\bar{f} \in \underline{\underline{\mathcal{K}}}(\bar{a}, \bar{a}')$. These data must satisfy the following conditions:

(d) For any (domain/codomain compatible) 1-cells $p'' \in \underline{\underline{\mathcal{M}}}$ and $f, f' \in \underline{\underline{\mathcal{K}}}$ the diagram

$$\begin{tikzcd}
\overline{\mathrm{G}}(p'' \otimes f'_* \otimes f_*) \arrow[rr, "\rho_{(p''\otimes f'_*)f}"] \arrow[d, "\overline{\mathrm{G}}(p \otimes \mathrm{can})"', "\wr"] & & \overline{\mathrm{G}}(p'' \otimes f'_*) \otimes (\mathrm{G}f)_* \arrow[d, "\rho_{p''f'} \otimes (\mathrm{G}f)_*"] \\
\overline{\mathrm{G}}(p'' \otimes (f' \circ f)_*) \arrow[r, "\rho_{p''(f'\circ f)}"'] & \overline{\mathrm{G}}(p'') \otimes (\mathrm{G}(f' \circ f))_* & \overline{\mathrm{G}}(p'') \otimes (\mathrm{G}f')_* \otimes (\mathrm{G}f)_* \arrow[l, "\simeq", "\overline{\mathrm{G}}(p'') \otimes \mathrm{can}"']
\end{tikzcd}$$

commutes. We also stipulate that the comparison maps $\mu_{\bar{f}p}$ must satisfy the obvious dual rule.

(e) *For each 1-cell $p \in \underline{\underline{\mathcal{M}}}$ the diagram*

$$\begin{tikzcd}
\overline{\mathrm{G}}(p) \arrow[r, "\mathrm{can}", "\simeq"'] \arrow[d, "\overline{\mathrm{G}}(\mathrm{can})"', "\wr"] & \overline{\mathrm{G}}(p) \otimes (\mathrm{i}_{\mathrm{G}a})_* \arrow[d, "\overline{\mathrm{G}}(p)\otimes \mathrm{can}_*", "\wr"'] \\
\overline{\mathrm{G}}(p \otimes (\mathrm{i}_a)_*) \arrow[r, "\rho_{p\,\mathrm{i}_a}"'] & \overline{\mathrm{G}}(p) \otimes (\mathrm{G}\mathrm{i}_a)_*
\end{tikzcd}$$

commutes. We also require that a dual rule applies to comparison maps $\mu_{\mathrm{i}_{\bar{a}}\,p}$.

(f) *For any (domain/codomain compatible) 1-cells $\bar{f}', f \in \underline{\underline{\mathcal{K}}}$ and $p' \in \underline{\underline{\mathcal{M}}}$ the diagram*

$$\begin{tikzcd}
(\mathrm{G}\bar{f}')_* \otimes \overline{\mathrm{G}}(p' \otimes f_*) \arrow[r, "(\mathrm{G}\bar{f}')_* \otimes \rho_{p'f}"] \arrow[d, "\mu_{\bar{f}'(p'\otimes f_*)}"'] & (\mathrm{G}\bar{f}')_* \otimes \overline{\mathrm{G}}(p') \otimes (\mathrm{G}f)_* \arrow[d, "\mu_{\bar{f}'p'} \otimes (\mathrm{G}f)_*"'] \\
\overline{\mathrm{G}}(\bar{f}'_* \otimes p' \otimes f_*) \arrow[r, "\rho_{(\bar{f}'_*\otimes p')f}"] & \overline{\mathrm{G}}(\bar{f}'_* \otimes p') \otimes (\mathrm{G}f)_*
\end{tikzcd}$$

commutes.

Definition 1.5.3 (Transformations of Equipment Maps) *Such a transformation $\ddot{\Psi}\colon \ddot{\mathrm{G}} \longrightarrow \ddot{\mathrm{H}}$ consists of:*

(i) *A strong transformation $\Psi\colon \mathrm{G} \longrightarrow \mathrm{H}$.*

(ii) *For each $p\colon a \longrightarrow \bar{a} \in \underline{\underline{\mathcal{M}}}$ a 2-cell:*

$$\begin{tikzcd}
\mathrm{G}(a) \arrow[r, "(\Psi_a)_*"] \arrow[d, "\overline{\mathrm{G}}(p)"', "\shortmid" marking] \arrow[dr, phantom, "\overline{\Psi}_p \Uparrow"] & \mathrm{H}(a) \arrow[d, "\overline{\mathrm{H}}(p) \in \underline{\underline{\mathcal{N}}}", "\shortmid" marking] \\
\mathrm{G}(\bar{a}) \arrow[r, "(\Psi_{\bar{a}})_*"'] & \mathrm{H}(\bar{a})
\end{tikzcd}$$

These data must satisfy:

(iii) *For each 2-cell $\gamma\colon p \Rightarrow \dot{p} \in \underline{\underline{\mathcal{M}}}$ the following pasting equality holds:*

$$\begin{tikzcd}
\mathrm{G}(a) \arrow[rr, "(\Psi_a)_*"] \arrow[d, "\overline{\mathrm{G}}(p)"', "\shortmid" marking] \arrow[d, phantom, shift left=4, "\overset{\overline{\mathrm{G}}(\gamma)}{\Rightarrow}"] & & \mathrm{H}(a) \arrow[d, "\overline{\mathrm{H}}(\dot{p})", "\shortmid" marking] \\
\mathrm{G}(\bar{a}) \arrow[rr, "(\Psi_{\bar{a}})_*"'] & \Uparrow \overline{\Psi}_{\dot{p}} & \mathrm{H}(\bar{a})
\end{tikzcd}
\quad \overline{\mathrm{G}}(\dot{p}) \quad = \quad
\begin{tikzcd}
\mathrm{G}(a) \arrow[rr, "(\Psi_a)_*"] \arrow[d, "\overline{\mathrm{G}}(p)"', "\shortmid" marking] & \overline{\Psi}_p \Uparrow & \mathrm{H}(a) \arrow[d, "\overline{\mathrm{H}}(p)"', "\shortmid" marking] \arrow[d, phantom, shift left=4, "\overset{\overline{\mathrm{H}}(\gamma)}{\Rightarrow}"] \\
\mathrm{G}(\bar{a}) \arrow[rr, "(\Psi_{\bar{a}})_*"'] & & \mathrm{H}(\bar{a})
\end{tikzcd}
\quad \overline{\mathrm{H}}(\dot{p})$$

(iv) For 1-cells $f\colon a \longrightarrow a' \in \underline{\underline{\mathcal{K}}}$ and $p'\colon a' \longrightarrow\!\!\!\!\!\!\!+\;\; \bar{a}' \in \underline{\underline{\mathcal{M}}}$ the pasting equality

$$
\begin{tikzcd}
\mathrm{G}(a) \arrow[r, "(\Psi_a)_*"] \arrow[dr, "(\mathrm{G}f)_*"'] \arrow[dd, "\overline{\mathrm{G}}(p'\otimes f_*)"', "\shortmid" marking] & \mathrm{H}(a) \arrow[dr, "(\mathrm{H}f)_*"] \arrow[d, phantom, "\overset{\cong}{(\Psi_f)_*}"] & \\
\arrow[r, phantom, "\overset{\rho_{p'f}}{\Rightarrow}"] & \mathrm{G}(a') \arrow[r, "(\Psi_{a'})_*"'] \arrow[dl, "\overline{\mathrm{G}}(p')"', "\shortmid" marking] \arrow[d, phantom, "\Uparrow \overline{\Psi}_p"] & \mathrm{H}(a') \arrow[dl, "\overline{\mathrm{H}}(p')", "\shortmid" marking] \\
\mathrm{G}(\bar{a}') \arrow[r, "(\Psi_{\bar{a}'})_*"'] & \mathrm{H}(\bar{a}') &
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{G}(a) \arrow[r, "(\Psi_a)_*"] \arrow[dd, "\overline{\mathrm{G}}(p'\otimes f_*)"', "\shortmid" marking] \arrow[ddr, phantom, "\overline{\Psi}_{p'\otimes f_*} \Uparrow"] & \mathrm{H}(a) \arrow[dr, "(\mathrm{H}f)_*"] \arrow[dd, "\overline{\mathrm{H}}(p'\otimes f_*)"', "\shortmid" marking] & \\
& \arrow[r, phantom, "\overset{\rho_{p'f}}{\Rightarrow}"] & \mathrm{H}(a') \arrow[dl, "\overline{\mathrm{H}}(p')", "\shortmid" marking] \\
\mathrm{G}(\bar{a}') \arrow[r, "(\Psi_{\bar{a}'})_*"'] & \mathrm{H}(\bar{a}') &
\end{tikzcd}
$$

in $\underline{\underline{\mathcal{N}}}$ holds. The dual rule involving the 2-cells $\mu_{\bar{f}p}$ of $\ddot{\mathrm{G}}$ and $\ddot{\mathrm{H}}$ must also hold true.

Definition 1.5.4 *Modifications $\ddot{\alpha}\colon \ddot{\Psi} \Rightarrow \ddot{\Phi}$ in $\underline{\underline{\mathcal{E}Map}}$ consist of a strong transformation $\alpha\colon \Psi \Rightarrow \Phi$ satisfying the pasting condition*

$$
\begin{tikzcd}
\mathrm{G}(a) \arrow[r, shift left=2, "(\Psi_a)_*"] \arrow[r, shift right=2, "(\Phi_a)_*"'] \arrow[r, phantom, "\Uparrow (\alpha_a)_*"] \arrow[d, "\overline{\mathrm{G}}(p)"', "\shortmid" marking] \arrow[dr, phantom, "\Uparrow \overline{\Phi}_p"] & \mathrm{H}(a) \arrow[d, "\overline{\mathrm{H}}(p)", "\shortmid" marking] \\
\mathrm{G}(\bar{a}) \arrow[r, "(\Phi_{\bar{a}})_*"'] & \mathrm{H}(\bar{a})
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{G}(a) \arrow[r, "(\Psi_a)_*"] \arrow[d, "\overline{\mathrm{G}}(p)"', "\shortmid" marking] \arrow[dr, phantom, "\Uparrow \overline{\Psi}_p"] & \mathrm{H}(a) \arrow[d, "\overline{\mathrm{H}}(p)", "\shortmid" marking] \\
\mathrm{G}(\bar{a}) \arrow[r, shift left=2, "(\Psi_{\bar{a}})_*"] \arrow[r, shift right=2, "(\Phi_{\bar{a}})_*"'] \arrow[r, phantom, "\Uparrow (\alpha_{\bar{a}})_*"] & \mathrm{H}(\bar{a})
\end{tikzcd}
$$

for each 1-cell $p\colon a \longrightarrow\!\!\!\!\!\!\!+\;\; \bar{a} \in \underline{\underline{\mathcal{M}}}$.

Proof. (of proposition 1.5.1) Given an equipment map

$$(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \xrightarrow{\ddot{\mathrm{G}}} (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$$

clauses (a) and (b) of its definition provide us with the actions, on horizontal and vertical cells, of a homomorphism

$$\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \xrightarrow{\tilde{\mathrm{G}}} \mathrm{Sq}(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*) \tag{1.70}$$

in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{S*}$, for which the action on squares

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\Uparrow \lambda"] & a' \arrow[d, "p'"] \\ \bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \end{tikzcd} \longmapsto \begin{tikzcd} \mathrm{G}(a) \arrow[r, "\mathrm{G}(f)"] \arrow[d, "\overline{\mathrm{G}}(p)"'] \arrow[dr, phantom, "\Uparrow \mathrm{G}_S(\lambda)"] & \mathrm{G}(a') \arrow[d, "\overline{\mathrm{G}}(p')"] \\ \mathrm{G}(\bar{a}) \arrow[r, "\mathrm{G}(\bar{f})"'] & \mathrm{G}(\bar{a}') \end{tikzcd}$$

is given by the composite:

$$(\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(p) \xRightarrow{\mu_{\bar{f}p}} \overline{\mathrm{G}}(\bar{f}_* \otimes p) \xRightarrow{\overline{\mathrm{G}}(\lambda)} \overline{\mathrm{G}}(p' \otimes f) \xRightarrow{\rho_{p'f}} \overline{\mathrm{G}}(p') \otimes (\mathrm{G}f)_* \tag{1.71}$$

The naturality of $\mu_{\bar{f}p}$ and $\rho_{p'f}$ in both variables ensures that $\tilde{\mathrm{G}}$ defined in this way preserves the actions of horizontal and vertical 2-cells on squares. Straightforward diagram chases serve to establish the two remaining cylinder conditions (v) and (vi) in the definition of horizontal maps on page 104. To demonstrate (v) we use conditions (d) and (f) of definition 1.5.2 and (vi) follows from clause (e).

To prove the converse consider the double bicategory $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{–})_*)$, we have canonical squares

$$\begin{tikzcd} a \arrow[r, "\mathrm{i}_a"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\Theta_{\bar{f}p}}{\cong}"] & a \arrow[d, "\bar{f}_* \otimes p"] \\ \bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "f"] \arrow[d, "p' \otimes f_*"'] \arrow[dr, phantom, "\overset{\Pi_{p'f}}{\cong}"] & a' \arrow[d, "p'"] \\ \bar{a}' \arrow[r, "\mathrm{i}_{\bar{a}'}"'] & \bar{a}' \end{tikzcd}$$

for each pair of 2-cells $p \in \underline{\underline{\mathcal{M}}}, \bar{f} \in \underline{\underline{\mathcal{K}}}$ and $f \in \underline{\underline{\mathcal{K}}}, p' \in \underline{\underline{\mathcal{M}}}$. With the help of these we may usefully describe the connection between squares and vertical 2-cells in $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{–})_*)$, there is a bijection

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\underline{\lambda|}"] & a' \arrow[d, "p'"] \\ \bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \end{tikzcd} \longleftrightarrow \begin{tikzcd} a \arrow[d, bend right, "\bar{f}_* \otimes p"'] \arrow[d, bend left, "p' \otimes f_*"] \arrow[d, phantom, "\overset{\lambda}{\Rightarrow}"] \\ \bar{a}' \end{tikzcd}$$

such that the canonical 2-cells

$$\mathrm{can}\colon f \circ \mathrm{i}_a \overset{\cong}{\Longrightarrow} f \qquad\qquad \mathrm{can}\colon \mathrm{i}_{\bar{a}'} \circ \bar{f} \overset{\cong}{\Longrightarrow} \bar{f}$$

in $\underline{\underline{\mathcal{K}}}$ constitute the components of a (horizontal) cylinder:

$$(\Pi_{p'f} \underset{H}{*} \lambda) \circ \Theta_{\bar{f}p} \Longrightarrow_H \lambda \tag{1.72}$$

Here we hope that no confusion arises in using λ in the codomain for the square and in the domain for the vertical 2-cell $\bar{f}_* \otimes p \Rightarrow p' \otimes f_*$.

Given a horizontal map as in (1.70) use G and $\overline{\mathrm{G}}$ to denote its actions on horizontal and vertical cells respectively, so as to agree with the notation introduced for equipment maps, furthermore define 2-cells $\mu_{\bar{f}p}$ and $\rho_{p'f}$ to be the composites

$$\begin{aligned}
(\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(p) &\xRightarrow{\mathrm{G}_S(\Theta_{\bar{f}p})} \overline{\mathrm{G}}(\bar{f}_* \otimes p) \otimes (\mathrm{Gi}_a)_* \xRightarrow[\sim]{\overline{\mathrm{G}}(\bar{f}_*\otimes p)\otimes\mathrm{can}} \overline{\mathrm{G}}(\bar{f}_* \otimes p) \otimes \mathrm{i}_{\mathrm{G}a} \\
&\xRightarrow[\sim]{\mathrm{can}} \overline{\mathrm{G}}(\bar{f}_* \otimes p) \text{ and} \\
\overline{\mathrm{G}}(p' \otimes f_*) &\xRightarrow[\sim]{\mathrm{can}} \mathrm{i}_{\mathrm{G}\bar{a}'} \otimes \overline{\mathrm{G}}(p' \otimes f_*) \xRightarrow[\sim]{\mathrm{can}\otimes\overline{\mathrm{G}}(p'\otimes f_*)} (\mathrm{Gi}_{\bar{a}'})_* \otimes \overline{\mathrm{G}}(p' \otimes f_*) \\
&\xRightarrow{\mathrm{G}_S(\Pi_{p'f})} \overline{\mathrm{G}}(p') \otimes (\mathrm{G}f)_*
\end{aligned} \tag{1.73}$$

respectively. As a horizontal map we know that $\tilde{\mathrm{G}}$ preserves the actions of vertical and horizontal 2-cells on squares. Therefore applying it to the cylinder in (1.72) and appealing to the cylinder condition satisfied by the compositional comparisons of $\mathrm{G}\colon \underline{\underline{\mathcal{K}}} \longrightarrow \underline{\underline{\mathcal{L}}}$ ((v) page 104) we see that the pair

$$\begin{aligned}
\mathrm{G}(f) \circ \mathrm{G}(\mathrm{i}_a) &\xRightarrow[\sim]{\mathrm{can}} \mathrm{G}(f \circ \mathrm{i}_a) \xRightarrow[\sim]{\mathrm{G(can)}} \mathrm{G}(f) \\
\mathrm{G}(\mathrm{i}_{\bar{a}'}) \circ \mathrm{G}(\bar{f}) &\xRightarrow[\sim]{\mathrm{can}} \mathrm{G}(\mathrm{i}_{\bar{a}'} \circ \bar{f}) \xRightarrow[\sim]{\mathrm{G(can)}} \mathrm{G}(\bar{f})
\end{aligned}$$

constitute a cylinder:

$$(\mathrm{G}_S(\Pi_{p'f}) \underset{H}{*} \overline{\mathrm{G}}(\lambda)) \circ \mathrm{G}_S(\Theta_{\bar{f}p}) \Longrightarrow_H \mathrm{G}_S(\lambda)$$

Starting with this cylinder a straightforward diagram chase demonstrates that with $\mu_{\bar{f}p}$ and $\rho_{p'f}$ defined as in (1.73) the square $\mathrm{G}_S(\lambda)$ is given by the composite in (1.71). Notice that these families are the unique ones which satisfy both condition (e) of definition 1.5.2 and our property relating them to the action of $\tilde{\mathrm{G}}$ on squares.

The first step in demonstrating that the data $(\overline{\mathrm{G}}, \mathrm{G}, \mu, \rho)$ derived from $\tilde{\mathrm{G}}$ (as above) satisfies the conditions required of an equipment map is to notice that μ and ρ are natural in both variables since $\tilde{\mathrm{G}}$ preserves actions of 2-cells on squares. In order to establish the remaining three conditions consider the following equalities in $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$:

(i) For 1-cells $p'' \in \underline{\underline{\mathcal{M}}}$ and $f', f \in \underline{\underline{\mathcal{K}}}$ as in 1.5.2(d) we have

$$\mathrm{can}_1 \underset{V}{*} (\Pi_{p''f'} \circ \Pi_{(p''\otimes f'_*)f}) = \Pi_{p''(f'\circ f)} \underset{H}{*} (p'' \otimes \mathrm{can}_2)$$

where the 2-cells denoted "can" are the canonical cells:

$$\mathrm{i}_{\bar{a}''} \circ \mathrm{i}_{\bar{a}''} \xRightarrow[\sim]{\mathrm{can}_1}_H \mathrm{i}_{\bar{a}''} \ \in \underline{\underline{\mathcal{K}}} \qquad\qquad f'_* \otimes f_* \xRightarrow[\sim]{\mathrm{can}_2}_V (f' \circ f)_* \ \in \underline{\underline{\mathcal{M}}}$$

A similar identity holds amongst the squares $\Theta_{\bar{f}p}$.

(ii) For a 1-cell $p \in \underline{\underline{\mathcal{M}}}$ as in 1.5.2(e) we have

$$\Pi_{p\,\mathrm{i}_a} = \mathrm{i}^h_p \underset{H}{*} \mathrm{can}$$

where "can" denotes the canonical isomorphism:

$$p \otimes (\mathrm{i}_a)_* \xRightarrow[\sim]{\mathrm{can}} p$$

A similar identity holds for squares $\Theta_{\mathrm{i}_{\bar{a}}p}$.

(iii) For 1-cells $\bar{f}', f \in \underline{\underline{\mathcal{K}}}$ and $p' \in \underline{\underline{\mathcal{M}}}$ as in 1.5.2(f) the pair of canonical isomorphisms

$$\mathrm{i}_{a'} \circ f \xRightarrow{\sim} f \xRightarrow{\sim} f \circ \mathrm{i}_a$$
$$\bar{f}' \circ \mathrm{i}_{\bar{a}'} \xRightarrow{\sim} \bar{f}' \xRightarrow{\sim} \mathrm{i}_{\bar{a}''} \circ \bar{f}'$$

in $\underline{\underline{\mathcal{K}}}$ comprise the components of a cylinder:

$$\Theta_{\bar{f}'p'} \circ \Pi_{p'f} \xRightarrow{\sim}_H \Pi_{(\bar{f}'_* \otimes p')f} \circ \Theta_{\bar{f}'(p' \otimes f)}$$

Each is easily verified and follows directly from the coherence properties of the homomorphism $(\text{-})_* \colon \underline{\underline{\mathcal{K}}} \longrightarrow \underline{\underline{\mathcal{M}}}$.

On examining these properties and recalling the definitions of $\mu_{\bar{f}p}$ and $\rho_{p'f}$ in terms of $\mathrm{G}_S(\Theta_{\bar{f}p})$ and $\mathrm{G}_S(\Pi_{p'f})$ we see a strong resemblance to conditions (d)–(f) of definition 1.5.2. Indeed it is clear that by applying $\tilde{\mathrm{G}}$ to each one then arguing as we did on page 129 (for the cylinder in (1.72)) we obtain equations in $\mathrm{Sq}(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$ which, with a little elementary diagram chasing, yield each of the required conditions (d)–(f) in turn.

We have already demonstrated (page 129) that on applying both constructions described above to a horizontal map we regain the structure we started with. Conversely it is easy to show, using condition (e) of definition 1.5.2, that the same is true if we begin with an equipment map instead, it follows that equipment maps correspond to homomorphisms in $\underline{\underline{\underline{\mathcal{E}Map}}}$. We leave it up to the reader to verify that the structures of definitions 1.5.3 and 1.5.4 do indeed describe the transformations and modifications of $\underline{\underline{\underline{\mathcal{E}Map}}}$, we hope that the method for doing so should be clear. We also leave it up to the reader identify the various compositions of $\underline{\underline{\underline{\mathcal{E}Map}}}$ in terms of the descriptions given in this proposition, but pause to point out that they all present themselves naturally from the information given. □

We will also need a concrete description of the homomorphisms of $\underline{\underline{\underline{\mathcal{E}Mor}}}$:

Lemma 1.5.5 *Homomorphisms in $\underline{\underline{\mathcal{E}Mor}}$, called equipment morphisms, admit a description with the flavour of the last proposition. These consist of an equipment map $\ddot{\mathrm{G}}\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$ and a morphism $\overline{\mathrm{G}}\colon \underline{\underline{\mathcal{M}}} \longrightarrow \underline{\underline{\mathcal{N}}}$ enriching $\ddot{\mathrm{G}}$s action on the homsets of $\underline{\underline{\mathcal{M}}}$, satisfying:*

(i) For any (domain/codomain compatible) 1-cells $\bar{p}', p' \in \underline{\underline{\mathcal{M}}}$ and $f \in \underline{\underline{\mathcal{K}}}$ the diagram

$$\begin{tikzcd}
\overline{\mathrm{G}}(\bar{p}') \otimes \overline{\mathrm{G}}(p' \otimes f_*) \arrow[r, "\overline{\mathrm{G}}(\bar{p}') \otimes \rho_{p'f}"] \arrow[d, "\mathrm{can}"'] & \overline{\mathrm{G}}(\bar{p}') \otimes \overline{\mathrm{G}}(p') \otimes (\mathrm{G}f)_* \arrow[d, "\mathrm{can} \otimes (\mathrm{G}f)_*"] \\
\overline{\mathrm{G}}(\bar{p}' \otimes p' \otimes f_*) \arrow[r, "\rho_{(\bar{p}'\otimes p')f}"'] & \overline{\mathrm{G}}(\bar{p}' \otimes p') \otimes (\mathrm{G}f)_*
\end{tikzcd}$$

commutes. We also insist that the maps $\mu_{\bar{f}p}$ satisfy the obvious dual rule.

(ii) For each $f\colon a \longrightarrow a' \in \underline{\underline{\mathcal{K}}}$ we have a commutative diagram:

$$\begin{tikzcd}
 & (\mathrm{G}f)_* \otimes \overline{\mathrm{G}}(\mathrm{i}_a) \arrow[r, "\mu_{f\,\mathrm{i}_a}"] & \overline{\mathrm{G}}(f_* \otimes \mathrm{i}_a) \arrow[dr, "\overline{\mathrm{G}}(\mathrm{can})", "\cong"'] & \\
(\mathrm{G}f)_* \arrow[ur, "(\mathrm{G}f)_* \otimes \mathrm{can}"] \arrow[dr, "\mathrm{can} \otimes (\mathrm{G}f)_*"'] & & & \overline{\mathrm{G}}(f_*) \arrow[dl, "\overline{\mathrm{G}}(\mathrm{can})", "\cong"'] \\
 & \overline{\mathrm{G}}(\mathrm{i}_{a'}) \otimes (\mathrm{G}f)_* & \overline{\mathrm{G}}(\mathrm{i}_{a'} \otimes f_*) \arrow[l, "\rho_{\mathrm{i}_{a'}f}"] &
\end{tikzcd}$$

(iii) For any (domain/codomain compatible) 1-cells $\bar{f} \in \underline{\underline{\mathcal{K}}}$ and $\bar{p}', p \in \underline{\underline{\mathcal{M}}}$ we have a commutative diagram:

$$\begin{tikzcd}
\overline{\mathrm{G}}(\bar{p}' \otimes \bar{f}_*) \otimes \overline{\mathrm{G}}(p) \arrow[r, "\rho_{\bar{p}'\bar{f}} \otimes \overline{\mathrm{G}}(p)"] \arrow[d, "\mathrm{can}"', "\wr"] & \overline{\mathrm{G}}(\bar{p}') \otimes (\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(p) \arrow[d, "\overline{\mathrm{G}}(\bar{p}') \otimes \mu_{\bar{f}p}"] \\
\overline{\mathrm{G}}(\bar{p}' \otimes \bar{f}_* \otimes p) & \overline{\mathrm{G}}(\bar{p}') \otimes \overline{\mathrm{G}}(\bar{f}_* \otimes p) \arrow[l, "\simeq", "\mathrm{can}"']
\end{tikzcd}$$

Proof. Under the bijection established in the last proposition the equipment map $\ddot{\mathrm{G}}$ corresponds to some horizontal map

$$\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \xrightarrow{\tilde{\mathrm{G}}} \mathrm{Sq}(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$$

and the morphism $\overline{\mathrm{G}}: \underline{\underline{\mathcal{M}}} \longrightarrow \underline{\underline{\mathcal{N}}}$ is exactly an enrichment of the action of $\tilde{\mathrm{G}}$ on vertical cells. All that remains is to demonstrate that the conditions in the statement of this lemma correspond to the cylinder conditions given in the definition of the homomorphisms in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$

It should be clear, once we take into account the definition of $\tilde{\mathrm{G}}$s action on squares as given in (1.71), that condition (ii) above is a literal restatement of condition (b) on page 116. Furthermore a simple diagram chase involving (i) and (iii) establishes condition (a) (of the same definition), therefore $\overline{\mathrm{G}}$ does enrich $\tilde{\mathrm{G}}$ to a homomorphism in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$.

To show the converse (eg. verify (i) and (iii) once we know that $\overline{\mathrm{G}}$ enriches $\tilde{\mathrm{G}}$ to $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$) consider the following identities in $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ (which are easily verified):

(i)′ For 1-cells $\bar{p}', p' \in \underline{\underline{\mathcal{M}}}$ and $f \in \underline{\underline{\mathcal{K}}}$ as in condition (i) of this lemma we have

$$(\mathrm{i}^h_{\bar{p}'} \otimes \Pi_{p'f}) \underset{H}{*} \mathrm{can} = \Pi_{(\bar{p}' \otimes p')f}$$

where "can" denotes the associativity isomorphism:

$$(\bar{p}' \otimes p') \otimes f_* \overset{\text{can}}{\underset{\cong}{\Longrightarrow}} \bar{p}' \otimes (p' \otimes f_*)$$

A dual property holds amongst squares $\Theta_{\bar{f}p}$.

(iii)′ For 1-cells $\bar{p}', p \in \underline{\underline{\mathcal{M}}}$ and $\bar{f} \in \underline{\underline{\mathcal{K}}}$ as in condition (iii) of this lemma we have

$$(\Pi_{\bar{p}'\bar{f}} \otimes \Theta_{\bar{f}p}) \underset{H}{*} \mathrm{can} = \mathrm{i}^h_{\bar{p}' \otimes (\bar{f}_* \otimes p)}$$

where the vertical 2-cell named "can" is the canonical associativity:

$$\bar{p}' \otimes (\bar{f}_* \otimes p) \overset{\text{can}}{\underset{\cong}{\Longrightarrow}}_V (\bar{p}' \otimes \bar{f}_*) \otimes p$$

Following the argument at the end of the proof of proposition 1.5.1, notice that the resemblance between these identities and conditions (i) and (iii) becomes clear as soon as we recall the definitions of $\mu_{\bar{f}p}$ and $\rho_{p'f}$ in terms of $\mathrm{G}_S(\Theta_{\bar{f}p})$ and $\mathrm{G}_S(\Pi_{p'f})$. Simply apply the homomorphism $\tilde{\mathrm{G}} \in \underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ to each identity then argue as we did before, using the cylinder conditions on page 116, to obtain (i) and (iii) from (i)′ and (iii)′ respectively. □

Let us look in a little more detail at these equipment morphisms:

Lemma 1.5.6 *Suppose that* $\ddot{\mathrm{G}}$ *is an equipment morphism (as in the last lemma) then each 2-cell*

$$\overline{\mathrm{G}}(p' \otimes f_*) \xRightarrow{\rho_{p'f}} \overline{\mathrm{G}}(p') \otimes (\mathrm{G}f)_*$$

is an isomorphism.

Proof. For each 1-cell $f \in \underline{\underline{\mathcal{K}}}$ define the 2-cell

$$(\mathrm{G}f)_* \xRightarrow{\quad \nu_f \quad} \overline{\mathrm{G}}(f_*)$$

in $\underline{\underline{\mathcal{N}}}$ to be the composite;

$$(\mathrm{G}f)_* \xRightarrow{(\mathrm{G}f)_* \otimes \mathrm{can}} (\mathrm{G}f)_* \otimes \overline{\mathrm{G}}(\mathrm{i}_a) \xRightarrow{\mu_{f\,\mathrm{i}_a}} \overline{\mathrm{G}}(f_* \otimes \mathrm{i}_a) \xRightarrow[\sim]{\overline{\mathrm{G}}(\mathrm{can})} \overline{\mathrm{G}}(f_*)$$

which we use to get another 2-cell

$$\overline{\mathrm{G}}(p') \otimes (\mathrm{G}f)_* \xRightarrow{\rho^{-1}_{p'f}} \overline{\mathrm{G}}(p' \otimes f_*)$$

for each 1-cell $p' \in \underline{\underline{\mathcal{M}}}$, given by the composite:

$$\overline{\mathrm{G}}(p') \otimes (\mathrm{G}f)_* \xRightarrow{\overline{\mathrm{G}}(p') \otimes \nu_f} \overline{\mathrm{G}}(p') \otimes \overline{\mathrm{G}}(f_*) \xRightarrow{\mathrm{can}} \overline{\mathrm{G}}(p' \otimes f_*) \tag{1.74}$$

Straightforward diagram chases now demonstrate that conditions (i) and (ii) on $\ddot{\mathrm{G}}$ (see statement of lemma 1.5.5) imply that $\rho_{p'f} \bullet \rho^{-1}_{p'f} = \mathrm{i}_{\overline{\mathrm{G}}(p') \otimes (\mathrm{G}f)_*}$ and condition (iii) implies the other identity $\rho^{-1}_{p'f} \bullet \rho_{p'f} = \mathrm{i}_{\overline{\mathrm{G}}(p' \otimes f_*)}$. □

This proof of the previous lemma opens up the possibility of a more useful description of equipment morphisms:

Lemma 1.5.7 *Given equipments* $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$ *and* $(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (-)_*)$ *and a pair* $(\overline{\mathrm{G}}, \mathrm{G})$ *consisting of a homomorphism* $\mathrm{G}\colon \underline{\underline{\mathcal{K}}} \longrightarrow \underline{\underline{\mathcal{L}}}$ *and a morphism* $\overline{\mathrm{G}}\colon \underline{\underline{\mathcal{M}}} \longrightarrow \underline{\underline{\mathcal{L}}}$ *with the same actions on 0-cells then there is a bijection between:*

(A) *Families of 2-cells*

$$\overline{\mathrm{G}}(p' \otimes f_*) \xRightarrow{\rho_{p'f}} \overline{\mathrm{G}}(p') \otimes (\mathrm{G}f)_* \qquad p' \in \underline{\underline{\mathcal{M}}}, f \in \underline{\underline{\mathcal{K}}}$$

$$(\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(p) \xRightarrow{\mu_{\bar{f}p}} \overline{\mathrm{G}}(\bar{f}_* \otimes p) \qquad p \in \underline{\underline{\mathcal{M}}}, \bar{f} \in \underline{\underline{\mathcal{K}}}$$

natural in all variables and satisfying conditions (i)-(iii) of lemma 1.5.5.

(B) *A family of 2-cells*

$$(\mathrm{G}f)_* \xrightarrow{\nu_f} \overline{\mathrm{G}}(f_*) \qquad (f \in \underline{\underline{\mathcal{K}}})$$

natural in the variable f and satisfying the condition that each 2-cell $\rho^{-1}_{p'f}$ derived from ν_f as in display (1.74) is an isomorphism.

Proof. We set this bijection up as follows:

- **(A) $\mapsto$ (B)** Given the families $\rho_{p'f}$ and $\mu_{\bar{f}p}$ of (A) define ν_f as in lemma 1.5.6, which ensures naturality in the variable f and that each $\rho^{-1}_{p'f}$ is invertible.
- **(B) $\mapsto$ (A)** Given the family ν_f ($f \in \underline{\underline{\mathcal{K}}}$) as in (B), define $\rho^{-1}_{p'f}$ from ν_f as in lemma 1.5.5 and these are isomorphisms (by assumption) therefore we get a family of their inverses $\rho_{p'f}$, similarly we define $\mu_{\bar{f}p}$ to be the composite:

$$(\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(p) \xrightarrow{\nu_{\bar{f}} \otimes \overline{\mathrm{G}}(p)} \overline{\mathrm{G}}(\bar{f}_*) \otimes \overline{\mathrm{G}}(p) \xrightarrow{\text{can}} \overline{\mathrm{G}}(\bar{f}_* \otimes p)$$

The naturality of ν_f ensures that families defined in this way are natural in both variables, and once we have translated conditions (i)-(iii) of lemma 1.5.5 so as to refer to $\rho^{-1}_{p'f}$ (rather than $\rho_{p'f}$) they too may be established by easy diagram chases.

We leave the (easy) demonstration that these two constructions are mutual inverses up to the reader. □

Proposition 1.5.8 *Under the bijection of the last lemma equipment morphisms*

$$(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \xrightarrow{\ddot{\mathrm{G}}} (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$$

correspond to triples $(\overline{\mathrm{G}}, \mathrm{G}, \nu)$*. These consist of a homomorphism*

$$\underline{\underline{\mathcal{K}}} \xrightarrow{\mathrm{G}} \underline{\underline{\mathcal{L}}}$$

a morphism

$$\underline{\underline{\mathcal{M}}} \xrightarrow{\overline{\mathrm{G}}} \underline{\underline{\mathcal{N}}}$$

with the same action on 0-cells as G*, and a family of 2-cells*

$$(\mathrm{G}f)_* \xrightarrow{\nu_f} \overline{\mathrm{G}}(f_*)$$

natural in $f \in \underline{\underline{\mathcal{K}}}$ and satisfying:

(c)′ Each 2-cell $\rho^{-1}_{p'f}$ defined as in (1.74) from ν_f is an isomorphism.

(d)′ For each (compatible) pair of 1-cells $f, f' \in \underline{\underline{\mathcal{K}}}$ the following diagram commutes:

$$\begin{tikzcd}
(\mathrm{G}f')_* \otimes (\mathrm{G}f)_* \arrow[r, "\nu_{f'} \otimes \nu_f"] \arrow[d, "\mathrm{can}"', "\wr"] & \overline{\mathrm{G}}(f'_*) \otimes \overline{\mathrm{G}}(f_*) \arrow[d, "\mathrm{can}"] \\
(\mathrm{G}(f' \circ f))_* \arrow[r, "\nu_{f' \circ f}"'] & \overline{\mathrm{G}}((f' \circ f)_*)
\end{tikzcd}$$

(e)′ For each 0-cell $a \in \underline{\underline{\mathcal{K}}}$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{i}_{\mathrm{G}a} \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{can}"', "\wr"] & \overline{\mathrm{G}}(\mathrm{i}_a) \arrow[d, "\overline{\mathrm{G}}(\mathrm{can})", "\wr"'] \\
(\mathrm{Gi}_a)_* \arrow[r, "\nu_{\mathrm{i}_a}"'] & \overline{\mathrm{G}}((\mathrm{i}_a)_*)
\end{tikzcd}$$

Proof. To prove this first convert conditions (d)-(f) of definition 1.5.2 to refer to $\rho^{-1}_{p'f}$ rather than $\rho_{p'f}$ itself. Now substituting the definitions of ρ^{-1} and μ in terms of ν into condition (f) we see that it holds without having to assume (d)′ or (e)′, therefore it remains to prove that under the bijection of the last lemma (d)′ $\Leftrightarrow$ (d) and (e)′ $\Leftrightarrow$ (e)

The forward implications follow by substituting the expressions for ρ^{-1} and μ in terms of ν into (d) and (e) then performing simple diagram chases involving (d)′ and (e)′ respectively. Conversely let $p = \mathrm{i}_a$ in the portions of (d) and (e) relating to μ, then apply the definition of ν in terms of μ easy diagram chases to reduce them to (d)′ and (e)′ in turn. $\square$

It may also be of use to give some indication of how to describe the transformations and modifications of $\underline{\underline{\underline{\mathcal{E}Mor}}}$ in the manner of the last proposition, we leave verifications of these up to the reader:

transformations: $(\overline{\Psi}, \Psi)\colon (\overline{\mathrm{G}}, \mathrm{G}, \nu) \longrightarrow (\overline{\mathrm{H}}, \mathrm{H}, \xi)$ in $\underline{\underline{\underline{\mathcal{E}Mor}}}$ consisting of a strong transformation $\Psi\colon \mathrm{G} \longrightarrow \mathrm{H}$ and an optransformation $\overline{\Psi}\colon \overline{\mathrm{G}} \longrightarrow \overline{\mathrm{H}}$ satisfying:

- For each 0-cell $a \in \underline{\underline{\mathcal{K}}}$ we have $\overline{\Psi}_a = (\Psi_a)_*$ and

- For each 1-cell $f\colon a \longrightarrow a' \in \underline{\underline{\mathcal{K}}}$ the following pasting identity holds:

$$\begin{tikzcd}[column sep=large]
\mathrm{G}(a) \arrow[rr, "(\Psi_a)_*", "\shortmid" marking] \arrow[d, shift right=4ex, "(\mathrm{G}f)_*"', "\shortmid" marking] \arrow[d, "\overline{\mathrm{G}}(f_*)", "\shortmid" marking] \arrow[d, phantom, shift right=2ex, "\overset{\nu_f}{\Rightarrow}"] & {} \arrow[d, phantom, "\Uparrow \overline{\Psi}_{f_*}"] & \mathrm{H}(a) \arrow[d, "\overline{\mathrm{H}}(f_*)", "\shortmid" marking] \\
\mathrm{G}(a') \arrow[rr, "(\Psi_{a'})_*"', "\shortmid" marking] & {} & \mathrm{H}(a')
\end{tikzcd}
=
\begin{tikzcd}[column sep=large]
\mathrm{G}(a) \arrow[rr, "(\Psi_a)_*", "\shortmid" marking] \arrow[d, "(\mathrm{G}f)_*"', "\shortmid" marking] & {} \arrow[d, phantom, "\overset{(\Psi_f)_*}{\cong}"] & \mathrm{H}(a) \arrow[d, shift right=4ex, "(\mathrm{H}f)_*"', "\shortmid" marking] \arrow[d, "\overline{\mathrm{H}}(f_*)", "\shortmid" marking] \arrow[d, phantom, shift right=2ex, "\overset{\xi_f}{\Rightarrow}"] \\
\mathrm{G}(a') \arrow[rr, "(\Psi_{a'})_*"', "\shortmid" marking] & {} & \mathrm{H}(a')
\end{tikzcd}$$

modifications: $\alpha\colon (\overline{\Psi}, \Psi) \Rightarrow (\overline{\Phi}, \Phi)$ consisting of 2-cells $\{\alpha_a\colon \Psi_a \Rightarrow \Phi_a\}_{a\in 0\text{-cell}(\underline{\underline{\mathcal{K}}})}$ such that:

- The family $\{\alpha_a\colon \Psi_a \Rightarrow \Phi_a\}_{a\in 0\text{-cell}(\underline{\underline{\mathcal{K}}})}$ is a modification from Ψ to Φ and
- The family $\{(\alpha_a)_*\colon \overline{\Phi}_a \Rightarrow \overline{\Psi}_a\}_{a\in 0\text{-cell}(\underline{\underline{\mathcal{K}}})}$ is a modification from $\overline{\Phi}$ to $\overline{\Psi}$.

While the descriptions given above of equipment maps and morphisms all seem highly asymmetrical with respect to the process of taking dual equipments, as presented in definition 1.2.13, that perception is no more than an illusion. Our choice to display these structures in this way is intended to illustrate that most of our theory works for weak equipments. For the remainder of this section we assume that our equipments are not weak, so let us briefly examine a more symmetrical description of equipment morphisms:

Definition 1.5.9 (The Vertical Dual of a Double Bicategory)
If $\underline{\underline{\tilde{\mathcal{D}}}} = (\mathcal{A}, \underline{\underline{\mathcal{H}}}, \underline{\underline{\mathcal{V}}}, \mathcal{S})$ is a double bicategory then we may construct a (vertical) dual $\underline{\underline{\tilde{\mathcal{D}}}}^{\mathrm{vop}}$ by "reflecting squares through a horizontal axis". More formally $\underline{\underline{\tilde{\mathcal{D}}}}^{\mathrm{vop}} \stackrel{\mathrm{def}}{=} (\mathcal{A}, \underline{\underline{\mathcal{H}}}^{\mathrm{co}}, \underline{\underline{\mathcal{V}}}^{\mathrm{op}}, \mathcal{S})$, we reverse the orientation of vertical 1-cells and correspondingly swap the rôles of $\mathrm{dom}_V(\lambda)$ and $\mathrm{cod}_V(\lambda)$ for each square λ, in other words there is a correspondence of squares:

$$\begin{tikzcd}
\bar{a} \arrow[r, "\bar{f}"] \arrow[d, "p^{\mathrm{op}}"', "\shortmid" marking] \arrow[dr, phantom, "\lambda^{\mathrm{vop}}\lrcorner"] & \bar{a}' \arrow[d, "(p')^{\mathrm{op}}", "\shortmid" marking] \\
a \arrow[r, "f"'] & a'
\end{tikzcd}
\in \underline{\underline{\tilde{\mathcal{D}}}}^{\mathrm{vop}} \longleftrightarrow
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "p"', "\shortmid" marking] \arrow[dr, phantom, "\lambda\lrcorner"] & a' \arrow[d, "p'", "\shortmid" marking] \\
\bar{a} \arrow[r, "\bar{f}"'] & \bar{a}'
\end{tikzcd}
\in \underline{\underline{\tilde{\mathcal{D}}}}$$

Clearly we endow $\underline{\underline{\tilde{\mathcal{D}}}}^{\mathrm{vop}}$ with the horizontal and vertical compositions of squares possessed by $\underline{\underline{\tilde{\mathcal{D}}}}$. When carrying the actions of 2-cells on squares over from $\underline{\underline{\tilde{\mathcal{D}}}}$ notice that we need to take the dual $\underline{\underline{\mathcal{H}}}^{\mathrm{co}}$ (reversing 2-cells) as the bicategory of horizontal cells in $\tilde{\underline{\underline{\mathcal{D}}}}^{\mathrm{vop}}$ thereby ensuring that the actions of horizontal 2-cells on vertically reflected squares is correct.

Looking at the bicategories of cylinders derived from $\tilde{\underline{\underline{\mathcal{D}}}}^{\mathrm{vop}}$ we see that we have canonical strict isomorphisms

$$\mathrm{Cyl}_H(\tilde{\underline{\underline{\mathcal{D}}}}^{\mathrm{vop}}) \cong (\mathrm{Cyl}_H\,\tilde{\underline{\underline{\mathcal{D}}}})^{\mathrm{co}} \qquad\qquad \mathrm{Cyl}_V(\tilde{\underline{\underline{\mathcal{D}}}}^{\mathrm{vop}}) \cong (\mathrm{Cyl}_V\,\tilde{\underline{\underline{\mathcal{D}}}})^{\mathrm{op}}$$

therefore the pair of strict homomorphisms

$$\begin{tikzcd} \mathrm{Cyl}_H(\tilde{\underline{\underline{\mathcal{D}}}}^{\mathrm{vop}}) \arrow[r, shift left, "\mathrm{dom}_V^{\mathrm{vop}}"] \arrow[r, shift right, "\mathrm{cod}_V^{\mathrm{vop}}"'] & \underline{\underline{\mathcal{K}}}^{\mathrm{co}} \end{tikzcd}$$

used in the definition of $\underline{\underline{\underline{\mathcal{H}oriz}}}_S$ is strictly isomorphic to:

$$\begin{tikzcd} (\mathrm{Cyl}_H\,\tilde{\underline{\underline{\mathcal{D}}}})^{\mathrm{co}} \arrow[r, shift left, "(\mathrm{cod}_V)^{\mathrm{co}}"] \arrow[r, shift right, "(\mathrm{dom}_V)^{\mathrm{co}}"'] & \underline{\underline{\mathcal{K}}}^{\mathrm{co}} \end{tikzcd}$$

We may exploit this fact to demonstrate that vertical duality may be made into a (strongly) bicategory enriched functor. Consider first the duality of bicategories $(\text{–})^{\mathrm{co}}$, it is not true that there exists an enriched functor

$$\underline{\underline{\underline{\mathcal{H}om}}}_S \xrightarrow{(\text{–})^{\mathrm{co}}} \underline{\underline{\underline{\mathcal{H}om}}}_S$$

since in changing the polarity of 2-cells in each bicategory we also reverse the orientation of modifications. We may fix this fault by restricting ourselves to the enriched subcategory $\underline{\underline{\underline{\mathcal{H}om}}}_G$ of $\underline{\underline{\underline{\mathcal{H}om}}}_S$ with the same objects, homomorphisms and transformations but only those modifications which are isomorphisms. Of course we may do this for any strongly bicategory enriched category $\underline{\underline{\mathcal{A}}}$ thereby obtaining an enriched subcategory denoted $\underline{\underline{\mathcal{A}}}_G$, the subscript indicates that each bicategory $\underline{\underline{\mathcal{A}}}(A, A')$ is locally groupoidal. Notice that the distinction between the strong and weak forms of bicategory enrichment disappears once we assume that each homset is locally groupoidal. Moving to the subcategory $\underline{\underline{\mathcal{A}}}_G$ makes no difference to the study of biadjunctions in $\underline{\underline{\mathcal{A}}}$, since all modifications we encounter in that endeavour are already isomorphisms. So we have an involutive enriched functor:

$$\underline{\underline{\underline{\mathcal{H}om}}}_G \xrightarrow{(\text{–})^{\mathrm{co}}} \underline{\underline{\underline{\mathcal{H}om}}}_G\ .$$

This preserves strict homomorphisms therefore we get another enriched functor:

$$\underline{\underline{\underline{\mathcal{H}om}}}_G^{\mathbb{P}} \xrightarrow{\{(\text{–})^{\mathrm{co}}\}^{\mathbb{P}}} \underline{\underline{\underline{\mathcal{H}om}}}_G^{\mathbb{P}}\ .$$

Referring to definition 1.4.7, which describes $\underline{\underline{\underline{\mathcal{H}oriz}}}_G$ in terms of $\underline{\underline{\underline{\mathcal{H}om}}}_G$, and using the isomorphism $\mathrm{Cyl}_H(\tilde{\underline{\underline{\mathcal{D}}}}^{\mathrm{vop}}) \cong (\mathrm{Cyl}_H \tilde{\underline{\underline{\mathcal{D}}}})^{\mathrm{co}}$ it is clear that $\{(\text{-})^{\mathrm{co}}\}^{\mathbb{P}}$ provides us with an involutive enriched endo-functor:

$$\underline{\underline{\underline{\mathcal{H}oriz}}}_G \xrightarrow{(\text{-})^{\mathrm{vop}}} \underline{\underline{\underline{\mathcal{H}oriz}}}_G \ .$$

With no further comment it is immediate that this functor extends to an enriched involution on each of the categories $\underline{\underline{\underline{\mathcal{H}oriz}}}_{G*}$, $\underline{\underline{\underline{\mathcal{H}oriz}}}_{GM}$, $\underline{\underline{\underline{\mathcal{H}oriz}}}_{GC}$ and $\underline{\underline{\underline{\mathcal{H}oriz}}}_{GH}$ (subcategories of $\underline{\underline{\underline{\mathcal{H}oriz}}}_{S*}$, $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$, $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SC}$ and $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH}$ respectively). □

In terms of equipments the importance of vertical duals lies in:

Lemma 1.5.10 *There is a bijection between the squares of* $(\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*))^{\mathrm{vop}}$ *and* $\mathrm{Sq}((\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}})$*, where* $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}}$ *is the dual equipment described in definition 1.2.13. Identifying squares via that correspondence we get the following equality of double bicategories:*

$$\mathrm{Sq}((\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}}) = (\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*))^{\mathrm{vop}} \tag{1.75}$$

Proof. First notice that $(\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*))^{\mathrm{vop}}$ and $\mathrm{Sq}((\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}})$ possess the same bicategories of horizontal and vertical cells, $\underline{\underline{\mathcal{K}}}^{\mathrm{co}}$ and $\underline{\underline{\mathcal{M}}}^{\mathrm{op}}$ respectively. A square in $\mathrm{Sq}((\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}})$

$$\begin{tikzcd} \bar{a} \arrow[r, "\bar{f}"] \arrow[d, "p^{\mathrm{op}}"', "\shortmid" marking] \arrow[dr, phantom, "\hat{\lambda}\Uparrow"] & \bar{a}' \arrow[d, "(p')^{\mathrm{op}}", "\shortmid" marking] \\ a \arrow[r, "f"'] & a' \end{tikzcd} \tag{1.76}$$

is simply a 2-cell $\hat{\lambda}\colon p \otimes f^* \Rightarrow \bar{f}^* \otimes p'$ in $\underline{\underline{\mathcal{M}}}$, so let $\lambda\colon \bar{f}_* \otimes p \Rightarrow p' \otimes f_*$ be the mate of that 2-cell under the adjunctions $f_* \dashv f^*$ and $\bar{f}_* \dashv \bar{f}^*$ in $\underline{\underline{\mathcal{M}}}$. Of course λ is a square

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "p"', "\shortmid" marking] \arrow[dr, phantom, "\lambda\Uparrow"] & a' \arrow[d, "p'", "\shortmid" marking] \\ \bar{a} \arrow[r, "\bar{f}"'] & \bar{a}' \end{tikzcd}$$

in $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$, which has a vertical reflection λ^{vop} in $(\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*))^{\mathrm{vop}}$ which we may identify with the square $\hat{\lambda}$ that we started with in (1.76).

To establish the equality in (1.75) we only need to check that our bijection between the squares of $(\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*))^{\mathrm{vop}}$ and $\mathrm{Sq}((\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}})$ respects the actions on, and compositions of the squares in each double bicategory. This is no more than routine verification, simply apply the usual properties of mates and recall that the action of the homomorphism $(\text{-})^*$ on 2-cells, as well as its canonical isomorphisms, are derived from those of $(\text{-})_*$ by taking mates. $\square$

Corollary 1.5.11 *The dual equipment construction of definition 1.2.13 extends naturally to an involutive endo-functor on each of the bicategory enriched categories* $\underline{\underline{\mathcal{E}Map}}_G$, $\underline{\underline{\mathcal{E}Mor}}_G$, $\underline{\underline{\mathcal{E}coMor}}_G$ *and* $\underline{\underline{\mathcal{E}Hom}}_G$.

Proof. These enriched categories are defined in terms of $\underline{\underline{\mathcal{H}oriz}}_{G*}$, $\underline{\underline{\mathcal{H}oriz}}_{GM}$, $\underline{\underline{\mathcal{H}oriz}}_{GC}$ and $\underline{\underline{\mathcal{H}oriz}}_{GH}$ (respectively) using the "Sq" construction on equipments. We already know that vertical duality of double bicategories $(\text{-})^{\mathrm{vop}}$ extends to an involutive enriched functor on each of these latter categories therefore it follows that the equality in (1.75) ensures the same is true for $(\text{-})^{\mathrm{op}}$ on the corresponding enriched categories of equipments.

One important consequence of this lemma is that any property of equipment maps or morphisms relating to colimit cylinders may be translated to a dual property with respect to limit cylinders. $\square$

Lemma 1.5.12 *Suppose we are given an equipment map*

$$(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \xrightarrow{\ddot{\mathrm{G}} = (\overline{\mathrm{G}}, \mathrm{G}, \rho, \mu)} (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$$

then a second map

$$(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}} \xrightarrow{\ddot{\mathrm{G}}' = (\overline{\mathrm{G}}^{\mathrm{op}}, \mathrm{G}^{\mathrm{co}}, \rho', \mu')} (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)^{\mathrm{op}}$$

is the dual of the first, under the duality functor constructed above, iff the 2-cells

$$\overline{\mathrm{G}}(p) \otimes (\mathrm{G}f)^* \xRightarrow{\mu'_{pf}} \overline{\mathrm{G}}(p \otimes f^*)$$
$$\overline{\mathrm{G}}(\bar{f}^* \otimes p') \xRightarrow{\rho'_{\bar{f}p'}} (\mathrm{G}\bar{f})^* \otimes \overline{\mathrm{G}}(p')$$

correspond to the composites

$$\overline{\mathrm{G}}(p) \xRightarrow{\overline{\mathrm{G}}(p \otimes \eta_f)} \overline{\mathrm{G}}(p \otimes f^* \otimes f_*) \xRightarrow{\rho_{(p\otimes f^*)f}} \overline{\mathrm{G}}(p \otimes f^*) \otimes (\mathrm{G}f)_*$$
$$(\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(\bar{f}^* \otimes p') \xRightarrow{\mu_{\bar{f}(\bar{f}^*\otimes p')}} \overline{\mathrm{G}}(\bar{f}_* \otimes \bar{f}^* \otimes p') \xRightarrow{\overline{\mathrm{G}}(\epsilon_{\bar{f}} \otimes p')} \overline{\mathrm{G}}(p')$$

under the adjunctions $(\mathrm{G}f)_* \dashv (\mathrm{G}f)^*$ *and* $(\mathrm{G}\bar{f})_* \dashv (\mathrm{G}\bar{f})^*$ *respectively.*

In the statement of this lemma we follow the usual convention that if $f\colon a \longrightarrow a'$ is a 1-cell in $\underline{\underline{\mathcal{K}}}$ then $\eta_f\colon \mathrm{i}_a \Rightarrow f^* \otimes f_*$ and $\epsilon_f\colon f_* \otimes f^* \Rightarrow \mathrm{i}_{a'}$ are used to denote the unit and counit of the adjunction $f_* \dashv f^*$ in $\underline{\underline{\mathcal{M}}}$.

Proof. The equipment maps given correspond to horizontal maps

$$(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \xrightarrow{\tilde{\mathrm{G}}} (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$$
$$(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}} \xrightarrow{\tilde{\mathrm{G}}'} (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)^{\mathrm{op}}$$

and we know that the process of taking mates provides a natural bijection between the squares of $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ and $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}}$ ($(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$ and $(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)^{\mathrm{op}}$). On examining the proofs of the last couple of lemmas we see that $\ddot{\mathrm{G}}$ and $\ddot{\mathrm{G}}'$ are dual iff under the correspondences provided by taking mates the actions of $\tilde{\mathrm{G}}$ and $\tilde{\mathrm{G}}'$ on squares coincide.

Suppose first that the condition on ρ' and μ' hold, and let $\hat{\lambda}$ be a square in $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}}$ as in (1.76). Its mate λ is given by

$$\begin{aligned}\bar{f}_* \otimes p &\xrightarrow{\bar{f}_* \otimes p \otimes \eta_f} \bar{f}_* \otimes p \otimes f^* \otimes f_* \xrightarrow{\bar{f}_* \otimes \hat{\lambda} \otimes f_*} \bar{f}_* \otimes \bar{f}^* \otimes p' \otimes f_* \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \xrightarrow{\epsilon_{\bar{f}} \otimes p' \otimes f_*} p' \otimes f_*\end{aligned}$$

to which we apply $\tilde{\mathrm{G}}$, the action on squares of which we recall is determined by ρ and μ via the composite shown in (1.71). A routine diagram chase, involving the coherence properties of $\ddot{\mathrm{G}}$ as an equipment map, demonstrates that we may re-express $\mathrm{G}_S(\lambda)$ as the composite:

$$\begin{aligned}(\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(p) &\xrightarrow{(\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(p \otimes \eta_f)} (\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(p \otimes f^* \otimes f_*) \\ &\xrightarrow{(\mathrm{G}\bar{f})_* \otimes \rho_{(p \otimes f^*)f}} (\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(p \otimes f^*) \otimes (\mathrm{G}f)_* \\ &\xrightarrow{(\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(\hat{\lambda}) \otimes (\mathrm{G}f)_*} (\mathrm{G}\bar{f})_* \otimes \overline{\mathrm{G}}(\bar{f}^* \otimes p') \otimes (\mathrm{G}f)_* \\ &\xrightarrow{\mu_{\bar{f}(\bar{f}^* \otimes p')} \otimes (\mathrm{G}f)_*} \overline{\mathrm{G}}(\bar{f}_* \otimes \bar{f}^* \otimes p') \otimes (\mathrm{G}f)_* \\ &\xrightarrow{\overline{\mathrm{G}}(\epsilon_{\bar{f}} \otimes p') \otimes (\mathrm{G}f)_*} \overline{\mathrm{G}}(p') \otimes (\mathrm{G}f)_*\end{aligned}$$

Therefore, so long as the condition of the statement holds, it is clear that the mate of this composite is simply the square $\mathrm{G}'_S(\hat{\lambda})$ defined using ρ' and μ' as in the dual of (1.76).

Conversely suppose the equipment maps $\tilde{\mathrm{G}}$ and $\tilde{\mathrm{G}}'$ have corresponding actions on squares. The 2-cell μ'_{pf} may be obtained, as in the proof of lemma 1.5.1, by applying $\tilde{\mathrm{G}}'$ to the canonical square:

$$\begin{tikzcd}
\bar{a} \arrow[r, "\mathrm{i}_{\bar{a}}"] \arrow[d, "p^{\mathrm{op}}"'] \arrow[dr, phantom, "\overset{\Theta'_{pf}}{\cong}"] & \bar{a} \arrow[d, "(p \otimes f^*)^{\mathrm{op}}"] \\
a \arrow[r, "f"'] & a'
\end{tikzcd}$$

in $\mathrm{Sq}((\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}})$. Since the actions on squares of the two horizontal maps coincide with respect to mates we may calculate $\mathrm{G}'_S(\Theta'_{pf})$ by first taking the mate of Θ'_{pf}

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\Uparrow \check{\Theta}'_{pf}"] & a' \arrow[d, "p \otimes f^*"] \\
\bar{a} \arrow[r, "\mathrm{i}_{\bar{a}}"'] & \bar{a}
\end{tikzcd}$$

which is given by

$$(\mathrm{i}_{\bar{a}})_* \otimes p \xRightarrow[\sim]{\text{can}} p \xRightarrow{p \otimes \eta_f} p \otimes f^* \otimes f_*$$

then apply G_S (as defined using ρ and μ) to get a square which, with a little diagram chasing, we may show to be equal to the composite

$$\begin{aligned}
(\mathrm{Gi}_{\bar{a}})_* \otimes \overline{\mathrm{G}}(p) \xRightarrow[\sim]{\text{can}} \overline{\mathrm{G}}(p) &\xRightarrow{\overline{\mathrm{G}}(p \otimes \eta_f)} \overline{\mathrm{G}}(p \otimes f^* \otimes f_*) \\
&\xRightarrow{\rho_{(p \otimes f^*)f}} \overline{\mathrm{G}}(p \otimes f^*) \otimes (\mathrm{G}f)_*
\end{aligned}$$

finally it follows that $\mathrm{G}'_S(\Theta'_{pf})$ is the mate of this last square. Once we have reminded ourselves of how μ'_{pf} is derived from $\mathrm{G}'_S(\Theta'_{pf})$ this last composite reveals that the postulated relationship between μ' and ρ holds; a dual verification establishes the relationship between ρ' and μ. $\square$

Proposition 1.5.13 *Suppose* $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ *and* $(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$ *are equipments which are not weak then equipment morphisms* $\ddot{\mathrm{G}}\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$ *admit a description whereby they consist of:*

(i) A homomorphism $\mathrm{G}\colon \underline{\underline{\mathcal{K}}} \longrightarrow \underline{\underline{\mathcal{L}}}$ *and a morphism* $\overline{\mathrm{G}}\colon \underline{\underline{\mathcal{M}}} \longrightarrow \underline{\underline{\mathcal{N}}}$ *with the same actions on 0-cells.*

(ii) Families of 2-cells

$$(\mathrm{G}f)_* \xRightarrow{\nu_f} \overline{\mathrm{G}}(f_*)$$
$$(\mathrm{G}f)^* \xRightarrow{\nu'_f} \overline{\mathrm{G}}(f^*)$$

natural in the 1-cell $f \in \underline{\underline{\mathcal{K}}}$ and both satisfying conditions (d)′ and (e)′ of proposition 1.5.8 with respect to the canonical isomorphisms of the homomorphisms $(\text{-})_*$ and $(\text{-})^*$ respectively.

(iii) For each 1-cell $f\colon a \longrightarrow a' \in \underline{\underline{\mathcal{K}}}$ the following diagram commutes

$$\begin{tikzcd}
\mathrm{i}_{\mathrm{G}a} \arrow[r, Rightarrow, "\eta_{\mathrm{G}f}"] \arrow[d, Rightarrow, "\mathrm{can}"'] & (\mathrm{G}f)^* \otimes (\mathrm{G}f)_* \arrow[r, Rightarrow, "\nu'_f \otimes \nu_f"] & \overline{\mathrm{G}}(f^*) \otimes \overline{\mathrm{G}}(f_*) \arrow[d, Rightarrow, "\mathrm{can}"] \\
\overline{\mathrm{G}}(\mathrm{i}_a) \arrow[rr, Rightarrow, "\overline{\mathrm{G}}(\eta_f)"'] & & \overline{\mathrm{G}}(f^* \otimes f_*)
\end{tikzcd}$$

as does a dual one concerning the counits of $f_* \dashv f^*$ and $(\mathrm{G}f)_* \dashv (\mathrm{G}f)^*$.

Proof. Since this proposition is not of great importance to the remainder of the narrative we give no more than a few lines of sketch proof.

Given an equipment morphism $\ddot{\mathrm{G}}\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$ it may be described in terms of a family of 2-cells $\nu_f\colon (\mathrm{G}f)_* \Rightarrow \overline{\mathrm{G}}(f_*)$ as demonstrated by proposition 1.5.8. It has a dual $\ddot{\mathrm{G}}^{\mathrm{op}}\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}} \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)^{\mathrm{op}}$ which we describe in terms of another family $\nu'_f\colon (\mathrm{G}f)^* \Rightarrow \overline{\mathrm{G}}(f^*)$, and the relationship between a map and its dual given in the last lemma establishes condition (iii) relating these two families.

Conversely suppose we were given the data in the statement of this proposition. First show that $(\overline{\mathrm{G}}, \mathrm{G}, \nu)$ is an equipment morphism from $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ to $(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$, to do so all we need do is verify condition (c)′ of proposition 1.5.8, but this can be shown to follow from condition (iii) relating ν and ν'. In an identical fashion we show that $(\overline{\mathrm{G}}^{\mathrm{op}}, \mathrm{G}^{\mathrm{co}}, \nu')$ is an equipment morphism from $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)^{\mathrm{op}}$ to $(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)^{\mathrm{op}}$, now use condition (iii) again alongside the last lemma to demonstrate that these two are in fact duals. $\square$

When equipments are not weak we get a particularly simple method for constructing biadjoints in $\underline{\underline{\mathcal{E}Map}}$, derived directly from the one-sided universal property for biadjoints in $\underline{\underline{\mathcal{H}oriz}}_{\mathcal{S}}$ as given in proposition 1.4.8:

Theorem 1.5.14 *Given an equipment morphism*

$$(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \xrightarrow{\ddot{\mathrm{G}}} (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$$

and a left biadjoint $\mathrm{F}\colon \underline{\underline{\mathcal{L}}} \longrightarrow \underline{\underline{\mathcal{K}}}$ to the homomorphism $\mathrm{G}\colon \underline{\underline{\mathcal{K}}} \longrightarrow \underline{\underline{\mathcal{L}}}$ (with unit $\Psi\colon \mathrm{I}_{\underline{\underline{\mathcal{L}}}} \longrightarrow \mathrm{GF}$ and counit $\Phi\colon \mathrm{FG} \longrightarrow \mathrm{I}_{\underline{\underline{\mathcal{K}}}}$) then tfae:

(i) $\ddot{\mathrm{G}}$ *has a left biadjoint* $\ddot{\mathrm{F}}$ *in* $\underline{\underline{\mathcal{E}Map}}$. *In that case, since* $\ddot{\mathrm{G}}$ *is an equipment morphism, there is an induced enrichment of* $\ddot{\mathrm{F}}$ *to an equipment comorphism, cf. proposition 1.4.11.*

(ii) For each pair of 0-cells $a, \bar{a} \in \underline{\underline{\mathcal{K}}}$ *the functor*

$$\underline{\underline{\mathcal{M}}}(a, \bar{a}) \xrightarrow{\overline{\mathrm{G}}} \underline{\underline{\mathcal{N}}}(\mathrm{G}(a), \mathrm{G}(\bar{a}))$$

has a left adjoint.

(iii) For each pair of 0-cells $b, \bar{b} \in \underline{\underline{\mathcal{L}}}$ *the functor*

$$\underline{\underline{\mathcal{M}}}(\mathrm{F}(b), \mathrm{F}(\bar{b})) \xrightarrow{(\Psi_{\bar{b}})^* \otimes \overline{\mathrm{G}}(-) \otimes (\Psi_b)_*} \underline{\underline{\mathcal{N}}}(b, \bar{b})$$

has a left adjoint.

Proof. This theorem turns on the observation that a square

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "p"', "\shortmid" marking] \arrow[dr, phantom, "\Uparrow \lambda"] & a' \arrow[d, "p'", "\shortmid" marking] \\
\bar{a} \arrow[r, "\bar{a}"'] & \bar{a}'
\end{tikzcd}$$

in $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*)$ corresponds, under the adjunction $\bar{f}_* \dashv \bar{f}^*$, to a 2-cell:

$$\begin{tikzcd}
a \arrow[r, "f_*", "\shortmid" marking] \arrow[d, "p"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\hat{\lambda}}{\Rightarrow}"] & a' \arrow[d, "p'", "\shortmid" marking] \\
\bar{a} & \bar{a}' \arrow[l, "\bar{f}^*", "\shortmid" marking]
\end{tikzcd}$$

Now examine the action on squares of the horizontal map corresponding to $\ddot{\mathrm{G}}$, by applying lemma 1.5.12 and a simple diagram chase we see that the 2-cell

$$\begin{tikzcd}
\mathrm{G}(a) \arrow[r, "(\mathrm{G}f)_*", "\shortmid" marking] \arrow[d, "\overline{\mathrm{G}}(p)"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\widehat{\mathrm{G}_{\underline{S}}(\lambda)}}{\Rightarrow}"] & \mathrm{G}(a') \arrow[d, "\overline{\mathrm{G}}(p')", "\shortmid" marking] \\
\mathrm{G}(\bar{a}) & \mathrm{G}(\bar{a}') \arrow[l, "(\mathrm{G}\bar{f})^*", "\shortmid" marking]
\end{tikzcd}$$

is given by the composite

$$\begin{aligned}\overline{\mathrm{G}}(p) \xRightarrow{\overline{\mathrm{G}}(\hat{\lambda})} \overline{\mathrm{G}}(\bar{f}^* \otimes p \otimes f_*) &\xRightarrow{\rho_{(\bar{f}^*\otimes p)f}} \overline{\mathrm{G}}(\bar{f}^* \otimes p) \otimes (\mathrm{G}f)_* \\ &\xRightarrow{\rho'_{\bar{f}p} \otimes (\mathrm{G}f)_*} (\mathrm{G}\bar{f})^* \otimes \overline{\mathrm{G}}(p) \otimes (\mathrm{G}f)_*\end{aligned}$$

where $\rho'_{\bar{f}p}$ is a structural 2-cell of the dual morphism $\ddot{\mathrm{G}}^{\mathrm{op}}$. Notice that $\ddot{\mathrm{G}}$ and $\ddot{\mathrm{G}}^{\mathrm{op}}$ are equipment morphisms, therefore lemma 1.5.6 tells us that $\rho_{(\bar{f}^*\otimes p)f}$ and $\rho'_{\bar{f}p}$ are both isomorphisms.

It follows that in this context we may express the one sided universal property described in proposition 1.4.8 as a bijection

$$\begin{tikzcd} b \arrow[r, "(\mathrm{G}f\circ\Psi_b)_*", "\shortmid"{marking}] \arrow[d, "q"', "\shortmid"{marking}] \arrow[dr, phantom, "\overset{\hat{\lambda}}{\Rightarrow}"] & \mathrm{G}(a) \arrow[d, "\overline{\mathrm{G}}(p)", "\shortmid"{marking}] \\ \bar{b} & \mathrm{G}(\bar{a}) \arrow[l, "(\mathrm{G}\bar{f}\circ\Psi_{\bar{b}})^*", "\shortmid"{marking}] \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} \mathrm{F}(b) \arrow[r, "f_*", "\shortmid"{marking}] \arrow[d, "\overline{\mathrm{F}}(q)"', "\shortmid"{marking}] \arrow[dr, phantom, "\overset{\tilde{\lambda}}{\Rightarrow}"] & a \arrow[d, "p", "\shortmid"{marking}] \\ \mathrm{F}(\bar{b}) & \bar{a} \arrow[l, "\bar{f}^*", "\shortmid"{marking}] \end{tikzcd} \tag{1.77}$$

satisfying the condition that the following diagram commutes:

$$\begin{tikzcd} q \arrow[r, "\hat{\Psi}_q"] \arrow[d, "\hat{\lambda}"'] & (\Psi_{\bar{a}})^* \otimes \overline{\mathrm{GF}}(q) \otimes (\Psi_a)_* \arrow[d, "(\Psi_{\bar{a}})^* \otimes \overline{\mathrm{G}}(\tilde{\lambda}) \otimes (\Psi_a)_*"] \\ (\mathrm{G}\bar{f}\circ\Psi_{\bar{a}})^* \otimes \overline{\mathrm{G}}(p) \otimes (\mathrm{G}f\circ\Psi_a)_* \arrow[r, "\cong", "\mathrm{can}"'] & (\Psi_{\bar{a}})^* \otimes \overline{\mathrm{G}}(\bar{f}^* \otimes p \otimes f_*) \otimes (\Psi_a)_* \end{tikzcd} \tag{1.78}$$

Here $\hat{\Psi}_q$ corresponds to the square $\overline{\Psi}_q\colon (\Psi_{\bar{a}})_* \otimes q \Rightarrow \overline{\mathrm{GF}}(q) \otimes (\Psi_a)_*$ and the 2-cell marked "can" is the obvious one to be obtained from $\rho_{(\bar{f}^*\otimes p)f}$, $\rho'_{\bar{f}p}$ and the canonical isomorphisms $(\Psi_{\bar{a}})^* \otimes (\mathrm{G}\bar{f})^* \cong (\mathrm{G}\bar{f}\circ\Psi_{\bar{a}})^*$, $(\mathrm{G}f)_* \otimes (\Psi_a)_* \cong (\mathrm{G}f\circ\Psi_a)_*$.

The theorem now becomes straightforward:

(ii)⇒(iii): The functor mentioned in (iii) is the composite

$$\underline{\underline{\mathcal{M}}}(\mathrm{F}(b),\mathrm{F}(\bar{b})) \xrightarrow{\overline{\mathrm{G}}} \underline{\underline{\mathcal{N}}}(\mathrm{GF}(b),\mathrm{GF}(\bar{b})) \xrightarrow{(\Psi_{\bar{b}})^* \otimes - \otimes (\Psi_b)_*} \underline{\underline{\mathcal{N}}}(b,\bar{b})$$

but $((\Psi_{\bar{b}})^* \otimes - \otimes (\Psi_b)_*)$ has a left adjoint $((\Psi_{\bar{b}})_* \otimes - \otimes (\Psi_b)^*)$, condition (ii) ensures that $\overline{\mathrm{G}}$ also possesses one therefore it follows that their composite does.

(iii)⇒(i): Let $\overline{\mathrm{F}}\colon \underline{\underline{\mathcal{N}}}(b,\bar{b}) \longrightarrow \underline{\underline{\mathcal{M}}}(\mathrm{F}(b),\mathrm{F}(\bar{b}))$ be the left adjoint postulated in (iii) with counit consisting of components $\hat{\Psi}_q\colon q \Rightarrow (\Psi_{\bar{a}})^* \otimes \overline{\mathrm{GF}}(q) \otimes (\Psi_a)_*$. Since the 2-cell "can" in (1.78) is an isomorphism it is quite clear that this data satisfies our re-expressed version of the one sided universal property for $\ddot{\mathrm{F}} \dashv_b \ddot{\mathrm{G}}$.

(i)⇒(ii) If we set $a = \mathrm{F}(b)$, $\bar{a} = \mathrm{F}(\bar{b})$, $f = \mathrm{i}_{\mathrm{F}(b)}$ and $\bar{f} = \mathrm{i}_{\mathrm{F}(\bar{b})}$ in (1.77) that bijection clearly reduces to one which simply demonstrates an adjunction

$$\begin{tikzcd} \underline{\underline{\mathcal{M}}}(\mathrm{F}(b), \mathrm{F}(\bar{b})) \arrow[r, shift right, "(\Psi_{\bar{b}})^* \otimes \overline{\mathrm{G}}(\text{-}) \otimes (\Psi_b)_*"'] \arrow[r, phantom, "\perp"] & \underline{\underline{\mathcal{N}}}(b, \bar{b}) \arrow[l, shift right, "\overline{\mathrm{F}}"'] \end{tikzcd}$$

which has unit with components $\hat{\Psi}_q$. Dually the one sided property with respect to the counit $\ddot{\Phi}\colon \ddot{\mathrm{F}}\ddot{\mathrm{G}} \longrightarrow \mathrm{I}$ gives an adjunction

$$\begin{tikzcd} \underline{\underline{\mathcal{N}}}(\mathrm{G}(a), \mathrm{G}(\bar{a})) \arrow[r, shift right, "(\Phi_{\bar{a}})_* \otimes \overline{\mathrm{F}}(\text{-}) \otimes (\Phi_a)^*"'] \arrow[r, phantom, "\top"] & \underline{\underline{\mathcal{M}}}(a, \bar{a}) \arrow[l, shift right, "\overline{\mathrm{G}}"'] \end{tikzcd}$$

the counit of which has components given by composites:

$$(\Phi_{\bar{a}})_* \otimes \overline{\mathrm{FG}}(p) \otimes (\Phi_a)^* \xRightarrow{\overline{\Phi}_p \otimes (\Phi_a)^*} p \otimes (\Phi_a)_* \otimes (\Phi_a)^* \xRightarrow{p \otimes \epsilon_{\Phi_a}} p \tag{1.79}$$

□

Observation 1.5.15 In some cases it is useful to have an explicit description of the comorphism structure induced on the left biadjoint $\ddot{\mathrm{F}}$ which we constructed in the last theorem. For each pair of 0-cells $a, \bar{a} \in \underline{\underline{\mathcal{M}}}$ let

$$\underline{\underline{\mathcal{N}}}(\mathrm{G}(a), \mathrm{G}(\bar{a})) \xrightarrow{\overline{\mathrm{L}}_{a\bar{a}}} \underline{\underline{\mathcal{M}}}(a, \bar{a})$$

be the left adjoint to $\overline{\mathrm{G}}$, as introduced in theorem 1.5.14(ii), and for each pair of 1-cells $r\colon \mathrm{G}(a) \longrightarrow\!\!\!\!\!+\!\!> \mathrm{G}(\bar{a})$ and $\bar{r}\colon \mathrm{G}(\bar{a}) \longrightarrow\!\!\!\!\!+\!\!> \mathrm{G}(\tilde{a})$ in $\underline{\underline{\mathcal{N}}}$ define a 2-cell

$$\overline{\mathrm{L}}_{a\tilde{a}}(\bar{r} \otimes r) \xRightarrow{\delta_{\bar{r}r}} \overline{\mathrm{L}}_{\bar{a}\tilde{a}}(\bar{r}) \otimes \overline{\mathrm{L}}_{a\bar{a}}(r) \tag{1.80}$$

corresponding under the adjunction $\overline{\mathrm{L}}_{a\tilde{a}} \dashv \overline{\mathrm{G}}$ to the composite

$$\bar{r} \otimes r \xRightarrow{\eta_{\bar{r}} \otimes \eta_r} \overline{\mathrm{G}}\overline{\mathrm{L}}_{\bar{a}\tilde{a}}(\bar{r}) \otimes \overline{\mathrm{G}}\overline{\mathrm{L}}_{a\bar{a}}(r) \xRightarrow{\text{can}} \overline{\mathrm{G}}(\overline{\mathrm{L}}_{\bar{a}\tilde{a}}(\bar{r}) \otimes \overline{\mathrm{L}}_{a\bar{a}}(r))$$

where $\eta_{\bar{r}}$ and η_r are unit components. Now consider any 1-cell $q\colon b \longrightarrow\!\!\!\!\!+\!\!> \bar{b}$ in $\underline{\underline{\mathcal{N}}}$, it follows from the proof of last theorem that $\overline{\mathrm{F}}(q) \cong \overline{\mathrm{L}}_{(\mathrm{F}b)(\mathrm{F}\bar{b})}((\Psi_{\bar{b}})_* \otimes q \otimes (\Psi_b)^*)$. With a little checking, which we leave up to the reader, it is clear that if $\bar{q}\colon \bar{b} \longrightarrow\!\!\!\!\!+\!\!> \tilde{b} \in \underline{\underline{\mathcal{N}}}$ is another 1-cell then the comparison 2-cell $\overline{\mathrm{F}}(\bar{q} \otimes q) \Rightarrow \overline{\mathrm{F}}(\bar{q}) \otimes \overline{\mathrm{F}}(q)$ induced as in proposition 1.4.11 is given by the composite:

$$\begin{aligned} \overline{\mathrm{L}}((\Psi_{\tilde{b}})_* \otimes \bar{q} \otimes q \otimes (\Psi_b)^*) &\xRightarrow{\overline{\mathrm{L}}(\cdot \otimes \eta_{\Psi_{\bar{b}}} \otimes \cdot)} \overline{\mathrm{L}}((\Psi_{\tilde{b}})_* \otimes \bar{q} \otimes (\Psi_{\bar{b}})^* \otimes (\Psi_{\bar{b}})_* \otimes q \otimes (\Psi_b)^*) \\ &\xRightarrow{\quad\delta\quad} \overline{\mathrm{L}}((\Psi_{\tilde{b}})_* \otimes \bar{q} \otimes (\Psi_{\bar{b}})^*) \otimes \overline{\mathrm{L}}((\Psi_{\bar{b}})_* \otimes q \otimes (\Psi_b)^*) \end{aligned} \tag{1.81}$$

□

To finish off this section we present a few biadjunctions between equipments of matrices and spans. In the next section we show that the equipment of monads construction given in example 1.2.5 may be given the structure of a bicategory enriched functor. This we apply to the biadjoints constructed here to obtain those which encapsulate the notions of change of base for enriched and internal category theory.

Notice that in many of the examples of equipments $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ that we meet the bicategory $\underline{\underline{\mathcal{K}}}$ is a mere category. The full sub-category of $\underline{\underline{\underline{\mathcal{E}Map}}}$ on these equipments is simply a 2-category, which we denote by using two underlines $\underline{\underline{\mathcal{E}Map}}$ (similarly we have $\underline{\underline{\mathcal{E}Mor}}$, $\underline{\underline{\mathcal{E}Hom}}$ etc.). Biadjoints become adjoints in the usual 2-categorical sense and all of the theory presented here restricts to this context, so it might seem that there were simplifications to be gained by developing the theory only in this situation. In fact these really make little difference to the conceptual complexity of the job at hand, furthermore in the next section we will introduce one important case in which the full strength of biadjoints is crucial.

Example 1.5.16 (Change of Base for Matrices) If $\underline{\underline{\mathcal{B}}}$ and $\underline{\underline{\mathcal{C}}}$ are bicategories satisfying the conditions of example 1.2.2 then we may form equipments of matrices $(\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (\text{-})_\circ)$ and $(\underline{\underline{\mathcal{C}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{C}}}|}, (\text{-})_\circ)$. It is of importance to know how these equipments may be related once we know that we have a comorphism

$$\underline{\underline{\mathcal{C}}} \xrightarrow{\quad \mathrm{F} \quad} \underline{\underline{\mathcal{B}}}$$

such that each of its actions on homsets

$$\underline{\underline{\mathcal{C}}}(c, \bar{c}) \xrightarrow{\quad \mathrm{F}_{c\bar{c}} \quad} \underline{\underline{\mathcal{B}}}(\mathrm{F}(c), \mathrm{F}(\bar{c}))$$

has a right adjoint $\mathrm{R}_{c\bar{c}}$. Of course, since the sets of 0-cells $|\underline{\underline{\mathcal{B}}}|$ and $|\underline{\underline{\mathcal{B}}}|$ are small, we have an adjunction

$$\begin{tikzcd} \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|} \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "\mathrm{F}_\#"'] & \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{C}}}|} \arrow[l, shift right=1.5ex, "\mathrm{F}^\#"'] \end{tikzcd} \tag{1.82}$$

where $\mathrm{F}^\#$ and $\mathrm{F}_\#$ are respectively "composition with" and "pullback along" the map $|\mathrm{F}|\colon |\underline{\underline{\mathcal{C}}}| \longrightarrow |\underline{\underline{\mathcal{B}}}|$. Using $\mathrm{F}^\#$ we propose first to define an equipment comorphism:

$$(\underline{\underline{\mathcal{C}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{C}}}|}, (\text{-})_\circ) \xrightarrow{\quad (\overline{\mathrm{F}}^\#, \mathrm{F}^\#, \nu) \quad} (\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (\text{-})_\circ)$$

The comorphism $\overline{\mathrm{F}}^\#$ acts on matrices and their transformations "pointwise", in other words if $m = \{m_{x\bar{x}}\}\colon (X, \alpha) \longrightarrow\!\!\!\!\!\!\circ\;\; (\bar{X}, \bar{\alpha})$ is a matrix then $\overline{\mathrm{F}}^\#(\{m_{x\bar{x}}\}) = \{\mathrm{F}(m_{x\bar{x}})\}$, of course we must provide $\overline{\mathrm{F}}^\#$ with compositional comparison 2-cells. Notice that F preserves local coproducts, since its actions on each homset have right adjoints, so given a second matrix $\bar{m}\colon (\bar{X}, \bar{\alpha}) \multimapdotboth (\tilde{X}, \tilde{\alpha})$ the 2-cell

$$\mathrm{can}\colon \overline{\mathrm{F}}^{\#}(\bar{m} \circledast m) \Rightarrow \overline{\mathrm{F}}^{\#}(\bar{m}) \circledast \overline{\mathrm{F}}^{\#}(m)$$

may be defined to have components

$$\mathrm{F}\Big(\coprod_{\bar{x}\in\bar{X}} (\bar{m}_{\bar{x}\tilde{x}} \otimes m_{x\bar{x}})\Big) \xRightarrow{\ \sim\ } \coprod_{\bar{x}\in\bar{X}} \mathrm{F}(\bar{m}_{\bar{x}\tilde{x}} \otimes m_{x\bar{x}}) \xRightarrow{\ \coprod_{\bar{x}\in\bar{X}}(\mathrm{can})\ } \coprod_{\bar{x}\in\bar{X}} \mathrm{F}(\bar{m}_{\bar{x}\tilde{x}}) \otimes \mathrm{F}(m_{x\bar{x}})$$

for $x \in X$ and $\tilde{x} \in \tilde{X}$. As for identity comparisons we define them at the same time as the family $\nu_g\colon \overline{\mathrm{F}}^{\#}(g_\circ) \Rightarrow (\mathrm{F}^{\#}g)_\circ$, since if $g = \mathrm{i}_{(X,\alpha)}$ then ν_g is a 2-cell $\overline{\mathrm{F}}^{\#}(\mathrm{i}_{(X,\alpha)}) \Rightarrow \mathrm{i}_{\mathrm{F}^{\#}(X,\alpha)}$. Each entry of the matrix $(\mathrm{F}^{\#}g)_\circ$ is either an identity $\mathrm{i}_{\mathrm{F}c} \in \underline{\underline{\mathcal{B}}}(\mathrm{F}c, \mathrm{F}c)$ or a terminal object $0 \in \underline{\underline{\mathcal{B}}}(\mathrm{F}c, \mathrm{F}\bar{c})$, the corresponding entry of $\overline{\mathrm{F}}^{\#}(g_\circ)$ is then $\mathrm{F}(\mathrm{i}_c) \in \underline{\underline{\mathcal{B}}}(\mathrm{F}c, \mathrm{F}c)$ or $\mathrm{F}(0) \in \underline{\underline{\mathcal{B}}}(\mathrm{F}c, \mathrm{F}\bar{c})$ respectively. A moments reflection reveals that we have canonical 2-cells

$$\begin{array}{c}
\mathrm{F}(\mathrm{i}_c) \xRightarrow{\quad\mathrm{can}\quad} \mathrm{i}_{\mathrm{F}c} \\
\square \\
\mathrm{F}(0) \xRightarrow[\quad\sim\quad]{} 0
\end{array}$$

where the first one is the canonical identity comparison of the comorphism F and the second simply expresses the fact that F preserves local terminal objects because each of its actions on homsets has a right adjoint. These may now be used as the components of the comparison $\nu_g\colon \overline{\mathrm{F}}^{\#}(g_\circ) \Rightarrow (\mathrm{F}^{\#}g)_\circ$. Checking that $\overline{\mathrm{F}}^{\#}$ satisfies the coherence conditions for comorphisms and ν_g (duals of) the first two conditions of proposition 1.5.8 is now straightforward.

The last condition of that proposition stipulates that each 2-cell

$$\overline{\mathrm{F}}^{\#}(\bar{g}_\circ \circledast m) \xRightarrow{\mathrm{can}} \overline{\mathrm{F}}^{\#}(\bar{g}_\circ) \circledast \overline{\mathrm{F}}^{\#}(m) \xRightarrow{\ \nu_g \circledast \overline{\mathrm{F}}^{\#}(m)\ } (\overline{\mathrm{F}}^{\#}\bar{g})_\circ \circledast \overline{\mathrm{F}}^{\#}(m)$$

is an isomorphism. Writing out the components of that 2-cell in detail we get

$$\begin{aligned}
\mathrm{F}\Big(\coprod_{\bar{x}\in\bar{X}} ((\bar{g}_\circ)_{\bar{x}\bar{x}'} \otimes m_{x\bar{x}})\Big) \xRightarrow{\ \sim\ } \coprod_{\bar{x}\in\bar{X}} \mathrm{F}((\bar{g}_\circ)_{\bar{x}\bar{x}'} \otimes m_{x\bar{x}}) &\xRightarrow{\ \coprod \mathrm{can}\ } \coprod_{\bar{x}\in\bar{X}} \mathrm{F}((\bar{g}_\circ)_{\bar{x}\bar{x}'}) \otimes \mathrm{F}(m_{x\bar{x}}) \\
&\xRightarrow{\ \coprod_{\bar{x}\in\bar{X}}(\nu_g)_{\bar{x}\bar{x}'} \otimes \mathrm{F}(m_{x\bar{x}})\ } \coprod_{\bar{x}\in\bar{X}} ((\mathrm{F}^{\#}\bar{g})_\circ)_{\bar{x}\bar{x}'} \otimes \mathrm{F}(m_{x\bar{x}})
\end{aligned}$$

which is a coproduct of factors of two types:

$$\begin{array}{c}
\mathrm{F}(\mathrm{i}_{\bar{c}} \otimes m) \xRightarrow{\mathrm{can}} \mathrm{F}(\mathrm{i}_{\bar{c}}) \otimes \mathrm{F}(m) \xRightarrow{\ \mathrm{can} \otimes \mathrm{F}(m)\ } \mathrm{i}_{\mathrm{F}\bar{c}} \otimes \mathrm{F}(m) \\
\mathrm{F}(0 \otimes m) \xRightarrow{\mathrm{can}} \mathrm{F}(0) \otimes \mathrm{F}(m) \xRightarrow{\ \square \otimes \mathrm{F}(m)\ } 0 \otimes \mathrm{F}(m)
\end{array}$$

Both are isomorphisms, the first by the coherence conditions on the comorphism F and the second because both its domain and codomain are isomorphic to the terminal object. It follows that we have constructed our first example of an equipment comorphism.

Given the existence of the adjunction in (1.82) the dual of theorem 1.5.14 implies that $(\overline{\mathrm{F}}^{\#}, \mathrm{F}^{\#}, \nu)$ has a right adjoint in $\underline{\underline{\mathcal{E}Map}}$ iff the action of $\overline{\mathrm{F}}^{\#}$ on each homset has a right adjoint. Of course the action of $\overline{\mathrm{F}}^{\#}$ on the homset $\underline{\underline{\mathcal{C}}}$–Mat$((X,\alpha),(\bar{X},\bar{\alpha}))$ is constructed by applying the comorphism F "pointwise" and by assumption each functor $\mathrm{F}_{c\bar{c}}$ has a right adjoint $\mathrm{R}_{c\bar{c}}$. By applying these pointwise on the matrices in $\underline{\underline{\mathcal{B}}}$–Mat$((X,|\mathrm{F}|\circ\alpha),(\bar{X},|\mathrm{F}|\circ\bar{\alpha}))$ we obtain the required right adjoint.

To recap, under the conditions given we get an adjoint pair of equipment maps

$$(\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (\text{-})_\circ) \underset{(\overline{\mathrm{F}}_{\#}, \mathrm{F}_{\#})}{\overset{(\overline{\mathrm{F}}^{\#}, \mathrm{F}^{\#})}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} (\underline{\underline{\mathcal{C}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{C}}}|}, (\text{-})_\circ)$$

where $(\overline{\mathrm{F}}^{\#}, \mathrm{F}^{\#})$ is an equipment comorphism; it follows therefore that we get an induced equipment morphism structure on $(\overline{\mathrm{F}}_{\#}, \mathrm{F}_{\#})$. If the comorphism F, from which this adjunction is derived, is in fact a homomorphism it follows that $(\overline{\mathrm{F}}^{\#}, \mathrm{F}^{\#})$ is an equipment homomorphism and $(\overline{\mathrm{F}}^{\#}, \mathrm{F}^{\#}) \dashv (\overline{\mathrm{F}}_{\#}, \mathrm{F}_{\#})$ becomes an adjunction in $\underline{\underline{\mathcal{E}Mor}}$. □

Example 1.5.17 (Change of Base for Spans) Suppose that $\underline{\mathcal{E}}$ and $\underline{\mathcal{F}}$ are categories with finite limits and $\mathrm{F}\colon \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{F}}$ is any functor (not necessarily left exact). Given a span $(s_0, S, s_1) \in \mathrm{Span}(\underline{\mathcal{E}})(A,B)$ we may simply apply F to it to get another span $(\mathrm{F}(s_0), \mathrm{F}(S), \mathrm{F}(s_1)) \in \mathrm{Span}(\underline{\mathcal{F}})(\mathrm{F}(A), \mathrm{F}(B))$, of course this is just the action on objects of a functor:

$$\mathrm{Span}(\underline{\mathcal{E}})(A,B) \xrightarrow{\overline{\mathrm{F}}} \mathrm{Span}(\underline{\mathcal{F}})(\mathrm{F}(A),\mathrm{F}(B))$$

Notice that for each map $f\colon A \longrightarrow B \in \underline{\mathcal{E}}$ this functor preserve the representable spans $f_\circ = (f, A, \mathrm{i}_A)$ and $f^\circ = (\mathrm{i}_A, A, f)$ on the nose, in the sense that $(\mathrm{F}f)_\circ = \overline{\mathrm{F}}(f_\circ)$ and $(\mathrm{F}f)_\circ = \overline{\mathrm{F}}(f_\circ)$, in particular this means that they strictly preserve all identity spans. Composition of spans is formed by taking pullback, so we are immediately provided with a compositional comparison 2-cell

$$\overline{\mathrm{F}}((t_1,T,t_0) \underset{\bullet}{\times} (s_1,S,s_0)) \Rightarrow \overline{\mathrm{F}}(t_1,T,t_0) \underset{\bullet}{\times} \overline{\mathrm{F}}(s_1,S,s_0)$$

which is the unique map

$$\mathrm{F}(T \underset{B}{\times} S) \longrightarrow \mathrm{F}(T) \underset{\mathrm{F}(B)}{\times} \mathrm{F}(S)$$

induced by the universal property of the pullback $F(T) \underset{F(B)}{\times} F(S)$ and the commutative square

$$\begin{tikzcd} F(T \underset{B}{\times} S) \arrow[r, "F(\pi_S)"] \arrow[d, "F(\pi_T)"'] & F(S) \arrow[d, "F(s_0)"] \\ F(T) \arrow[r, "F(t_1)"'] & F(B) \end{tikzcd}$$

obtained by applying the functor F to the pullback square defining $T \underset{B}{\times} S$. These 2-cells clearly satisfy the coherence conditions making $\overline{F}$ into a (normal) comorphism and, since representables are preserved on the nose by $\overline{F}$, little effort is required to verify (duals of) the conditions in proposition 1.5.13 demonstrating that the pair $(\overline{F}, F)$ is an equipment comorphism:

$$(\mathrm{Span}(\underline{\mathcal{E}}), \underline{\mathcal{E}}, (\text{-})_\circ) \xrightarrow{(\overline{F}, F)} (\mathrm{Span}(\underline{\mathcal{F}}), \underline{\mathcal{F}}, (\text{-})_\circ)$$

We may also enrich any natural transformation $\Psi\colon F \Rightarrow F'\colon \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{F}}$ to a transformation $(\overline{\Psi}, \Psi)\colon (\overline{F}, F) \Rightarrow (\overline{F}', F')$ in $\underline{\underline{\mathcal{E}coMor}}$. For each span $(s_0, S, s_1) \in \mathrm{Span}(\underline{\mathcal{E}})(A, B)$ an enriching 2-cell $\overline{\Psi}_S\colon (\Psi_B)_\circ \underset{\bullet}{\times} \overline{F}(S) \Rightarrow \overline{F}'(S) \underset{\bullet}{\times} (\Psi_A)_\circ$ corresponds under the adjunction $(\Psi_A)_\circ \dashv (\Psi_A)^\circ$ to a 2-cell $\hat{\Psi}_S\colon (\Psi_B)_\circ \underset{\bullet}{\times} \overline{F}(S) \underset{\bullet}{\times} (\Psi_A)^\circ \Rightarrow \overline{F}'(S)$, and the composite in the domain of $\hat{\Psi}_S$ is easily seen to be isomorphic to the span $(\Psi_B \circ F(s_0), F(S), \Psi_A \circ F(s_1))$. We may therefore take $\hat{\Psi}_S$ to be the map $\Psi_S\colon F(S) \longrightarrow F'(S)$ which is a suitable map of spans since the naturality of Ψ ensures that the following diagram commutes:

$$\begin{tikzcd} F(B) \arrow[d, "\Psi_B"'] & F(S) \arrow[l, "F(s_0)"'] \arrow[r, "F(s_1)"] \arrow[d, "\Psi_S"'] & F(A) \arrow[d, "\Psi_A"] \\ F'(B) & F'(S) \arrow[l, "F'(s_0)"] \arrow[r, "F'(s_1)"'] & F'(A) \end{tikzcd}$$

Checking that this data does indeed enrich Ψ to a 2-cell in $\underline{\underline{\mathcal{E}coMor}}$ is routine and left up to the reader.

It should be clear, and is easy to verify, that we have constructed a canonical 2-functor

$$\underline{\underline{\mathrm{CAT}}}_f \xrightarrow{\mathrm{Span}} \underline{\underline{\mathcal{E}coMor}}$$

where $\underline{\underline{\mathrm{CAT}}}_f$ is the 2-category of categories possessing finite limits with all functors and natural transformations between them. It follows therefore that any adjoint pair

$$\begin{tikzcd} \underline{\mathcal{E}} \arrow[rr, bend right, "\mathrm{F}_{\#}"'] \arrow[rr, phantom, "\perp"] & & \underline{\mathcal{F}} \arrow[ll, bend right, "\mathrm{F}^{\#}"'] \end{tikzcd}$$

gives rise to an adjunction

$$\begin{tikzcd} (\mathrm{Span}(\underline{\mathcal{E}}), \underline{\mathcal{E}}, (\text{-})_\circ) \arrow[rr, bend right, "(\overline{\mathrm{F}}_{\#}, \mathrm{F}_{\#})"'] \arrow[rr, phantom, "\perp"] & & (\mathrm{Span}(\underline{\mathcal{F}}), \underline{\mathcal{F}}, (\text{-})_\circ \arrow[ll, bend right, "(\overline{\mathrm{F}}^{\#}, \mathrm{F}^{\#})"'] \end{tikzcd}$$

in $\underline{\underline{\mathcal{E}coMor}}$. Notice that if a functor F is left exact then the corresponding comorphism $\overline{\mathrm{F}}$ is in fact a homomorphism, it follows that the 2-functor "Span" restricts to

$$\underline{\underline{\mathrm{LEX}}} \xrightarrow{\mathrm{Span}} \underline{\underline{\mathcal{E}Hom}}$$

where $\underline{\underline{\mathrm{LEX}}}$ is the sub-2-category of $\underline{\underline{\mathrm{CAT}}}_f$ on the finite limit preserving functors.

Example 1.5.18 (Change of Base for Relations) Carboni, Kelly and Wood's work on change of base [12] may be considered to be a special case of the theory developed here wherein all equipments $(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (\text{-})_*)$ have $\underline{\mathcal{K}}$ a category and $\underline{\underline{\mathcal{M}}}$ a poset enriched category. It follows therefore that all the work they present there carries over to our context, and we refer the reader to that report for further details. □

Example 1.5.19 (Cartesian Bicategories)[4] In their classical form, cartesian bicategories arise in the work of Carboni and Walters [13] as a pivotal part of their characterisation of those order enriched categories which arise as a category of relations in some regular category. Their original definition is couched in a somewhat ad-hoc form and its elucidation into a more conceptually elegant form provided Carboni, Kelly and Wood with one of their primary motivations for the development of the framework presented in [12].

As discussed in the last example, we may reframe the presentation of cartesian bicategories given in [12] within the more general theory of change of base given herein. To do so we start by considering a strongly bicategory enriched category $\underline{\underline{\underline{\mathcal{A}}}}$ which possesses all finite products in the usual enriched sense. In other words, $\underline{\underline{\underline{\mathcal{A}}}}$ has:

- a terminal object $1 \in \underline{\underline{\underline{\mathcal{A}}}}$ for which each $\underline{\underline{\underline{\mathcal{A}}}}(A, 1)$ is isomorphic to the terminal bicategory $\underline{\underline{1}}$ (which has only a single cell at each dimension 0,1 and 2), and

[4]this is a new example which was not present in the original 1992 version of this work.

- for each pair of objects $A, B \in \underline{\underline{\underline{\mathcal{A}}}}$ a product object $A \times B \in \underline{\underline{\underline{\mathcal{A}}}}$ and a pair of homomorphisms $\pi_A \colon A \times B \longrightarrow A$ and $\pi_B \colon A \times B \longrightarrow B$ composition with which induces an isomorphism $\underline{\underline{\underline{\mathcal{A}}}}(C, A \times B) \cong \underline{\underline{\underline{\mathcal{A}}}}(C, A) \times \underline{\underline{\underline{\mathcal{A}}}}(C, B)$ of bicategories.

Now we say that an object $A \in \underline{\underline{\underline{\mathcal{A}}}}$ is *cartesian* if the homomorphisms $! \colon A \longrightarrow 1$ (unique) and $\Delta \colon A \longrightarrow A \times A$ (diagonal) have right biadjoints in $\underline{\underline{\underline{\mathcal{A}}}}$.

Notice here that one might naturally investigate a more liberal finite product notion for (strongly) bicategory enriched categories, under which the isomorphisms above are replaced by biequivalences. However, we will not need these more general notions because our primary examples $\underline{\underline{\underline{\mathcal{E}Map}}}$, $\underline{\underline{\underline{\mathcal{E}Mor}}}$, $\underline{\underline{\underline{\mathcal{E}coMor}}}$ and $\underline{\underline{\underline{\mathcal{E}Hom}}}$ all possess finite products in the stronger sense above. Indeed, in each of these we construct such finite products of equipments *pointwise* in the category $\underline{\underline{\underline{\mathcal{H}om}}}_S$ of bicategories and homomorphisms, wherein such products are given in the obvious (strictly algebraic) way as discussed in observation 1.4.3 and corollary 1.4.4. In other words, the equipment $(\underline{\underline{1}}, \underline{\underline{1}}, \mathrm{I}_{\underline{\underline{1}}})$ is their terminal object and $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \times (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*) = (\underline{\underline{\mathcal{M}}} \times \underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{K}}} \times \underline{\underline{\mathcal{L}}}, (\text{-})_* \times (\text{-})_*)$ is the binary product of a pair of equipments, as the reader may readily verify.

One might now ask for a more explicit description of the cartesian objects in the various (strongly) bicategory enriched categories of equipments. To do this, we start by observing that the canonical homomorphisms $\ddot{!} \colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{\underline{1}}, \underline{\underline{1}}, \mathrm{I}_{\underline{\underline{1}}})$ and $\ddot{\Delta} \colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \times (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ in $\underline{\underline{\underline{\mathcal{E}Hom}}}$ are mapped to the corresponding canonical homomorphisms in $\underline{\underline{\underline{\mathcal{E}Mor}}}$, $\underline{\underline{\underline{\mathcal{E}coMor}}}$ and $\underline{\underline{\underline{\mathcal{E}Map}}}$ by the various forgetful functors that apply between these categories. It follows that we can creep up to an explicit understanding of the cartesian objects in $\underline{\underline{\underline{\mathcal{E}Hom}}}$ by starting with the corresponding characterisation in $\underline{\underline{\underline{\mathcal{E}Map}}}$. This is provided immediately by theorem 1.5.14, an appropriate dual of which applies routinely to show that if $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ is an equipment then:

- The unique homomorphism $\ddot{!} \colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{\underline{1}}, \underline{\underline{1}}, \mathrm{I}_{\underline{\underline{1}}})$ has a right biadjoint in $\underline{\underline{\underline{\mathcal{E}Map}}}$ if and only if

  (i) the bicategory $\underline{\underline{\mathcal{K}}}$ has a *bi-terminal* object, in the sense that there is a 0-cell $1 \in \underline{\underline{\mathcal{K}}}$ with the property that each hom-category $\underline{\underline{\mathcal{K}}}(a, 1)$ is equivalent to the terminal category $\underline{1}$, and
  (ii) the hom-category $\underline{\underline{\mathcal{M}}}(1, 1)$ has a terminal object.

  Here the first clause of this characterisation is simply an explicit restatement of condition that the unique homomorphism of bicategories $! \colon \underline{\underline{\mathcal{K}}} \longrightarrow \underline{\underline{1}}$ should have a right biadjoint, and its second clause simply corresponds to equivalent condition (iii) of the statement of theorem 1.5.14. Notice then that equivalent condition (ii) of that theorem implies that we may infer from the conditions above that *every* hom-category $\underline{\underline{\mathcal{M}}}(a, \bar{a})$ has a terminal object.

- The diagonal homomorphism $\ddot{\Delta}\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)\times(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ in $\underline{\underline{\underline{\mathcal{E}Hom}}}$ has a right biadjoint in $\underline{\underline{\underline{\mathcal{E}Map}}}$ if and only if
  - (i) each pair of 0-cells $a, b \in \underline{\underline{\mathcal{K}}}$ has a *binary bi-product*, in the sense that there exists a 0-cell $a \times b$ and a pair of 1-cells $\pi_a\colon a \times b \longrightarrow a$ and $\pi_b\colon a \times b \longrightarrow b$ in $\underline{\underline{\mathcal{K}}}$ composition with which induces an equivalence of categories $\underline{\underline{\mathcal{K}}}(c, a \times b) \simeq \underline{\underline{\mathcal{K}}}(c, a) \times \underline{\underline{\mathcal{K}}}(c, b)$ for each 0-cell $c \in \underline{\underline{\mathcal{K}}}$, and
  - (ii) each hom-category $\underline{\underline{\mathcal{M}}}(a, \bar{a})$ admits all binary products.

At this point it is worth mentioning that if we are given a bicategory $\underline{\underline{\mathcal{M}}}$ then we may construct a corresponding equipment $(\underline{\underline{\mathcal{M}}}, \mathrm{map}(\underline{\underline{\mathcal{M}}})^{\mathrm{co}}, \mathrm{inc})$ simply by taking $\mathrm{map}(\underline{\underline{\mathcal{M}}})$ to be the locally full sub-bicategory of $\underline{\underline{\mathcal{M}}}$ containing those 1-cells $p\colon a \longrightarrow b$ which possess a right adjoint $p^*\colon b \longrightarrow a$. These 1-cells are often called the *maps* of $\underline{\underline{\mathcal{M}}}$, hence the nomenclature $\mathrm{map}(\underline{\underline{\mathcal{M}}})$ for this sub-bicategory. Now if we apply the above characterisations to this particular equipment we find that it is a cartesian object in $\underline{\underline{\underline{\mathcal{E}Map}}}$ if an only if $\underline{\underline{\mathcal{M}}}$ is a *precartesian bicategory* in the sense of Carboni, Kelly, Walters and Wood [11] definition 3.1. Consequently, we call the cartesian objects in $\underline{\underline{\underline{\mathcal{E}Map}}}$ *precartesian equipments.*

Now suppose that the equipment $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ is precartesian and adopt the notation $\ddot{\mathrm{I}} = (1, \bar{\mathrm{I}})$ to denote the right biadjoint to the unique equipment map $!\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{1}, \underline{1}, \mathrm{I}_{\underline{1}})$ in $\underline{\underline{\underline{\mathcal{E}Map}}}$ and $\ddot{\otimes} = (\times, \overline{\otimes})$ to denote the right biadjoint to the diagonal $\ddot{\Delta}\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \times (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ in $\underline{\underline{\underline{\mathcal{E}Map}}}$. Then we may apply corollary 1.4.12 to show that $\ddot{\mathrm{I}}$ and $\ddot{\otimes}$ admit enrichments to equipment morphisms which are unique for the property that they make the biadjunctions $\ddot{!} \dashv_b \ddot{\mathrm{I}}$ and $\ddot{\Delta} \dashv_b \ddot{\otimes}$ in $\underline{\underline{\underline{\mathcal{E}Map}}}$ into biadjunctions in $\underline{\underline{\underline{\mathcal{E}Mor}}}$. In other words, we have shown that the precartesian equipments are also the cartesian objects of $\underline{\underline{\underline{\mathcal{E}Mor}}}$, a result which subsumes and generalises those presented in propositions 3.15 and 3.18 of [11].

Finally, we might ask for a characterisation of the cartesian objects in $\underline{\underline{\underline{\mathcal{E}Hom}}}$. We know that the enriched forgetful functor $\underline{\underline{\underline{\mathcal{E}Hom}}} \longrightarrow \underline{\underline{\underline{\mathcal{E}Mor}}}$ preserves finite products, the canonical maps $\ddot{!}$ and $\ddot{\Delta}$ and any biadjunctions, so it also preserves cartesian objects. Conversely, returning to the comments at the top of page 124 we also find that $\underline{\underline{\underline{\mathcal{E}Hom}}} \longrightarrow \underline{\underline{\underline{\mathcal{E}Mor}}}$ acts in a full and faithful manner on modifications and transformations and in a faithful manner on homomorphisms. Consequently, the biadjunctions $\ddot{!} \dashv_b \ddot{\mathrm{I}}$ and $\ddot{\Delta} \dashv_b \ddot{\otimes}$ in $\underline{\underline{\underline{\mathcal{E}Mor}}}$ become biadjoints in $\underline{\underline{\underline{\mathcal{E}Hom}}}$ if and only if $\ddot{\mathrm{I}}$ and $\ddot{\otimes}$ happen to be homomorphisms of equipments.

When interpreted for equipments of the form $(\underline{\underline{\mathcal{M}}}, \mathrm{map}(\underline{\underline{\mathcal{M}}})^{\mathrm{co}}, \mathrm{inc})$ this last condition is precisely that given in definition 4.1 of of Carboni, Kelly, Walters and Wood [11]. In other words, we have shown that a bicategory $\underline{\underline{\mathcal{M}}}$ is a *cartesian bicategory* in their sense if and only if the corresponding equipment $(\underline{\underline{\mathcal{M}}}, \mathrm{map}(\underline{\underline{\mathcal{M}}})^{\mathrm{co}}, \mathrm{inc})$ is a cartesian object in $\underline{\underline{\underline{\mathcal{E}Hom}}}$. For that reason we call the cartesian objects of $\underline{\underline{\underline{\mathcal{E}Hom}}}$ *cartesian equipments.*

It is now interesting to observe that results of section 2 of [11] may be generalised to apply in any (strongly) bicategory enriched category $\underline{\underline{\underline{\mathcal{A}}}}$. In essence, all we need do is use the enriched variant of Yoneda's lemma to embed $\underline{\underline{\underline{\mathcal{A}}}}$ in its enriched category of presheaves and then apply those results, which follow by elementary means in $\mathcal{H}om_S$, in a pointwise fashion. So we find that any cartesian object in $\underline{\underline{\underline{\mathcal{A}}}}$ supports the structure of a *symmetric monoidal object* in there, where this latter notion generalises that of *symmetric monoidal bicategory* (as studied by Day and Street [16] and McCrudden [35]) in the most obvious way to our enriched context. As a consequence we find that precartesian and cartesian equipments carry the structure of symmetric monoidal objects in $\underline{\underline{\underline{\mathcal{E}Mor}}}$ and $\underline{\underline{\underline{\mathcal{E}Hom}}}$ respectively. $\square$

1.6 The Equipment of Monads Construction as an Enriched Functor.

The principle purpose of this section is to look in greater detail at the equipment of monads construction introduced in example 1.2.5. Since the construction of $\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ) = (\mathrm{Mon}\,\underline{\underline{\mathcal{M}}}, \mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}), (\text{-})_*)$ relies on the bicategory $\underline{\underline{\mathcal{M}}}$ having local stable coequalisers of reflexive pairs we will assume that condition for those equipments we meet in this section.

In the next few lemmas we establish a practical description of the double bicategory $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$ which we will then apply to establishing the enriched functoriality of the equipment of monads construction $\mathcal{M}\mathrm{on}(\text{-})$:

Lemma 1.6.1 *There exists a canonical forgetful equipment morphism:*

$$\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ) \xrightarrow{(\overline{\mathrm{U}}, \mathrm{U})} (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$$

Proof. It is quite clear that there exist forgetful maps

$$\begin{array}{ccc} \mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}) & \xrightarrow{\mathrm{U}} & \underline{\underline{\mathcal{K}}} \\ (a, \mathbb{A}) & \longmapsto & a \\ (f,\lambda)\big\downarrow & \longmapsto & \big\downarrow f \\ (a', \mathbb{A}') & \longmapsto & a' \end{array} \qquad \begin{array}{ccc} \mathrm{Mon}(\underline{\underline{\mathcal{M}}}) & \xrightarrow{\overline{\mathrm{U}}} & \underline{\underline{\mathcal{M}}} \\ (a, \mathbb{A}) & \longmapsto & a \\ (l_p, p, r_p)\big\downarrow & \longmapsto & \big\downarrow p \\ (\bar{a}, \overline{\mathbb{A}}) & \longmapsto & \bar{a} \end{array}$$

the first of which is a strict homomorphism, since horizontal composition of 1-cells in $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}})$ is achieved by first composing underlying 1-cells in $\underline{\underline{\mathcal{K}}}$. The second is a morphism with canonical 2-cells

$$\mathrm{i}_{\mathrm{U}(a,\mathbb{A})} = \mathrm{i}_a \xrightarrow{\eta_a} \mathbb{A} = \overline{\mathrm{U}}(\mathrm{i}_{(a,\mathbb{A})})$$
$$\overline{\mathrm{U}}(l_{\bar{p}}, \bar{p}, r_{\bar{p}}) \circledast \overline{\mathrm{U}}(l_p, p, r_p) = \bar{p} \circledast p \xrightarrow{q_{\bar{p}p}} \bar{p} \otimes p = \overline{\mathrm{U}}((l_{\bar{p}}, \bar{p}, r_{\bar{p}}) \otimes (l_p, p, r_p))$$

which are respectively the unit of the monad $(a, \mathbb{A})$ and the canonical map displaying $\bar{p} \otimes p$ as a quotient of $\bar{p} \circledast p$.

According to lemma 1.5.7 we also need to provide 2-cells relating these two morphisms, if (f, λ) is a 1-cell in $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}})$ define the 2-cell $\nu_{(f,\lambda)}$ in $\underline{\underline{\mathcal{M}}}$ to be:

$$(\mathrm{U}(f,\lambda))_\circ = f_\circ \xrightarrow{\eta_{a'} \circledast f_\circ} \mathbb{A}' \circledast f_\circ = \overline{\mathrm{U}}(f,\lambda)_*$$

It is a matter of routine verification to show that this family is natural in (f, λ) and satisfies conditions (d)′ and (e)′ of lemma 1.5.7. All that remains is to establish condition (c)$'$ but, with $\nu_{(f,\lambda)}$ defined as above, the 2-cell $\rho^{-1}_{p'(f,\lambda)}$ is given by the composite

$$\overline{\mathrm{U}}(l_{p'},p',r_{p'}) \circledast (\mathrm{U}(f,\lambda))_\circ = p' \circledast f_\circ \xrightarrow{p' \circledast \eta_{a'} \circledast f_\circ} p' \circledast \mathbb{A}' \circledast f_\circ \xrightarrow{q} p' \otimes (\mathbb{A}' \circledast f_\circ) = \overline{\mathrm{U}}((l_{p'},p',r_{p'}) \otimes (f,\lambda)_*) \tag{1.83}$$

which we have to show is an isomorphism. Recall from lemma 1.2.6(i) that there is an equivariant isomorphism $p' \otimes (\mathbb{A}' \circledast f_\circ) \cong p' \circledast f_\circ$ which is defined so as to make the diagram

$$\begin{tikzcd}
p' \circledast \mathbb{A}' \circledast f_\circ \arrow[r, "q"] \arrow[dr, "r_{p'} \circledast f_\circ"'] & p' \otimes (\mathbb{A}' \circledast f_\circ) \\
& p' \circledast f_\circ \arrow[u, "\wr\ \mathrm{iso}"']
\end{tikzcd}$$

commute. Of course we know that $(r_{p'} \circledast f_\circ) \bullet (p' \circledast \eta_{a'} \circledast f_\circ)$ is simply the identity on $p' \circledast f_\circ$ therefore the 2-cell in (1.83) is the isomorphism "iso" above. $\square$

This lemma allows us to describe the squares of $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$ more concretely:

Lemma 1.6.2 *Fixing 1-cells*

$$\begin{array}{rcl}
(f,\lambda),(\bar{f},\bar{\lambda}) & \in & \mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}) \\
(l_p,p,r_p),(l_{p'},p',r_{p'}) & \in & \mathrm{Mon}(\underline{\underline{\mathcal{M}}})
\end{array}$$

the action of $(\overline{\mathrm{U}},\mathrm{U})$ *on squares*

$$\begin{tikzcd}
(a,\mathbb{A}) \arrow[r, "(f,\lambda)"] \arrow[d, "(l_p,p,r_p)"'] \arrow[dr, phantom, "\Uparrow \omega"] & (a',\mathbb{A}') \arrow[d, "(l_{p'},p',r_{p'})"] & & a \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\Uparrow \mathrm{U}_S(\omega)"] & a' \arrow[d, "p'"] \\
(\bar{a},\overline{\mathbb{A}}) \arrow[r, "(\bar{f},\bar{\lambda})"'] & (\bar{a}',\overline{\mathbb{A}}') & \longmapsto & \bar{a} \arrow[r, "\bar{f}"'] & \bar{a}'
\end{tikzcd}$$

gives a bijection between squares $\omega \in \mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$ *and those squares* $\theta \in (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$*, as on the right of the diagram, satisfying the conditions that:*

- *the pair* $(l_p, l_{p'})$ *constitute a cylinder* $\bar{\lambda} \circledast \theta \Rightarrow_V \bar{\lambda}$ *and*
- *the pair* $(r_p, r_{p'})$ *constitute a cylinder* $\theta \circledast \lambda \Rightarrow_V \lambda$*.*

(Here we are using $\circledast$ *for vertical composition of squares in* $\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$*.)*

Proof. This is a direct corollary of lemma 1.2.6. We know that the square $\mathrm{U}_S(\omega)$ is the unique 2-cell such that

$$\begin{tikzcd}
\bar{f}_\circ \circledast p \arrow[r, "\eta_{\bar{a}'} \circledast \bar{f}_\circ \circledast p"] \arrow[d, "\mathrm{U}_S(\omega)"'] & \overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p \arrow[r, "q"] & (\bar{f}, \bar{\lambda}) \otimes p \arrow[d, "\omega"] \\
p' \circledast f_\circ \arrow[r, "p' \circledast \eta_{a'} \circledast f_\circ"'] & p' \circledast \mathbb{A}' f_\circ \arrow[r, "q"'] & p' \otimes (f, \lambda)_*
\end{tikzcd}$$

commutes. Consider the two horizontal composites in this diagram, the lower one was shown (during the proof of the last lemma) to be equal to the canonical equivariant isomorphism $p' \circledast f_\circ \cong p' \otimes (f, \lambda)_*$ of lemma 1.2.6(i), but what about the upper one? Referring to the proof of lemma 1.2.6(ii) we see that it is exactly composition with this map which provides us with a bijection:

$$\frac{(\bar{f}, \bar{\lambda})_* \otimes p \longrightarrow q}{\bar{f}_\circ \circledast p \longrightarrow q} \qquad \begin{array}{l}\text{(equivariant)}\\ \text{(satisfying conds. in (1.26))}\end{array}$$

It follows that there is a correspondence

$$\frac{(\bar{f}, \bar{\lambda})_* \otimes p \xrightarrow{\ \omega\ } p' \otimes (f, \lambda)}{\bar{f}_\circ \circledast p \xrightarrow{\ \mathrm{U}_S(\omega)\ } p' \circledast f} \qquad \begin{array}{l}\text{(equivariant)}\\ \text{(satisfying conds. in (1.26))}\end{array}$$

and recasting the conditions of (1.26) in terms of the left and right actions on $p' \circledast f_\circ$, as given in lemma 1.2.6(i), we get:

$$\begin{tikzcd}
\bar{f}_\circ \circledast p \circledast \mathbb{A} \arrow[r, "\mathrm{U}_S(\omega) \circledast \mathbb{A}"] \arrow[d, "\bar{f}_\circ \circledast r_p"'] & p' \circledast f_\circ \circledast \mathbb{A} \arrow[r, "p' \circledast \lambda"] \arrow[d, phantom, "(\mathrm{a})'"] & p' \circledast \mathbb{A}' \circledast f_\circ \arrow[d, "r_{p'} \circledast f_\circ"] \\
\bar{f}_\circ \circledast p \arrow[rr, "\mathrm{U}_S(\omega)"'] & & p' \circledast f_\circ
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{f}_\circ \circledast \overline{\mathbb{A}} \circledast p \arrow[r, "\bar{\lambda} \circledast p"] \arrow[d, "\bar{f}_\circ \circledast l_p"'] & \overline{\mathbb{A}}' \circledast \bar{f}_\circ \circledast p \arrow[r, "\overline{\mathbb{A}}' \circledast \mathrm{U}_S(\omega)"] \arrow[d, phantom, "(\mathrm{b})'"] & \overline{\mathbb{A}}' \circledast p' \circledast f_\circ \arrow[d, "l_{p'} \circledast f_\circ"] \\
\bar{f}_\circ \circledast p \arrow[rr, "\mathrm{U}_S(\omega)"'] & & p' \circledast f_\circ
\end{tikzcd}$$

These are simply expanded forms of the cylinder conditions given in the statement of this lemma. By a slight abuse of terminology we will often refer to $\mathrm{U}_S(\omega)$ as the square *underlying* ω in $\mathrm{Sq}(\underline{\mathcal{M}}, \underline{\mathcal{K}}, (-)_\circ)$. $\square$

The last lemma allows us to interpret the squares of $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$ in one of two ways depending on whether we wish to think of them as 1-cells of $\mathrm{Cyl}_H(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)))$ or $\mathrm{Cyl}_V(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)))$. First let us think in terms of $\mathrm{Cyl}_H(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)))$:

We already know that we may identify monads in $\underline{\underline{\mathcal{M}}}$ with bicategorical morphisms

$$\mathtt{1} \xrightarrow{(a,\mathbb{A})} \underline{\underline{\mathcal{M}}}$$

where $\mathtt{1}$ is the discrete one object category, but it is less commonly noted that we may extend this sort of description to bimodules. Let $\mathtt{2}$ be the ordinal "2" as a category (cf. (2.10) later on) with two objects $0, 1$ and a single non identity map $m\colon 0 \longrightarrow 1$ and consider a morphism:

$$\mathtt{2} \xrightarrow{\ulcorner p \urcorner} \underline{\underline{\mathcal{M}}}$$

Composing this with the canonical functors $\ulcorner 0 \urcorner, \ulcorner 1 \urcorner \colon \mathtt{1} \longrightarrow \mathtt{2}$ we get monads $(a, \mathbb{A})$ and $(\bar{a}, \overline{\mathbb{A}})$, beyond these the remaining structure of $\ulcorner p \urcorner$ consists of a 1-cell $p = \ulcorner p \urcorner(m)\colon a \longrightarrow a'$ and compositional comparisons:

$$\begin{aligned} p \circledast \mathbb{A} = \ulcorner p \urcorner(m) \circledast \ulcorner p \urcorner(\mathrm{i}_a) &\xRightarrow{r_p} \ulcorner p \urcorner(m \circ \mathrm{i}_a) = p \\ \overline{\mathbb{A}} \circledast p = \ulcorner p \urcorner(\mathrm{i}_{\bar{a}}) \circledast \ulcorner p \urcorner(m) &\xRightarrow{l_p} \ulcorner p \urcorner(\mathrm{i}_{\bar{a}} \circ m) = p \end{aligned}$$

A moments reflection reveals that the conditions that this data satisfies as (part of) the structure of a morphism are exactly the ones which (l_p, p, r_p) must satisfy as a bimodule $(a, \mathbb{A}) \longrightarrow\!\!\!\!\!\!\!+\;\; (\bar{a}, \overline{\mathbb{A}})$.

Having described bimodules in this way it should be clear that the last lemma simply establishes that squares in $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$ correspond to 1-cells in the bicategory $\mathcal{B}icat_H(\mathtt{2}, \mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$, as defined in observation 1.4.2. To make this a little more apparent consider the functors $\ulcorner 0 \urcorner$ and $\ulcorner 1 \urcorner$ which give rise to strict homomorphisms

$$\mathcal{B}icat_H(\mathtt{2}, \mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)) \underset{\text{-}\circ\ulcorner 1 \urcorner}{\overset{\text{-}\circ\ulcorner 0 \urcorner}{\rightrightarrows}} \mathcal{B}icat_H(\mathtt{1}, \mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)) \tag{1.84}$$

by pre-composition (cf lemma 1.6.3 later on). Given a 1-cell

$$\ulcorner \omega \urcorner \colon \ulcorner p \urcorner \longrightarrow \ulcorner p' \urcorner \in \mathcal{B}icat_H(\mathtt{2}, \mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$$

we already know that $\ulcorner p \urcorner$ and $\ulcorner p' \urcorner$ are bimodules which, along with the images in $\mathcal{B}icat_H(\mathtt{1}, \mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)) \overset{\text{def}}{=} \mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}})$ of $\ulcorner \gamma \urcorner$ under the homomorphisms $\text{-} \circ \ulcorner 0 \urcorner$ and $\text{-}\ulcorner 1 \urcorner$, constitute the boundary of a square in $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$. The remaining data encapsulated in $\ulcorner \omega \urcorner$ takes the form of a square in $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$ satisfying the conditions given in the statement of lemma 1.6.2.

Horizontal composition of the squares in $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$ is easy to interpret in this context, $\mathrm{U}\colon \mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}) \longrightarrow \underline{\underline{\mathcal{K}}}$ is a strict homomorphism therefore the cylinder conditions on it as part of a horizontal map

$$\tilde{\mathrm{U}}\colon \mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)) \longrightarrow \mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$$

reduce to the simple equations $\mathrm{U}_S(\omega' \circ \omega) = \mathrm{U}_S(\omega') \circ \mathrm{U}_S(\omega)$ and $\mathrm{U}_S(\mathrm{i}^h_{(l_p,p,r_p)}) = \mathrm{i}^h_p$. In other words we horizontally compose squares in $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$ by horizontally composing their underlying squares in $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$. In a similar fashion the fact that $\tilde{\mathrm{U}}$ preserves the actions of horizontal and vertical 2-cells on squares in $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$ implies that those actions are calculated on underlying squares as in $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$. Notice that these facts simply demonstrate that the representation of lemma 1.6.2 sets up a canonical strict isomorphism (of bicategories):

$$\mathrm{Cyl}_H(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))) \cong \mathcal{B}icat_H(2, \mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)) \tag{1.85}$$

On the other hand re-interpreting the squares of $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$ as 1-cells in $\mathrm{Cyl}_V(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)))$ provides us with a description of their vertical composition. On considering ω as a 1-cell $(f,\lambda) \longrightarrow (\bar{f},\bar{\lambda})$ in here it becomes natural to think of (f,λ) and $(\bar{f},\bar{\lambda})$ as monads in $\mathrm{Cyl}_V(\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$ and in this paradigm the description of ω afforded by lemma 1.6.2 reveals that it is simply a bimodule.

Showing that $\mathrm{Cyl}_V(\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$ has stable local coequalisers of reflexive pairs if $\underline{\underline{\mathcal{M}}}$ does demonstrates that we may tensorially compose its bimodules. Suppose that

$$\begin{tikzcd} \omega_0 \arrow[r, shift left, "{(\alpha_0,\alpha'_0)}"] \arrow[r, shift right, "{(\alpha_1,\alpha'_1)}"'] & \omega_1 \end{tikzcd}$$

is such a pair in $\mathrm{Cyl}_V(\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))(f,\bar{f})$ then pairing components we get reflexive pairs with coequalisers:

$$\begin{tikzcd} p_0 \arrow[r, shift left, "\alpha_0"] \arrow[r, shift right, "\alpha_1"'] & p_1 \arrow[r, "\beta"] & p_2 \end{tikzcd} \qquad \begin{tikzcd} p'_0 \arrow[r, shift left, "\alpha'_0"] \arrow[r, shift right, "\alpha'_1"'] & p'_1 \arrow[r, "\beta'"] & p'_2 \end{tikzcd}$$

The cylinder conditions on (α_0,α'_0) and (α_1,α'_1) imply that we have a serially commutative diagram

$$\begin{tikzcd}[column sep=large]
\bar{f}_* \circledast p_0 \arrow[r, shift left, "\bar{f}_* \circledast \alpha_0"] \arrow[r, shift right, "\bar{f}_* \circledast \alpha_1"'] \arrow[d, "\omega_0"'] & \bar{f}_* \circledast p_1 \arrow[r, "\bar{f}_* \circledast \beta"] \arrow[d, "\omega_1"] & \bar{f}_* \circledast p_2 \arrow[d, dotted, "\omega_2"] \\
p'_0 \circledast f_* \arrow[r, shift left, "\alpha'_0 \circledast f_*"] \arrow[r, shift right, "\alpha'_1 \circledast f_*"'] & p'_1 \circledast f_* \arrow[r, "\beta' \circledast f_*"'] & p'_2 \circledast f_*
\end{tikzcd}$$

both horizontal forks are coequalisers, since $\bar{f}_* \circledast -$ and $- \circledast f_*$ preserve coequalisers of reflective pairs, and that ensures the existence of the unique map ω_2. It is routine to check that the cylinder $(\beta, \beta')\colon \omega_1 \Rightarrow_V \omega_2$ presents the square ω_2 as the coequaliser of (α_0, α'_0) and (α_1, α'_1). The stability of these coequalisers is immediate and (by definition) they are preserved by the (action on homsets of the) homomorphisms $\mathrm{dom}_H, \mathrm{cod}_H\colon \mathrm{Cyl}_V(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ) \longrightarrow \underline{\underline{\mathcal{M}}}$.

Lets return to vertical composition of squares in $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ))$, given a second

$$\begin{tikzcd}[column sep=huge, row sep=large]
(\bar{a}, \overline{\mathbb{A}}) \arrow[r, "(\bar{f}, \bar{\lambda})"] \arrow[d, "(l_{\bar{p}}, \bar{p}, r_{\bar{p}})"', "\shortmid" marking] \arrow[dr, phantom, "\Uparrow \bar{\omega}"] & (\bar{a}', \overline{\mathbb{A}}') \arrow[d, "(l_{\bar{p}'}, \bar{p}', r_{\bar{p}'})", "\shortmid" marking] \\
(\tilde{a}, \widetilde{\mathbb{A}}) \arrow[r, "(\tilde{f}, \tilde{\lambda})"'] & (\tilde{a}', \widetilde{\mathbb{A}}')
\end{tikzcd}$$

one of the cylinder conditions on $\overline{\mathrm{U}}$, as the vertical action of a homomorphism $\tilde{\mathrm{U}} \in \underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$, states that the pair of canonical 2-cells

$$\overline{\mathrm{U}}(l_{\bar{p}}, \bar{p}, r_{\bar{p}}) \circledast \overline{\mathrm{U}}(l_p, p, r_p) \overset{\mathrm{can}}{\Longrightarrow_V} \overline{\mathrm{U}}((l_{\bar{p}}, \bar{p}, r_{\bar{p}}) \otimes (l_p, p, r_p))$$

$$\overline{\mathrm{U}}(l_{\bar{p}'}, \bar{p}', r_{\bar{p}'}) \circledast \overline{\mathrm{U}}(l_{p'}, p', r_{p'}) \overset{\mathrm{can}}{\Longrightarrow_V} \overline{\mathrm{U}}((l_{\bar{p}'}, \bar{p}', r_{\bar{p}'}) \otimes (l_{p'}, p', r_{p'}))$$

constitute a cylinder $\mathrm{U}_S(\bar{\omega}) \circledast \mathrm{U}_S(\omega) \Rightarrow_V \mathrm{U}_S(\bar{\omega} \otimes \omega)$. In other words, notationally identifying the squares ω and $\bar{\omega}$ with their underlying squares in $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ)$, the quotient maps

$$\bar{p} \circledast p \xrightarrow{\quad q \quad} \bar{p} \otimes p$$

$$\bar{p}' \circledast p \xrightarrow{\quad q' \quad} \bar{p}' \otimes p'$$

form a cylinder $\bar{\omega} \circledast \omega \Rightarrow_V \bar{\omega} \otimes \omega$. Returning to the construction we gave for coequalisers in $\mathrm{Cyl}_V(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ)$, we see that the fork

$$\begin{tikzcd}[column sep=huge]
\bar{\omega} \circledast \bar{\lambda} \circledast \omega \arrow[r, shift left, "(r_{\bar{p}}, r_{\bar{p}'}) \circledast \omega"] \arrow[r, shift right, "\bar{\omega} \circledast (l_p, l_{p'})"'] & \bar{\omega} \circledast \omega \arrow[r, "(q, q')"] & \bar{\omega} \otimes \omega
\end{tikzcd}$$

is a coequaliser in $\mathrm{Cyl}_V(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ)((f,\lambda),(\tilde{f},\tilde{\lambda}))$, which of course simply shows that vertical composition of squares in $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ))$ coincides with their composite as 1-cells in $\mathrm{Mon}(\mathrm{Cyl}_V(\mathrm{Sq}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ)))$. Using $\tilde{\mathrm{U}}$ we may also show that the vertical identity in $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ))$ on the horizontal 1-cell (f,λ) has underlying square $\lambda\in\mathrm{Sq}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ)$. To summarise these results the correspondence of lemma 1.6.2 gives rise to a canonical strict isomorphism of bicategories:

$$\mathrm{Cyl}_V(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ)))\cong\mathrm{Mon}(\mathrm{Cyl}_V(\mathrm{Sq}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ)))\ . \tag{1.86}$$

This completes a description of the double bicategory $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ))$ of far greater use in practice, in future we will assume that its squares are *always* given in terms of their underlying ones in $\mathrm{Sq}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ)$.

Our analysis of $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ))$ immediately gives us a method for constructing an enriched functor:

$$\underline{\underline{\mathcal{E}Mor}}\xrightarrow{\ \mathcal{M}\mathrm{on}(\text{-})\ }\underline{\underline{\mathcal{E}Map}}\ .$$

Recall that $\underline{\underline{\mathcal{E}Map}}$ was constructed from $\underline{\underline{\mathcal{H}oriz}}_S$ (via the Sq(-) construction) and that in turn was derived from $\underline{\underline{\mathcal{H}om}}_S^{\mathbb{P}}$ as in definition 1.4.7. By those definitions we know that

$$\begin{aligned}
&\underline{\underline{\mathcal{E}Map}}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ),\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{N}}},\underline{\underline{\mathcal{L}}},(\text{-})_\circ))\\
&\qquad=\ \underline{\underline{\mathcal{H}oriz}}_S(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ)),\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{N}}},\underline{\underline{\mathcal{L}}},(\text{-})_\circ)))\\
&\qquad=\ \underline{\underline{\mathcal{H}om}}_S^{\mathbb{P}}(\mathcal{H}or(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ))),\mathcal{H}or(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{N}}},\underline{\underline{\mathcal{L}}},(\text{-})_\circ)))
\end{aligned}$$

where $\mathcal{H}or(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ)))$ is the diagram

$$\begin{tikzcd}\mathrm{Cyl}_H(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ))) \arrow[r, shift left, "\mathrm{dom}_V"] \arrow[r, shift right, "\mathrm{cod}_V"'] & \mathrm{Mnd}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}})\end{tikzcd} \tag{1.87}$$

which, under the description of $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}},\underline{\underline{\mathcal{K}}},(\text{-})_\circ))$ given above, is equal to the diagram in (1.84). This formulation opens up the possibility of constructing $\mathcal{M}\mathrm{on}(\text{-})$ by an application of:

Lemma 1.6.3 *Let $\mathbb{C}$ be a small category and*

$$\mathbb{C}^{\mathrm{op}}\xrightarrow{\ \mathrm{D}\ }\underline{\mathcal{B}icat}$$

a functor, where $\underline{\mathcal{B}icat}$ is the category of (small) bicategories and morphisms, then there exists an enriched functor

$$\underline{\underline{\mathcal{H}oriz}}_{SM}\xrightarrow{\ \mathcal{B}icat_H(\mathrm{D}(*),\text{-})\ }\underline{\underline{\mathcal{H}om}}_S^{\mathbb{C}}$$

such that for each object $\underline{\underline{\tilde{\mathcal{D}}}} \in \underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ *the functor*

$$\mathbb{C} \xrightarrow{\mathcal{B}icat_H(\mathrm{D}(*), \underline{\underline{\tilde{\mathcal{D}}}})} \underline{\mathcal{SH}om}$$

maps an object $c \in \mathbb{C}$ *to the bicategory* $\mathcal{B}icat_H(\mathrm{D}(c), \underline{\underline{\tilde{\mathcal{D}}}})$ *as defined in observation 1.4.2.*

Proof. Here we restrict ourselves to constructing $\mathcal{B}icat_H(\mathrm{D}(*), -)$ only on the subcategory $\underline{\underline{\underline{\mathcal{H}oriz}}}_{NM}$ of $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ obtained by restricting ourselves to those homomorphisms $\tilde{\mathrm{G}}: \underline{\underline{\tilde{\mathcal{D}}}} \longrightarrow \underline{\underline{\tilde{\mathcal{D}}}}'$ for which $\mathrm{G}_H: \underline{\mathcal{H}} \longrightarrow \underline{\mathcal{H}}'$ is *normal* (preserves identities "on the nose"). This choice prevents us from getting embroiled in the inessential technical details which the full version requires, and which obscure the fundamentals behind this construction. In fact in the cases we apply this lemma to later on the homomorphisms G_H are all functors and so therefore normal, although in any case we may always replace a homomorphism in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SM}$ with one in $\underline{\underline{\underline{\mathcal{H}oriz}}}_{NM}$.

First let us look to see what it means to give an enriched functor

$$\underline{\underline{\mathcal{A}}} \xrightarrow{\mathcal{F}} \underline{\underline{\mathcal{B}}}^{\mathbb{B}}$$

where $\underline{\underline{\mathcal{A}}}$ and $\underline{\underline{\mathcal{B}}}$ are arbitrary (strongly) bicategory enriched categories. Evaluating this functor at an object $A \in \underline{\underline{\mathcal{A}}}$ we get a diagram

$$\mathbb{C} \xrightarrow{\mathcal{F}(A, -)} \underline{\underline{\mathcal{B}}}_{\mathrm{st}}$$

that is for each object $c \in \mathbb{C}$ we get an object $\mathcal{F}(A, c) \in \underline{\underline{\mathcal{B}}}$ and for each map $\gamma: c \longrightarrow c' \in \mathbb{C}$ a strict homomorphism $\mathcal{F}(A, \gamma): \mathcal{F}(A, c) \longrightarrow \mathcal{F}(A, c')$ such that $\mathcal{F}(A, \mathrm{i}_c) = \mathrm{I}_{\mathcal{F}(A,c)}$ and $\mathcal{F}(A, \gamma' \circ \gamma) = \mathcal{F}(A, \gamma') \circ \mathcal{F}(A, \gamma)$. The action of $\mathcal{F}$ on the homset

$$\underline{\underline{\mathcal{A}}}(A, A') \xrightarrow{\mathcal{F}} \underline{\underline{\mathcal{B}}}^{\mathbb{C}}(\mathcal{F}(A), \mathcal{F}(A')) = \int_{c \in \mathbb{C}} \underline{\underline{\mathcal{B}}}(\mathcal{F}(A, c), \mathcal{F}(A', c))$$

corresponds to a family of bicategory homomorphisms

$$\underline{\underline{\mathcal{A}}}(A, A') \xrightarrow{\mathcal{F}(-, c)} \underline{\underline{\mathcal{B}}}(\mathcal{F}(A, c), \mathcal{F}(A', c))$$

with the property that for each map $\gamma \in \mathbb{C}$ the following diagram commutes:

$$\begin{tikzcd}
\underline{\underline{\mathcal{A}}}(A, A') \arrow[r, "\mathcal{F}(-, c)"] \arrow[d, "\mathcal{F}(-, c')"'] & \underline{\underline{\mathcal{B}}}(\mathcal{F}(A, c), \mathcal{F}(A', c)) \arrow[d, "\mathcal{F}(A', \gamma) \circ -"] \\
\underline{\underline{\mathcal{B}}}(\mathcal{F}(A, c'), \mathcal{F}(A', c')) \arrow[r, "- \circ \mathcal{F}(A, \gamma)"'] & \underline{\underline{\mathcal{B}}}(\mathcal{F}(A, c), \mathcal{F}(A', c'))
\end{tikzcd} \tag{1.88}$$

It is easily verified that $\mathcal{F}$ is an enriched functor iff for each $c \in \mathbb{C}$ the homomorphisms $\mathcal{F}(-,c)$ are the homset actions of a functor $\mathcal{F}(-,c)\colon \underline{\underline{\mathcal{A}}} \longrightarrow \underline{\underline{\mathcal{B}}}$. The condition in (1.88) then simply state that the strict homomorphisms $\mathcal{F}(A,\gamma)$ are the components of a natural transformation $\mathcal{F}(-,\gamma)\colon \mathcal{F}(-,c) \longrightarrow \mathcal{F}(-,c')$.

This process is clearly reversible therefore enriched functors $\mathcal{F}\colon \underline{\underline{\mathcal{A}}} \longrightarrow \underline{\underline{\mathcal{B}}}^{\mathbb{C}}$ correspond to (honest) functors:

$$\mathbb{C} \xrightarrow{\hat{\mathcal{F}}} \left[\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{B}}}\right]_{\mathrm{st}}$$

Here $\left[\underline{\underline{\mathcal{A}}}, \underline{\underline{\mathcal{B}}}\right]_{\mathrm{st}}$ denotes the category of enriched functors $\mathcal{G}\colon \underline{\underline{\mathcal{A}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ and (enriched) strict natural transformations $\Gamma\colon \mathcal{G} \Rightarrow \mathcal{G}'$, that is those such that each homomorphism $\Gamma_A\colon \mathcal{G}(A) \longrightarrow \mathcal{G}'(A) \in \underline{\underline{\mathcal{B}}}$ is strict.

Returning to the example in question we may identify any bicategory $\underline{\underline{\mathcal{B}}}$ with a "horizontally discrete" double bicategory in which $\underline{\underline{\mathcal{B}}}$ itself is the bicategory of vertical cells and the only horizontal cells or squares are identities. For a pair of bicategories $\underline{\underline{\mathcal{B}}}$, $\underline{\underline{\mathcal{B}}}'$ the bicategory $\underline{\underline{\mathcal{H}oriz}}_{NM}(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{B}}}')$ is simply the discrete one on the set $\underline{\mathcal{B}icat}(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{B}}}')$, and it follows that we may identify the category $\underline{\mathcal{B}icat}$ with the full subcategory of $\underline{\underline{\mathcal{H}oriz}}_{NM}$ on the horizontally discrete double bicategories.

On restricting our attention to $\underline{\underline{\mathcal{H}oriz}}_{NM}$ and making the identification of the last paragraph we see that

$$\mathcal{B}icat_H(\underline{\underline{\mathcal{B}}}, \underline{\underline{\tilde{\mathcal{D}}}}) = \underline{\underline{\mathcal{H}oriz}}_{NM}(\underline{\underline{\mathcal{B}}}, \underline{\underline{\tilde{\mathcal{D}}}})$$

for each bicategory $\underline{\underline{\mathcal{B}}}$ and double bicategory $\underline{\underline{\tilde{\mathcal{D}}}}$. It is this which motivates our adoption of $\underline{\underline{\mathcal{H}oriz}}_{NM}$ since $\underline{\underline{\mathcal{H}oriz}}_{SM}(\underline{\underline{\mathcal{B}}}, \underline{\underline{\tilde{\mathcal{D}}}})$ is only *biequivalent* to $\mathcal{B}icat_H(\underline{\underline{\mathcal{B}}}, \underline{\underline{\tilde{\mathcal{D}}}})$. The last ingredient is to notice that for any homomorphism $\tilde{\mathrm{G}}\colon \underline{\underline{\tilde{\mathcal{D}}}} \longrightarrow \underline{\underline{\tilde{\mathcal{D}}}}' \in \underline{\underline{\mathcal{H}oriz}}_{NM}$ we may show that each homomorphism of bicategories

$$\underline{\underline{\mathcal{H}oriz}}_{NM}(\underline{\underline{\tilde{\mathcal{D}}}}', \underline{\underline{\tilde{\mathcal{E}}}}) \xrightarrow{-\circ \tilde{\mathrm{G}}} \underline{\underline{\mathcal{H}oriz}}_{NM}(\underline{\underline{\tilde{\mathcal{D}}}}, \underline{\underline{\tilde{\mathcal{E}}}})$$

is *strict*. A 1-cell $\tilde{\Psi}\colon \tilde{\mathrm{F}} \longrightarrow \tilde{\mathrm{F}}' \in \underline{\underline{\mathcal{H}oriz}}_{NM}(\underline{\underline{\tilde{\mathcal{D}}}}', \underline{\underline{\tilde{\mathcal{E}}}})$ consists of families

$$\{\Psi_a \in \underline{\underline{\mathcal{H}}}'\}_{a\in\mathcal{A}} \qquad \left\{\Psi_{a'} \underset{H}{\otimes} \mathrm{F}_H(f) \overset{\Psi_f}{\cong} \mathrm{F}'_H(f) \underset{H}{\otimes} \Psi_a \in \underline{\underline{\mathcal{H}}}'\right\}_{f\in\underline{\underline{\mathcal{H}}}} \qquad \left\{\overline{\Psi}_p \in \mathcal{S}'\right\}_{p\in\underline{\underline{\mathcal{V}}}}$$

satisfying some mutual compatibility conditions, in terms of which the 1-cells of $\underline{\underline{\mathcal{H}oriz}}_{NM}(\underline{\underline{\tilde{\mathcal{D}}}}', \underline{\underline{\tilde{\mathcal{E}}}})$ are (horizontally) composed "pointwise" using horizontal composition in $\underline{\underline{\tilde{\mathcal{D}}}}'$. Correspondingly the composite $\tilde{\Psi} \circ \tilde{\mathrm{G}} \in \underline{\underline{\mathcal{H}oriz}}_{NM}(\underline{\underline{\tilde{\mathcal{D}}}}, \underline{\underline{\tilde{\mathcal{E}}}})$ is formed by "re-indexing" families along the actions of $\tilde{\mathrm{G}}$ on 0-cells and vertical & horizontal 1-cells, it follows from these descriptions that $(\tilde{\Phi}\otimes\tilde{\Psi})\circ\tilde{\mathrm{G}} = (\tilde{\Phi}\circ\tilde{\mathrm{G}})\otimes(\tilde{\Psi}\circ\tilde{\mathrm{G}})$. Of course identity 1-cells in $\underline{\underline{\mathcal{H}oriz}}_{NM}(\underline{\underline{\tilde{\mathcal{D}}}}', \underline{\underline{\tilde{\mathcal{E}}}})$ simply consist of families of horizontal identities, which are clearly preserved by re-indexing; as a result we have demonstrated that $- \circ \tilde{\mathrm{G}}$ is indeed strict.

This is all we need to establish the existence of the required functor since for each object $c \in \mathbb{C}$ we have a representable enriched functor

$$\underline{\underline{\underline{\mathcal{H}oriz}}}_{NM} \xrightarrow{\underline{\underline{\underline{\mathcal{H}oriz}}}_{NM}(\mathrm{D}(c), -)} \underline{\underline{\underline{\mathcal{H}om}}}_{S}$$

which maps each double bicategory $\underline{\underline{\tilde{\mathcal{D}}}}$ to the bicategory $\mathcal{B}icat_H(\mathrm{D}(c), \underline{\underline{\tilde{\mathcal{D}}}})$. Furthermore for each map $\gamma \colon c \longrightarrow c' \in \mathbb{C}$ there is a representable (enriched) natural transformation

$$\underline{\underline{\underline{\mathcal{H}oriz}}}_{NM}(\mathrm{D}(c), -) \xrightarrow{\underline{\underline{\underline{\mathcal{H}oriz}}}_{NM}(\mathrm{D}(\gamma), -)} \underline{\underline{\underline{\mathcal{H}oriz}}}_{NM}(\mathrm{D}(c'), -)$$

the components of which are the strict homomorphisms:

$$\underline{\underline{\underline{\mathcal{H}oriz}}}_{NM}(\mathrm{D}(c), A) \xrightarrow{- \circ \mathrm{D}(\gamma)} \underline{\underline{\underline{\mathcal{H}oriz}}}_{NM}(\mathrm{D}(c'), A)$$

All of this provides us with a functor

$$\mathbb{C} \xrightarrow{\left(\underline{\underline{\underline{\mathcal{H}oriz}}}_{NM}(\mathrm{D}(*), -)\right)^{\wedge}} \left[\underline{\underline{\underline{\mathcal{H}oriz}}}_{NM}, \underline{\underline{\underline{\mathcal{H}om}}}_{S}\right]_{\mathrm{st}}$$

which corresponds as above to an enriched functor

$$\underline{\underline{\underline{\mathcal{H}oriz}}}_{NM} \xrightarrow{\underline{\underline{\underline{\mathcal{H}oriz}}}_{NM}(\mathrm{D}(*), -)} \underline{\underline{\underline{\mathcal{H}om}}}_{S}^{\mathbb{C}}$$

the natural candidate for $\mathcal{B}icat_H(\mathrm{D}(*), -)$. $\square$

As promised at the top of page 160, lemma 1.6.3 can be applied to the problem of constructing the enriched functor $\mathcal{M}\mathrm{on}(-)$; giving the immediate corollary:

Corollary 1.6.4 *The equipment of monads construction of example 1.2.5 extends to an enriched functor:*

$$\underline{\underline{\underline{\mathcal{E}Mor}}} \xrightarrow{\mathcal{M}\mathrm{on}(-)} \underline{\underline{\underline{\mathcal{E}Map}}}$$

For each equipment morphism $\ddot{\mathrm{G}} \colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ) \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (-)_\circ)$ *in* $\underline{\underline{\underline{\mathcal{E}Mor}}}$ *the diagram*

$$\begin{tikzcd}
\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ) \arrow[r, "\mathcal{M}\mathrm{on}(\ddot{\mathrm{G}})"] \arrow[d, "(\overline{\mathrm{U}}, \mathrm{U})"'] & \mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (-)_\circ) \arrow[d, "(\overline{\mathrm{U}}, \mathrm{U})"] \\
(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ) \arrow[r, "\ddot{\mathrm{G}}"] & (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (-)_\circ)
\end{tikzcd} \tag{1.89}$$

commutes in $\underline{\underline{\underline{\mathcal{E}Map}}}$.

Proof. Take for $\mathbb{C}$ the category $\mathbb{P}$ (2 parallel arrows) and let $\mathrm{D}\colon \mathbb{P}^{\mathrm{op}} \longrightarrow \underline{\mathcal{B}icat}$ be the functor which carries the objects of $\mathbb{P}^{\mathrm{op}}$ to the categories $\mathtt{1}$ and $\mathtt{2}$ and its maps to the functors $\ulcorner 0 \urcorner, \ulcorner 1 \urcorner\colon \mathtt{1} \longrightarrow \mathtt{2}$. Now consider the composite enriched functor

$$\underline{\underline{\underline{\mathcal{E}Mor}}} \xrightarrow{\mathrm{Sq}(\text{-})} \underline{\underline{\underline{\mathcal{H}oriz}}}_{SM} \xrightarrow{\mathcal{B}icat_H(\mathrm{D}(*),\text{-})} \underline{\underline{\underline{\mathcal{H}om}}}_S^{\mathbb{P}} .$$

This maps each equipment $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$ to the diagram in (1.84), which in turn is equal to that in (1.87). It follows, by the definition of $\underline{\underline{\underline{\mathcal{E}Map}}}$ from $\underline{\underline{\underline{\mathcal{H}om}}}_S^{\mathbb{P}}$ of which we reminded ourselves before the last lemma, that the enriched functor above factors through

$$\underline{\underline{\underline{\mathcal{E}Map}}} \xrightarrow{\mathrm{Sq}(\text{-})} \underline{\underline{\underline{\mathcal{H}oriz}}}_{S*} \xrightarrow{\mathcal{H}or(\text{-})} \underline{\underline{\underline{\mathcal{H}om}}}_S^{\mathbb{P}}$$

yielding an enriched functor $\mathcal{M}\mathrm{on}(\text{-})$ extending the equipment of monads construction.

We may demonstrate a more general result than the simple commutativity of (1.89); the forgetful functors $(\overline{\mathrm{U}}, \mathrm{U})\colon \mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ) \longrightarrow (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$ are the components of a strict natural transformation from $\mathcal{M}\mathrm{on}\colon \underline{\underline{\underline{\mathcal{E}Mor}}} \longrightarrow \underline{\underline{\underline{\mathcal{E}Map}}}$ to the forgetful functor $(\text{-})_\diamond\colon \underline{\underline{\underline{\mathcal{E}Mor}}} \longrightarrow \underline{\underline{\underline{\mathcal{E}Map}}}$. Checking this result is simply a matter of unravelling (a little) the definition we gave of $\mathcal{M}\mathrm{on}(\text{-})$ and we leave it up to the reader. For example all that the property in (1.89) says is that $\mathcal{M}\mathrm{on}(\ddot{\mathrm{G}})$ acts on the structures in $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ))$ as $\ddot{\mathrm{G}}$ does on those that underlie them in $\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$. $\square$

Proposition 1.6.5 *The enriched functor of corollary 1.6.4 lifts along the forgetful functor* $(\text{-})_\diamond\colon \underline{\underline{\underline{\mathcal{E}Mor}}} \longrightarrow \underline{\underline{\underline{\mathcal{E}Map}}}$ *to*

$$\underline{\underline{\underline{\mathcal{E}Mor}}} \xrightarrow{\mathcal{M}\mathrm{on}(\text{-})} \underline{\underline{\underline{\mathcal{E}Mor}}}$$

which also has the property that the diagram (1.89) commutes in $\underline{\underline{\underline{\mathcal{E}Mor}}}$ *for each equipment morphism* $\ddot{\mathrm{G}}$. *If* $\ddot{\mathrm{G}}\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ) \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_\circ)$ *is in* $\underline{\underline{\underline{\mathcal{E}Hom}}}$ *and the (honest) homomorphism* $\overline{\mathrm{G}}\colon \underline{\underline{\mathcal{M}}} \longrightarrow \underline{\underline{\mathcal{N}}}$ *preserves local coequalisers of reflexive pairs then* $\mathcal{M}\mathrm{on}(\ddot{\mathrm{G}})$ *is in* $\underline{\underline{\underline{\mathcal{E}Hom}}}$ *as well.*

Proof. This proposition is essentially no more than the observation that we may extend the "bicategory of modules" construction to a functor:

$$\underline{\mathcal{B}icat} \xrightarrow{\mathrm{Mon}(\text{-})} \underline{\mathcal{B}icat}$$

Suppose that $\overline{\mathrm{G}}\colon \underline{\underline{\mathcal{M}}} \longrightarrow \underline{\underline{\mathcal{N}}}$ is a morphism then a 0-cell of $\mathrm{Mon}(\underline{\underline{\mathcal{M}}})$ is a morphism $\mathbb{A}\colon \mathtt{1} \longrightarrow \underline{\underline{\mathcal{M}}}$ which we may map to the 0-cell in $\mathrm{Mon}(\underline{\underline{\mathcal{N}}})$ corresponding to the composite morphism $(\overline{\mathrm{G}} \circ \mathbb{A})$. Similarly a 1-cell in $\mathrm{Mon}(\underline{\underline{\mathcal{M}}})$ is a morphism $\ulcorner p \urcorner\colon \mathtt{2} \longrightarrow \underline{\underline{\mathcal{M}}}$ which we map to $(\overline{\mathrm{G}} \circ \ulcorner p \urcorner)$, it is easily checked that $\overline{\mathrm{G}}$ then preserves equivariant maps. Since these actions clearly preserve the identities of $\mathrm{Mon}(\underline{\underline{\mathcal{M}}})$ "on the nose" it remains only to provide $\mathrm{Mon}(\mathrm{G})$ with compositional comparison 2-cells.

It will help to make a little more explicit the action of $\mathrm{Mon}(\overline{\mathrm{G}})$ on a bimodule $(l_p, p, r_p)\colon (a, \mathbb{A}) \longrightarrow (a', \mathbb{A}')$, the bimodule $\mathrm{Mon}(\overline{\mathrm{G}})(l_p, p, r_p)$ has underlying 1-cell $\overline{\mathrm{G}}(p)$ equipped with actions:

$$\overline{\mathrm{G}}(p) \circledast \overline{\mathrm{G}}(\mathbb{A}) \xrightarrow{\text{can}} \overline{\mathrm{G}}(p \circledast \mathbb{A}) \xrightarrow{\overline{\mathrm{G}}(r_p)} \overline{\mathrm{G}}(p)$$
$$\overline{\mathrm{G}}(\mathbb{A}') \circledast \overline{\mathrm{G}}(p) \xrightarrow{\text{can}} \overline{\mathrm{G}}(\mathbb{A}' \circledast p) \xrightarrow{\overline{\mathrm{G}}(l_p)} \overline{\mathrm{G}}(p)$$

Now if we are given a second bimodule $(l_{p'}, p', r_{p'})\colon (a', \mathbb{A}') \longrightarrow (a'', \mathbb{A}'')$ we have a serially commutative diagram

$$\begin{tikzcd}
\overline{\mathrm{G}}(p') \circledast \overline{\mathrm{G}}(\mathbb{A}') \circledast \overline{\mathrm{G}}(p) \arrow[r, shift left, "\overline{\mathrm{G}}(p') \circledast (\overline{\mathrm{G}}(l_p) \bullet \text{can})"] \arrow[r, shift right, "(\overline{\mathrm{G}}(r_{p'}) \bullet \text{can}) \circledast \overline{\mathrm{G}}(p)"'] \arrow[d, "\text{can}"'] & \overline{\mathrm{G}}(p') \circledast \overline{\mathrm{G}}(p) \arrow[r, "q"] \arrow[d, "\text{can}"'] & \overline{\mathrm{G}}(p') \otimes \overline{\mathrm{G}}(p) \arrow[d, dotted] \\
\overline{\mathrm{G}}(p' \circledast \mathbb{A}' \circledast p) \arrow[r, shift left, "\overline{\mathrm{G}}(p' \circledast l_p)"] \arrow[r, shift right, "\overline{\mathrm{G}}(r_{p'} \circledast p"'] &)\overline{\mathrm{G}}(p' \circledast p) \arrow[r, "\overline{\mathrm{G}}(q)"'] & \overline{\mathrm{G}}(p' \otimes p)
\end{tikzcd}$$

the upper fork of which is a coequaliser, this induces the dotted comparison arrow to the right. It is now straightforward to show that this is an equivariant map $\text{can}\colon \mathrm{Mon}(\overline{\mathrm{G}})(l_{p'}, p', r_{p'}) \otimes \mathrm{Mon}(\overline{\mathrm{G}})(l_p, p, r_p) \Rightarrow \mathrm{Mon}(\overline{\mathrm{G}})((l_{p'}, p', r_{p'}) \otimes (l_p, p, r_p))$ and that they collectively satisfy the coherence conditions required of the compositional comparisons of a morphism $\mathrm{Mon}(\overline{\mathrm{G}})$. Notice that if $\overline{\mathrm{G}}$ is a homomorphism then the maps marked "can" in the diagram are isomorphisms, furthermore if it preserves local reflexive coequalisers then the bottom fork is a coequaliser. It follows that under those conditions the dotted arrow is an isomorphism and so $\mathrm{Mon}(\overline{\mathrm{G}})$ is a homomorphism. Notice that $\mathrm{Mon}(\overline{\mathrm{G}})$ is defined precisely so that the diagram

$$\begin{tikzcd}
\mathrm{Mon}(\underline{\underline{\mathcal{M}}}) \arrow[r, "\mathrm{Mon}(\overline{\mathrm{G}})"] \arrow[d, "\overline{\mathrm{U}}"'] & \mathrm{Mon}(\underline{\underline{\mathcal{N}}}) \arrow[d, "\overline{\mathrm{U}}"] \\
\underline{\underline{\mathcal{M}}} \arrow[r, "\overline{\mathrm{G}}"] & \underline{\underline{\mathcal{N}}}
\end{tikzcd} \tag{1.90}$$

commutes.

Given an equipment morphism $\ddot{\mathrm{G}}\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ) \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (-)_\circ)$ we show that we can enrich $\mathcal{M}\mathrm{on}(\ddot{\mathrm{G}})$ to an equipment morphism by enriching the corresponding horizontal map

$$\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ)) \xrightarrow{\mathrm{Sq}(\mathcal{M}\mathrm{on}(\ddot{\mathrm{G}}))} \mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (-)_\circ))$$

to $\underline{\underline{\mathcal{H}oriz}}_{SM}$. The morphism $\mathrm{Mon}(\overline{\mathrm{G}})\colon \mathrm{Mon}(\underline{\underline{\mathcal{M}}}) \longrightarrow \mathrm{Mon}(\underline{\underline{\mathcal{N}}})$ enriches its action on vertical cells but it remains to check that this satisfies the usual compatibility conditions with respect to the action of $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\ddot{\mathrm{G}}))$ on squares. Those hold iff there exists a morphism M which makes the diagram

$$\begin{tikzcd}
\mathrm{Cyl}_V(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ))) \arrow[r, "\mathrm{M}"] \arrow[d, shift right, "\mathrm{dom}_H"'] \arrow[d, shift left, "\mathrm{cod}_H"] & \mathrm{Cyl}_V(\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (-)_\circ))) \arrow[d, shift right, "\mathrm{dom}_H"'] \arrow[d, shift left, "\mathrm{cod}_H"] \\
\mathrm{Mon}(\underline{\underline{\mathcal{M}}}) \arrow[r, "\mathrm{Mon}(\overline{\mathrm{G}})"] & \mathrm{Mon}(\underline{\underline{\mathcal{N}}})
\end{tikzcd}$$

commute serially while acting on 1-cells as $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\ddot{\mathrm{G}}))$ does on squares. Under the representation of $\mathrm{Sq}(\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ))$ we developed earlier in the section, and in particular the isomorphism in (1.86), it is clear that the parallel pair of strict homomorphisms to the left of this diagram is (essentially) equal to:

$$\begin{tikzcd}
\mathrm{Mon}(\mathrm{Cyl}_V(\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ))) \arrow[r, shift left, "\mathrm{Mon}(\mathrm{dom}_H)"] \arrow[r, shift right, "\mathrm{Mon}(\mathrm{cod}_H)"'] & \mathrm{Mon}(\underline{\underline{\mathcal{M}}})
\end{tikzcd}$$

The same result holds for the right of the diagram and it becomes immediately apparent that we may take for M the morphism obtained by applying $\mathrm{Mon}(-)$ to the morphism $\mathrm{Cyl}_V(\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ)) \longrightarrow \mathrm{Cyl}_V(\mathrm{Sq}(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (-)_\circ))$ constructed from $\ddot{\mathrm{G}}$.

We leave it up to the reader to check that $\mathcal{M}\mathrm{on}(-)$ carries transformations to those compatible with the morphism structures provided above. The remainder of this proposition is clear given the commutative diagram (1.90) and the result concerning the action of $\mathrm{Mon}(-)$ on homomorphisms.

Notice that if $\ddot{\mathrm{F}} \dashv_b \colon \ddot{\mathrm{U}}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ) \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (-)_\circ)$ is a biadjoint in $\underline{\underline{\underline{\mathcal{E}Mor}}}$ with $\ddot{\mathrm{F}}$ in $\underline{\underline{\underline{\mathcal{E}Hom}}}$ then by (a dual of) theorem 1.5.14 the action of $\overline{\mathrm{F}}$ on each homset

$$\underline{\underline{\mathcal{N}}}(b, b') \xrightarrow{\overline{\mathrm{F}}} \underline{\underline{\mathcal{M}}}(\mathrm{F}b, \mathrm{F}b')$$

has a right adjoint. It follows that each of these functors preserves coequalisers and therefore $\mathcal{M}\mathrm{on}(\ddot{\mathrm{F}})$ is in $\underline{\underline{\underline{\mathcal{E}Hom}}}$ □

Again in many of the cases of direct interest here we are only interested in equipments $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_\circ)$ with $\underline{\underline{\mathcal{K}}}$ a mere category, in this case the enriched functor derived in the last proposition restricts to a 2-functor:

$$\underline{\underline{\underline{\mathcal{E}Mor}}} \xrightarrow{\mathcal{M}\mathrm{on}(-)} \underline{\underline{\underline{\mathcal{E}Mor}}}$$

Applying $\mathcal{M}\mathrm{on}(-)$ to our examples of adjunctions between equipments of spans and matrices we get:

Example 1.6.6 (change of base for enriched categories) For a pair of bicategories $\underline{\underline{\mathcal{B}}}$, $\underline{\underline{\mathcal{C}}}$ with local stable (small) colimits and small sets of 0-cells along with a homomorphism

$$\underline{\underline{\mathcal{C}}} \xrightarrow{\quad \mathrm{F} \quad} \underline{\underline{\mathcal{B}}}$$

satisfying the local adjointness condition of example 1.5.16, we have already seen that we may construct an adjunction

$$\begin{tikzcd}(\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (\text{–})_\circ) \arrow[r, shift right, "(\overline{\mathrm{F}}_\#, \mathrm{F}_\#)"'] \arrow[r, phantom, "\perp"] & (\underline{\underline{\mathcal{C}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{C}}}|}, (\text{–})_\circ) \arrow[l, shift right, "(\overline{\mathrm{F}}^\#, \mathrm{F}^\#)"']\end{tikzcd}$$

in $\underline{\mathcal{E}\mathcal{M}or}$ with $(\overline{\mathrm{F}}^\#, \mathrm{F}^\#)$ an equipment homomorphism. Applying $\mathcal{M}\mathrm{on}(\text{–})$ to this we get a further adjunction

$$\begin{tikzcd}(\underline{\underline{\mathcal{B}}}\text{–Prof}, \underline{\underline{\mathcal{B}}}\text{–Cat}_1, (\text{–})_*) \arrow[r, shift right, "(\overline{\mathrm{F}}_\star, \mathrm{F}_\star)"'] \arrow[r, phantom, "\perp"] & (\underline{\underline{\mathcal{C}}}\text{–Prof}, \underline{\underline{\mathcal{C}}}\text{–Cat}_1, (\text{–})_*) \arrow[l, shift right, "(\overline{\mathrm{F}}^\star, \mathrm{F}^\star)"']\end{tikzcd}$$

where example 1.2.8 identifies the equipment $\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (\text{–})_\circ)$ as that of $\underline{\underline{\mathcal{B}}}$-enriched categories $(\underline{\underline{\mathcal{B}}}\text{–Prof}, \underline{\underline{\mathcal{B}}}\text{–Cat}_1, (\text{–})_*)$. By the comment at the end of proposition 1.6.5 we know that $(\overline{\mathrm{F}}^\star, \mathrm{F}^\star)$ is an equipment homomorphism.

On the bicategories of profunctors, as we shall see in the next section, the structure of such an (bi)adjoint pair is that of a local adjoint, in the sense of section 1.1. In fact the local adjoint associated with the adjunction constructed here is exactly that described in the first example of section 2 of [6]. They do not stipulate directly that the homomorphism F we start with should satisfy a local adjointness condition, requiring instead that it preserve local colimits and that $\underline{\underline{\mathcal{C}}}$ should be small.

Of course these conditions imply the local adjointness property via the general adjoint functor theorem, but more than that each homset of $\underline{\underline{\mathcal{C}}}$ is small and small cocomplete so by a well know result of Freyd they must all be *preorders*. This clearly restricts the applicability of the result in [6] radically, a better size requirement might be to insist that each homset of $\underline{\underline{\mathcal{C}}}$ has a small generating set. Now we simply apply the special adjoint functor theorem to the homset actions of F and infer the local adjointness property. □

Example 1.6.7 (change of base for internal categories) Suppose $\underline{\mathcal{E}}$ and $\underline{\mathcal{F}}$ are (locally small) categories with finite limits and coequalisers of reflexive pairs stable under pullback, and let

$$\begin{tikzcd}\underline{\mathcal{E}} \arrow[r, shift right, "\mathrm{F}_\#"'] \arrow[r, phantom, "\perp"] & \underline{\mathcal{F}} \arrow[l, shift right, "\mathrm{F}^\#"']\end{tikzcd}$$

be an adjunction in $\underline{\underline{\mathrm{LEX}}}$, or in other words a geometric morphism. Applying the 2-functor "Span" considered in example 1.5.17 to this we get an adjunction

$$(\mathrm{Span}(\underline{\mathcal{E}}), \underline{\mathcal{E}}, (-)_\circ) \underset{(\overline{\mathrm{F}}_\#, \mathrm{F}_\#)}{\overset{(\overline{\mathrm{F}}^\#, \mathrm{F}^\#)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} (\mathrm{Span}(\underline{\mathcal{F}}), \underline{\mathcal{F}}, (-)_\circ$$

in $\underline{\underline{\mathcal{E}Hom}}$ to which (in turn) we apply $\mathcal{M}\mathrm{on}(-)$ obtaining

$$(\mathrm{Prof}(\underline{\mathcal{E}}), \mathrm{Cat}(\underline{\mathcal{E}})_1, (-)_*) \underset{(\overline{\mathrm{F}}_\star, \mathrm{F}_\star)}{\overset{(\overline{\mathrm{F}}^\star, \mathrm{F}^\star)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} (\mathrm{Prof}(\underline{\mathcal{F}}), \mathrm{Cat}(\underline{\mathcal{F}})_1, (-)_*)$$

in $\underline{\underline{\mathcal{E}Mor}}$. Here we have exploited the identification of $\mathcal{M}\mathrm{on}(\mathrm{Span}(\underline{\mathcal{E}}), \underline{\mathcal{E}}, (-)_\circ)$ as the equipment $(\mathrm{Prof}(\underline{\mathcal{E}}), \mathrm{Cat}(\underline{\mathcal{E}})_1, (-)_*)$ which was provided by example 1.2.9. The comment at the end of the proof of proposition 1.6.5 demonstrates that $(\overline{\mathrm{F}}^\star, \mathrm{F}^\star)$ is an equipment homomorphism. □

As we pointed out in observation 1.2.10 we are often not so much interested in equipments like $(\underline{\underline{\mathcal{B}}}\text{–Prof}, \underline{\underline{\mathcal{B}}}\text{–Cat}_1, (-)_*)$ or $(\mathrm{Prof}(\underline{\mathcal{E}}), \mathrm{Cat}(\underline{\mathcal{E}})_1, (-)_*)$ rather than in their *repletions* $\underline{\underline{\mathcal{B}}}$–Equip and $\mathrm{Equip}(\underline{\mathcal{E}})$, wherein we replace the skeletons $\underline{\underline{\mathcal{B}}}\text{–Cat}_1$ and $\mathrm{Cat}(\underline{\mathcal{E}})_1$ with the corresponding 2-categories of categories, functors and natural transformations. This process is however not quite so well behaved with respect to equipment morphisms, in particular there can be problems if the morphism $(\overline{\mathrm{G}}, \mathrm{G}, \nu)$ in question has comparison maps $\nu_f\colon (\mathrm{G}f)_* \Rightarrow \overline{\mathrm{G}}(f_*)$ which are not all isomorphisms. Since we have certainly seen examples of such morphisms, for instance that of lemma 1.6.1, the following definition is far from being vacuous:

Definition 1.6.8 (preservation of representables) An equipment morphism

$$(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (-)_*) \xrightarrow{(\overline{\mathrm{G}}, \mathrm{G}, \nu)} (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (-)_*)$$

is said to *preserves representables* iff the 2-cell $\nu_f\colon (\mathrm{G}f)_* \Rightarrow \overline{\mathrm{G}}(f_*)$ is an isomorphism for each 1-cell $f \in \underline{\underline{\mathcal{K}}}$.

The author has no reason to believe that it is generally true that the preservation of representables by $(\overline{\mathrm{G}}, \mathrm{G}, \nu)$ implies that the same holds for its dual $(\overline{\mathrm{G}}^{\mathrm{op}}, \mathrm{G}^{\mathrm{co}}, \nu')$, although no natural counter examples have presented themselves. Often we will say that $(\overline{\mathrm{G}}, \mathrm{G}, \nu)$ preserves both *left* and *right* representables if both it and its dual preserve representables.

Lemma 1.6.9 *Let* $\underline{\underline{\underline{\mathcal{E}Mor}}}_{\mathrm{pr}}$ *be the (locally full) enriched subcategory of* $\underline{\underline{\underline{\mathcal{E}Mor}}}$ *on those equipment morphisms which preserve representables. There exists a canonical enriched functor*

$$\underline{\underline{\underline{\mathcal{E}Mor}}}_{\mathrm{pr}} \xrightarrow{(\text{-})_{\mathrm{rep}}} \underline{\underline{\underline{\mathcal{E}Mor}}}_{\mathrm{pr}}$$

extending the operation of repletion described in observation 1.2.10.

Proof. Given a morphism $(\overline{\mathrm{G}}, \mathrm{G}, \nu)\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$ in $\underline{\underline{\underline{\mathcal{E}Mor}}}_{\mathrm{pr}}$ we define a homomorphism

$$\underline{\underline{\mathcal{K}}}_* \xrightarrow{\mathrm{G}_*} \underline{\underline{\mathcal{L}}}$$

which acts as G does on the 0- and 1-cells of $\underline{\underline{\mathcal{K}}}_*$. A 2-cell $\alpha\colon f \Rightarrow g$ of $\underline{\underline{\mathcal{K}}}_*$ corresponds to $\bar{\alpha}\colon g_* \Rightarrow f_*$ in $\underline{\underline{\mathcal{M}}}$ so we map it to the cell corresponding to the composite

$$(\mathrm{G}g)_* \overset{\nu_g}{\underset{\sim}{\Longrightarrow}} \overline{\mathrm{G}}(g_*) \overset{\overline{\mathrm{G}}(\bar{\alpha})}{\Longrightarrow} \overline{\mathrm{G}}(f_*) \overset{\nu_f^{-1}}{\underset{\sim}{\Longrightarrow}} (\mathrm{G}f)_*$$

thus defining a functor $\mathrm{G}_*\colon \underline{\underline{\mathcal{K}}}_*(a, a') \longrightarrow \underline{\underline{\mathcal{L}}}_*(\mathrm{G}a, \mathrm{G}a')$ for each pair of 0-cells. Now by applying the axioms satisfied by our ν_fs and the coherence properties of G and $(\text{-})_*$ as homomorphisms it is easy to show that the isomorphisms in $\underline{\underline{\mathcal{L}}}_*$ corresponding to

$$(\mathrm{G}(f' \circ f))_* \overset{\mathrm{can}_*}{\underset{\sim}{\Longrightarrow}} (\mathrm{G}(f') \circ \mathrm{G}(f))_*$$
$$(\mathrm{G}(\mathrm{i}_a))_* \overset{\mathrm{can}_*}{\underset{\sim}{\Longrightarrow}} (\mathrm{i}_{\mathrm{G}a})_*$$

satisfy the conditions required of the 2-cellular structure of a homomorphism G_*. We have defined the action on 2-cells precisely so as to ensure that the family ν_f is a natural transformation $(\mathrm{G}_*\text{-})_* \Rightarrow \overline{\mathrm{G}}(\text{-})_*$ on each homset $\underline{\underline{\mathcal{K}}}_*(a, a')$, which is all we need to verify in order to show that $(\overline{\mathrm{G}}, \mathrm{G}, \nu)_{\mathrm{rep}} \overset{\mathrm{def}}{=} (\overline{\mathrm{G}}, \mathrm{G}_*, \nu)$ is an equipment morphism $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)_{\mathrm{rep}} \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)_{\mathrm{rep}}$.

We leave it up to the reader to provide the actions of $(\text{-})_{\mathrm{rep}}$ on modifications and transformations, which are straightforward, and then to verify that this does indeed define an enriched functor. □

Now to check that the adjunctions in examples 1.6.6 and 1.6.7 are in fact in $\underline{\underline{\underline{\mathcal{E}Mor}}}_{\mathrm{pr}}$:

Lemma 1.6.10 *The enriched functor* $\mathcal{M}\mathrm{on}(\text{-})$ *carries any equipment morphism to one which preserves both left and right representables.*

Proof. Let $(\overline{\mathrm{G}}, \mathrm{G}, \nu)$ be an equipment morphism and for convenience adopt the notation $(\overline{\mathrm{G}}^m, \mathrm{G}^m, \nu^m)$ for $\mathcal{M}\mathrm{on}(\overline{\mathrm{G}}, \mathrm{G}, \nu)$. Now consider the commutative diagram

(1.89) of corollary 1.6.4 which we know by proposition 1.6.5 holds in $\underline{\underline{\underline{\mathcal{E}Mor}}}$ as well. For a 1-cell $(f, \lambda)\colon (a, \mathbb{A}) \longrightarrow (a', \mathbb{A}')$ in $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}})$ it follows that we have:

$$\begin{array}{rcccl} \overline{\mathrm{U}} \circ \overline{\mathrm{G}}^m((f,\lambda)_*) & = & \overline{\mathrm{G}} \circ \overline{\mathrm{U}}((f,\lambda)_*) & = & \overline{\mathrm{G}}(\mathbb{A}' \circledast f_\circ) \\ (\mathrm{U} \circ \mathrm{G}^m(f,\lambda))_\circ & = & (\mathrm{G} \circ \mathrm{U}(f,\lambda))_\circ & = & (\mathrm{G}f)_\circ \end{array}$$

Furthermore we also have two comparison maps at this representable, one for each composite in (1.89), which must be equal in order for that diagram to commute, giving the commutative diagram:

$$\begin{tikzcd} (\mathrm{G}f)_\circ \arrow[r, "\eta_{\mathrm{G}a} \circledast (\mathrm{G}f)_\circ"] \arrow[d, "\nu_f"'] & \overline{\mathrm{G}}(\mathbb{A}') \circledast (\mathrm{G}f)_\circ \arrow[d, "\nu^m_{(f,\lambda)}"] \\ \overline{\mathrm{G}}(f_\circ) \arrow[r, "\overline{\mathrm{G}}(\eta_a \circledast f_\circ)"'] & \overline{\mathrm{G}}(\mathbb{A}' \circledast f_\circ) \end{tikzcd} .$$

Applying the functor $\overline{\mathrm{G}}(\mathbb{A}') \circledast -$ to this diagram and using the equivariance of $\nu^m_{(f,\lambda)}$ on the left we may infer that its underlying map is simply $\rho^{-1}_{\mathbb{A}'f}$ derived from ν_f as in (1.74), and this is of course an isomorphism as required.

That the dual of $\mathcal{M}\mathrm{on}(\overline{\mathrm{G}}, \mathrm{G}, \nu)$ also preserves representables follows easily on observing that the processes of taking duals commutes with the equipment of monads construction. $\square$

With no further verification the last two lemmas give:

Corollary 1.6.11 *We may apply the repletion construction to the adjunctions of examples 1.6.6 and 1.6.7 to give biadjoints*

$$\begin{tikzcd} \underline{\underline{\mathcal{B}}}\text{–Equip} \arrow[r, shift right=1.5ex, "(\overline{\mathrm{F}}_\star, \mathrm{F}_\star)"'] \arrow[r, phantom, "\perp_b"] & \underline{\underline{\mathcal{C}}}\text{–Equip} \arrow[l, shift right=1.5ex, "(\overline{\mathrm{F}}^\star, \mathrm{F}^\star)"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Equip}(\underline{\underline{\mathcal{E}}}) \arrow[r, shift right=1.5ex, "(\overline{\mathrm{F}}_\star, \mathrm{F}_\star)"'] \arrow[r, phantom, "\perp_b"] & \mathrm{Equip}(\underline{\underline{\mathcal{F}}}) \arrow[l, shift right=1.5ex, "(\overline{\mathrm{F}}^\star, \mathrm{F}^\star)"'] \end{tikzcd}$$

in $\underline{\underline{\underline{\mathcal{E}Mor}}}$ *with left biadjoint* $(\overline{\mathrm{F}}^\star, \mathrm{F}^\star)$ *an equipment homomorphism in both cases. The underlying biadjunctions* $\mathrm{F}^\star \dashv \mathrm{F}_\star$ *of these are in fact 2-adjunctions.* $\square$

The analogy with geometric morphisms in topos theory is striking: change of base is naturally expressible as a (bi)adjoint pair of maps in a suitably well chosen bicategory enriched category. Even the stipulation that inverse image functors should preserve finite limits is mirrored by the fact that the map $\ddot{\mathrm{F}}^\star$ is an equipment homomorphism. If the reader needs convincing of this he/she should return to considering example 1.2.3 in which we considered change of base for equipments of spans.

To round off this section we give a slightly more general result concerning the construction of a left biadjoint to an equipment morphism $\mathcal{M}\mathrm{on}(\ddot{\mathrm{G}})$ in the case when the left biadjoint to $\ddot{\mathrm{G}}$ is not an equipment homomorphism. Notice that in the statement of the lemma after the following definition we only consider equipments $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$ in $\underline{\mathcal{E}Map}$, in other words those in which the bicategory $\underline{\underline{\mathcal{K}}}$ is merely a category. This is not an essential requirement but simply prevents us from getting embroiled in a detailed description of the evident bicategorical versions of Beck's precise monadicity theorem and Butler's adjoint lifting results.

Definition 1.6.12 Given an equipment $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$ define the bicategory of *graphs* $\mathrm{Grph}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$ to be that formed by taking the equaliser of the pair of strict homomorphisms:

$$\mathrm{Cyl}_H(\mathrm{Sq}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)) \underset{\mathrm{cod}_V}{\overset{\mathrm{dom}_V}{\rightrightarrows}} \underline{\underline{\mathcal{K}}}$$

More explicitly this has:

0-cells: are endo-1-cells $\mathbb{G}\colon a \longrightarrow\!\!\!\!\!\!\!+\;\; a$ in $\underline{\underline{\mathcal{M}}}$;

1-cells: pairs $(f, \lambda)\colon (a, \mathbb{G}) \longrightarrow (a', \mathbb{G}')$ consisting of a 1-cell $f\colon a \longrightarrow a' \in \underline{\underline{\mathcal{K}}}$ and a 2-cell $\lambda\colon f_\circ \circledast \mathbb{G} \Rightarrow \mathbb{G}' \circledast f_\circ$ in $\underline{\underline{\mathcal{M}}}$;

2-cells: $\alpha\colon (f, \lambda) \Rightarrow (f', \lambda')$ consisting of a 2-cell $\alpha\colon f \Rightarrow f'$ in $\underline{\underline{\mathcal{K}}}$ such that the cylinder condition $\lambda \bullet (\alpha_\circ \circledast \mathbb{G}) = (\mathbb{G}' \circledast \alpha_\circ) \bullet \lambda'$ holds.

It is clear that there is a canonical forgetful strict homomorphism

$$\mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ) \xrightarrow{\mathrm{U}_{(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}})}} \mathrm{Grph}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)\ . \tag{1.91}$$

Notice also that $\mathrm{Grph}(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_\circ)$ is a (2-)category if $\underline{\underline{\mathcal{K}}}$ is. For instance consider the equipment $(\mathrm{Span}(\underline{\mathcal{E}}), \underline{\mathcal{E}}, (\text{-})_\circ)$ of example 1.2.3. For this $\mathrm{Grph}(\mathrm{Span}(\underline{\mathcal{E}}), \underline{\mathcal{E}}, (\text{-})_\circ)$ is simply the usual category of (non reflexive) graphs in $\underline{\mathcal{E}}$ and the functor in (1.91) is the canonical forgetful one from $\mathrm{Cat}(\underline{\mathcal{E}})$. $\square$

Proposition 1.6.13 *Let*

$$(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (\text{-})_\circ) \xrightarrow{\ddot{\mathrm{G}} = (\overline{\mathrm{G}}, \mathrm{G}, \nu)} (\underline{\underline{\mathcal{N}}}, \underline{\mathcal{L}}, (\text{-})_\circ)$$

be an equipment morphism in $\underline{\underline{\mathcal{E}Mor}}$ *(so* $\underline{\mathcal{K}}$ *and* $\underline{\mathcal{L}}$ *are categories and* G *a functor) with a left adjoint* $\ddot{\mathrm{F}}$ *in* $\underline{\underline{\mathcal{E}Map}}$*. Assume further that, as in the last section,* $\underline{\underline{\mathcal{M}}}$ *and* $\underline{\underline{\mathcal{N}}}$ *have local stable coequalisers of reflexive pairs ensuring that we may form* $\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (\text{-})_\circ)$ *and* $\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{N}}}, \underline{\mathcal{L}}, (\text{-})_\circ)$*.*

If we adopt the further suppositions that

(i) for both equipments $(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (\text{-})_\circ)$ *and* $(\underline{\underline{\mathcal{N}}}, \underline{\mathcal{L}}, (\text{-})_\circ)$ *the forgetful functor in (1.91) has a left adjoint,*

(ii) the forgetful functor $\mathrm{U}_{(\underline{\underline{\mathcal{N}}}, \underline{\mathcal{L}})}$ *is of descent type, in other words its Eilenberg-Moore comparison functor is fully faithful,*

(iii) the category $\mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (\text{-})_\circ)$ *has coequalisers of reflexive pairs,*

then the equipment morphism

$$\mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (\text{-})_\circ) \xrightarrow{\mathcal{M}\mathrm{on}(\ddot{\mathrm{G}})} \mathcal{M}\mathrm{on}(\underline{\underline{\mathcal{N}}}, \underline{\mathcal{L}}, (\text{-})_\circ)$$

has a left adjoint in $\underline{\underline{\mathcal{E}Map}}$*.*

Proof. This is just a simple application of Butler's left adjoint lifting theorem. For a precise statement of the version we use here see [2] section 3.7 theorem 3(b). Notice that we may weaken their conditions slightly by replacing (b)(vi) with one which only requires coequalisers of *reflexive* pairs to be present in the codomain of the functor for which we are constructing a left adjoint.

First we construct a left adjoint to the functor:

$$\mathrm{Mnd}(\ddot{\mathrm{G}})\colon \mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (\text{-})_\circ) \longrightarrow \mathrm{Mnd}(\underline{\underline{\mathcal{N}}}, \underline{\mathcal{L}}, (\text{-})_\circ)$$

Examining the definition of the bicategory enriched category $\underline{\underline{\mathcal{E}Map}}$ it is clear that we may extend the bicategory of graphs construction to an enriched functor

$$\underline{\underline{\mathcal{E}Map}} \xrightarrow{\mathrm{Grph}(\text{-})} \underline{\underline{\mathcal{H}om}}_S$$

which restricts to a 2-functor:

$$\underline{\underline{\mathcal{E}Map}} \xrightarrow{\mathrm{Grph}(\text{-})} \underline{\underline{\mathrm{Cat}}}$$

We could not do this for the Mnd(-) construction since equipment maps have no morphism structure on bicategories of "profunctors", $\underline{\underline{\mathcal{M}}}$ and $\underline{\underline{\mathcal{N}}}$, so do not necessarily preserve monads. Applying this 2-functor to the adjunction $\ddot{\mathrm{F}} \dashv \ddot{\mathrm{G}}$ we get an adjunction $\mathrm{Grph}(\ddot{\mathrm{F}}) \dashv \mathrm{Grph}(\ddot{\mathrm{G}})\colon \mathrm{Grph}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (-)_\circ) \longrightarrow \mathrm{Grph}(\underline{\underline{\mathcal{N}}}, \underline{\mathcal{L}}, (-)_\circ)$ in $\underline{\underline{\mathrm{Cat}}}$ and the square of functors

$$\begin{tikzcd}
\mathrm{Mnd}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (-)_\circ) \arrow[r, "\mathrm{Mnd}(\ddot{\mathrm{G}})"] \arrow[d, "\mathrm{U}"'] & \mathrm{Mnd}(\underline{\underline{\mathcal{N}}}, \underline{\mathcal{L}}, (-)_\circ) \arrow[d, "\mathrm{U}"] \\
\mathrm{Grph}(\underline{\underline{\mathcal{M}}}, \underline{\mathcal{K}}, (-)_\circ) \arrow[r, "\mathrm{Grph}(\ddot{\mathrm{G}})"] & \mathrm{Grph}(\underline{\underline{\mathcal{N}}}, \underline{\mathcal{L}}, (-)_\circ)
\end{tikzcd}$$

commutes. Conditions (i)-(iii) in the statement of this lemma now ensure that we may apply Butler's left adjoint lifting theorem, as stated in [2], to get a left adjoint to $\mathrm{Mnd}(\ddot{\mathrm{G}})$.

According to theorem 1.5.14 all that remains is to construct a left adjoint to

$$\mathrm{Mon}(\underline{\underline{\mathcal{M}}})((a, \mathbb{A}), (a', \mathbb{A}')) \xrightarrow{\mathrm{Mon}(\overline{\mathrm{G}})(-)} \mathrm{Mon}(\underline{\underline{\mathcal{N}}})((\mathrm{G}a, \overline{\mathrm{G}}\mathbb{A}), (\mathrm{G}a', \overline{\mathrm{G}}\mathbb{A}'))$$

for each pair of monads $(a, \mathbb{A})$ and $(a', \mathbb{A}')$. Now the diagram of functors

$$\begin{tikzcd}
\mathrm{Mon}(\underline{\underline{\mathcal{M}}})((a, \mathbb{A}), (a', \mathbb{A}')) \arrow[r, "\mathrm{Mon}(\overline{\mathrm{G}})(-)"] \arrow[d, "\mathrm{U}"'] & \mathrm{Mon}(\underline{\underline{\mathcal{N}}})((\mathrm{G}a, \overline{\mathrm{G}}\mathbb{A}), (\mathrm{G}a', \overline{\mathrm{G}}\mathbb{A}')) \arrow[d, "\mathrm{U}"] \\
\underline{\underline{\mathcal{M}}}(a, a') \arrow[r, "\overline{\mathrm{G}}(-)"] & \underline{\underline{\mathcal{N}}}(\mathrm{G}a, \mathrm{G}a')
\end{tikzcd}$$

commutes, wherein the forgetful maps to left and right are monadic and each hom-set of $\mathrm{Mon}(\underline{\underline{\mathcal{M}}})$ and $\mathrm{Mon}(\underline{\underline{\mathcal{N}}})$ has coequalisers of reflexive pairs (cf. [9] section 4). Of course the functor $\overline{\mathrm{G}}$ at the bottom of this diagram has a left adjoint by theorem 1.5.14, since $\ddot{\mathrm{G}}$ has a left adjoint in $\underline{\underline{\mathcal{E}Map}}$, and we apply Butler's theorem again to lift this to the left adjoint required. □

Observation 1.6.14 Section 2 of [5] demonstrates that if $\underline{\underline{\mathcal{B}}}$ is a bicategory as in example 1.2.2 with local stable coequalisers of reflexive pairs then for the equipment $(\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (-)_\circ)$ constructed therein:

- the forgetful functor of (1.91) is monadic,
- $\mathrm{Grph}(\underline{\underline{\mathcal{B}}}\text{–Mat}, \underline{\mathrm{Set}}/_{|\underline{\underline{\mathcal{B}}}|}, (-)_\circ)$ has all coequalisers.

The same results hold for the equipment $(\mathrm{Span}(\underline{\mathcal{E}}), \underline{\mathcal{E}}, (-)_\circ)$ described in example 1.2.3 so long as $\underline{\mathcal{E}}$ is locally cartesian closed with coequalisers of reflexive pairs and a natural numbers object. For details of the proof of this result see [28] section 6.4. It follows therefore that we may always apply proposition 1.6.13 for an equipment morphism (with left adjoint in $\underline{\underline{\mathcal{E}Map}}$) between any of these equipments, since conditions (i)-(iii) are then always satisfied.

With this in mind it might seem that we have wasted time in establishing the existence of the enriched functor $\mathcal{M}\mathrm{on}(\text{-})$ as a route to the construction of the adjoints in examples 1.6.6 and 1.6.7, since we could have just proceeded to the last result and used that instead. In fact the principle problem with this sort of approach is that in practical situations it is often not enough just to have a left adjoint in $\underline{\underline{\mathcal{E}Map}}$, we also need to know if it preserves representables or is an equipment homomorphism. This sort of question is hard to answer unless we are provided with an explicit description of the map under examination, which use of the functor $\mathcal{M}\mathrm{on}(\text{-})$ gives us.

As an example of a case in which lemma 1.6.13 is necessary we have:

Example 1.6.15 (essential geometric morphisms) *Suppose that $\underline{\mathcal{E}}$ and $\underline{\mathcal{F}}$ are locally cartesian closed categories with reflexive coequalisers and an NNO and we are given an essential geometric morphism*

$$\begin{tikzcd} \underline{\mathcal{E}} \arrow[r, shift left=4, "f_!"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "f_*"'] & \underline{\mathcal{F}} \arrow[l, "f^*"'] \end{tikzcd} \tag{1.92}$$

then we may construct another one

$$\begin{tikzcd} \mathrm{Equip}(\underline{\mathcal{E}})_1 \arrow[r, shift left=4, "\ddot{\mathrm{F}}_!"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "\ddot{\mathrm{F}}_\star"'] & \mathrm{Equip}(\underline{\mathcal{F}})_1 \arrow[l, "\ddot{\mathrm{F}}^\star"'] \end{tikzcd}$$

in $\underline{\underline{\mathcal{E}Map}}$ where $\ddot{\mathrm{F}}^\star$ is an equipment homomorphism.

Proof. Applying the functor

$$\underline{\underline{\mathrm{CAT}}}_f \xrightarrow{\mathrm{Span}} \underline{\underline{\mathcal{E}coMor}}$$

of example 1.5.17 to the adjunctions in (1.92) to get

$$\begin{tikzcd} (\mathrm{Span}(\underline{\mathcal{E}}), \underline{\mathcal{E}}, (\text{-})_\circ) \arrow[r, shift left=4, "\ddot{\mathrm{F}}."] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "\ddot{\mathrm{F}}_\#"'] & (\mathrm{Span}(\underline{\mathcal{F}}), \underline{\mathcal{F}}, (\text{-})_\circ) \arrow[l, "\ddot{\mathrm{F}}^\#"'] \end{tikzcd}$$

in $\underline{\underline{\mathcal{E}coMor}}$. The functors f_* and f^* are left exact therefore $\ddot{\mathrm{F}}_\#$ and $\ddot{\mathrm{F}}^\#$ are equipment homomorphisms to which we apply the 2-functor $\mathcal{M}\mathrm{on}(\text{-})$ to get the adjoint pair

$\ddot{\mathrm{F}}^{\star} \dashv \ddot{\mathrm{F}}_{\star}\colon \mathrm{Equip}(\underline{\mathcal{E}}) \longrightarrow \mathrm{Equip}(\underline{\mathcal{F}})$ in $\underline{\underline{\mathcal{E}Mor}}$, as in example 1.6.7, where $\ddot{\mathrm{F}}^{\star}$ is an equipment homomorphism. Now applying lemma 1.6.13 we lift the left adjoint of $\ddot{\mathrm{F}}^{\#}$ to $\ddot{\mathrm{F}}_{!} \dashv \ddot{\mathrm{F}}^{\star}$. Of course the morphism structure of $\ddot{\mathrm{F}}^{*}$ induces a comorphism one on $\ddot{\mathrm{F}}_{!}$ giving an adjoint in $\underline{\underline{\mathcal{E}coMor}}$, but it is not clear that we can apply the repletion construction (lemma 1.6.9) to this since we have not provided ourselves with any way of checking that $\ddot{\mathrm{F}}_{!}$ preserves representables. $\square$

1.7 Colimits and Change of Base

In this final section on change of base we examine the relationship between biadjunctions in $\underline{\underline{\mathcal{E}Map}}$ and the local adjoints introduced in section 1.1. We may then apply corollary 1.1.10 to prove results about the preservation of colimit cylinders (cf. observation 1.2.11) by the "direct image" part of a biadjunction between equipments. This result will be applied in the next chapter where we represent certain kinds of enriched category as internal ones allowing us to use a version of the Grothendieck construction in the analysis of enriched colimits.

Theorem 1.7.1 *Let*

$$\ddot{\mathrm{G}} = (\overline{\mathrm{G}}, \mathrm{G}, \nu)\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$$

be an equipment morphism with a left biadjoint in $\underline{\underline{\mathcal{E}Map}}$, by proposition 1.4.11 we get an induced comorphism structure on this making it an equipment comorphism

$$\ddot{\mathrm{F}} = (\overline{\mathrm{F}}, \mathrm{F}, \xi)\colon (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*) \longrightarrow (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \ .$$

Adopt $\ddot{\Psi}$ and $\ddot{\Phi}$ for the unit and counit of this, with α and β triangle isomorphisms chosen so that $(\ddot{\mathrm{G}}, \ddot{\mathrm{F}}, \ddot{\Psi}, \ddot{\Phi}, \alpha, \beta)$ is a locally adjoint biadjunction. For this data we have:

(i) There is a local adjunction $\overline{\mathrm{F}} \dashv\!\!\mid \overline{\mathrm{G}}\colon \underline{\underline{\mathcal{M}}} \longrightarrow \underline{\underline{\mathcal{N}}}$ mediated by a unit and counit as in theorem 1.1.6.

(ii) Suppose that

$$\begin{tikzcd}
\mathrm{F}(b) \arrow[rr, "\overline{\mathrm{F}}(p)", "|" marking] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] & \mathrm{F}(b') \arrow[dl, "\mathrm{colim}(\overline{\mathrm{F}}p, f)"] \\
& b &
\end{tikzcd} \tag{1.93}$$

is a colimit cylinder in $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ and our biadjunction satisfies

(a) $(\overline{\mathrm{G}}, \mathrm{G}, \nu)$ preserves representables,

(b) For each 1-cell $r\colon b' \longrightarrow\!\!\!\!\!\!\mid\;\; \mathrm{G}(a) \in \underline{\underline{\mathcal{N}}}$ the compositional comparison map

$$\overline{\mathrm{F}}(r \otimes p) \overset{\mathrm{can}}{\Longrightarrow} \overline{\mathrm{F}}(r) \otimes \overline{\mathrm{F}}(p)$$

is an isomorphism.

then the pasting

$$\begin{tikzcd} b \arrow[rr, "p", "\shortmid"{marking}] \arrow[d, "\Psi_b"'] \arrow[drr, phantom, "\overset{\overline{\Psi}_p}{\Leftarrow}"] & & b' \arrow[d, "\Psi_{b'}"] \\ \mathrm{GF}(b) \arrow[rr, "\overline{\mathrm{GF}}(p)", "\shortmid"{marking}] \arrow[dr, "\mathrm{G}(f)"'] \arrow[rr, phantom, shift right=4ex, "\overset{\kappa'}{\Leftarrow}"] & & \mathrm{GF}(b') \arrow[dl, "\mathrm{G}(\mathrm{colim}(\overline{\mathrm{F}}p, f))"] \\ & \mathrm{G}(a) & \end{tikzcd} \tag{1.94}$$

is a colimit cylinder as well; here κ' *is the composite:*

$$(\mathrm{G}(\mathrm{colim}(\overline{\mathrm{F}}p, f)))_* \otimes \overline{\mathrm{GF}}(p) \xRightarrow{\mu} \overline{\mathrm{G}}(\mathrm{colim}(\overline{\mathrm{F}}p, f)_* \otimes \overline{\mathrm{F}}(p)) \xRightarrow{\overline{\mathrm{G}}(\kappa)} \overline{\mathrm{G}}(f_*)$$
$$\xRightarrow[\sim]{\nu_f^{-1}} (\mathrm{G}f)_*$$

Proof.

(i) Returning to proposition 1.4.11 we see that the comorphism structure on $\ddot{\mathrm{F}}$ as induced by the morphism structure on $\ddot{\mathrm{G}}$ is defined precisely to ensure that the conditions in (1.53) and (1.54) hold. On examining these they clearly do no more than stipulate that the pairs of families $(\{(\Psi_b)_*\}, \{\overline{\Psi}_q\})$ and $(\{(\Phi_a)_*\}, \{\overline{\Phi}_p\})$ satisfy the conditions required of generalised optransformations $\overline{\Psi}\colon \mathrm{I}_{\underline{\underline{\mathcal{N}}}} \longrightarrow \overline{\mathrm{GF}}$ and $\overline{\Phi}\colon \overline{\mathrm{FG}} \longrightarrow \mathrm{I}_{\underline{\underline{\mathcal{M}}}}$ respectively (cf. definition 1.1.2).

Define the families of 2-cells $\{\bar{\alpha}_a\}$ and $\{\bar{\beta}_b\}$ required by theorem 1.1.6 to be the composites:

$$\mathrm{i}_{\mathrm{G}a} \xRightarrow[\sim]{\alpha_*} (\mathrm{G}\Phi_a)_* \otimes (\Phi_{\mathrm{G}a})_* \xRightarrow{\nu_{\Phi_a} \otimes (\Phi_{\mathrm{G}a})_*} \overline{\mathrm{G}}((\Phi_a)_*) \otimes (\Phi_{\mathrm{G}a})_*$$
$$(\Phi_{\mathrm{F}b})_* \otimes \overline{\mathrm{F}}((\Psi_b)_*) \xRightarrow{(\Phi_{\mathrm{F}b})_* \otimes \xi_{\Psi_b}} (\Phi_{\mathrm{F}b})_* \otimes (\mathrm{F}\Psi_b)_* \xRightarrow[\sim]{\beta_*} \mathrm{i}_{\mathrm{F}b}\ .$$

It is easily verified that these families satisfy conditions (1.13) and $(1.13)^{\mathrm{coop}}$ directly from the fact that α and β are modifications in $\underline{\underline{\mathcal{E}Map}}$; furthermore (1.14) and $(1.14)^{\mathrm{coop}}$ also hold since they are direct translations of the conditions in (1.37) on $(\ddot{\mathrm{G}}, \ddot{\mathrm{F}}, \ddot{\Psi}, \ddot{\Phi}, \alpha, \beta)$ as a locally adjoint biadjunction. With this we complete the proof that $\overline{\mathrm{F}} \dashv\!\mid \overline{\mathrm{G}}$ as mediated by unit $\overline{\Psi}$ and counit $\overline{\Phi}$.

(ii) Condition (b) on the comorphism $\overline{\mathrm{F}}$ ensures that we may apply corollary 1.1.10 for the local adjunction $\overline{\mathrm{F}} \dashv\!\mid \overline{\mathrm{G}}$ to the colimit cylinder in (1.93), which is a right Kan extension diagram by definition (cf observation 1.2.11). In this way we demonstrate that $\overline{\mathrm{G}}(\mathrm{colim}(\overline{\mathrm{F}}p, f)_*) \otimes (\Psi_b)_*$ is the right Kan extension of

$\overline{\mathrm{G}}(f_*) \otimes (\Psi_a)_*$ along p. Applying condition (a) we get isomorphisms

$$\begin{array}{rcl}\overline{\mathrm{G}}(\mathrm{colim}(\overline{\mathrm{F}}p, f)_*) \otimes (\Psi_b)_* & \cong & (\mathrm{G}(\mathrm{colim}(\overline{\mathrm{F}}p, f)) \circ \Psi_b)_* \\ \overline{\mathrm{G}}(f_*) \otimes (\Psi_a)_* & \cong & (\mathrm{G}(f) \circ \Psi_a)_*\end{array}$$

and on composing these with an appropriate instance of (1.23) we get the cylinder in (1.94). □

Example 1.7.2 Applying theorem 1.7.1(i) to the biadjoint

$$\begin{tikzcd} \underline{\underline{\mathcal{B}}}\text{–Equip} \arrow[r, shift right, "\ddot{\mathrm{F}}_\star"'] \arrow[r, phantom, "\perp"] & \underline{\underline{\mathcal{C}}}\text{–Equip} \arrow[l, shift right, "\ddot{\mathrm{F}}^\star"'] \end{tikzcd}$$

of corollary 1.6.11 we get the local adjunction for change of base described in [6]. □

Notice that for the adjoint pairs in corollary 1.6.11 conditions (a) and (b) hold for all colimit cylinders as in (1.93), since $(\overline{\mathrm{F}}^\star, \mathrm{F}^*)$ is an equipment homomorphism in each case, so we may apply the theorem above to any of these. By taking duals we get an identical result for limit cylinders demonstrating that change of base for enriched and internal category theory, when formulated in this way, is nicely behaved with respect to a large class of weighted limits and colimits.

Notation 1.7.3 Suppose *Expr1* and *Expr2* are expressions *conditionally* defining objects of some category $\underline{\mathcal{C}}$, then the notation *Expr1* $\overset{\cong}{\Rightarrow}$ *Expr2* means: if *Expr1* defines an object of $\underline{\mathcal{C}}$ then so does *Expr2* and the two are isomorphic.

We also have *Expr1* $\overset{\cong}{\Leftarrow}$ *Expr2* in which the existential implication goes in the opposite direction, and *Expr1* $\overset{\cong}{\Leftrightarrow}$ *Expr2* meaning (*Expr1* $\overset{\cong}{\Rightarrow}$ *Expr2*) $\wedge$ (*Expr1* $\overset{\cong}{\Leftarrow}$ *Expr2*).

As an example of the use of this notation we may re-express the gist of theorem 1.7.1(ii) which simply says that under the conditions given

$$\Big(\mathrm{colim}(\overline{\mathrm{F}}(p), f)\Big)^{\vee} \overset{\cong}{\Rightarrow} \mathrm{colim}(p, \check{f})$$

where the $\vee$ symbol is used to denote right hand (bi)adjoint transposition under $\mathrm{F} \dashv_b \mathrm{G}$. The intended meaning is: if $\mathrm{colim}(\overline{\mathrm{F}}(p), f)$ exists in $(\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*)$ then $\mathrm{colim}(p, \check{f})$ exists in $(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$ and the two sides of the relation are isomorphic.

Had we started with a 1-cell $g\colon b \longrightarrow \mathrm{G}(a) \in \underline{\underline{\mathcal{L}}}$ rather than $f\colon \mathrm{F}(b) \longrightarrow a \in \underline{\underline{\mathcal{K}}}$ we might re-express theorem 1.7.1(ii) as:

$$\mathrm{colim}(\overline{\mathrm{F}}(p), \hat{g}) \overset{\cong}{\Rightarrow} (\mathrm{colim}(p, g))^{\wedge}$$

again the symbol $\wedge$ is used to denote left hand (bi)adjoint transposition under $\mathrm{F} \dashv_b \mathrm{G}$. □

As we have already mentioned, in the next chapter we will become interested in representing one kind of category theory as another, in that case we study enriched categories as internal ones. In this sort of situation the following definition and its attendant lemmas become important:

Definition 1.7.4 (equipment inclusions) We say that a biadjoint pair $\ddot{\mathrm{F}} \dashv_b \ddot{\mathrm{G}}$ as in the last theorem is an *inclusion* if

(i) $(\overline{\mathrm{G}}, \mathrm{G}, \nu)$ preserves both left and right representables.

(ii) G is locally an equivalence, in other words its action on each homset

$$\underline{\underline{\mathcal{K}}}(a, a') \xrightarrow{\mathrm{G}} \underline{\underline{\mathcal{L}}}(\mathrm{G}a, \mathrm{G}a')$$

is an equivalence.

(iii) $\overline{\mathrm{G}}$ is locally fully faithful,

(iv) If the 0-cell $b' \in \underline{\underline{\mathcal{L}}}$ is in the full image of G, that is to say there exists $a \in \underline{\underline{\mathcal{K}}}$ and an equivalence $\mathrm{G}(a) \simeq b$, and $p\colon b \longrightarrow b'$ and $p'\colon b' \longrightarrow b''$ are 1-cells in $\underline{\underline{\mathcal{M}}}$ then the compositional comparison map

$$\overline{\mathrm{F}}(p' \otimes p) \xRightarrow{\text{can}} \overline{\mathrm{F}}(p') \otimes \overline{\mathrm{F}}(p)$$

is an isomorphism.

From this definition it is clear that the class of inclusions is closed under composition and taking duals. $\square$

Lemma 1.7.5 *Conditions (ii) and (iii) of definition 1.7.4 are together equivalent to:*

(ii)′ For each 0-cell $a \in \underline{\underline{\mathcal{K}}}$ the component $\Phi_a\colon \mathrm{FG}(a) \longrightarrow a$ of the counit $\ddot{\Phi}$ is an equivalence in $\underline{\underline{\mathcal{K}}}$.

(iii)′ For each 1-cell $p \in \underline{\underline{\mathcal{M}}}$ the 2-cell $\overline{\Phi}_p\colon (\Phi_{a'})_* \otimes \overline{\mathrm{FG}}(p) \Rightarrow p \otimes (\Phi_a)_*$ is an isomorphism.

Proof. The equivalence of (ii) and (ii)′ is no more than a bicategorical version of the well known fact that an adjunction has right adjoint part which is fully faithful iff its counit is an isomorphism.

In the presence of (ii)′ we may prove that (iii) is equivalent to (iii)′ as follows: Theorem 1.5.14(ii) tells us that the action of $\overline{\mathrm{G}}$ on each homset

$$\underline{\underline{\mathcal{M}}}(a, \bar{a}) \xrightarrow{\overline{\mathrm{G}}} \underline{\underline{\mathcal{N}}}(\mathrm{G}(a), \mathrm{G}(\bar{a}))$$

has a left adjoint which, in the course of the proof of that theorem, we discover is simply the functor $(\Phi_{\bar{a}})_* \otimes \overline{\mathrm{F}}(-) \otimes (\Phi_a)^*$, furthermore the counit of this adjunction has component at $p \in \underline{\underline{\mathcal{M}}}(a, \bar{a})$ given in terms of $\overline{\Phi}_p$ in (1.79). Now Φ_a is an equivalence in $\underline{\underline{\mathcal{K}}}$, a property which is preserved by application of the homomorphism $(-)_*$ therefore $(\Phi_a)_* \dashv (\Phi_a)^*$ is an adjoint equivalence in $\underline{\underline{\mathcal{M}}}$ and so has counit ϵ_{Φ_a} which is an isomorphism. Of course we know that the action of $\overline{\mathrm{G}}$ on $\underline{\underline{\mathcal{M}}}(a, \bar{a})$ is fully faithful iff for each $p \in \underline{\underline{\mathcal{M}}}(a, \bar{a})$ the composite in (1.79) is an isomorphism; but bearing in mind what we know about Φ_a and ϵ_{Φ_a} this holds iff each $\overline{\Phi}_p$ is an isomorphism. $\square$

Example 1.7.6 *Let $\underline{\underline{\mathcal{B}}}$ and $\underline{\underline{\mathcal{C}}}$ be bicategories and $\mathrm{F}: \underline{\underline{\mathcal{C}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ a homomorphism, all satisfying the conditions of example 1.6.6, and suppose further that*

- *the function $|\mathrm{F}|: |\underline{\underline{\mathcal{C}}}| \longrightarrow |\underline{\underline{\mathcal{B}}}|$ (action of F on 0-cells) is surjective,*
- *for each pair of 0-cells $c, c' \in \underline{\underline{\mathcal{C}}}$ the right adjoint*

$$\underline{\underline{\mathcal{B}}}(\mathrm{F}c, \mathrm{F}c') \xrightarrow{\mathrm{R}_{cc'}} \underline{\underline{\mathcal{C}}}(c, c')$$

  to the local action of F is fully faithful;

then the (bi)adjunction $\ddot{\mathrm{F}}^\star \dashv_b \ddot{\mathrm{F}}_\star: \underline{\underline{\mathcal{B}}}$–Equip $\longrightarrow \underline{\underline{\mathcal{C}}}$–Equip of example 1.6.11 is an inclusion.

Proof. We already know that $\ddot{\mathrm{F}}^\star$ is an equipment homomorphism and $\ddot{\mathrm{F}}_\star$ preserves left and right representables (by lemma 1.6.10) so all that remains is to verify conditions (ii) and (iii) of definition 1.7.4. Examining the construction of $\ddot{\mathrm{F}}^\star \dashv \ddot{\mathrm{F}}_\star$ it is clear that if $\mathbb{A}$ is a $\underline{\underline{\mathcal{B}}}$-enriched category with underlying set of objects (X, α) then the category $\mathrm{F}^\star\mathrm{F}_\star(\mathbb{A})$ has:

- Set of objects $\{(x, c) : x \in X, c \in |\underline{\underline{\mathcal{C}}}|$ and $\alpha(x) = \mathrm{F}(c)\}$ with projection taking (x, c) to $c \in |\underline{\underline{\mathcal{C}}}|$,
- $\mathrm{F}^\star\mathrm{F}_\star(\mathbb{A})((x, c), (x', c')) = \mathrm{F}_{cc'}\mathrm{R}_{cc'}(\mathbb{A}(x, x'))$.

On examining the 1-cellular component of the counit of $\ddot{\mathrm{F}}^\star \dashv \ddot{\mathrm{F}}_\star$ at $\mathbb{A}$ we see that it maps an object $(x, c) \in \mathrm{F}^\star\mathrm{F}_\star(\mathbb{A})$ to $x \in \mathbb{A}$, and this is surjective since $|\mathrm{F}|$ is. Its action on the homset $\mathrm{F}^\star\mathrm{F}_\star(\mathbb{A})((x, c), (x', c'))$ is simply the component $\epsilon_{\mathbb{A}(x,x')}: \mathrm{F}_{cc'}\mathrm{R}_{cc'}(\mathbb{A}(x, x')) \Rightarrow \mathbb{A}(x, x')$ of the counit of $\mathrm{F}_{cc'} \dashv \mathrm{R}_{cc'}$, each of which is an isomorphism since every $\mathrm{R}_{cc'}$ is fully faithful. Therefore $\Phi_{\mathbb{A}}: \mathrm{F}^\star\mathrm{F}_\star(\mathbb{A}) \longrightarrow \mathbb{A}$, being essentially surjective and fully faithful, is an equivalence in the 2-category $\underline{\underline{\mathcal{B}}}$–Cat.

A similar calculation shows that if $p: \mathbb{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{B}$ is a profunctor in $\underline{\underline{\mathcal{B}}}$–Prof then the component of $\overline{\mathrm{F}}^\star\overline{\mathrm{F}}_\star(p)$ between $(x, c) \in \mathrm{F}^\star\mathrm{F}_\star\mathbb{A}$ and $(\bar{x}, \bar{c}) \in \mathrm{F}^\star\mathrm{F}_\star\mathbb{B}$ is simply $\mathrm{F}_{c\bar{c}}\mathrm{R}_{c\bar{c}}(p_{x\bar{x}})$. Since $\Phi_{\mathbb{B}}$ is an equivalence, implying that $(\Phi_{\mathbb{B}})_* \dashv (\Phi_{\mathbb{B}})^*$ is an adjoint equivalence, it is clear that the 2-cell $\overline{\Phi}_p: (\Phi_{\mathbb{B}})_* \otimes \overline{\mathrm{F}}^\star\overline{\mathrm{F}}_\star(p) \Rightarrow p \otimes (\Phi_{\mathbb{A}})_*$ is an isomorphism iff the corresponding cell $\hat{\Phi}_p: \overline{\mathrm{F}}^\star\overline{\mathrm{F}}_\star(p) \Rightarrow (\Phi_{\mathbb{B}})^* \otimes p \otimes (\Phi_{\mathbb{A}})_*$ is one. Between the objects (x, c) and $(\bar{x}, \bar{c})$ as above the profunctor $(\Phi_{\mathbb{B}})^* \otimes p \otimes (\Phi_{\mathbb{A}})_*$ has component

$p_{x\bar{x}}$ and the 2-cell $\hat{\Phi}_p$ consists of counits $\epsilon_{p_{x\bar{x}}}\colon \mathrm{F}_{c\bar{c}}\mathrm{R}_{c\bar{c}}(p_{x\bar{x}}) \Rightarrow p_{x\bar{x}}$, these are all isomorphisms, since each $\mathrm{R}_{c\bar{c}}$ is fully faithful, therefore $\hat{\Phi}_p$ is an equivariant isomorphism as required.

Notice that if $|\mathrm{F}|\colon |\underline{\underline{\mathcal{C}}}| \longrightarrow |\underline{\underline{\mathcal{B}}}|$ an isomorphism then for each $\underline{\underline{\mathcal{B}}}$-category $\mathbb{A}$ the component $\Phi_{\mathbb{A}}\colon \mathrm{F}^\star\mathrm{F}_\star(\mathbb{A}) \longrightarrow \mathbb{A}$ is in fact an isomorphism, not just an equivalence. This implies that the adjunction $\ddot{\mathrm{F}}^\star \dashv \ddot{\mathrm{F}}_\star\colon \underline{\underline{\mathcal{B}}}\text{–Equip} \longrightarrow \underline{\underline{\mathcal{C}}}\text{–Equip}$ in $\underline{\underline{\mathcal{E}Mor}}$ is an inclusion between the non-replete equipments of $\underline{\underline{\mathcal{B}}}$- and $\underline{\underline{\mathcal{C}}}$-enriched categories. $\square$

For inclusions we have some stronger results with respect to colimit cylinders.

Lemma 1.7.7 *Let* $\ddot{\mathrm{F}} \dashv_b \ddot{\mathrm{G}}\colon (\underline{\underline{\mathcal{M}}}, \underline{\underline{\mathcal{K}}}, (\text{-})_*) \longrightarrow (\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (\text{-})_*)$ *be an inclusion then*

(i) Let the 0-cell $b' \in \underline{\underline{\mathcal{L}}}$ be in the full image of G *and assume there exists a right Kan extension*

$$\begin{tikzcd}
\mathrm{F}(b) \arrow[rr, "\overline{\mathrm{F}}(p)", "\shortmid" marking] \arrow[dr, "f_*"', "\shortmid" marking] & \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] & \mathrm{F}(b') \arrow[dl, "f_* \Leftarrow \overline{\mathrm{F}}(p)", "\shortmid" marking] \\
& a &
\end{tikzcd}$$

in $\underline{\underline{\mathcal{M}}}$. Then we may apply theorem 1.7.1(ii) to any colimit cylinder as in (1.93) and strengthen it to the two way conditional:

$$\left(\mathrm{colim}(\overline{\mathrm{F}}(p), f)\right)^{\vee} \Leftrightarrows \mathrm{colim}(p, \check{f}) \ .$$

(ii) If

$$\begin{tikzcd}
a \arrow[rr, "r", "\shortmid" marking] \arrow[dr, "q"', "\shortmid" marking] & \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] & a' \arrow[dl, "q \Leftarrow r", "\shortmid" marking] \\
& a'' &
\end{tikzcd} \tag{1.95}$$

is a right Kan extension diagram in $\underline{\underline{\mathcal{M}}}$ then the triangle

$$\begin{tikzcd}
\mathrm{G}(a) \arrow[rr, "\overline{\mathrm{G}}(r)", "\shortmid" marking] \arrow[dr, "\overline{\mathrm{G}}(q)"', "\shortmid" marking] & \arrow[d, phantom, "\overset{\overline{\mathrm{G}}(\kappa)}{\Leftarrow}"] & \mathrm{G}(a') \arrow[dl, "\overline{\mathrm{G}}(q \Leftarrow r)", "\shortmid" marking] \\
& \mathrm{G}(a'') &
\end{tikzcd} \tag{1.96}$$

in $\underline{\underline{\mathcal{N}}}$ is one as well. Here we divine the meaning of $\overline{\mathrm{G}}(\kappa)$ from its context as described in the comment following corollary 1.1.10. Taking $q = f_*$ for some 1-cell $f\colon a \longrightarrow a''$ in $\underline{\underline{\mathcal{K}}}$ and still assuming the existence of the right Kan extension (1.95) we get the two way conditional:

$$\mathrm{G}(\mathrm{colim}(r, f)) \Leftrightarrows \mathrm{colim}(\overline{\mathrm{G}}(r), \mathrm{G}(f)) \ .$$

Proof.

(i) Since $b' \in \underline{\underline{\mathcal{L}}}$ is in the full image of G conditions (i) and (iv) on $\ddot{\mathrm{F}} \dashv \ddot{\mathrm{G}}$ as an inclusion imply that we may apply theorem 1.7.1(ii) to give:

$$\left(\mathrm{colim}(\overline{\mathrm{F}}(p), f)\right)^{\vee} \overset{\Rightarrow}{\cong} \mathrm{colim}(p, \check{f})$$

Conversely suppose $\mathrm{colim}(p, \check{f})$ exists in $(\underline{\underline{\mathcal{N}}}, \underline{\underline{\mathcal{L}}}, (-)_*)$, then by applying corollary 1.1.10 to the local adjunction $\overline{\mathrm{F}} \dashv\!\!\mid \overline{\mathrm{G}}$ we see that

$$\overline{\mathrm{G}}(f_* \Leftarrow \overline{\mathrm{F}}(p)) \otimes (\Psi_{b'})_* \cong (\overline{\mathrm{G}}(f_*) \otimes (\Psi_b)_*) \Leftarrow p$$

but $\ddot{\mathrm{G}}$ preserves representables therefore $\overline{\mathrm{G}}(f_*) \otimes (\Psi_b)_* \cong (\mathrm{G}(f) \circ \Psi_b)_* \cong \check{f}_*$ and by definition $\mathrm{colim}(p, \check{f})_* \cong \check{f}_* \Leftarrow p$. Now consider the left (bi)adjoint transpose of $\mathrm{colim}(p, \check{f})$, we have $\mathrm{colim}(p, \check{f}) \cong \mathrm{G}\left(\mathrm{colim}(p, \check{f})\right)^{\wedge} \circ \Psi_{b'}$ and putting this all together we get

$$\begin{aligned} \overline{\mathrm{G}}(f_* \Leftarrow \overline{\mathrm{F}}(p)) \otimes (\Psi_{b'})_* &\cong \mathrm{colim}(p, \check{f})_* \\ &\cong (\mathrm{G}(\mathrm{colim}(p, \check{f})^{\wedge}) \circ \Psi_{b'})_* \\ &\cong \overline{\mathrm{G}}(\mathrm{colim}(p, \check{f})^{\wedge})_* \otimes (\Psi_{b'})_* \end{aligned}$$

where the last isomorphism follows from the assumption that $\ddot{\mathrm{G}}$ preserves representables. We may infer that $\Psi_{b'}$ is an equivalence from the fact that b' is in the full image of G and we already know that $\overline{\mathrm{G}}$ is locally fully faithful. Therefore the functor $\overline{\mathrm{G}}(-) \otimes (\Psi_{b'})_*$ on $\underline{\underline{\mathcal{M}}}(\mathrm{F}(b'), a)$ is fully faithful and so $(\mathrm{colim}(p, \check{f})^{\wedge})_* \cong f_* \Leftarrow p$ identifying $\mathrm{colim}(p, \check{f})^{\wedge}$ as $\mathrm{colim}(\overline{\mathrm{F}}p, f)$ or on taking right adjoint transposes we get the reverse conditional:

$$\left(\mathrm{colim}(\overline{\mathrm{F}}(p), f)\right)^{\vee} \overset{\Leftarrow}{\cong} \mathrm{colim}(p, \check{f}) \ .$$

(ii) If the triangle in (1.93) is a right Kan extension then so is the pasting of

$$\begin{tikzcd} \mathrm{FG}(a) \arrow[rr, "\overline{\mathrm{FG}}(r)", "\shortmid" marking] \arrow[d, "\Phi_a"'] \arrow[drr, phantom, "\overset{\overline{\Phi}_r}{\cong}"] & & \mathrm{FG}(a') \arrow[d, "\Phi_{a'}"] \\ a \arrow[rr, "r", "\shortmid" marking] \arrow[dr, "q"', "\times" marking] \arrow[rr, phantom, shift right=6ex, "\overset{\kappa}{\Leftarrow}"] & & a' \arrow[dl, "q \Leftarrow r", "\times" marking] \\ & a'' & \end{tikzcd}$$

since each Φ_a is an equivalence and Φ_p is an isomorphism (cf. lemma 1.7.5). The 0-cell $\overline{\mathrm{G}}(a')$ is (of course) in the full image of G and so, since $\ddot{\mathrm{F}} \dashv_b \ddot{\mathrm{G}}$

satisfies condition (iv) of definition 1.7.4, we may apply corollary 1.1.10 to the Kan extension above. In this case it is straightforward to show that the bottom triangle of (1.23) composed with the isomorphisms $\overline{\mathrm{G}}(q) \otimes (\mathrm{G}\Phi_a)_* \cong \overline{\mathrm{G}}(q \otimes (\Phi_a)_*)$ and $\overline{\mathrm{G}}(q \Leftarrow r) \otimes (\mathrm{G}\Phi_{a'})_* \cong \overline{\mathrm{G}}((q \Leftarrow r) \otimes (\Phi_{a'})_*)$ is equal to the pasting:

$$\begin{tikzcd}
\mathrm{GFG}(a) \arrow[rr, "\overline{\mathrm{GFG}}(r)", "|" marking] \arrow[d, "\mathrm{G}(\Phi_a)"'] \arrow[drr, phantom, "\overset{\mathrm{G}_S(\overline{\Phi}_r)}{\Leftarrow}"] & & \mathrm{GFG}(a') \arrow[d, "\mathrm{G}(\Phi_{a'})"] \\
\mathrm{G}(a) \arrow[rr, "\overline{\mathrm{G}}(r)", "|" marking] \arrow[dr, "\overline{\mathrm{G}}(q)"', "|" marking] \arrow[rr, phantom, bend right=40, "\overset{\overline{\mathrm{G}}(\kappa)}{\Leftarrow}"] & & \mathrm{G}(a') \arrow[dl, "\overline{\mathrm{G}}(q \Leftarrow r)", "|" marking] \\
& \mathrm{G}(a'') &
\end{tikzcd}$$

Now compose this with the square $\overline{\Psi}_{\overline{\mathrm{G}}(r)}$ to obtain the Kan extension in (1.23), but notice that the components $\mathrm{i}_{\mathrm{G}(a)} \cong \mathrm{G}(\Phi_a) \circ \Psi_{\mathrm{G}(a)}$ and $\mathrm{i}_{\mathrm{G}(a')} \cong \mathrm{G}(\Phi_{a'}) \circ \Psi_{\mathrm{G}(a')}$ of the triangle isomorphism $\alpha\colon \mathrm{i}_{\ddot{\mathrm{G}}} \cong \ddot{\mathrm{G}}\ddot{\Phi} \circ \ddot{\Psi}\ddot{\mathrm{G}}$ constitute a cylinder $\mathrm{i}^h_{\overline{\mathrm{G}}(r)} \cong \overline{\mathrm{G}}_S(\overline{\Phi}_r) \circ \overline{\Psi}_{\overline{\mathrm{G}}(r)}$ reducing the resulting diagram to that in (1.96). The remainder of the result follows as in part (i). □

While we have developed the theory of change of base in terms of bicategory enriched categories and biadjunctions, the actual examples we have encountered really only need to be interpreted in terms of some 2-category of equipments and (traditional) adjunctions therein. We should really give some justification for the more involved bicategorical theory described here, apart from the mere fact that in truth it is not really any harder to set up and study. The next example describes a case in which biadjunctions of equipment maps have underlying biadjunctions which cannot in general be reduced to adjunctions. We recall notions in enriched category theory closely related to the usual ones of "site" and "sheaf" and prove a "comparison lemma" by which we can construct biadjunctions in $\underline{\underline{\mathcal{E}Mor}}$ between equipments of "sheaves".

Due to space restrictions much of this is in the form of a sketch, the references give more insight to the background and we leave it up to the reader to fill in any detail. First fix a bicategory $\underline{\mathcal{B}}$ with a small set of 0-cells and all (small) stable local colimits and we review the following definitions:

Definition 1.7.8 (Cauchy bimodules) Cauchy bimodules in $\underline{\mathcal{B}}$–Prof are those 1-cells $p\colon \mathbb{A} \looparrowright \mathbb{A}'$ with a left adjoint $p_\bullet$. The gist of [49] is that these are precisely the bimodules which are weights for *absolute* colimits, in other words those colimits which (when they exist) are preserved by all functors. □

Definition 1.7.9 (family of Cauchy coverings) Let $\mathrm{J}(-)$ be a function which assigns to each 0-cell $u \in \underline{\underline{\mathcal{B}}}$ a set $\mathrm{J}(u)$ of Cauchy bimodules $p\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!+\; u$ in $\underline{\underline{\mathcal{B}}}$–Prof (here we identify the 0-cell $u \in \underline{\underline{\mathcal{B}}}$ with the trivial one object $\underline{\underline{\mathcal{B}}}$-category over u). Under this identification the bicategory $\underline{\underline{\mathcal{B}}}$ becomes the full sub-bicategory of $\underline{\underline{\mathcal{B}}}$–Prof on the trivial one object categories.

We say that a general bimodule $q\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!+\; \mathbb{B}$ in $\underline{\underline{\mathcal{B}}}$–Prof is J-*covering* if for each object $y \in \mathbb{B}$, which we assume lies over the 0-cell $\beta(y) \in \underline{\underline{\mathcal{B}}}$, the bimodule

$$\mathbb{A} \xrightarrow[\quad]{q(-,y)} \beta(y)$$

is in $\mathrm{J}(\beta(y))$. Notice that any J-covering module has a left adjoint in $\underline{\underline{\mathcal{B}}}$–Prof, in other words it is Cauchy.

The family $\mathrm{J}(-)$ is a *family of (Cauchy) covers* iff

- For each object x in a (small) $\underline{\underline{\mathcal{B}}}$-enriched category $\mathbb{A}$ the representable

  $$\mathbb{A} \xrightarrow[\quad]{x^* = \mathbb{A}(-,x)} \alpha(x)$$

  is in $\mathrm{J}(\alpha(x))$,
- If $q\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!+\; \mathbb{B}$ is J-covering and $p\colon \mathbb{B} \longrightarrow\!\!\!\!\!\!+\; u$ is in $\mathrm{J}(u)$ then $p \otimes q$ is in $\mathrm{J}(u)$ as well. □

Definition 1.7.10 For each (small) $\underline{\underline{\mathcal{B}}}$-category $\mathbb{A}$ we may define a $\underline{\underline{\mathcal{B}}}$-category $\mathcal{P}_{\mathrm{J}}(\mathbb{A})$ with

objects: over the 1-cell $u \in \underline{\underline{\mathcal{B}}}$ are the bimodules $p\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!+\; u$ in $\mathrm{J}(u)$,

homsets: for objects $p\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!+\; u$ and $\bar{p}\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!+\; \bar{u}$ the homset $\mathcal{P}_{\mathrm{J}}(\mathbb{A})(p, \bar{p})$ is defined to be the 1-cell $\bar{p} \otimes p_\bullet\colon u \longrightarrow\!\!\!\!\!\!+\; \bar{u}$ in $\underline{\underline{\mathcal{B}}}$. We use the units and counits of the adjunctions $p_\bullet \dashv p$ to furnish us with identities and compositions making this data into a $\underline{\underline{\mathcal{B}}}$-enriched category.

In general this $\underline{\underline{\mathcal{B}}}$-category may not be small without further assumptions on $\underline{\underline{\mathcal{B}}}$. In particular if we adopt the stipulation that each homset of $\underline{\underline{\mathcal{B}}}$ is *locally presentable*, which we shall do from now on, then [27] demonstrates that $\mathcal{P}_{\mathrm{J}}(\mathbb{A})$ is always small, and is therefore an object of our equipment $\underline{\underline{\mathcal{B}}}$–Equip.

For more detail on the $\underline{\underline{\mathcal{B}}}$-categories $\mathcal{P}_{\mathrm{J}}(\mathbb{A})$ we refer the reader to [4], principle amongst these for our purposes are:

(i) The conditions on $\mathrm{J}(-)$ as a family of covers imply that the set of J-covering modules in $\underline{\underline{\mathcal{B}}}$–Prof is closed under both identities and composition, making them the 1-cells of a locally full sub-bicategory J–$\mathcal{C}$ov. The first of those conditions also ensures that for each functor $f\colon \mathbb{A} \longrightarrow \mathbb{B}$ in $\underline{\underline{\mathcal{B}}}$–Cat the right representable $f^*\colon \mathbb{B} \to\!\!\!\!\!\!+\!\!> \mathbb{A}$ is in J–$\mathcal{C}$ov. It follows therefore that the homomorphism $(-)^*\colon \underline{\underline{\mathcal{B}}}$–Cat $\longrightarrow \underline{\underline{\mathcal{B}}}$–Prof$^{\mathrm{op}}$ restricts to

$$\underline{\underline{\mathcal{B}}}\text{–Cat} \xrightarrow{\ (-)^*\ } \text{J–}\mathcal{C}\text{ov}^{\mathrm{op}}$$

for which the natural equivalence

$$\frac{\mathbb{B} \longrightarrow \mathcal{P}_{\mathrm{J}}(\mathbb{A}) \qquad \text{in } \underline{\underline{\mathcal{B}}}\text{–Cat}}{\mathbb{A} \longrightarrow\!\!\!\!\!\!\!\!+\;\;\; \mathbb{B} \qquad \text{in J–}\mathcal{C}\text{ov}}$$

demonstrates that $\mathcal{P}_{\mathrm{J}}(-)$ extends to a homomorphism

$$\text{J–}\mathcal{C}\text{ov}^{\mathrm{op}} \xrightarrow{\ \mathcal{P}_{\mathrm{J}}(-)\ } \underline{\underline{\mathcal{B}}}\text{–Cat}$$

which is right biadjoint to $(-)^*$.

(ii) The unit of this biadjunction is the Yoneda embedding $\mathcal{Y}\colon \mathbb{A} \longrightarrow \mathcal{P}_{\mathrm{J}}(\mathbb{A}) \in \underline{\underline{\mathcal{B}}}$–Cat which maps an object $x \in \mathbb{A}$ to the representable $x^* = \mathbb{A}(-,x) \in \mathcal{P}_{\mathrm{J}}(\mathbb{A})$. The counit is simply the left representable $\mathcal{Y}_*\colon \mathbb{A} \to\!\!\!\!\!\!+\!\!> \mathcal{P}_{\mathrm{J}}(\mathbb{A})$, which is in J–$\mathcal{C}$ov since $\mathcal{Y}_*(-,p) = p(-)$ for each $p \in \mathcal{P}_{\mathrm{J}}(\mathbb{A})$, furthermore $\mathcal{Y}_* \dashv \mathcal{Y}^*$ is an adjoint equivalence in J–$\mathcal{C}$ov therefore the homomorphism $\mathcal{P}_{\mathrm{J}}(-)$ is a local equivalence.

(iii) A (small) $\underline{\underline{\mathcal{B}}}$-category is in the full image of $\mathcal{P}_{\mathrm{J}}(-)$ iff it admits all colimits weighted by bimodules in the family of Cauchy covers J$(-)$, we say that such a category is J-*cocomplete.* In fact the $\underline{\underline{\mathcal{B}}}$-category $\mathcal{P}_{\mathrm{J}}(\mathbb{A})$ is the free J-cocompletion of $\mathbb{A}$. We now have two ways of describing the bicategory J–$\mathcal{C}$ov, either as it was introduced, in which case we think of its 1-cells as being J-*functional relations,* or as the full sub-bicategory of $\underline{\underline{\mathcal{B}}}$–Cat on the J-cocomplete categories which we may view as J-*sheaves.* The analogy with sheaves is taken a step further in [55] where traditional sheaves are cast into our context.

Definition 1.7.11 (sites) In this context a site $(\underline{\underline{\mathcal{B}}}, \mathrm{J})$ consists of

- A bicategory $\underline{\underline{\mathcal{B}}}$ satisfying all the conditions we introduced above, that is it has a small set of 0-cells, homsets which are locally presentable and local colimits which are stable.
- A family of Cauchy covers J$(-)$ on $\underline{\underline{\mathcal{B}}}$.

For a site $(\underline{\underline{\mathcal{B}}}, \mathrm{J})$ we may define an *equipment of sheaves*, denoted $(\underline{\underline{\mathcal{B}}}, \mathrm{J})$–$\mathcal{S}$hf, to be the triple (J–$\mathcal{C}ov^{\mathrm{op}}$, $\underline{\underline{\mathcal{B}}}$–Prof, $(-)_\bullet$) where $(-)_\bullet\colon$ J–$\mathcal{C}$ov$^{\mathrm{coop}} \longrightarrow \underline{\underline{\mathcal{B}}}$–Prof is the canonical homomorphism which takes each J-covering bimodule $p\colon \mathbb{A} \to\!\!\!\!\!\!+\!\!> \mathbb{B}$ to its left adjoint $p_\bullet\colon \mathbb{B} \to\!\!\!\!\!\!+\!\!> \mathbb{A}$ in $\underline{\underline{\mathcal{B}}}$–Prof. □

Definition 1.7.12 (cocontinuous homomorphism of sites) For sites $(\underline{\underline{\mathcal{B}}}, \mathrm{J})$ and $(\underline{\underline{\mathcal{C}}}, \mathrm{K})$ a homomorphism $\mathrm{F}\colon \underline{\underline{\mathcal{C}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ which preserves local colimits automatically satisfies the local adjointness condition of lemma 1.6.6, since each homset of $\underline{\underline{\mathcal{C}}}$ has a small generator and all small colimits. We may therefore apply that lemma (and the repletion construction) to get a (2-)adjunction

$$\begin{tikzcd} \underline{\underline{\mathcal{B}}}\text{–Equip} \arrow[r, shift right, "(\overline{\mathrm{F}}_\star, \mathrm{F}_\star)"'] \arrow[r, phantom, "\perp_b"] & \underline{\underline{\mathcal{C}}}\text{–Equip} \arrow[l, shift right, "(\overline{\mathrm{F}}^\star, \mathrm{F}^\star)"'] \end{tikzcd} \tag{1.97}$$

in $\underline{\underline{\underline{\mathcal{E}Mor}}}$, where $\ddot{\mathrm{F}}^\star$ is an equipment homomorphism. For each 0-cell $u \in \underline{\underline{\mathcal{C}}}$ the homomorphism $\overline{\mathrm{F}}^\star\colon \underline{\underline{\mathcal{C}}}\text{–Prof} \longrightarrow \underline{\underline{\mathcal{B}}}\text{–Prof}$ carries a bimodule $p\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\mapsto u$ to one with codomain $\mathrm{F}(u)$, accordingly it makes sense to say that F is a *cocontinuous* homomorphism from $(\underline{\underline{\mathcal{C}}}, \mathrm{K})$ to $(\underline{\underline{\mathcal{B}}}, \mathrm{J})$ iff

(i) F preserves local colimits,
(ii) The homomorphism $\overline{\mathrm{F}}^\star$ carries any bimodule in $\mathrm{K}(u)$ to one in $\mathrm{J}(\mathrm{F}(u))$. □

With all these definitions under our belt the following theorem is surprisingly easy to establish:

Theorem 1.7.13 (the comparison lemma) *Any cocontinuous homomorphism of sites* $\mathrm{F}\colon (\underline{\underline{\mathcal{C}}}, \mathrm{K}) \longrightarrow (\underline{\underline{\mathcal{C}}}, \mathrm{J})$ *gives rise to a biadjoint pair*

$$\begin{tikzcd} (\underline{\underline{\mathcal{B}}}, \mathrm{J})\text{–}\mathcal{S}\mathrm{hf} \arrow[r, shift right, "\ddot{\mathrm{F}}_\star"'] \arrow[r, phantom, "\perp_b"] & (\underline{\underline{\mathcal{C}}}, \mathrm{K})\text{–}\mathcal{S}\mathrm{hf} \arrow[l, shift right, "\ddot{\mathrm{F}}^\star"'] \end{tikzcd}$$

in $\underline{\underline{\underline{\mathcal{E}Mor}}}$, *where* $\ddot{\mathrm{F}}^\star$ *is an equipment homomorphism.*

Proof. Return to the biadjoint pair in (1.97), the homomorphism

$$\overline{\mathrm{F}}^\star\colon \underline{\underline{\mathcal{C}}}\text{–Prof} \longrightarrow \underline{\underline{\mathcal{B}}}\text{–Prof}$$

acts as F pointwise on bimodules, in other words if $p\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathbb{B}$ is a bimodule in $\underline{\underline{\mathcal{C}}}$–Prof with components $p(x,y) \in \underline{\underline{\mathcal{C}}}$ then $\overline{\mathrm{F}}^\star(p)$ has components $\mathrm{F}(p(x,y)) \in \underline{\underline{\mathcal{C}}}$. Simply checking left actions it follows that $\overline{\mathrm{F}}^\star(p(\text{-},y)) = \overline{\mathrm{F}}^\star(p)(\text{-},y)\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\mapsto \beta(y)$ for each object $y \in \mathbb{B}$ so, since F is cocontinuous, any 1-cell $p \in \mathrm{K}\text{–}\mathcal{C}\mathrm{ov}$ maps to $\overline{\mathrm{F}}^\star(p) \in \mathrm{J}\text{–}\mathcal{C}\mathrm{ov}$ and we may restrict $\overline{\mathrm{F}}^\star$ to a homomorphism:

$$\mathrm{K}\text{–}\mathcal{C}\mathrm{ov} \xrightarrow{\tilde{\mathrm{F}}} \mathrm{J}\text{–}\mathcal{C}\mathrm{ov}$$

Together these two give an equipment homomorphism

$$(\underline{\underline{\mathcal{C}}}, \mathrm{K})\text{–}\mathcal{S}\mathrm{hf} \xrightarrow{(\overline{\mathrm{F}}^\star, \tilde{\mathrm{F}}^{\mathrm{op}})} (\underline{\underline{\mathcal{B}}}, \mathrm{J})\text{–}\mathcal{S}\mathrm{hf}$$

which preserves representables strictly. Of course theorem 1.5.14 tells us that the action of $\overline{\mathrm{F}}^\star$ on each homset has a right adjoint, since $\ddot{\mathrm{F}}^\star$ has a right biadjoint in $\underline{\underline{\underline{\mathcal{E}Mor}}}$, and so by that theorem all that remains in showing that $(\overline{\mathrm{F}}^\star, \tilde{\mathrm{F}}^{\mathrm{op}})$ also has a right biadjoint in $\underline{\underline{\underline{\mathcal{E}Mor}}}$ is to check that the homomorphism of bicategories $\tilde{\mathrm{F}}^{\mathrm{op}}$ has one itself.

Here is where we apply theorem 1.7.1 for the biadjunction in (1.97), suppose that the $\underline{\underline{\mathcal{B}}}$-category $\mathbb{B}$ is J-complete and $p\colon \mathbb{A} \mathrel{\longrightarrow\!\!\!\!\!\!\!+\;} u$ is a bimodule in $\mathrm{K}(u)$. For any $\underline{\underline{\mathcal{C}}}$-functor $g\colon \mathbb{A} \longrightarrow \mathrm{F}_\star(\mathbb{B}$ theorem 1.7.1(ii) gives the conditional congruence

$$\mathrm{colim}(\overline{\mathrm{F}}^\star(p), \hat{g}) \overset{\sim}{\Rightarrow} (\mathrm{colim}(p,g))^\wedge$$

F is cocontinuous so $\overline{\mathrm{F}}^\star(p)$ is in $\mathrm{J}(\mathrm{F}(u))$ and the colimit on the left exists ($\mathbb{B}$ is J-cocomplete) implying that the one on the right does as well. It follows that $\mathrm{F}_\star(\mathbb{B})$ is K-cocomplete and so is in the full image of $\mathcal{P}_{\mathrm{K}}(\text{–})$ therefore the unit $\mathcal{Y}\colon \mathrm{F}_\star(\mathbb{B}) \longrightarrow \mathcal{P}_{\mathrm{K}}(\mathrm{F}_\star(\mathbb{B}))$ is an equivalence in $\underline{\underline{\mathcal{C}}}$–Cat.

Any $\underline{\underline{\mathcal{B}}}$-category $\mathcal{P}_{\mathrm{J}}(\mathbb{B})$ is J-cocomplete and so we may always apply the last result to them, which we do in the following sequence of natural equivalences:

$$\begin{array}{ccr}
 & \tilde{\mathrm{F}}^{\mathrm{op}}(\mathbb{A}) = \mathrm{F}^\star(\mathbb{A}) \mathrel{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\;} \mathbb{B} & \text{in } \mathrm{J}\text{–}\mathcal{C}\mathrm{ov}^{\mathrm{op}} \\
(\text{–})^* \dashv_b \mathcal{P}_{\mathrm{J}} & \overline{} & \\
 & \mathrm{F}^\star(\mathbb{A}) \longrightarrow \mathcal{P}_{\mathrm{J}}(\mathbb{B}) & \text{in } \underline{\underline{\mathcal{B}}}\text{–Cat} \\
\mathrm{F}^\star \dashv_b \mathrm{F}_\star & \overline{} & \\
 & \mathbb{A} \longrightarrow \mathrm{F}_\star\mathcal{P}_{\mathrm{J}}(\mathbb{B}) \simeq \mathcal{P}_{\mathrm{K}}\mathrm{F}_\star\mathcal{P}_{\mathrm{J}}(\mathbb{B}) & \text{in } \underline{\underline{\mathcal{C}}}\text{–Cat} \\
(\text{–})^* \dashv_b \mathcal{P}_{\mathrm{K}} & \overline{} & \\
 & \mathbb{A} \mathrel{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\;} \mathrm{F}_\star\mathcal{P}_{\mathrm{J}}(\mathbb{B}) & \text{in } \mathrm{K}\text{–}\mathcal{C}\mathrm{ov}^{\mathrm{op}}
\end{array}$$

These demonstrate that $\tilde{\mathrm{F}}^{\mathrm{op}}\colon \mathrm{K}\text{–}\mathcal{C}\mathrm{ov} \longrightarrow \mathrm{J}\text{–}\mathcal{C}\mathrm{ov}$ has a right biadjoint $\mathrm{F}_\star\mathcal{P}_{\mathrm{J}}(\text{–})$, and as we indicated above all that remains is to apply theorem 1.5.14 to obtain the required right biadjoint in $\underline{\underline{\underline{\mathcal{E}Mor}}}$. $\square$

With a little bit of further elucidation we may apply this result to the classical theory of *stacks* as presented in [50], wherein it is shown that the stack condition is simply a cocompleteness requirement with respect to a family of Cauchy covers. This allows us to obtain a deeper insight into the interactions between stacks and geometric morphisms of the toposes over which they sit, but this example will have to be left for later work.

Chapter 2

Double Limits.

2.1 The context.

The purpose of this chapter is to use a little of the material in the last chapter to examining Paré's work on "Double" limits in 2-category theory, which he presented to the Bangor International Category Theory Meeting in the summer of 1989 (see [38]). We will analyse a class of 2-limits which he defined in this context and dubbed the "Persistent" ones, in particular demonstrating that it is in fact identical to the more widely known class of "Flexible" ones. In doing this we modify and recast the constructions in Power and Robinson's characterisation of PIE limits (see [43]).

In essence Paré's contention is that we should think of 2-categories as a particular kind of double category. These are simply categories internal to $\underline{\mathrm{CAT}}$, first studied by Charles Ehresmann and in fact used by Kelly and Street as a preliminary to the definition of a 2-category that they give in their well known expository article [32]. For him the principle advantage of this representation is that we pass from enriched category theory, in which there is in general no Grothendieck construction, to the internal theory for which there is. As we shall see later in this chapter we may define a natural conical limit notion, parameterised by double categories, in such a way that every weighted 2-limit may be reduced to one of these *double* limits.

One of the principle problems that Paré encountered in carrying out this program was in demonstrating a way of "reversing" the Grothendieck construction. Not only do we wish to know how to calculate a weighted 2-limit as a double limit but also how to derive a weight from a given double category which parameterises the same limits. Weaker results are recounted in [38], for instance the fact that a 2-category possesses all (small) weighted limits iff it admits all (small) double limits, but we really need more. In 2-category theory we are rarely interested in considering all limits, but rather a sub-class which behave well with respect to the weakened structures that 2-categories permit, equivalences being a good example. If we are to take double categories seriously it seems necessary to translate classes of limits defined in terms of them into traditional classes of weights and back again. With this achieved we might hope to go further and use double categories in constructing and analysing well behaved *closed* classes of 2-limits.

In making all these ideas more precise we might start by following the lead of earlier chapters and adopt an equipment of internal categories $\mathrm{Equip}(\underline{\mathrm{CAT}}) = (\mathrm{Cat}(\underline{\mathrm{CAT}}), \mathrm{Prof}(\underline{\mathrm{CAT}}), (\text{-})_*)$, in which we might solve our problem as a simple exercise in the calculus of profunctors. Sadly, its not quite that easy: in order to define the tensorial composition of $\mathrm{Prof}(\underline{\mathrm{CAT}})$ we require reflexive coequalisers to be stable under pullback, a condition which $\underline{\mathrm{CAT}}$ fails to satisfy.

In fact all is not lost: by taking nerves we represent $\underline{\mathrm{CAT}}$ as a full reflective subcategory of $[\mathbf{\Delta}^{\mathrm{op}}, \underline{\mathrm{SET}}]$, the category of simplicial sets, which as a presheaf topos supports the definition of an equipment of internal categories $\mathrm{Equip}([\mathbf{\Delta}^{\mathrm{op}}, \underline{\mathrm{SET}}])$. So, as long as we establish the existence of a well behaved change of base between this and 2–EQUIP, we may make all of our calculations in terms of (internal) simplicial categories reflecting the results back into the more familiar 2-context at will. This chapter is devoted carrying this prescription through, initially describing the solution for enrichments over a wide variety of categories, and then as we near the end of the chapter, specialising to double categories and examining the class of *persistent* limits.

As usual we will fix three set theoretic universes $\underline{\mathrm{Set}} \subset \underline{\mathrm{SET}} \subset \underline{\mathcal{SET}}$, the sets of which we refer to as small, large and huge respectively. It will also be useful to reserve the notations $\underline{\mathrm{Cat}}$ and $\underline{\mathrm{CAT}}$ for the categories of small and large categories respectively (with two underlinings if we mean the corresponding 2-categories). For most of this chapter we will be considering the internalisation of categories enriched over a cartesian closed locally presentable category, so recall the definition of a locally presentable category (for more detail and proofs of the following theorems on such categories see [18] and [47]):

Definition 2.1.1 *A (large and locally small) category $\underline{\mathcal{A}}$ is locally presentable iff*

- *(a) It is small complete and cocomplete.*
- *(b) There exists a small regular cardinal α and a small strongly generating set of α-presentable objects.*

Also recall that a *Gabriel theory* $\mathbb{J} = (\underline{\mathcal{C}}, J)$ consists of a small category $\underline{\mathcal{C}}$ and a function J which assigns to each object $u \in \underline{\mathcal{C}}$ a set $J(u)$ of cocones in $\underline{\mathcal{C}}$ with vertex U. Given such a theory its category of models $\mathrm{Mod}(\mathbb{J}, \underline{\mathcal{S}})$ in a category $\underline{\mathcal{S}}$ is the full sub-category of $[\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{S}}]$ on those presheaves which map each cocone in $\mathbb{J}$ to a limiting cone in $\underline{\mathcal{S}}$. The fundamental theorem of locally presentable categories is:

Theorem 2.1.2 *If $\underline{\mathcal{A}}$ is a (large and locally small) category then it is locally presentable iff there exists a gabriel theory $\mathbb{J}$ with $\underline{\mathcal{A}} \simeq \mathrm{Mod}(\mathbb{J}, \underline{\mathrm{Set}})$. Furthermore $\mathbb{J}$ may be picked such that each representable is a model (or equivalently each cocone in J is colimiting).* $\square$

So for the remainder of this chapter we will fix a gabriel theory $\mathbb{J}$, with each of its cocones colimiting, and adopt the notations:

$$\begin{array}{rclcrcl} \underline{\mathcal{A}} & \stackrel{\text{def}}{=} & \mathrm{Mod}(\mathbb{J}, \underline{\mathrm{Set}}) & \qquad & \widetilde{\underline{\mathcal{C}}} & \stackrel{\text{def}}{=} & [\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Set}}] \\ \underline{\mathcal{A}}_{\mathrm{l}} & \stackrel{\text{def}}{=} & \mathrm{Mod}(\mathbb{J}, \underline{\mathrm{SET}}) & & \widetilde{\underline{\mathcal{C}}}_{\mathrm{l}} & \stackrel{\text{def}}{=} & [\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{SET}}] \end{array}$$

Our interest in the categories $\underline{\mathcal{A}}_{\mathrm{l}}$ and $\widetilde{\underline{\mathcal{C}}}_{\mathrm{l}}$, of large models and presheaves, is of course due to the fact that we will be interested in "large" enriched categories. For example if the theory $\mathbb{J}$ is that of "categories and functors" then $\underline{\mathcal{A}}$ and $\underline{\mathcal{A}}_{\mathrm{l}}$ are the categories $\underline{\mathrm{Cat}}$ and $\underline{\mathrm{CAT}}$, of small and large categories respectively, with $\widetilde{\underline{\mathcal{C}}}$ and $\widetilde{\underline{\mathcal{C}}}_{\mathrm{l}}$ the categories of small and large simplicial sets. In this case $\underline{\mathcal{A}}$-enriched categories are just locally small 2-categories, it is necessary to consider $\underline{\mathcal{A}}_{\mathrm{l}}$-categories in order to capture the theory of large 2-categories. Two theorems which will hold some importance for us are:

Theorem 2.1.3 *If $\underline{\mathcal{A}}$ is (large and locally small and) locally presentable then so is* $\mathrm{Mod}(\mathbb{J}, \underline{\mathcal{A}})$. $\square$

Theorem 2.1.4 *If each representable is a $\mathbb{J}$-model then tfae*

(i) $\mathrm{Mod}(\mathbb{J}, \underline{\mathrm{Set}})$ *is cartesian closed,*

(ii) for all presheaves $F, P \in [\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Set}}]$ if F is a model of $\mathbb{J}$ so is F^P,

(iii) the left adjoint of $\mathrm{Mod}(\mathbb{J}, \underline{\mathrm{Set}}) \hookrightarrow [\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Set}}]$ *preserves finite products,*

(iv) for all small complete, cartesian closed. large and locally small categories $\underline{\mathcal{A}}$, $\mathrm{Mod}(\mathbb{J}, \underline{\mathcal{A}})$ *is cartesian closed.*

(v) for all $\underline{\mathcal{A}}$ as in (iv) and $F, P \in [\mathbb{C}^{\mathrm{op}}, \underline{\mathcal{A}}]$, F is a $\mathbb{J}$-model implies that F^P is,

(vi) for all cartesian closed, locally presentable $\underline{\mathcal{A}}$, the left adjoint to the functor $\mathrm{Mod}(\mathbb{J}, \underline{\mathcal{A}}) \hookrightarrow [\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{A}}]$ *preserves finite products.* $\square$

We will assume from here on that $\underline{\mathcal{A}}$ satisfies the equivalent conditions of the last theorem, in particular (iii) will be important for the construction of a change of base between $\underline{\mathcal{A}}$-enriched and $[\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Set}}]$-enriched categories. From section 2.5 on we will also require an extra condition:

Condition 2.1.5 *The base of each cone of $\mathbb{J}$ is a diagram on a non-empty connected category.*

In framing this condition we are motivated by the well known fact that in $\underline{\mathrm{Set}}$ coproducts commute with the limits of diagrams on non-empty connected categories (or *connected limits* for short), and so it follows that if $\mathbb{J}$ satisfies condition 2.1.5 then the embedding $\underline{\mathcal{A}} \hookrightarrow \widetilde{\underline{\mathcal{C}}}$ preserves all (small) coproducts.

We say that an object $A \in \underline{\mathcal{A}}$ is *indecomposable* when the representable $\underline{\mathcal{A}}(A,\text{-})$ preserves all (small) coproducts. The proof of the following theorem is a slight modification of any one of the traditional proofs of theorem 2.1.2:

Theorem 2.1.6 *A (large and locally small) category $\underline{\mathcal{A}}$ is equivalent to* $\mathrm{Mod}(\mathbb{J}, \underline{\mathrm{Set}})$ *for some Gabriel theory $\mathbb{J}$ satisfying condition 2.1.5 iff*

- *$\underline{\mathcal{A}}$ is small cocomplete*
- *There exists some regular cardinal α such that $\underline{\mathcal{A}}$ has a strong generator consisting of indecomposable, α-presentable objects.* $\square$

Of course we will primarily interested in the theory of categories. Its gabriel theory is $(\mathbf{\Delta}, J_{\mathrm{cat}})$ where $\mathbf{\Delta}$ is the category of nonzero finite ordinals and order preserving functions, in which $[n]$ denotes the ordinal $n+1$, and:

$$J_{\mathrm{cat}}([0]) = J_{\mathrm{cat}}([1]) = \emptyset$$

$$J_{\mathrm{cat}}([n])\ (n > 1) \quad \text{consists of the single cocone} \quad \begin{tikzcd} {[n-2]} \arrow[r, "\delta_{n-1}"] \arrow[d, "\delta_0"'] & {[n-1]} \arrow[d, dotted, "\delta_0"] \\ {[n-1]} \arrow[r, dotted, "\delta_n"'] & {[n]} \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

Here $\delta_r : [n-1] \longrightarrow [n]$ denotes the unique monic without r in its image, and each of these cocones are pushouts as indicated so, since we know that $\underline{\mathrm{Cat}}$ is cartesian closed, this theory satisfies the conditions of theorem 2.1.4. Furthermore because pushouts are cones on connected diagrams it also satisfies the supplementary condition 2.1.5.

In this chapter we will assume that all the equipments we use are in the 2-category $\underline{\underline{\mathcal{E}Map}}$, in other words their bicategories of functors are simply plain categories. In sections 2.2 to 2.5 we study colimits in $\underline{\mathcal{A}}$-enriched categories using the internal category theory of $\widetilde{\underline{\mathcal{C}}}$ via a change of base adjoint in $\underline{\underline{\mathcal{E}Map}}$. Before going further we'll recall and fix the precise definitions of the equipments of interest to us here:

$\underline{\mathcal{A}}$–Equip: the equipment of "small $\underline{\mathcal{A}}$-categories". To elaborate,

$$\underline{\mathcal{A}}\text{–Equip} = (\underline{\mathcal{A}}\text{–Cat}, \underline{\mathcal{A}}\text{–Prof}, (\text{-})_*)$$

where $\underline{\mathcal{A}}$–Cat is the category of $\underline{\mathcal{A}}$-enriched categories with a *small* set of objects and functors between them while $\underline{\mathcal{A}}$–Prof is the bicategory with the same 0-cells and 1-cells which are $\underline{\mathcal{A}}$-enriched profunctors. We also have a related equipment of $\underline{\widetilde{\mathcal{C}}}$-enriched categories $\underline{\widetilde{\mathcal{C}}}$–Equip, both of these are *large* equipments.

Clearly these are special cases of the equipments of $\underline{\underline{\mathcal{B}}}$-enriched categories for a bicategory $\underline{\underline{\mathcal{B}}}$ as introduced in the last chapter. Simply think of $\underline{\mathcal{A}}$ (or $\underline{\widetilde{\mathcal{C}}}$) as the homset of a bicategory with a single 0-cell and tensorial composition given by product in $\underline{\mathcal{A}}$.

$\underline{\mathcal{A}}_{\mathrm{l}}$–EQUIP: the equipment of "large $\underline{\mathcal{A}}$-categories". Again

$$\underline{\mathcal{A}}_{\mathrm{l}}\text{–EQUIP} = (\underline{\mathcal{A}}_{\mathrm{l}}\text{–CAT}, \underline{\mathcal{A}}_{\mathrm{l}}\text{–PROF}, (\text{-})_*)$$

where $\underline{\mathcal{A}}_{\mathrm{l}}$–CAT is the category of $\underline{\mathcal{A}}_{\mathrm{l}}$-enriched categories with possibly *large* sets of objects and functors between them while $\underline{\mathcal{A}}_{\mathrm{l}}$–PROF is the bicategory of $\underline{\mathcal{A}}_{\mathrm{l}}$-profunctors. This and the related equipment $\underline{\widetilde{\mathcal{C}}}_{\mathrm{l}}$–PROF are both *huge* equipments.

$\underline{\mathcal{A}}$–CAT: the category of "locally small $\underline{\mathcal{A}}$-categories". This is simply the category of $\underline{\mathcal{A}}$-categories with a possibly *large* set of objects and functors between these. This is a *huge* category. There is no corresponding bicategory $\underline{\mathcal{A}}$–PROF since $\underline{\mathcal{A}}$ does not necessarily admit the large colimits needed to define the tensorial composition of two $\underline{\mathcal{A}}$-profunctors between $\underline{\mathcal{A}}$-categories with large sets of objects.

Equip($\underline{\widetilde{\mathcal{C}}}$): the equipment of categories internal to $\underline{\widetilde{\mathcal{C}}}$:

$$\mathrm{Equip}(\underline{\widetilde{\mathcal{C}}}) = (\mathrm{Cat}(\underline{\widetilde{\mathcal{C}}}), \mathrm{Prof}(\underline{\widetilde{\mathcal{C}}}), (\text{-})_*)$$

One of our primary interests will be the theory of *closed* classes of weights for colimits and so we recall a few important definitions and facts in the study of $\underline{\mathcal{A}}$-colimits. For more detail see Albert and Kelly [1] and Kelly [30]:

A Small Weight on a small $\underline{\mathcal{A}}$-category $\underline{\mathbf{A}}$ is an $\underline{\mathcal{A}}$-profunctor $X\colon \underline{\mathbf{A}} \longrightarrow\!\!\!\!\!+\!\!> \underline{\mathbf{1}}$ in $\underline{\mathcal{A}}$–Equip, where $\underline{\mathbf{1}}$ is the category with a single object and "homset" $1 \in \underline{\mathcal{A}}$. These are sometimes referred to as *presheaves* and are the parameterising objects for small $\underline{\mathcal{A}}$-colimits.

$\mathcal{P}(\underline{\mathbf{A}})$ is the locally small $\underline{\mathcal{A}}$-category of small weights on $\underline{\mathbf{A}}$ and is constructed as follows:

objects: the small weights on $\underline{\mathbf{A}}$.

"homsets": $\mathcal{P}(\underline{\mathbf{A}})(X,Y)$ is the object of $\underline{\mathcal{A}}$ underlying the right Kan extension

$$\begin{tikzcd} \underline{\mathbf{A}} \arrow[rr, "X"] \arrow[dr, "Y"'] & \arrow[d, phantom, "\overset{\alpha_{(X,Y)}}{\Leftarrow}"] & \underline{\mathbf{1}} \arrow[dl, "\mathcal{P}(\underline{\mathbf{A}})(X,Y)"] \\ & \underline{\mathbf{1}} & \end{tikzcd}$$

in $\underline{\mathcal{A}}$–Prof, or more explicitly the end:

$$\mathcal{P}(\underline{\mathbf{A}})(X,Y) \cong \int_a Y(a)^{X(a)}$$

composition & identities correspond under the universal property of right Kan extensions in $\underline{\mathcal{A}}$–Prof to the maps

$$1 \otimes X \xrightarrow{\cong} X$$

and

$$\mathcal{P}(\underline{\mathbf{A}})(Y,Z) \otimes \mathcal{P}(\underline{\mathbf{A}})(X,Y) \otimes X \xrightarrow{\mathcal{P}(\underline{\mathbf{A}})(Y,Z)\otimes \alpha_{(X,Y)}} \mathcal{P}(\underline{\mathbf{A}})(Y,Z) \otimes Y \xrightarrow{\alpha_{(Y,Z)}} Z$$

respectively.

As in the traditional setting there is a fully faithful $\underline{\mathcal{A}}$-functor

$$\mathcal{Y}\colon \underline{\mathbf{A}} \hookrightarrow \mathcal{P}(\underline{\mathbf{A}}),$$

the Yoneda embedding, which takes an object $a \in \underline{\mathbf{A}}$ to the corresponding representable a^*. Often we will identify $\underline{\mathbf{A}}$ with its image under $\mathcal{Y}$. $\mathcal{P}(\underline{\mathbf{A}})$ admits all small $\underline{\mathcal{A}}$-colimits and the Yoneda embedding displays it as the universal small colimit completion of $\underline{\mathbf{A}}$.

A Class of Weights is a function $\mathcal{X}(-)$ which assigns to each small $\underline{\mathcal{A}}$-category a (possibly large) set of small weights on it. We order classes of weights by inclusion.

A colimit $\mathrm{colim}\,(W,\Gamma)$ of a diagram $\Gamma\colon \underline{\mathbf{A}} \longrightarrow \underline{\mathbf{C}}$ weighted by $W \in \mathcal{X}(\underline{\mathbf{A}})$ is called an $\mathcal{X}$–colimit. Correspondingly an $\underline{\mathcal{A}}$-category is $\mathcal{X}$–cocomplete iff it admits all $\mathcal{X}$–colimits, and a functor between such categories is $\mathcal{X}$–cocontinuous iff it preserves each $\mathcal{X}$–colimit.

The Closure $\mathcal{X}^*(-)$ of a class of weights $\mathcal{X}(-)$ is the largest class such that

- $\underline{\mathbf{C}}$ is $\mathcal{X}^*$–cocomplete iff it is $\mathcal{X}$–cocomplete
- $f\colon \underline{\mathbf{C}} \longrightarrow \underline{\mathbf{D}}$ is $\mathcal{X}^*$–cocontinuous iff it is $\mathcal{X}$–cocontinuous.

Closure is clearly inflationary, idempotent and intersection preserving.

A Closed Class of weights $\mathcal{X}(-)$ is one which satisfies $\mathcal{X}(-) = \mathcal{X}^*(-)$.

Given a class of weights $\mathcal{X}(-)$ let $\mathcal{X}[\underline{\mathbf{A}}]$ denote the full sub-$\underline{\mathcal{A}}$-category of $\mathcal{P}(\underline{\mathbf{A}})$ on those weights in $\mathcal{X}(\underline{\mathbf{A}})$. The principal result of [1] states that $\mathcal{X}^*[\underline{\mathbf{A}}]$ is the $\mathcal{X}$–colimit closure of $\underline{\mathbf{A}}$ in $\mathcal{P}(\underline{\mathbf{A}})$. It follows that we may construct $\mathcal{X}^*(-)$ by transfinite induction (in $\underline{\mathrm{SET}}$) as follows:

$$\begin{aligned}
\mathcal{X}^*_0(\underline{\mathbf{A}}) &= \underline{\mathbf{A}} \subset \mathcal{P}(\underline{\mathbf{A}}) \\
\mathcal{X}^*_{\alpha^+}(\underline{\mathbf{A}}) &= \mathcal{X}^*_\alpha(\underline{\mathbf{A}}) \cup \left\{ W \in \mathcal{P}(\underline{\mathbf{A}}) \;\middle|\; \begin{array}{l} \exists \text{ a functor } \Gamma\colon \underline{\mathbf{D}} \longrightarrow \mathcal{X}^*_\alpha[\underline{\mathbf{A}}] \text{ and} \\ V \in \mathcal{X}(\underline{\mathbf{D}}) \text{ with } W \cong \operatorname{colim}(V,\Gamma) \end{array} \right\} \\
\mathcal{X}^*_\lambda(\underline{\mathbf{A}}) &= \textstyle\bigcup_{\alpha<\lambda} \mathcal{X}^*_\alpha(\underline{\mathbf{A}}) \text{ for each limit ordinal } \lambda.
\end{aligned}$$

2.2 Internalising $\underline{\mathcal{A}}$-enriched categories.

In this section we are interested in representing $\underline{\mathcal{A}}$-enriched categories as internal ones. Our first intuition might be to consider categories internal to $\underline{\mathcal{A}}$, but as we observed at the start of the chapter we may be unable to define any tensor product of profunctors in here. This would certainly impede the abstract development of category theory that we have presented in earlier chapters.

We choose to take a slightly different course and represent $\underline{\mathcal{A}}$-categories as categories internal to $\widetilde{\underline{\mathcal{C}}}$. Our first step in this objective is to construct an inclusion of equipments

$$\begin{tikzcd} \underline{\mathcal{A}}\text{–Equip} \arrow[r, shift right, "\ddot{\mathrm{F}}_\star"'] \arrow[r, phantom, "\perp"] & \widetilde{\underline{\mathcal{C}}}\text{–Equip} \arrow[l, shift right, "\ddot{\mathrm{F}}^\star"'] \end{tikzcd} \tag{2.1}$$

(in $\underline{\underline{\mathcal{E}Mor}}$) as described in definition 1.7.4, to achieve this we return to the adjoint pair:

$$\begin{tikzcd} \underline{\mathcal{A}} \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \widetilde{\underline{\mathcal{C}}} \arrow[l, shift right, "\mathrm{F}"'] \end{tikzcd}$$

We know (by theorem 2.1.4) that F preserves finite products, so on thinking of $\underline{\mathcal{A}}$ and $\widetilde{\underline{\mathcal{C}}}$ as bicategories (each with a solitary 0-cell) F becomes a homomorphism satisfying the local adjointness property of example 1.6.6 and thus providing us with the adjunction of (2.1). Recalling example 1.7.6 it becomes clear that this adjunction is an equipment inclusion, since F cannot fail to be an isomorphism on 0-cells and its (local) right adjoint is simply the fully faithful embedding of $\underline{\mathcal{A}}$ into $\widetilde{\underline{\mathcal{C}}}$.

Now we examine the process of internalising $\widetilde{\underline{\mathcal{C}}}$-categories. Let $\Delta\colon \underline{\mathrm{Set}} \longrightarrow \widetilde{\underline{\mathcal{C}}}$ be the fully faithful "constant presheaf" functor, which we also think of as copower with the terminal object of $\widetilde{\underline{\mathcal{C}}}$, with left and right adjoints $\varinjlim_{\mathbb{C}}$ and $\varprojlim_{\mathbb{C}}$ respectively. We refer to the objects in the replete image of Δ as "discrete $\underline{\mathcal{C}}$-sets". The next lemma and its corollary demonstrates an inclusion of the equipment of $\widetilde{\underline{\mathcal{C}}}$-enriched categories into that of categories internal to $\widetilde{\underline{\mathcal{C}}}$:

Lemma 2.2.1 *There is an adjunction*

$$\begin{tikzcd} (\widetilde{\underline{\mathcal{C}}}\text{–Mat}, \underline{\mathrm{Set}}, (\text{-})_\circ) \arrow[r, hook, shift right, "\ddot{\Delta} = (\overline{\Delta}, \Delta)"'] \arrow[r, phantom, "\perp"] & (\mathrm{Span}(\widetilde{\underline{\mathcal{C}}}), \widetilde{\underline{\mathcal{C}}}, (\text{-})_\circ) \arrow[l, shift right, "\varinjlim_{\mathbb{C}}"'] \end{tikzcd}$$

in $\underline{\underline{\mathcal{E}Map}}$ with $\ddot{\Delta}$ an equipment homomorphism, Δ fully faithful and $\overline{\Delta}$ a local equivalence.

Proof. We already have an adjunction $\varinjlim_{\mathbb{C}} \dashv \Delta\colon \underline{\mathrm{Set}} \longrightarrow \widetilde{\underline{\mathcal{C}}}$ with Δ fully faithful so all that remains is to provide a homomorphism

$$\widetilde{\underline{\mathcal{C}}}\text{–Mat} \xrightarrow{\ \overline{\Delta}\ } \mathrm{Span}(\widetilde{\underline{\mathcal{C}}})$$

which is a local equivalence and comparison 2-cells $\nu_f\colon (\Delta f)_\circ \cong \overline{\Delta}(f_\circ)$, theorem 1.5.14 would then provide us with the remaining structure for a left adjoint in $\underline{\underline{\mathcal{E}Map}}$ simply because each $\overline{\Delta}\colon \underline{\widetilde{\mathcal{C}}}\text{–Mat}(X, X') \longrightarrow \text{Span}(\underline{\widetilde{\mathcal{C}}})(\Delta(X), \Delta(X'))$ is an equivalence and so has a left adjoint.

Given a matrix $m\colon X \longrightarrow X' \in \underline{\widetilde{\mathcal{C}}}$–Mat we get a span

$$\begin{tikzcd}[column sep=small] & \coprod_{x\in X, x'\in A'} m(x,x') \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ \Delta(X') & & \Delta(X) \end{tikzcd}$$

where p_0 is the unique map induced by the cone with components

$$\coprod_{x\in X} m(x,x') \longrightarrow 1 \xrightarrow{\ulcorner x' \urcorner} \Delta(X')$$

for each $x' \in X'$, here $\ulcorner x' \urcorner$ is the canonical inclusion of 1 into the copower $\Delta(X')$ corresponding to x'. The function p_1 is defined analogously giving a span as shown, and this makes an ideal candidate for $\overline{\Delta}(m)$. This construction is clearly functorial on the homset $\underline{\widetilde{\mathcal{C}}}$–Mat$(X, X')$ and if $m'\colon X' \longrightarrow X''$ is another matrix we define a comparison isomorphism $\overline{\Delta}(m') \underset{\Delta X'}{\times} \overline{\Delta}(m) \cong \overline{\Delta}(m' \circledast m)$ by:

$$\begin{aligned} \coprod_{\substack{x\in X\\ x''\in X''}} (m' \circledast m)(x, x'') &\cong \coprod_{\substack{x\in X\\ x''\in X''}} \left(\coprod_{x'\in X'} m'(x', x'') \times m(x, x') \right) \\ &\cong \left(\coprod_{\substack{x'\in X'\\ x''\in X''}} m'(x', x'') \right) \underset{\Delta(X')}{\times} \left(\coprod_{\substack{x\in X\\ x'\in X'}} m(x, x') \right). \end{aligned}$$

Now suppose $f\colon X \longrightarrow X'$ is a function in $\underline{\text{Set}}$ then $\coprod_{x,x'} f_\circ(x, x') \cong \Delta(X)$, under which isomorphism the span $\overline{\Delta}(f_\circ)$ becomes $(\Delta f)_\circ$ as required, providing us with an isomorphism ν_f. The identity comparison isomorphism $\mathrm{i}_{\Delta(X)} \cong \overline{\Delta}(\mathrm{i}_X)$ is defined identically to the maps ν_{i_X}. Checking that this information satisfies the coherence properties of an equipment homomorphism $(\overline{\Delta}, \Delta, \nu_f)$ is straightforward.

Given a span $S\colon \Delta(X) \longrightarrow \Delta(X')$ we define a matrix $\overline{\Delta}^{-1}(S)\colon X \longrightarrow X'$ by:

$$\overline{\Delta}^{-1}(S)(x, x') = (\ulcorner x' \urcorner)^\circ \underset{\Delta(X')}{\times} S \underset{\Delta(X)}{\times} (\ulcorner x \urcorner)_\circ \text{ for } x \in X, x' \in X'$$

Of course coproducts in $\underline{\widetilde{\mathcal{C}}}$ are disjoint and universal allowing us to demonstrate the existence of isomorphisms $S \cong \overline{\Delta}\overline{\Delta}^{-1}(S)$ for each such span S and $\overline{\Delta}^{-1}\overline{\Delta}(m) \cong m$ for each matrix $m\colon X \longrightarrow X'$, with this we establish the local equivalence property for $\overline{\Delta}$ as required. $\square$

Corollary 2.2.2 *There is an inclusion of equipments*

$$\begin{tikzcd}(\underline{\widetilde{\mathcal{C}}}\text{–Prof}, \underline{\widetilde{\mathcal{C}}}\text{–Cat}, (\text{–})_*) \arrow[r, hook, shift right, "\ddot{\mathrm{G}}_\star = (\overline{\mathrm{G}}_\star, \mathrm{G}_\star)"'] \arrow[r, phantom, "\perp"] & (\mathrm{Prof}(\underline{\widetilde{\mathcal{C}}}), \mathrm{Cat}(\underline{\widetilde{\mathcal{C}}}), (\text{–})_*) \arrow[l, shift right, "\ddot{\mathrm{G}}^\star = (\overline{\mathrm{G}}^\star, \mathrm{G}^\star)"']\end{tikzcd} \tag{2.2}$$

where $\ddot{\mathrm{G}}$ *is an equipment homomorphism and* $\overline{\mathrm{G}}$ *is a local equivalence.*

Proof. We know that

$$\begin{aligned}(\underline{\widetilde{\mathcal{C}}}\text{–Prof}, \underline{\widetilde{\mathcal{C}}}\text{–Cat}, (\text{–})_*) &= \mathcal{M}\mathrm{on}(\underline{\widetilde{\mathcal{C}}}\text{–Mat}, \underline{\mathrm{Set}}, (\text{–})_\circ) \\ (\mathrm{Prof}(\underline{\widetilde{\mathcal{C}}}), \mathrm{Cat}(\underline{\widetilde{\mathcal{C}}}), (\text{–})_*) &= \mathcal{M}\mathrm{on}(\mathrm{Span}(\underline{\widetilde{\mathcal{C}}}), \underline{\widetilde{\mathcal{C}}}, (\text{–})_\circ)\end{aligned}$$

and noted in observation 1.6.14 that the equipments of matrices and spans on the right of these equalities satisfy conditions which allow us to apply theorem 1.6.13 to the adjunction of the last lemma. Doing this we obtain another adjoint pair as in (2.2).

We know that $\ddot{\Delta}$ is an equipment homomorphism with $\Delta\colon \underline{\mathrm{Set}} \longrightarrow \underline{\widetilde{\mathcal{C}}}$ fully faithful and $\overline{\Delta}\colon \underline{\widetilde{\mathcal{C}}}\text{–Mat} \longrightarrow \mathrm{Span}(\underline{\widetilde{\mathcal{C}}})$ a local equivalence, so by examining the construction of $\ddot{\mathrm{G}}_\star = \mathcal{M}\mathrm{on}(\ddot{\Delta})$ it is plain that it shares these properties, and of course preserves representables.

All that remains is to examine the comorphism structure induced on $\ddot{\mathrm{G}}^\star$, and show that for 1-cells $q\colon \mathbb{B} \longrightarrow\!\!\!\!\!+\!\!> \mathbb{B}'$ and $q'\colon \mathbb{B}' \longrightarrow\!\!\!\!\!+\!\!> \mathbb{B}''$ in $\mathrm{Prof}(\underline{\widetilde{\mathcal{C}}})$ the comparison $\ddot{\mathrm{G}}^\star(q' \otimes q) \Rightarrow \ddot{\mathrm{G}}^\star(q') \otimes \ddot{\mathrm{G}}^\star(q)$ is an isomorphism when $\mathbb{A}'$ is in the replete image of $\ddot{\mathrm{G}}_\star$. This 2-cell is identified in observation 1.5.15, but notice that in this case the maps of (1.80) are all isomorphisms simply because each homset action of $\overline{\mathrm{G}}_\star$ is an equivalence. Finally the component $\Psi_{\mathbb{B}'}$ of the unit (of $\mathrm{G}^\star \dashv \mathrm{G}_\star$) is an equivalence exactly when $\mathbb{B}'$ is in the replete image of $\ddot{\mathrm{G}}_\star$, implying that the unit of $(\Phi_{\mathbb{B}'})_* \dashv (\Phi_{\mathbb{B}'})^*$ is isomorphic and therefore so is the comparison 2-cell as expressed in (1.81). $\square$

Composing the representation of the last lemma with that in (2.1) we get the desired inclusion

$$\begin{tikzcd}\underline{\mathcal{A}}\text{–Equip} \arrow[r, shift right, "\ddot{\mathrm{I}}_\star"'] \arrow[r, phantom, "\perp"] & \mathrm{Equip}(\underline{\widetilde{\mathcal{C}}}) \arrow[l, shift right, "\ddot{\mathrm{I}}^\star"']\end{tikzcd} \tag{2.3}$$

of $\underline{\mathcal{A}}$-enriched categories into those internal to $\underline{\widetilde{\mathcal{C}}}$. For the remainder of this chapter we will reserve the symbols $\ddot{\Psi} = (\Psi, \overline{\Psi})$ and $\ddot{\Phi} = (\Phi, \overline{\Phi})$ to denote the unit and counit of this adjunction in $\underline{\mathcal{EM}ap}$.

It is a matter of a straightforward argument in the theory of locally presentable categories to show that if the category of small $\mathbb{J}$-models is cartesian closed then so is the category of large such models. That result ensures that we may again derive the large variant of the inclusion of equipments shown in (2.1). This may then be used to construct an inclusion of huge equipments

$$\begin{tikzcd} \underline{\mathcal{A}}_{\mathrm{l}}\text{-EQUIP} \arrow[rr, bend right, "\ddot{\mathrm{I}}_\star"'] \arrow[rr, phantom, "\perp"] & & \mathrm{Equip}(\underline{\widetilde{\mathcal{C}}}_{\mathrm{l}}) \arrow[ll, bend right, "\ddot{\mathrm{I}}^\star"'] \end{tikzcd}$$

extending the one displayed as equation (2.3) above.

2.3 Colimits and the Grothendieck construction.

Our primary interest in representing enriched categories as internal ones is that in doing so we regain the traditional Grothendieck construction. This section describes the construction; we also show that it has the expected properties, then use it to reduce weighted limits to conical ones.

In this section take $\underline{\mathcal{E}}$ to be an arbitrary locally cartesian closed category. Let $\mathbb{A}$ and $\mathbb{B}$ be categories internal to $\underline{\mathcal{E}}$ and $X\colon \mathbb{A} \looparrowright \mathbb{B}$ be a 1-cell in $\mathrm{Prof}(\underline{\mathcal{E}})$. Define a category $\mathbb{G}(X)$ in $\mathrm{Cat}(\underline{\mathcal{E}})$ as follows

- $\mathbb{G}(X)_0 = X$, where by a slight abuse of notation we confuse the profunctor X with its underlying object in $\underline{\mathcal{E}}$.
- $\mathbb{G}(X)_1$ is shown in the following pullback diagram

$$\begin{tikzcd}
\mathbb{G}(X)_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \arrow[d, "r_X"] \\
\mathbb{B}_1 \underset{\mathbb{B}_0}{\times} X \arrow[r, "l_X"'] & X
\end{tikzcd}$$

in other words it is given symbolically by

$$\mathbb{G}(X)_1 = \left\{ \langle \bar{x}, \alpha, \beta, x \rangle \,\middle|\, \begin{array}{l} x, \bar{x} \in X, \alpha \in \mathbb{A}_1, \beta \in \mathbb{B}_1 \text{ such that} \\ d_0\alpha = x_1(\bar{x}) \wedge d_1\beta = x_0(x) \wedge \bar{x} \cdot \alpha = \beta \cdot x \end{array} \right\}$$

- We define maps

$$\begin{tikzcd}
\mathbb{G}(X)_1 \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] & \mathbb{G}(X)_0 \arrow[l, "i"']
\end{tikzcd}$$

in informal notation by

$$\begin{array}{rcl} \mathbb{G}(X)_1 & \xrightarrow{d_0} & \mathbb{G}(X)_0 \\ \langle \bar{x}, \alpha, \beta, x \rangle & \longmapsto & \bar{x} \end{array} \quad , \quad \begin{array}{rcl} \mathbb{G}(X)_1 & \xrightarrow{d_1} & \mathbb{G}(X)_0 \\ \langle \bar{x}, \alpha, \beta, x \rangle & \longmapsto & x \end{array}$$

$$\begin{array}{rcl} \mathbb{G}(X)_0 & \xrightarrow{i} & \mathbb{G}(X)_1 \\ x & \longmapsto & \langle x, ix_1(x), ix_0(x), x \rangle \end{array}$$

- Lastly we have a composition

$$\begin{array}{rcl} \mathbb{G}(X)_1 \underset{\mathbb{G}(X)_0}{\times} \mathbb{G}(X)_1 & \xrightarrow{\quad\circ\quad} & \mathbb{G}(X)_1 \\ \langle \langle \tilde{x}, \alpha', \beta', \bar{x} \rangle, \langle \bar{x}, \alpha, \beta, x \rangle \rangle & \longmapsto & \langle \tilde{x}, \alpha' \circ \alpha, \beta' \circ \beta, x \rangle \end{array}$$

again given informally, which is well defined because

$$\tilde{x} \cdot (\alpha' \circ \alpha) = (\tilde{x} \cdot \alpha') \cdot \alpha = (\beta' \cdot \bar{x}) \cdot \alpha = \beta' \cdot (\bar{x} \cdot \alpha) = \beta' \cdot (\beta \cdot x) = (\beta' \circ \beta) \cdot x$$

It is easily checked that these make $\mathbb{G}(X)$ into a category in $\mathrm{Cat}(\underline{\mathcal{E}})$ furthermore we have canonical projection functors $\bar{x}_0 \colon \mathbb{G}(X) \longrightarrow \mathbb{B}$ and $\bar{x}_1 \colon \mathbb{G}(X) \longrightarrow \mathbb{A}$.

Of course this construction gives us the object action of a functor

$$\mathbb{G} \colon \mathrm{Prof}(\underline{\mathcal{E}})(\mathbb{A}, \mathbb{B}) \longrightarrow \mathrm{Span}(\mathrm{Cat}(\underline{\mathcal{E}}))(\mathbb{A}, \mathbb{B}).$$

which takes an equivariant map $\Theta \colon X \longrightarrow Y$ to an internal functor $\mathbb{G}(\Theta)$ given by

- $\mathbb{G}(\Theta)_0 = \Theta$, the underlying map of Θ in $\underline{\mathcal{E}}$.
- $\mathbb{G}(\Theta)$'s action on morphisms is

$$\begin{array}{rcll} \mathbb{G}(\Theta)_1 \colon \mathbb{G}(X)_1 & \longrightarrow & \mathbb{G}(Y)_1 & \quad \text{given by} \\ \langle \bar{x}, \alpha, \beta, x \rangle & \longmapsto & \langle \Theta\bar{x}, \alpha, \beta, \Theta x \rangle & \end{array}$$

The equivariance of Θ ensures that this is a well defined internal functor. Indeed it is clear that this constitutes a map of spans $(\bar{x}_0, \mathbb{G}(X), \bar{x}_1)$ to $(\bar{y}_0, \mathbb{G}(Y), \bar{y}_1)$ in $\mathrm{Span}(\mathrm{Cat}(\underline{\mathcal{E}}))$, and that everything is nicely functorial.

Observation 2.3.1 *The functor $\mathbb{G}(\text{-})$ is full and faithful*

Proof. It is faithful since $\mathbb{G}(\Theta)_0 = \Theta$. For fullness let $F \colon \mathbb{G}(X) \longrightarrow \mathbb{G}(Y)$ be a functor of spans in $\mathrm{Span}(\mathrm{Cat}(\underline{\mathcal{E}}))$, then the action of F on objects gives a map of spans

$$\begin{tikzcd} & X \arrow[dl, "x_0"'] \arrow[dr, "x_1"] \arrow[dd, "F_0"] & \\ \mathbb{B}_0 & & \mathbb{A}_0. \\ & Y \arrow[ul, "y_0"] \arrow[ur, "y_1"'] & \end{tikzcd}$$

Now the diagrams

$$\begin{tikzcd} \mathbb{G}(X)_1 \arrow[r, "F_1"] \arrow[d, "d_0"'] & \mathbb{G}(Y)_1 \arrow[d, "d_0"] \\ \mathbb{G}(X)_0 \arrow[r, "F_0"'] & \mathbb{G}(Y)_0 \end{tikzcd} , \quad \begin{tikzcd} \mathbb{G}(X)_1 \arrow[r, "F_1"] \arrow[d, "d_1"'] & \mathbb{G}(Y)_1 \arrow[d, "d_1"] \\ \mathbb{G}(X)_0 \arrow[r, "F_0"'] & \mathbb{G}(Y)_0 \end{tikzcd} \text{ and } \begin{tikzcd} & \mathbb{G}(X)_1 \arrow[dl] \arrow[dr] \arrow[dd, "F_1"] & \\ \mathbb{B}_1 & & \mathbb{A}_1 \\ & \mathbb{G}(Y)_1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

all commute, so it follows that

$$(\forall\langle\bar{x},\alpha,\beta,x\rangle\in\mathbb{G}(X)_1)\,F_1\langle\bar{x},\alpha,\beta,x\rangle=\langle F_0(\bar{x}),\alpha,\beta,F_0(x)\rangle$$

and therefore if $\bar{x}\cdot\alpha=\beta\cdot x$ then $F_0(\bar{x})\cdot\alpha=\beta\cdot F_0(x)$, which entails that

$$\begin{array}{cccc}\text{setting} & x=\bar{x}\cdot\alpha,\ \beta=ix_0(x) & \Rightarrow & F_0(\bar{x})\cdot\alpha=F_0(\bar{x}\cdot\alpha)\\ \text{and} & \bar{x}=\beta\cdot x,\ \alpha=ix_1(\bar{x}) & \Rightarrow & F_0(\beta\cdot x)=\beta\cdot F_0(x).\end{array}$$

Putting this all together we have that F_0 is equivariant with $F=\mathbb{G}(F_0)$. □

A further property of the Grothendieck construction is:

Observation 2.3.2 $\mathbb{G}(-)$ *has a left adjoint*

$$\mathbf{L}\colon \mathrm{Span}(\mathrm{Cat}(\underline{\mathcal{E}}))(\mathbb{A},\mathbb{B})\longrightarrow\mathrm{Prof}(\underline{\mathcal{E}})(\mathbb{A},\mathbb{B})$$

given in terms of the calculus of profunctors by

$$\mathbf{L}\left(\begin{tikzcd} & \mathbb{E} \arrow[dl, "e_0"'] \arrow[dr, "e_1"] & \\ \mathbb{B} & & \mathbb{A}\end{tikzcd}\right)=(e_0)_*\otimes(e_1)^*\colon\mathbb{A}\longrightarrow\!\!\!\!\!\!\!+\;\;\mathbb{B}$$

Proof. More explicitly we may form the underlying span of $(e_0)_*\otimes(e_1)^*$ by taking the coequaliser of a pair

$$\mathbb{B}_1\underset{\mathbb{B}_0}{\times}\mathbb{E}_1\underset{\mathbb{A}_0}{\times}\mathbb{A}_1\ \underset{g}{\overset{f}{\rightrightarrows}}\ \mathbb{B}_1\underset{\mathbb{B}_0}{\times}\mathbb{E}_0\underset{\mathbb{A}_0}{\times}\mathbb{A}_1$$

where
$$\begin{array}{rcl}f\langle\beta,\epsilon,\alpha\rangle & = & \langle\beta,d_0(\epsilon),e_1(\epsilon)\circ\alpha\rangle\\ g\langle\beta,\epsilon,\alpha\rangle & = & \langle\beta\circ e_0(\epsilon),d_1(\epsilon),\alpha\rangle\end{array}$$

which we equip with actions induced by the canonical ones on $\mathbb{B}_1\underset{\mathbb{B}_0}{\times}\mathbb{E}_0\underset{\mathbb{A}_0}{\times}\mathbb{A}_1$.

So an equivariant map $m\colon(e_0)_*\otimes(e_1)^*\longrightarrow X$ corresponds to a map of spans

$$\mathbb{B}_1\underset{\mathbb{B}_0}{\times}\mathbb{E}_0\underset{\mathbb{A}_0}{\times}\mathbb{A}_1\xrightarrow{\bar{m}}X$$

such that for all $\langle\beta,e,\alpha\rangle\in\mathbb{B}_1\underset{\mathbb{B}_0}{\times}\mathbb{E}_0\underset{\mathbb{A}_0}{\times}\mathbb{A}_1$ and suitable $\alpha'\in\mathbb{A}_1$, $\beta'\in\mathbb{B}_1$ we have

(i) $\bar{m}\langle\beta,e,\alpha\circ\alpha'\rangle=\bar{m}\langle\beta,e,\alpha\rangle\cdot\alpha'$
(ii) $\bar{m}\langle\beta'\circ\beta,e,\alpha\rangle=\beta'\cdot\bar{m}\langle\beta,e,\alpha\rangle$
(iii) $\bar{m}\langle\beta,d_0(\epsilon),e_1(\epsilon)\circ\alpha\rangle=\bar{m}\langle\beta\circ e_0(\epsilon),d_1(\epsilon),\alpha\rangle$

Now (i) and (ii) ensure that $\bar{m}$ corresponds to a map of spans $\tilde{m}\colon\mathbb{E}_0\longrightarrow X$, where $\tilde{m}(e)=\bar{m}\langle ie_0(e),e,ie_1(e)\rangle$, and we regain $\bar{m}$ by letting $\bar{m}\langle\beta,e,\alpha\rangle=\beta\cdot\tilde{m}(e)\cdot\alpha$. Condition (iii) holds for $\bar{m}$ iff $\tilde{m}$ satisfies

$$\text{(iii)}'\ (\forall\epsilon\in\mathbb{E}_1)\{\tilde{m}(d_0(\epsilon))\cdot e_1(\epsilon)=e_0(\epsilon)\cdot\tilde{m}(d_1(\epsilon))\}\ .$$

Given such a map $\tilde{m}$ we can define a functor of spans

$$\check{m}: \mathbb{E} \longrightarrow \mathbb{G}(X)$$

with

$$\begin{array}{rcll} \check{m}_0(e) & = & \tilde{m}(e) & (\forall e \in \mathbb{E}_0) \\ \check{m}_1(\epsilon) & = & \langle \tilde{m}(d_0(\epsilon)), e_1(\epsilon), e_0(\epsilon), \tilde{m}(d_1(\epsilon))\rangle & (\forall \epsilon \in \mathbb{E}_1) \end{array}$$

which is well defined, since condition (iii)$'$ holds for $\tilde{m}$, and easily shown to be an internal functor. Conversely given such an $\check{m}$ then $\tilde{m} = \check{m}_0$ satisfies condition (iii)$'$ by the functoriality of $\check{m}$.

Therefore we obtain a bijection

$$\frac{\mathbf{L}(\mathbb{E}) \xrightarrow{\ m\ } X}{\mathbb{E} \xrightarrow{\ \check{m}\ } \mathbb{G}(x)}$$

between maps of spans, which is natural in X, and we have demonstrated that $\mathbb{G}$ has a left adjoint, which is indeed calculated as stated. □

Observation 2.3.3 Of course we may characterise spans in the full image of $\mathbb{G}$ as *two-sided discrete fibrations*. A well known case of this is the traditional description of the full image of:

$$\mathrm{Prof}(\underline{\mathcal{E}})(\mathbb{B}, 1) \xrightarrow{\ \mathbb{G}\ } \mathrm{Span}(\mathrm{Cat}(\underline{\mathcal{E}}))(\mathbb{B}, 1)$$

Here the category on the left is simply (equivalent to) the usual one of right $\mathbb{A}$-sets $r(\mathbb{A})$ and that on the right is the slice $\mathrm{Cat}(\underline{\mathcal{E}})/_{\mathbb{A}}$. An object $p: \mathbb{E} \longrightarrow \mathbb{B} \in \mathrm{Cat}(\underline{\mathcal{E}})/_{\mathbb{B}}$ is in the full image of this functor iff the square

$$\begin{tikzcd} \mathbb{E}_1 \arrow[r, "d_0"] \arrow[d, "p_1"'] & \mathbb{E}_0 \arrow[d, "p_0"] \\ \mathbb{B}_1 \arrow[r, "d_0"'] & \mathbb{B}_0 \end{tikzcd}$$

is a pullback in $\underline{\mathcal{E}}$. Functors sharing this property are called *discrete fibrations*, the full subcategory of which is denoted by $Dfib(\underline{\mathcal{E}})/_{\mathbb{B}}$. □

Armed with the Grothendieck construction we are now able to describe the process by which general weighted colimits may be reduced to conical ones in the internal category theory of $\underline{\mathcal{E}}$.

Let $X\colon \mathbb{A} \rightarrow\!\!\!\!\!\!+\; \mathbb{B}$ be a weight in Equip($\underline{\mathcal{E}}$) and Γ a diagram of $\mathbb{A}$ in $\mathbb{C}$. Then we know that $\operatorname{colim}(X,\Gamma) : \mathbb{B} \longrightarrow \mathbb{C}$ (if it exists) is the essentially unique functor such that

$$\operatorname{colim}(X,\Gamma)_* \cong (\Gamma_* \underset{\mathbb{B}}{\Leftarrow} X) .$$

$\mathbb{G}(-)$ is fully faithful so the counit $\varepsilon_X\colon \mathbf{L}\mathbb{G}X \longrightarrow X$ is an isomorphism. In other words if

$$\begin{tikzcd} & \mathbb{G}(X) \arrow[dl, "\bar{x}_0"'] \arrow[dr, "\bar{x}_1"] & \\ \mathbb{B} & & \mathbb{A} \end{tikzcd}$$

is the Grothendieck category of X then $X \cong (\bar{x}_0)_* \otimes (\bar{x}_1)^*$ and we have natural bijections:

$$\begin{array}{c}
Z \longrightarrow (\Gamma_* \underset{\mathbb{B}}{\Leftarrow} X) \\
\hline
Z \otimes (\bar{x}_0)_* \otimes (\bar{x}_1)^* \quad \cong \quad Z \otimes X \longrightarrow \Gamma_* \\
\hline
Z \otimes (\bar{x}_0)_* \longrightarrow \Gamma_* \otimes (\bar{x}_1)_* \quad \cong \quad (\Gamma \circ \bar{x}_1)_* \\
\hline
Z \longrightarrow (\Gamma \circ \bar{x}_1)_* \underset{\mathbb{G}(X)}{\Leftarrow} (\bar{x}_0)_*
\end{array}
\qquad (\bar{x}_1)_* \dashv (\bar{x}_1)^*$$

It follows that

$$\operatorname{colim}(X,\Gamma) \mathrel{\overset{\cong}{\leftrightarrows}} \operatorname{colim}((\bar{x}_0)_*, \Gamma \circ \bar{x}_1)$$

where the meaning of $A \mathrel{\overset{\cong}{\leftrightarrows}} B$ is described in notation comment 1.7.3.

This isomorphism simply says that in the presence of the Grothendieck construction we may reduce any colimit weighted by X to the *pointwise* or *enriched* left Kan extension of $\Gamma \circ \bar{x}_1$ along $\bar{x}_0$.

Given an arbitrary diagram $\Gamma\colon \mathbb{A} \longrightarrow \mathbb{C}$ in Cat($\underline{\mathcal{E}}$), the pointwise left Kan extension of Γ along the unique functor $\Box\colon \mathbb{A} \longrightarrow \mathbb{1}$ is known as the *(global) internal conical* colimit of the diagram Γ, which we denote by $\varinjlim_{\mathbb{A}} \Gamma$. In other words

$$\varinjlim_{\mathbb{A}} \Gamma\colon \mathbb{1} \longrightarrow \mathbb{C}$$

which corresponds to a global point of $\mathbb{C}_0$, is the essentially unique functor such that there exists a right (not left since $(-)_*$ is contra variant on 2-cells) Kan extension diagram

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "\Box_*", "|" marking] \arrow[dr, "\Gamma_*"', "/" marking] & \arrow[d, phantom, "\overset{\Gamma}{\Leftarrow}"] & \mathbb{1} \arrow[dl, "(\varinjlim_{\mathbb{A}} \Gamma)_*", "/" marking] \\ & \mathbb{C} & \end{tikzcd} \tag{2.4}$$

in $\mathrm{Prof}(\underline{\mathcal{E}})$. So if $X\colon \mathbb{A} \longrightarrow\!\!\!\!\!+\ \mathtt{1}$ is a weight then we see that

$$\mathrm{colim}\,(X,\Gamma) \leftrightarrows\cong \varinjlim_{\mathbb{G}(X)} (\Gamma \circ \bar{x}_1)$$

and in this sense we have succeeded in reducing all weighted colimits to conical ones. Of course when considering internal category theory we are not only interested in colimits weighted by profunctors with codomain $\mathtt{1}$, but these will be enough when we get round to considering enriched colimit internally.

Let us now turn to consider this material in the context of the previous two sections. To avoid confusion we will write categories in $\mathrm{Cat}(\widetilde{\underline{\mathcal{C}}})$ (or $\mathrm{Cat}(\widetilde{\underline{\mathcal{C}}}_{\mathrm{l}})$) as open face characters ($\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ etc.) and $\underline{\mathcal{A}}$-enriched categories as underlined bold face capitals ($\underline{\mathbf{A}}$, $\underline{\mathbf{B}}$, $\underline{\mathbf{C}}$ etc.).

Observation 2.3.4 Using the inclusion of equipments

$$\underline{\mathcal{A}}_{\mathrm{l}}\text{–EQUIP} \xrightarrow{\ \ddot{\mathrm{I}}^\star \dashv \ddot{\mathrm{I}}_\star\ } \mathrm{Equip}(\widetilde{\underline{\mathcal{C}}}_{\mathrm{l}})$$

as derived in section 2.2 and our work on the Grothendieck construction we may represent all weighted colimits in $\underline{\mathcal{A}}$-category theory as internal conical ones. Suppose $X\colon \underline{\mathbf{A}} \longrightarrow\!\!\!\!\!+\ \underline{\mathbf{1}}$ is a weight in $\underline{\mathcal{A}}$–Equip, and $\Theta\colon \underline{\mathbf{A}} \longrightarrow \underline{\mathbf{C}}$ a diagram in an arbitrary (possibly large) $\underline{\mathcal{A}}$-category $\underline{\mathbf{C}}$. By lemma 1.7.7(ii) we get

$$\mathrm{colim}\left(\bar{\mathrm{I}}_\star X, \mathrm{I}_\star\Theta\right) \leftrightarrows\cong \mathrm{I}_\star\,(\mathrm{colim}\,(X,\Theta))$$

then applying the Grothendieck construction to $\bar{\mathrm{I}}_\star X$ and noting that $\mathrm{I}_\star(\underline{\mathbf{1}}) \cong \mathtt{1}$ we obtain a span

$$\begin{tikzcd} & \mathbb{G}(\bar{\mathrm{I}}_\star) \arrow[dl, "\square"'] \arrow[dr, "p"] & \\ \mathtt{1} & & \mathrm{I}_\star(\underline{\mathbf{A}}) \end{tikzcd}$$

with $X \cong \square_* \otimes p^*$ therefore:

$$\varinjlim_{\mathbb{G}(\bar{\mathrm{I}}_\star X)} ((\mathrm{I}_\star\Theta)\circ p) \leftrightarrows\cong \mathrm{colim}\left(\bar{\mathrm{I}}_\star X, \mathrm{I}_\star\Theta\right).$$

Composing these bidirectional congruences we capture the $\underline{\mathcal{A}}$-colimit of Θ weighted by X as the internal conical colimit $\varinjlim_{\mathbb{G}(\bar{\mathrm{I}}_\star X)} ((\mathrm{I}_\star\Theta)\circ p)$

In the special case of 2-category theory this much was certainly known to Paré when he gave his talk [38], the difficulty then was reversing the process and constructing a 2-weight from a double category. Again we may resolve this problem using our inclusion, given a diagram $\Gamma\colon \mathbb{A} \longrightarrow \mathrm{I}_\star\underline{\mathbf{C}}$ in $\mathrm{Cat}(\widetilde{\underline{\mathcal{C}}}_{\mathrm{l}})$, with $\mathbb{A} \in \mathrm{Cat}(\widetilde{\underline{\mathcal{C}}})$, proposition 1.7.7(i) tells us that:

$$\mathrm{colim}\left(\bar{\mathrm{I}}^\star(\square_*), \hat{\Gamma}\right) \leftrightarrows\cong \left(\varinjlim_{\mathbb{A}} \Gamma\right)^{\wedge} \tag{2.5}$$

In other words if $\mathbb{A}$ is a category in $\mathrm{Cat}(\widetilde{\underline{\mathcal{C}}})$ then $\bar{\mathrm{I}}^\star(\square_*)\colon \mathrm{I}^\star\mathbb{A} \longrightarrow\!\!\!\!\!+\ \mathrm{I}^\star\mathtt{1} \cong \underline{\mathbf{1}}$ is the $\underline{\mathcal{A}}$-weight for the internal conical colimit $\varinjlim_{\mathbb{A}}$. $\square$

2.4 Colimits in categories internal to $\underset{\sim}{\mathcal{A}}$.

Before going any further we should check that Paré's notion of Double (co-)Limit, as adumbrated in [38], tallies with the conical colimit notion introduced in the last section.

In the last section (and observation 1.2.11) we encountered the close relationship between colimits and right Kan extensions of profunctors, so in order to get a better idea of what an internal conical colimit looks like we should think a little more about these extensions in $\mathrm{Prof}(\widetilde{\underset{\sim}{\mathcal{C}}}_1)$. Let

$$\begin{tikzcd} \mathbb{A} \arrow[r, "X", "\shortmid"{marking}] \arrow[dr, "Y"', "\shortmid"{marking}] & \mathbb{B} \\ & \mathbb{C} \end{tikzcd}$$

be profunctors in $\mathrm{Prof}(\widetilde{\underset{\sim}{\mathcal{C}}}_1)$ then observation 1.2.9 describes symbolically the right Kan extension $Y \underset{\mathbb{B}}{\Leftarrow} X$ of Y along X. It follows from (1.30) that a morphism $\delta\colon T \longrightarrow |Y \underset{\mathbb{B}}{\Leftarrow} X|$ in $\widetilde{\underset{\sim}{\mathcal{C}}}$, into the underlying $\underset{\sim}{\mathcal{C}}$-set of $Y \underset{\mathbb{B}}{\Leftarrow} X$, corresponds to:

a span $\begin{tikzcd} & T \arrow[dl, "t_0"'] \arrow[dr, "t_1"] & \\ \mathbb{B}_0 & & \mathbb{C}_0 \end{tikzcd}$ and a map $T \underset{\mathbb{B}_0}{\times} X \xrightarrow{\hat{\delta}} Y$ in $\mathrm{Span}(\widetilde{\underset{\sim}{\mathcal{C}}}_1)(\mathbb{A}_0, \mathbb{C}_0)$

such that the diagram $\begin{tikzcd} T \underset{\mathbb{B}_0}{\times} X \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \arrow[r, "\hat{\delta} \underset{\mathbb{A}_0}{\times} \mathbb{A}_1"] \arrow[d, "T \underset{\mathbb{B}_0}{\times} r_X"'] & Y \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \arrow[d, "r_Y"] \\ T \underset{\mathbb{B}_0}{\times} X \arrow[r, "\hat{\delta}"'] & Y \end{tikzcd}$ commutes.

In the particular case of (2.4) this reduces to a natural bijection between morphisms $\delta\colon T \longrightarrow |\Gamma_* \underset{1}{\Leftarrow} \Box_*|$ and pairs of maps

$$c\colon T \longrightarrow \mathbb{C}_0 \quad \text{and} \quad \hat{\delta}\colon T \times \mathbb{A}_0 \longrightarrow \mathbb{C}_1$$

such that

$$\begin{tikzcd} T \times \mathbb{A}_0 \arrow[r, "\pi_{\mathbb{A}_0}"] \arrow[d, "\hat{\delta}"'] \arrow[dr, phantom, "\text{(a)}"] & \mathbb{A}_0 \arrow[d, "\Gamma_0"] \\ \mathbb{C}_1 \arrow[r, "d_1"'] & \mathbb{C}_0 \end{tikzcd} , \quad \begin{tikzcd} T \times \mathbb{A}_0 \arrow[r, "\pi_T"] \arrow[d, "\hat{\delta}"'] \arrow[dr, phantom, "\text{(b)}"] & T \arrow[d, "c"] \\ \mathbb{C}_1 \arrow[r, "d_0"'] & \mathbb{C}_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \times \mathbb{A}_1 \arrow[r, "\varpi"] \arrow[d, "T \times d_1"'] \arrow[dr, phantom, "\text{(c)}"] & \mathbb{C}_1 \underset{\mathbb{C}_0}{\times} \mathbb{C}_1 \arrow[d, "\circ"] \\ T \times \mathbb{A}_0 \arrow[r, "\hat{\delta}"'] & \mathbb{C}_1 \end{tikzcd} \tag{2.6}$$

commute. The map ϖ in square (c) is given symbolically by

$$\begin{array}{ccc} T \times \mathbb{A}_1 & \xrightarrow{\quad\varpi\quad} & \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \\ \langle t, \alpha \rangle & \longmapsto & \langle \hat{\delta}(t, d_0\alpha), \Gamma_1\alpha \rangle \end{array}$$

which, assuming that square (a) commutes, is a well defined map into $\mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1$. We refer to such pairs $(c, \hat{\delta})$ as "cones below Γ defined at stage T".

From this description the following are apparent:

(i) Let $l\colon 1 \longrightarrow \mathbb{C}_0$ be a morphism in $\widetilde{\underline{\mathcal{C}}}_l$, then a natural transformation

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "\square"] \arrow[dr, "\Gamma"'] & \arrow[d, phantom, "\overset{\gamma}{\Rightarrow}"] & 1 \arrow[dl, "l"] \\ & \mathbb{C} & \end{tikzcd} \tag{2.7}$$

in $\mathrm{Cat}(\widetilde{\underline{\mathcal{C}}}_l)$ clearly corresponds to a morphism $\hat{\gamma}\colon \mathbb{A}_0 \longrightarrow \mathbb{C}_1$ such that

$$\text{(a)}\ \begin{tikzcd} & \mathbb{A}_0 \arrow[dl, "\hat{\gamma}"'] \arrow[d, "\Gamma_0"] \\ \mathbb{C}_1 \arrow[r, "d_1"'] & \mathbb{C}_0 \end{tikzcd}, \quad \begin{tikzcd} \mathbb{A}_0 \arrow[r, "\square"] \arrow[d, "\hat{\gamma}"'] \arrow[dr, phantom, "\text{(b)}"] & 1 \arrow[d, "l"] \\ \mathbb{C}_1 \arrow[r, "d_0"'] & \mathbb{C}_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{A}_1 \arrow[r, "\langle \hat{\gamma} d_0, \Gamma_1 \rangle"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\text{(c)}"] & \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \arrow[d, "\circ"] \\ \mathbb{A}_0 \arrow[r, "\hat{\gamma}"'] & \mathbb{C}_1 \end{tikzcd} \tag{2.8}$$

commute.

Of course these say no more than that $\hat{\gamma}$ is a cone with domain Γ and codomain l defined at 1.

(ii) The representable l_* has underlying span $\mathbb{C}_1 \times_{\mathbb{C}_0} 1$, so a morphism $\phi\colon T \longrightarrow |l_*|$ corresponds to some $\hat{\phi}\colon T \longrightarrow \mathbb{C}_1$ such that

$$\begin{tikzcd} T \arrow[r, "\hat{\phi}"] \arrow[d, "\square"'] & \mathbb{C}_1 \arrow[d, "d_1"] \\ 1 \arrow[r, "l"'] & \mathbb{C}_0 \end{tikzcd}$$

commutes. In other words this is simply a "map in $\mathbb{C}$ defined at T" with domain l.

(iii) The 2-cell in 2.7, induces a unique equivariant map $\bar{\gamma}\colon l_* \longrightarrow (\Gamma_* \underset{1}{\Leftarrow} \square_*)$, and it is a matter of a routine calculation that the composite

$$T \xrightarrow{\quad\phi\quad} |l_*| \xrightarrow{\quad|\bar{\gamma}|\quad} |\Gamma_* \underset{1}{\Leftarrow} \square_*|$$

corresponds to a pair

$$T \xrightarrow{\hat\phi} \mathbb{C}_1 \xrightarrow{d_0} \mathbb{C}_0 \qquad , \qquad \begin{array}{ccccc} T \times \mathbb{A}_0 & \longrightarrow & \mathbb{C}_1 \underset{\mathbb{C}_0}{\times} \mathbb{C}_1 & \xrightarrow{\circ} & \mathbb{C}_1 \\ \langle t, a \rangle & \longmapsto & \langle \hat\phi t, \hat\gamma a \rangle & \longmapsto & \hat\phi t \circ \hat\gamma a \end{array}$$

satisfying the equations of (2.6). In effect we are composing the "cone" of (i) with the "map" of (ii), and we denote this "cone" by $\phi \circ (l, \hat\gamma)$.

We now get a more concrete interpretation of $\varinjlim_{\mathbb{A}} \Gamma$. Suppose $\mathcal{G} \subset \underline{\widetilde{\mathcal{C}}}_1$ is a strongly generating class of objects in $\underline{\widetilde{\mathcal{C}}}$ then, by the definition of that notion, we know that $|\bar\gamma|$ is an isomorphism if and only if for each $T \in \mathcal{G}$ the function

$$\underline{\widetilde{\mathcal{C}}}_1(T, |l_*|) \xrightarrow{\underline{\widetilde{\mathcal{C}}}_1(T, |\bar\gamma|)} \underline{\widetilde{\mathcal{C}}}_1(T, |\Gamma_* \underset{1}{\Leftarrow} \Box_*|) \tag{2.9}$$

is a bijection. Of course an equivariant map $\bar\gamma$ is isomorphic in $\mathrm{Prof}(\underline{\widetilde{\mathcal{C}}}_1)(1, \mathbb{C})$ exactly when its underlying map $|\bar\gamma| \in \underline{\widetilde{\mathcal{C}}}_1$ is, therefore we may check $l \cong \varinjlim_{\mathbb{A}} \Gamma$ by demonstrating that each instance of (2.9) is a bijection. In more evocative language saying that $(l, \hat\gamma)$ is the colimiting cone simply means that each cone $(c, \hat\delta)$ defined at some $T \in \mathcal{G}$ factors as $\phi \circ (l, \hat\gamma)$ for a unique map ϕ of $\mathbb{C}$ which is also defined at stage T.

(iv) Re-interpreting (2.6) we see that we may construct $|\Gamma_* \underset{1}{\Leftarrow} \Box_*|$ as follows: Start with $\mathbb{C}_0 \times \mathbb{C}_1^{\mathbb{A}_0}$ and impose conditions 2.6(a),(b) via an equaliser

$$E \rightarrowtail \mathbb{C}_0 \times \mathbb{C}_1^{\mathbb{A}_0} \rightrightarrows \mathbb{C}_0^{\mathbb{A}_0} \times \mathbb{C}_0^{\mathbb{A}_0}$$

and obtain $|\Gamma_* \underset{1}{\Leftarrow} \Box_*|$ as the equaliser of a pair

$$E \rightrightarrows \mathbb{C}_1^{\mathbb{A}_1}$$

which imposes condition 2.6(c).

So if $\mathbb{C} \in \mathrm{Cat}(\underline{\mathcal{A}}_1)$ then by theorem 2.1.4 we know that $\mathbb{C}_0^{\mathbb{A}_0}, \mathbb{C}_1^{\mathbb{A}_1}$ are both objects of $\underline{\mathcal{A}}_1$. As this is closed in $\underline{\widetilde{\mathcal{C}}}_1$ under finite limits, it follows that E and $|\Gamma_* \underset{1}{\Leftarrow} \Box_*|$ are also objects of $\underline{\mathcal{A}}_1$.

This enables us to strengthen slightly the comment at the end of (iii). Under the condition that $\mathbb{C} \in \mathrm{Cat}(\underline{\mathcal{A}}_1)$, both $|\Gamma_* \underset{1}{\Leftarrow} \Box_*|$ and the underlying $\underline{\mathcal{C}}$-set of any representable $l_* \colon 1 \to\!\!\!\!\!+ \mathbb{C}$ is in $\underline{\mathcal{A}}_1$. It is therefore enough, in this case, to insist that $\mathcal{G}$ is a strong generator in $\underline{\mathcal{A}}_1$.

Finally we are in a position to give a simple description of our limit notion in the case of double categories.

When working with a double category $\mathbb{C}$ (an object in Cat($\underline{\mathrm{CAT}}$)) we refer to the objects and morphisms of $\mathbb{C}_0$ as its objects and *vertical* morphisms, and those of $\mathbb{C}_1$ as *horizontal* and *double* morphisms. We draw double cells as

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, dotted, "v"'] \arrow[dr, phantom, "\Downarrow\alpha"] & a' \arrow[d, dotted, "v'"] \\ \bar{a} \arrow[r, "\bar{h}"'] & \bar{a}' \end{tikzcd}$$

where v, v' are its horizontal domain and codomain with h, $\bar{h}$ its vertical ones, for which we introduce the notations $\mathrm{dom}_H(\alpha)$, $\mathrm{cod}_H(\alpha)$ and $\mathrm{dom}_V(\alpha)$, $\mathrm{cod}_V(\alpha)$ respectively. It will also be useful to adopt Paré's convention of displaying vertical morphisms as dotted arrows to distinguish them from horizontal ones.

The various category structures of $\mathbb{C}$ find their expression in this diagrammatic form as vertical and horizontal composite of (compatible) squares, each of these being associating and possessing identities. We use $\circ$ and $\bullet$ to denote horizontal and vertical composition respectively, reserving i and j for their corresponding identities. These must satisfy a "middle four interchange rule", in other words composing horizontally first and then vertically is the same as doing so vertically then horizontally, and a compatibility condition for identities expressed by $j_{i_c} = i_{j_c}$ for all objects $c \in \mathbb{C}$.

Take for our strong generator in $\underline{\mathrm{CAT}}$ the set $\mathcal{G} = \{\mathtt{1}, \mathtt{2}\}$, where

$$\begin{aligned} \mathtt{1} &= \boxed{\quad\bullet\quad} \\ \mathtt{2} &= \boxed{\bullet \longrightarrow \bullet} \end{aligned} \tag{2.10}$$

and let $\Gamma : \mathbb{A} \longrightarrow \mathbb{C}$ be a diagram of double categories.

A cone under Γ defined at $\mathtt{1}$ is exactly what Paré (in [38]) refers to as a "horizontal (co-)cone" and is given by the following data in $\mathbb{C}$

$$\Gamma(a) \xrightarrow{\delta_a} c$$

One for each object a of $\mathbb{A}$. These are natural with respect to the horizontal maps of $\mathbb{A}$.

$$\begin{tikzcd}
\Gamma(a) \arrow[dr, "\delta_a"] \arrow[dd, dotted, "\Gamma(v)"'] & \\
\arrow[r, phantom, "\Downarrow \delta_v"] & c \\
\Gamma(\bar{a}) \arrow[ur, "\delta_{\bar{a}}"'] &
\end{tikzcd}$$

One for each vertical morphism v of $\mathbb{A}$. Again these must be natural with respect to the double maps of $\mathbb{A}$, but they must also be compatible under vertical composition.

A moments reflection also reveals that what Paré calls a "double (co-)cone" is simply a cone under Γ defined at $\mathtt{2}$.

In [38] we find two conditions that must satisfied for a given horizontal cone to be limiting. These simply state that all other horizontal and double cones factor uniquely through the chosen cone, which is exactly what the concrete condition of (iii) requires. So our limiting concept coincides with Paré's, and we may apply our work on change of base to his notion, which he refers to as a *double (co-)limit*.

2.5 Closed Classes of $\underline{\mathcal{A}}$-Colimits

Having considered relationships between individual $\underline{\mathcal{A}}$-weights and categories internal to $\widetilde{\underline{\mathcal{C}}}$ (or $\underline{\mathcal{A}}$) it now makes sense to extend this to working with entire *closed* classes of weights. In this section we will provide a convenient method for defining such classes in terms of (large) sets of categories in $\mathrm{Cat}(\underline{\mathcal{A}})$, but before doing that we must first impose a new condition on $\underline{\mathcal{A}}$. For the remainder of this chapter we choose to apply condition 2.1.5 to $\mathbb{J}$, the Gabriel theory underlying $\underline{\mathcal{A}}$, viz:

The base of each cone of $\mathbb{J}$ is a diagram on a non-empty connected category.

The following observations motivate our adoption of this blanket assumption:

Observation 2.5.1 (i) In any category $\underline{\mathcal{B}}$, if $\ulcorner B \urcorner \colon \underline{\mathcal{D}} \longrightarrow \underline{\mathcal{B}}$ is a constant diagram on a non-empty connected category, then its limit is the object B itself. It follows that in the presence of condition 2.1.5 any discrete $\underline{\mathcal{C}}$-set is in $\underline{\mathcal{A}}$, furthermore in $\underline{\mathrm{Set}}$ connected limits commute with coproducts therefore $\underline{\mathcal{A}}$ is closed in $\widetilde{\underline{\mathcal{C}}}$ under small coproducts.

(ii) Returning to the construction of the equipment morphism

$$(\underline{\mathcal{A}}\text{–Prof}, \underline{\mathcal{A}}\text{–Cat}, (-)_*) \xrightarrow{\quad \ddot{\mathrm{I}}_\star \quad} (\mathrm{Prof}(\widetilde{\underline{\mathcal{C}}}), \mathrm{Cat}(\widetilde{\underline{\mathcal{C}}}), (-)_*)$$

we see that it may be obtained directly by applying the 2-functor $\mathcal{M}\mathrm{on}(-)$ of proposition 1.6.5 to an equipment homomorphism:

$$(\underline{\mathcal{A}}\text{–Mat}, \underline{\mathrm{Set}}, (-)_\circ) \xrightarrow{\quad \ddot{\mathrm{I}}_\# \quad} (\mathrm{Span}(\widetilde{\underline{\mathcal{C}}}), \widetilde{\underline{\mathcal{C}}}, (-)_\circ)$$

This carries $X \in \underline{\mathrm{Set}}$ to the discrete $\underline{\mathcal{C}}$-set $\Delta(X)$ and a matrix $m \colon X \longrightarrow\!\!\!\!\!\!\ominus\!\!> X'$ to the span

$$\Delta(X') \longleftarrow \coprod_{\substack{x \in X \\ x' \in X'}} m(x, x') \longrightarrow \Delta(X)$$

where the coproduct in question is that calculated in $\widetilde{\underline{\mathcal{C}}}$. In the presence of condition 2.1.5 the last observation implies that $\Delta(X)$, $\Delta(X')$ and $\coprod_{\substack{x \in X \\ x' \in X'}} m(x, x')$ are all in $\underline{\mathcal{A}}$, and so $\ddot{\mathrm{I}}_\#$ restricts to an equipment homomorphism

$$(\underline{\mathcal{A}}\text{–Mat}, \underline{\mathrm{Set}}, (-)_\circ) \xrightarrow{\quad \ddot{\mathrm{I}}_\# \quad} (\mathrm{Span}(\underline{\mathcal{A}}), \underline{\mathcal{A}}, (-)_\circ)$$

and for each pair of sets $X, X' \in \underline{\mathrm{Set}}$ the functor

$$\underline{\mathcal{A}}\text{–Mat}(X, X') \xrightarrow{\quad \bar{\mathrm{I}}_\# \quad} \mathrm{Span}(\underline{\mathcal{A}})(\Delta(X), \Delta(X'))$$

is an equivalence. Of course we could not simply apply $\mathcal{M}\mathrm{on}(-)$ to this since the local coequalisers in $\mathrm{Span}(\underline{\mathcal{A}})$ are not stable under pullback, although it is clear that all small colimits <u>are</u> stable under pullback along maps with codomain a *discrete* $\underline{\mathcal{C}}$-set.

The next two observations are a direct result of this one:

(iii) The action of $\ddot{\mathrm{I}}_\star$ on the category $\underline{\mathcal{A}}$–Cat restricts to

$$\mathrm{I}_\star\colon \underline{\mathcal{A}}\text{–Cat} \longrightarrow \mathrm{Cat}(\underline{\mathcal{A}})$$

the full image of which consists of those categories $\mathbb{A} \in \mathrm{Cat}(\underline{\mathcal{A}})$ such that $\mathbb{A}_0$ is a discrete $\underline{\mathcal{C}}$-set.

(iv) The action of $\ddot{\mathrm{I}}_\star$ on profunctors restricts to an equivalence

$$\bar{\mathrm{I}}_\star\colon \underline{\mathcal{A}}\text{–Prof}(\underline{\mathbf{A}}, \underline{\mathbf{B}}) \xrightarrow{\ \simeq\ } \mathrm{Prof}(\underline{\mathcal{A}})(\mathrm{I}_\star\underline{\mathbf{A}}, \mathrm{I}_\star\underline{\mathbf{B}})$$

wherein if $\mathbb{A}, \mathbb{B} \in \mathrm{Cat}(\underline{\mathcal{A}})$ then $\mathrm{Prof}(\underline{\mathcal{A}})(\mathbb{A}, \mathbb{B})$ is the usual category of profunctors and equivariant maps as defined in $\underline{\mathcal{A}}$. This is identical to the full subcategory of $\mathrm{Prof}(\widetilde{\underline{\mathcal{C}}})(\mathbb{A}, \mathbb{B})$ determined by those profunctors with underlying $\underline{\mathcal{C}}$-set in $\underline{\mathcal{A}}$.

(v) $\underline{\mathcal{A}}$ is closed in $\widetilde{\underline{\mathcal{C}}}$ under the finite limits used in the Grothendieck construction therefore the functor

$$\mathbb{G}\colon \mathrm{Prof}(\widetilde{\underline{\mathcal{C}}})(\mathbb{A}, \mathbb{B}) \longrightarrow \mathrm{Span}(\mathrm{Cat}(\widetilde{\underline{\mathcal{C}}}))(\mathbb{A}, \mathbb{B})$$

restricts to:

$$\mathbb{G}\colon \mathrm{Prof}(\underline{\mathcal{A}})(\mathbb{A}, \mathbb{B}) \longrightarrow \mathrm{Span}(\mathrm{Cat}(\underline{\mathcal{A}}))(\mathbb{A}, \mathbb{B})$$

The full image of $\mathrm{Prof}(\underline{\mathcal{A}})(\mathbb{A}, \mathbb{B})$ under $\mathbb{G}\,(-)$ consists of those two-sided discrete fibrations in $\mathrm{Span}(\mathrm{Cat}(\widetilde{\underline{\mathcal{C}}}))(\mathbb{A}, \mathbb{B})$ with total category in $\mathrm{Cat}(\underline{\mathcal{A}})$.

Furthermore these all hold when interpreted in <u>SET</u>, in other words they are true for the inclusion $\ddot{\mathrm{I}}^\star \dashv \ddot{\mathrm{I}}_\star\colon \underline{\mathcal{A}}_{\mathrm{l}}\text{–EQUIP} \longrightarrow \mathrm{Equip}(\widetilde{\underline{\mathcal{C}}}_{\mathrm{l}})$. □

As an example of the importance of condition 2.1.5 let us, for a moment, return to the situation at the end of section 2.3. We were given a weight $X\colon \underline{\mathbf{A}} \longrightarrow\!\!\!\!\!+\ \underline{\mathbf{1}}$ in $\underline{\mathcal{A}}$–Equip and a diagram $\Theta\colon \underline{\mathbf{A}} \longrightarrow \underline{\mathbf{C}}$, for which we saw that:

$$\mathrm{I}_\star(\mathrm{colim}\,(X, \Theta)) \cong \varinjlim_{\mathbb{G}(\bar{\mathrm{I}}_\star X)}((\mathrm{I}_\star\Theta) \circ \bar{x}_1)$$

Having applied our new blanket assumption, observation 2.5.1(iii) tells us that $\mathrm{I}_\star\underline{\mathbf{A}} \in \mathrm{Cat}(\underline{\mathcal{A}})$ and $\mathrm{I}_\star\underline{\mathbf{C}} \in \mathrm{Cat}(\underline{\mathcal{A}}_{\mathrm{l}})$ furthermore 2.5.1(iv), (v) together imply that $\mathbb{G}\left(\bar{\mathrm{I}}_\star X\right) \in \mathrm{Cat}(\underline{\mathcal{A}})$. So the internal conical colimit in the display above is of exactly the kind analysed in section 2.4. In particular when $\underline{\mathcal{A}} = \underline{\mathrm{Cat}}$ this is just the double co-limit of $(\mathrm{I}_\star\Theta) \circ \bar{x}_1$.

In recasting the theory of closed classes of $\underline{\mathcal{A}}$-colimits, our principal method will be to represent each $\underline{\mathcal{A}}$-category $\mathcal{P}(\underline{\mathbf{A}})$, the enriched category of presheaves on $\underline{\mathbf{A}}$ defined in section 2.1, as a full subcategory of some $\underline{\mathcal{A}}$-enriched version of the slice category $\mathrm{Cat}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$. As an indication of how to do this notice that the "honest" category underlying $\mathcal{P}(\underline{\mathbf{A}})$ is in fact just $\underline{\mathcal{A}}$–$\mathrm{Prof}(\underline{\mathbf{A}}, \underline{\mathbf{1}})$, but observation 2.5.1 points out that we may restrict the composite

$$\underline{\mathcal{A}}\text{–}\mathrm{Prof}(\underline{\mathbf{A}}, \underline{\mathbf{1}}) \xrightarrow{\bar{\mathrm{I}}_\star} \mathrm{Prof}(\widetilde{\underline{\mathcal{C}}})(\mathrm{I}_\star\underline{\mathbf{A}}, \mathrm{I}_\star\underline{\mathbf{1}}) \xrightarrow{\mathbb{G}} \mathrm{Span}(\mathrm{Cat}(\widetilde{\underline{\mathcal{C}}}))(\mathrm{I}_\star\underline{\mathbf{A}}, \mathrm{I}_\star\underline{\mathbf{1}}) \tag{2.11}$$

to a functor

$$\underline{\mathcal{A}}\text{–}\mathrm{Prof}(\underline{\mathbf{A}}, \underline{\mathbf{1}}) \xrightarrow[\sim]{\bar{\mathrm{I}}_\star} \mathrm{Prof}(\underline{\mathcal{A}})(\mathrm{I}_\star\underline{\mathbf{A}}, \mathrm{I}_\star\underline{\mathbf{1}}) \xrightarrow{\mathbb{G}} \mathrm{Span}(\mathrm{Cat}(\underline{\mathcal{A}}))(\mathrm{I}_\star\underline{\mathbf{A}}, \mathrm{I}_\star\underline{\mathbf{1}}) \tag{2.12}$$

which we call $\mathbb{G}_{\underline{\mathcal{A}}}$. Of course $\mathrm{I}_\star\underline{\mathbf{1}} \cong \mathbb{1}$ therefore $\mathrm{Span}(\mathrm{Cat}(\underline{\mathcal{A}}))(\mathrm{I}_\star\underline{\mathbf{A}}, \mathrm{I}_\star\underline{\mathbf{1}})$ is simply the category $\mathrm{Cat}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$. Furthermore, by observation 2.5.1, $\mathbb{G}_{\underline{\mathcal{A}}}$ is fully faithful with full image consisting of the discrete fibrations over $\mathrm{I}_\star\underline{\mathbf{A}}$. It seems natural therefore to take $\mathbb{G}_{\underline{\mathcal{A}}}$ as the action of our "$\underline{\mathcal{A}}$-enriched Grothendieck construction" on the underlying category of $\mathcal{P}(\underline{\mathbf{A}})$.

A point worth comment is that both of the functors in (2.11) admit left adjoints, as demonstrated in observation 2.3.2 for $\mathbb{G}$ and proposition 1.5.14(ii) for $\bar{\mathrm{I}}_\star$. So $\mathbb{G}_{\underline{\mathcal{A}}}$ has a left adjoint $\mathbf{L}_{\underline{\mathcal{A}}}$, constructed by restricting the composite of these two left adjoints.

Observation 2.5.2 It will be important later on to know more about $\underline{\mathcal{A}}$-colimits weighted by the profunctor obtained by applying $\mathbf{L}_{\underline{\mathcal{A}}}$ to a given object $p\colon \mathbb{E} \longrightarrow \mathrm{I}_\star\underline{\mathbf{A}}$ of $\mathrm{Cat}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$. Examining proposition 1.5.14 and observation 2.3.2 we see that this weight is calculated as follows

$$\begin{aligned}
\mathbf{L}_{\underline{\mathcal{A}}}\begin{pmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathrm{I}_\star\underline{\mathbf{A}}\end{pmatrix} &\cong \Phi_{\underline{\mathbf{1}}} \otimes \bar{\mathrm{I}}^\star(\square_* \otimes p^*) \otimes \Phi^*_{\underline{\mathbf{A}}} && \text{where } \Phi \text{ is the counit of } \ddot{\mathrm{I}}^\star \dashv \ddot{\mathrm{I}}_\star \\
&\cong \Phi_{\underline{\mathbf{1}}} \otimes \bar{\mathrm{I}}^\star(\square_*) \otimes (\mathrm{I}^\star p)^* \otimes \Phi^*_{\underline{\mathbf{A}}} && \text{since } \ddot{\mathrm{I}}^\star \text{ is a comorphism of equipments} \\
&\cong \Phi_{\underline{\mathbf{1}}} \otimes \bar{\mathrm{I}}^\star(\square_*) \otimes (\Phi_{\underline{\mathbf{A}}} \circ \mathrm{I}^\star p)^* \\
&\cong \Phi_{\underline{\mathbf{1}}} \otimes \bar{\mathrm{I}}^\star(\square_*) \otimes \hat{p}^*
\end{aligned}$$

but $\Phi_{\underline{1}}$ is an isomorphism so for any $\underline{\mathcal{A}}$-enriched functor $\Theta\colon \mathbf{A} \longrightarrow \underline{\mathbf{C}}$ we get:

$$\begin{aligned}
\operatorname{colim}\left(\mathbf{L}_{\underline{\mathcal{A}}}\left(\begin{array}{c}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathrm{I}_\star\underline{\mathbf{A}}\end{array}\right),\Theta\right) &\cong \operatorname{colim}\left(\bar{\mathrm{I}}^\star(\Box_*)\otimes\hat{p}^*,\Theta\right)\\
&\cong \operatorname{colim}\left(\bar{\mathrm{I}}^\star(\Box_*),\Theta\circ\hat{p}\right) && \text{since } \hat{p}_*\dashv\hat{p}^*\\
&\cong \left(\varinjlim_{\mathbb{E}}((\mathrm{I}_\star\Theta)\circ p)\right)^{\wedge} && \text{as shown in (2.5) at the end of section 2.3.}
\end{aligned}$$

The next few propositions establish some important properties of $\mathbf{L}_{\underline{\mathcal{A}}}\dashv\mathbb{G}_{\underline{\mathcal{A}}}$, leading to corollary 2.5.9 which says that it enriches to an $\underline{\mathcal{A}}$-adjunction.

Proposition 2.5.3 *The functor $\mathbb{G}_{\underline{\mathcal{A}}}\colon\mathcal{P}(\underline{\mathbf{A}})_0 \longrightarrow \mathrm{Cat}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$ preserves (small) colimits.*

Proof. Let $\mathbb{A}\overset{\text{def}}{=}\mathrm{I}_\star\underline{\mathbf{A}}$ and $r(\mathbb{A})\overset{\text{def}}{=}\mathrm{Prof}(\underline{\mathcal{A}})(\mathbb{A},\mathbb{1})$ then we know from observation 2.5.1(iv) that $r(\mathbb{A})\simeq\mathcal{P}(\underline{\mathbf{A}})_0$. Now $r(\mathbb{A})$ is the category of algebras for a monad on $\underline{\mathcal{A}}/_{\mathbb{A}_0}$ with functor part $(-\underset{\mathbb{A}_0}{\times}\mathbb{A}_1)$, and this preserves small colimits by observation 2.5.1(ii) and the fact that $\mathbb{A}_0$ is a discrete $\underline{\mathcal{C}}$-set. It follows that (small) colimits in $r(\mathbb{A})$ are calculated as in the underlying category $\underline{\mathcal{A}}/_{\mathbb{A}_0}$ (as usual), or in other words the forgetful functor $\mathrm{U}_{\mathbb{A}}\colon r(\mathbb{A})\longrightarrow\underline{\mathcal{A}}/_{\mathbb{A}_0}$ creates them.

Referring back to (2.12) we need to show that the restriction of $\mathbb{G}$ to

$$r(\mathbb{A})\overset{\text{def}}{=}\mathrm{Prof}(\underline{\mathcal{A}})(\mathbb{A},\mathbb{1})\xrightarrow{\ \mathbb{G}\ }\mathrm{Span}(\mathrm{Cat}(\underline{\mathcal{A}}))(\mathbb{A},\mathbb{1})\cong\mathrm{Cat}(\underline{\mathcal{A}})/_{\mathbb{A}}$$

preserves small colimits, but in fact it is enough to check this for the functor obtained by composing $\mathbb{G}$ with

$$\mathrm{Cat}(\underline{\mathcal{A}})/_{\mathbb{A}}\xrightarrow{\ \Sigma_{\mathbb{A}}\ }\mathrm{Cat}(\underline{\mathcal{A}})\xhookrightarrow{\ \mathrm{N}\ }[\mathbf{\Delta}^{\mathrm{op}},\underline{\mathrm{Set}}]$$

simply because $\Sigma_{\mathbb{A}}$ creates colimits and N (the "nerve" functor) is fully faithful. It is a matter of direct calculation to check that this composite functor maps a right $\mathbb{A}$-set $X\in r(\mathbb{A})$ to a simplicial set of the form

$$\cdots\cdots X\underset{\mathbb{A}_0}{\times}\mathbb{A}_1^{[n]}\ \begin{array}{c}\longrightarrow\\ \vdots\\ \longrightarrow\end{array}\ \cdots\cdots\ \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array}\ X\underset{\mathbb{A}_0}{\times}\mathbb{A}_1^{[2]}\ \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array}\ X\underset{\mathbb{A}_0}{\times}\mathbb{A}_1\ \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array}\ X$$

and an equivariant map $\theta\colon X\longrightarrow Y\in r(\mathbb{A})$ to the natural transformation depicted by:

$$\begin{array}{ccccccccc}
\cdots\cdots X\underset{\mathbb{A}_0}{\times}\mathbb{A}_1^{[n]} & \begin{array}{c}\longrightarrow\\ \vdots\\ \longrightarrow\end{array} & \cdots\cdots & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & X\underset{\mathbb{A}_0}{\times}\mathbb{A}_1^{[2]} & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & X\underset{\mathbb{A}_0}{\times}\mathbb{A}_1 & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & X\\
{\scriptstyle\theta\underset{\mathbb{A}_0}{\times}\mathbb{A}_1^{[n]}}\Big\downarrow & & & & {\scriptstyle\theta\underset{\mathbb{A}_0}{\times}\mathbb{A}_1^{[2]}}\Big\downarrow & & {\scriptstyle\theta\underset{\mathbb{A}_0}{\times}\mathbb{A}_1}\Big\downarrow & & {\scriptstyle\theta}\Big\downarrow\\
\cdots\cdots Y\underset{\mathbb{A}_0}{\times}\mathbb{A}_1^{[n]} & \begin{array}{c}\longrightarrow\\ \vdots\\ \longrightarrow\end{array} & \cdots\cdots & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & Y\underset{\mathbb{A}_0}{\times}\mathbb{A}_1^{[2]} & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & Y\underset{\mathbb{A}_0}{\times}\mathbb{A}_1 & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & Y
\end{array}\tag{2.13}$$

We should explain that in both of these:

$$\mathbb{A}_1^{[n]} \stackrel{\text{def}}{=} \underbrace{\mathbb{A}_1 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \underset{\mathbb{A}_0}{\times} \cdots\cdots\cdots \underset{\mathbb{A}_0}{\times} \mathbb{A}_1}_{n \text{ factors}}$$

So consider a diagram $\Gamma\colon \underline{\mathcal{B}} \longrightarrow r(\mathbb{A})$ with $\underline{\mathcal{B}}$ small, and $\amalg\colon \Gamma \stackrel{\cdot}{\longrightarrow} \varinjlim_{\underline{\mathcal{B}}} \Gamma$ a colimiting cone. The $\underline{\mathcal{C}}$-set $\mathbb{A}_0$ is discrete, therefore for each $n \in \mathbf{\Delta}$ we know that $(-\underset{\mathbb{A}_0}{\times} \mathbb{A}_1^{[n]})\colon \underline{\mathcal{A}}/_{\mathbb{A}_0} \longrightarrow \underline{\mathcal{A}}/_{\mathbb{A}_0}$ preserves colimits, by observation 2.5.1(ii). In particular

$$\left(\mathrm{U}_{\mathbb{A}}(\amalg) \underset{\mathbb{A}_0}{\times} \mathbb{A}_1^{[n]}\right) : (\mathrm{U}_{\mathbb{A}} \circ \Gamma) \underset{\mathbb{A}_0}{\times} \mathbb{A}_1^{[n]} \stackrel{\cdot}{\longrightarrow} \mathrm{U}_{\mathbb{A}}(\varinjlim_{\underline{\mathcal{B}}} \Gamma) \underset{\mathbb{A}_0}{\times} \mathbb{A}_1^{[n]} \tag{2.14}$$

is colimiting since the forgetful functor $\mathrm{U}_{\mathbb{A}}$ creates small colimits. Consulting our calculation of the action of $\mathrm{N} \circ \Sigma_{\mathbb{A}} \circ \mathbb{G}$ on right actions and equivariant maps, as displayed in (2.13), it is clear that the cone in (2.14) is simply that obtained by applying $\mathrm{ev}_n \circ \mathrm{N} \circ \Sigma_{\mathbb{A}} \circ \mathbb{G}$ to the cone $\amalg$, where

$$[\mathbf{\Delta}^{\mathrm{op}}, \underline{\mathrm{Set}}] \xrightarrow{\ \mathrm{ev}_n\ } \underline{\mathrm{Set}}$$

is the functor "evaluation at $n \in \mathbf{\Delta}$". Of course colimits in $[\mathbf{\Delta}^{\mathrm{op}}, \underline{\mathrm{Set}}]$ are calculated pointwise and it follows that $\mathrm{N} \circ \Sigma_{\mathbb{A}} \circ \mathbb{G}(\amalg)$ is colimiting thus so is $\mathbb{G}(\amalg)$ (in $\mathrm{Cat}(\underline{\mathcal{A}})/_{\mathbb{A}}$) as required. $\square$

We now proceed with the process of enriching $\mathbf{L}_{\underline{\mathcal{A}}} \dashv \mathbb{G}_{\underline{\mathcal{A}}}$ to an $\underline{\mathcal{A}}$-adjunction, by first providing an enrichment of the category $\mathrm{Cat}(\underline{\mathcal{A}})$. Of course, by theorem 2.1.4, $\mathrm{Cat}(\underline{\mathcal{A}})$ is a cartesian closed category, and so it has a natural enrichment to an $\mathrm{Cat}(\underline{\mathcal{A}})$-enriched category (cf. [30]). In future, in order to prevent confusion, we will reserve $\mathrm{Cat}(\underline{\mathcal{A}})$ for this $\mathrm{Cat}(\underline{\mathcal{A}})$-enriched category and use $\mathrm{Cat}_0(\underline{\mathcal{A}})$ for its unenriched (honest) underlying category. The next proposition describes explicitly an $\underline{\mathcal{A}}$-enriched version of $\mathrm{Cat}_0(\underline{\mathcal{A}})$, with the observation which follows showing how we may derive this $\underline{\mathcal{A}}$-enrichment, in a more abstract fashion, directly from the canonical $\mathrm{Cat}(\underline{\mathcal{A}})$-enrichment.

Proposition 2.5.4 *The category* $\mathrm{Cat}_0(\underline{\mathcal{A}})$ *has a natural* $\underline{\mathcal{A}}$*-enrichment* $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$*, under which it admits all* $\underline{\mathcal{A}}$*- tensors and cotensors.*

Proof. Let the "homset" $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})(\mathbb{A}, \mathbb{B}) \in \underline{\mathcal{A}}$ be the subobject of $\mathbb{B}_1^{\mathbb{A}_1} \times \mathbb{B}_0^{\mathbb{A}_0}$, given symbolically by:

$$\left\{ \begin{array}{l} \langle f_1, f_0 \rangle \in \\ \qquad \mathbb{B}_1^{\mathbb{A}_1} \times \mathbb{B}_0^{\mathbb{A}_0} \end{array} \left| \begin{array}{l} (\forall a \in \mathbb{A}_0)(i f_0(a) = f_1 i(a)) \wedge \\ (\forall \alpha \in \mathbb{A}_1)(d_0 f_1(\alpha) = f_0 d_0(\alpha) \wedge d_1 f_1(\alpha) = f_0 d_1(\alpha)) \wedge \\ (\forall \alpha, \alpha' \in \mathbb{A}_1)((d_1 \alpha' = d_0 \alpha) \Rightarrow (f_1(\alpha' \circ \alpha) = f_1(\alpha') \circ f_1(\alpha)) \end{array} \right. \right\}$$

Of course this is simply the "set" of functors from $\mathbb{A}$ to $\mathbb{B}$, as interpreted inside the category $\underline{\mathcal{A}}$.

Given these "homsets", it is clear that natural compositions and identities suggest themselves, and we adopt these in order to complete the construction of the $\underline{\mathcal{A}}$-category $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$. A morphism in the (honest) category underlying $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ is simply a global point of $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})(\mathbb{A}, \mathbb{B})$, and these clearly correspond to elements of $\mathrm{Cat}_0(\underline{\mathcal{A}})(\mathbb{A}, \mathbb{B})$ (i.e. functors from $\mathbb{A}$ to $\mathbb{B}$). Of course compositions and identities tally under this correspondence, and we see that $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ is indeed an $\underline{\mathcal{A}}$-enrichment of $\mathrm{Cat}_0(\underline{\mathcal{A}})$.

It remains to show that $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ admits all $\underline{\mathcal{A}}$- tensors and cotensors. If $A \in \underline{\mathcal{A}}$ then both of the endo-functors $*^A$ and $- \times A$ preserve limits, therefore if $\mathbb{A}$ is in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ we have categories $A \times \mathbb{C}$ and $\mathbb{C}^A$ given, in terms of truncated simplicial objects, by:

$$
\begin{tikzcd}
A \times \mathbb{C}_2 \arrow[r, shift left=2, "A\times\pi_1"] \arrow[r, "A\times\circ"] \arrow[r, shift right=2, "A\times\pi_2"'] & A \times \mathbb{C}_1 \arrow[r, shift left=2, "A\times d_0"] \arrow[r, shift right=2, "A\times d_1"'] & A \times \mathbb{C}_0 \arrow[l, "A\times i"']
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\mathbb{C}_2^A \arrow[r, shift left=2, "\pi_1^A"] \arrow[r, "\circ^A"] \arrow[r, shift right=2, "\pi_2^A"'] & \mathbb{C}_1^A \arrow[r, shift left=2, "d_0^A"] \arrow[r, shift right=2, "d_1^A"'] & \mathbb{C}_0^A \arrow[l, "i^A"']
\end{tikzcd}
$$

respectively. Now using the canonical isomorphisms

$$(\mathbb{B}_n^{\mathbb{A}_n})^A \cong \mathbb{B}_n^{\mathbb{A}_n \times A} \cong (\mathbb{B}_n^A)^{\mathbb{A}_n}$$

and preservation of the limits used to carve out each "homset" $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})(\mathbb{A}, \mathbb{B})$ as a subobject of $\mathbb{B}_1^{\mathbb{A}_1} \times \mathbb{B}_0^{\mathbb{A}_0}$, we may show that:

$$
\begin{aligned}
(\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})(\mathbb{A}, \mathbb{B}))^A &\cong \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})(A \times \mathbb{A}, \mathbb{B}) \\
&\cong \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})(\mathbb{A}, \mathbb{B}^A) \qquad \underline{\mathcal{A}}\text{-naturally in } \mathbb{A},\, \mathbb{B}.
\end{aligned}
$$

In other words $A \times \mathbb{C}$ is the $\underline{\mathcal{A}}$-tensor and $\mathbb{C}^A$ the $\underline{\mathcal{A}}$-cotensor of $\mathbb{C} \in \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ by $A \in \underline{\mathcal{A}}$. But, by theorem 2.1.3, we know that $\mathrm{Cat}(A)_0$ is locally presentable, and in particular it is both (small) complete and cocomplete. Therefore, by theorem 3.73 of [30] we see that $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ is (small) $\underline{\mathcal{A}}$-complete and $\underline{\mathcal{A}}$-cocomplete. □

Observation 2.5.5 Of course we could have presented the proof of the last proposition using our work on change of base from the last chapter. There is an adjoint pair

$$
\begin{tikzcd}
\mathrm{Cat}_0(\underline{\mathcal{A}}) \arrow[r, shift right=2, "(-)_0"'] \arrow[r, phantom, "\perp"] & \underline{\mathcal{A}} \arrow[l, shift right=2, "\mathrm{dis}"']
\end{tikzcd}
$$

where "$(-)_0$" denotes the functor taking an internal category $\mathbb{A} \in \mathrm{Cat}_0(\underline{\mathcal{A}})$ to its "set" of objects $\mathbb{A}_0$ and "dis" that mapping an object $A \in \underline{\mathcal{A}}$ to the corresponding discrete category. We consider $\mathrm{Cat}_0(\underline{\mathcal{A}})$ and $\underline{\mathcal{A}}$ to be the homsets of bicategories, each having a single 0-cell with tensorial composition given by product, furthermore the functor "dis" preserves products and therefore constitutes a homomorphism of these. The existence of a right adjoint $(-)_0$ ensures that this homomorphism satisfies the local adjointness condition of example 1.6.6 giving an adjoint pair

$$\begin{tikzcd} \mathrm{Cat}_0(\underline{\mathcal{A}})\text{–Equip} \arrow[r, shift right=1ex, "\ddot{\mathrm{F}}_\star"'] \arrow[r, phantom, "\perp"] & \underline{\mathcal{A}}\text{–Equip} \arrow[l, shift right=1ex, "\ddot{\mathrm{F}}^\star"'] \end{tikzcd}$$

in $\underline{\underline{\mathcal{E}Mor}}$, with $\ddot{\mathrm{F}}^\star$ an equipment homomorphism and $\ddot{\mathrm{F}}_\star$ preserving representables.

As we have already pointed out any monoidal closed category may be enriched over itself, and assuming that it was complete and cocomplete in the first place it becomes so as an enriched category (cf. [30]). In our case $\mathrm{Cat}(\underline{\mathcal{A}})$ denotes this $\mathrm{Cat}_0(\underline{\mathcal{A}})$-enriched category, to which we apply $\ddot{\mathrm{F}}_\star$ to get an $\underline{\mathcal{A}}$-category $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ clearly identical to that constructed in the last proposition. Now applying theorem 1.7.1 to $\ddot{\mathrm{F}}^\star \dashv \ddot{\mathrm{F}}_\star$ we show that $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ is $\underline{\mathcal{A}}$-cocomplete, since $\mathrm{Cat}(\underline{\mathcal{A}})$ is $\mathrm{Cat}_0(\underline{\mathcal{A}})$-cocomplete, furthermore we may use the same theorem and the dual adjunction

$$\begin{tikzcd} \mathrm{Cat}_0(\underline{\mathcal{A}})\text{–Equip}^{\mathrm{op}} \arrow[r, shift right=1ex, "(\ddot{\mathrm{F}}_\star)^{\mathrm{op}}"'] \arrow[r, phantom, "\perp"] & \underline{\mathcal{A}}\text{–Equip}^{\mathrm{op}} \arrow[l, shift right=1ex, "(\ddot{\mathrm{F}}^\star)^{\mathrm{op}}"'] \end{tikzcd}$$

and thereby demonstrate that $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ is $\underline{\mathcal{A}}$-complete as well.

Of course the only deficiency with this sort of approach is that we will need the explicit descriptions of $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$, along with that of its $\underline{\mathcal{A}}$-tensors and $\underline{\mathcal{A}}$-cotensors, furnished by the proof we gave.

Now to enrich the slices of $\mathrm{Cat}(\underline{\mathcal{A}})$:

Proposition 2.5.6 *If $\underline{\mathbf{C}}$ is an $\underline{\mathcal{A}}$-category and $C \in \underline{\mathbf{C}}$ an arbitrary object, then the slice category $\underline{\mathbf{C}}_0/_C$ has a natural enrichment to an $\underline{\mathcal{A}}$-category $\underline{\mathbf{C}}/_C$ such that $\left(\underline{\mathbf{C}}/_C\right)_0 \cong \underline{\mathbf{C}}_0/_C$. Furthermore we have the following completeness properties:*

(i) if $\underline{\mathbf{C}}$ admits all $\underline{\mathcal{A}}$-tensors then so does $\underline{\mathbf{C}}/_C$ and they are created by the functor $\Sigma_C\colon \underline{\mathbf{C}}/_C \longrightarrow \underline{\mathbf{C}}$ (which enriches to an $\underline{\mathcal{A}}$-functor).

(ii) if $\underline{\mathbf{C}}$ admits all $\underline{\mathcal{A}}$-cotensors and pullbacks (in the $\underline{\mathcal{A}}$-enriched sense) then $\underline{\mathbf{C}}/_C$ also admits all $\underline{\mathcal{A}}$-cotensors.

(iii)
- *$\underline{\mathbf{C}}$ is (small) $\underline{\mathcal{A}}$-complete $\Rightarrow$ $\underline{\mathbf{C}}/_C$ is.*
- *$\underline{\mathbf{C}}$ is (small) $\underline{\mathcal{A}}$-cocomplete $\Rightarrow$ $\underline{\mathbf{C}}/_C$ is and Σ_C creates these colimits.*

Proof. Construct $\underline{\mathbf{C}}/_C$ as follows:

objects: are pairs (X, p) with X an object of $\underline{\mathbf{C}}$ and $p: X \longrightarrow C$ a morphism in $\underline{\mathbf{C}}_0$ (in other words p is a global section of $\underline{\mathbf{C}}(X, C)$).

"homsets": given in the pullback:

$$\begin{tikzcd} \underline{\mathbf{C}}/_C\begin{pmatrix} X & Y \\ {\scriptstyle p}\downarrow, & {\scriptstyle q}\downarrow \\ C & C\end{pmatrix} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathbf{C}}(X, Y) \arrow[d, "\underline{\mathbf{C}}(X, q)"] \\ 1 \arrow[r, tail, "p"'] & \underline{\mathbf{C}}(X, C) \end{tikzcd} \tag{2.15}$$

composition & identities: are obtained by restricting those of $\underline{\mathbf{C}}$, to the subobjects $\underline{\mathbf{C}}/_C\begin{pmatrix} X & Y \\ {\scriptstyle p}\downarrow, & {\scriptstyle q}\downarrow \\ C & C\end{pmatrix} \rightarrowtail \underline{\mathbf{C}}(X, Y)$.

Notice that the fact that $\underline{\mathcal{A}}$ is cartesian closed, and not just monoidal closed, is of importance here. Crucially it means that the tensorial identity is the terminal object of $\underline{\mathcal{A}}$, from which it follows that the "homset" defined in (2.15) is indeed a subobject of $\underline{\mathbf{C}}(X, Y)$ and furthermore that the underlying category of $\underline{\mathbf{C}}/_C$ is $\underline{\mathbf{C}}_0/_C$. For the same reasons the existence of a canonical $\underline{\mathcal{A}}$-functor $\Sigma_C: \underline{\mathbf{C}}/_C \longrightarrow \underline{\mathbf{C}}$, enriching the traditional "sum" functor, is clear from the description of $\underline{\mathbf{C}}/_C$ given above.

Before proceeding we remind ourselves of some notation. Given an object $A \in \underline{\mathcal{A}}$ and a map $r: X \longrightarrow Y$ in $\underline{\mathbf{C}}_0(X, Y)$, or in other words a point $r: 1 \longrightarrow \underline{\mathbf{A}}(X, Y)$, adopt the notations $A \otimes X$ for the $\underline{\mathcal{A}}$-tensor of X by A, and $A \pitchfork Y$ for the $\underline{\mathcal{A}}$-cotensor of Y by A and define canonical maps in $\underline{\mathbf{A}}_0$:

$$\hat{r}: A \otimes X \longrightarrow Y \quad \text{given by} \quad 1 \xrightarrow{\cong} 1^A \xrightarrow{r^A} \underline{\mathbf{C}}(X, Y)^A \cong \underline{\mathbf{C}}(A \otimes X, Y)$$
$$\check{r}: X \longrightarrow A \pitchfork Y \quad \text{given by} \quad 1 \xrightarrow{\cong} 1^A \xrightarrow{r^A} \underline{\mathbf{C}}(X, Y)^A \cong \underline{\mathbf{C}}(X, A \pitchfork Y)$$

We should also recall what it means for an $\underline{\mathcal{A}}$-functor to create tensors:

Given $A \in \underline{\mathcal{A}}$ and $X \in \underline{\mathbf{C}}$ then a pair $(A \otimes X, \alpha)$, consisting of an object $A \otimes X \in \underline{\mathbf{C}}$ and an $\underline{\mathcal{A}}$-natural isomorphism

$$\alpha: \underline{\mathbf{C}}(X, \text{-})^A \xrightarrow{\cong} \underline{\mathbf{C}}(A \otimes X, \text{-}),$$

is said to *display* $A \otimes X$ as the tensor of X by A in $\underline{\mathbf{C}}$. We say that an $\underline{\mathcal{A}}$-functor $\Theta: \underline{\mathbf{C}} \longrightarrow \underline{\mathbf{D}}$ creates this tensor if for each pair $(A \otimes (\Theta X), \alpha')$ displaying a tensor of ΘX by A in $\underline{\mathbf{D}}$ there exists a unique pair $(\underline{\mathcal{A}} \otimes X, \alpha)$, displaying our tensor in $\underline{\mathbf{C}}$, such that:

- $\Theta(A \otimes X) = A \otimes (\Theta X)$ and

- For each $Y \in \underline{\mathbf{C}}$ the diagram

$$\begin{tikzcd}
\underline{\mathbf{C}}(X,Y)^A \arrow[r, "\alpha Y"] \arrow[d, "\Theta^A"'] & \underline{\mathbf{C}}(A\otimes X, Y) \arrow[d, "\Theta"] \\
\underline{\mathbf{D}}(\Theta X, \Theta Y)^A \arrow[r, "\alpha' \Theta Y"'] & \underline{\mathbf{D}}(A \otimes (\Theta X), \Theta Y)
\end{tikzcd} \tag{2.16}$$

  commutes.

We may now turn to establishing completeness properties (i)-(iii).

(i) Suppose $p : X \longrightarrow C$, $q : Y \longrightarrow C$ are objects of $\underline{\mathbf{C}}/_C$, then the "homset" $\underline{\mathbf{C}}/_C\left(\begin{smallmatrix} X \\ {\scriptstyle p}\downarrow \\ C\end{smallmatrix}, \begin{smallmatrix} Y \\ {\scriptstyle q}\downarrow \\ C\end{smallmatrix}\right)$ is given by the pullback in (2.15), to which we may apply the limit preserving functor $(\text{-})^A$ to obtain a diagram:

$$\begin{tikzcd}
\underline{\mathbf{C}}/_C\left(\begin{smallmatrix} X \\ {\scriptstyle p}\downarrow \\ C\end{smallmatrix}, \begin{smallmatrix} Y \\ {\scriptstyle q}\downarrow \\ C\end{smallmatrix}\right)^A \arrow[rr, tail] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & \underline{\mathbf{C}}(X,Y)^A \arrow[dd, "\underline{\mathbf{C}}(X,q)^A"'] \arrow[dr, "\alpha' Y", "\cong"'] & \\
& \underline{\mathbf{C}}/_C\left(\begin{smallmatrix} A\otimes X \\ {\scriptstyle \hat{p}}\downarrow \\ C\end{smallmatrix}, \begin{smallmatrix} Y \\ {\scriptstyle q}\downarrow \\ C\end{smallmatrix}\right) \arrow[rr, tail] \arrow[dd] & & \underline{\mathbf{C}}(A\otimes X, Y) \arrow[dd, "\underline{\mathbf{C}}(A\otimes X, q)"] \\
1^A \arrow[rr, tail, "p^A"' near end] \arrow[dr, "\cong"] & & \underline{\mathbf{C}}(X,C)^A \arrow[dr, "\alpha' C", "\cong"'] & \\
& 1 \arrow[rr, tail, "\hat{p}"'] & & \underline{\mathbf{C}}(A\otimes X, C)
\end{tikzcd}$$

This induces a unique iso $\alpha_q \colon \underline{\mathbf{C}}/_C\left(\begin{smallmatrix} X \\ {\scriptstyle p}\downarrow \\ C\end{smallmatrix}, \begin{smallmatrix} Y \\ {\scriptstyle q}\downarrow \\ C\end{smallmatrix}\right)^A \xrightarrow{\;\cong\;} \underline{\mathbf{C}}/_C\left(\begin{smallmatrix} A\otimes X \\ {\scriptstyle \hat{p}}\downarrow \\ C\end{smallmatrix}, \begin{smallmatrix} Y \\ {\scriptstyle q}\downarrow \\ C\end{smallmatrix}\right)$, completing the diagram to a commutative cube. It follows that these are the unique maps satisfying the condition in (2.16) for the functor Σ_C, and their $\underline{\mathcal{A}}$-naturality is entailed by that of α'. By substituting the terminal object $\mathrm{id}\colon C \longrightarrow C$ of $\underline{\mathbf{C}}/_C$ for $q\colon Y \longrightarrow C$ we see that $\hat{p}$ is indeed the only map making $A \otimes X$ into the required tensor, completing the proof that Σ_C creates all such.

(ii) Form the cotensor of $q\colon Y \longrightarrow C$ by A as the $\underline{\mathcal{A}}$-enriched pullback of $A \pitchfork q$ along $\check{\mathrm{id}}\colon C \longrightarrow A \pitchfork C$ in $\underline{\mathbf{C}}$, e.g:

$$\begin{tikzcd}
\overline{Y} \arrow[r] \arrow[d, "\bar{q}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \pitchfork Y \arrow[d, "A \pitchfork q"] \\
C \arrow[r, "\check{\mathrm{id}}"'] & A \pitchfork C
\end{tikzcd} \tag{2.17}$$

We may check that this truly is the cotensor we are interested in by considering the following diagram (for any given $p\colon X \longrightarrow C$ in $\underline{\mathbf{C}}/_C$):

$$\begin{tikzcd}
\underline{\mathbf{C}}/_C\left(\begin{smallmatrix} X \\ p\downarrow \\ C \end{smallmatrix}, \begin{smallmatrix} \overline{Y} \\ \bar{q}\downarrow \\ C \end{smallmatrix}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\text{(b)}"] & \underline{\mathbf{C}}(X, \overline{Y}) \arrow[r] \arrow[d, "\underline{\mathbf{C}}(X,\bar{q})"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\text{(a)}"] & \underline{\mathbf{C}}(X, A \pitchfork Y) \arrow[d, "\underline{\mathbf{C}}(X, A \pitchfork q)"'] \arrow[dr, "\cong"] \arrow[dr, phantom, "\text{(c)}"'] & \\
1 \arrow[r, "p"'] \arrow[dr, "\cong"] & \underline{\mathbf{C}}(X, C) \arrow[r, "\underline{\mathbf{C}}(X, \check{\mathrm{id}})"'] \arrow[d, phantom, "\text{(d)}"] & \underline{\mathbf{C}}(X, A \pitchfork C) \arrow[dr, "\cong"] & \underline{\mathbf{C}}(X, Y)^A \arrow[d, "\underline{\mathbf{C}}(X,q)^A"] \\
& 1^A \arrow[rr, "p^A"'] & & \underline{\mathbf{C}}(X, C)^A
\end{tikzcd} \tag{2.18}$$

We look at each of the squares in here in turn:

(a) Is obtained by applying the functor $\underline{\mathbf{C}}(X, -)$ to the square in (2.17), and so by the definition of $\underline{\mathcal{A}}$-pullback it is a pullback in $\underline{\mathcal{A}}$.

(b) Is the pullback square which defines the "homset" $\underline{\mathbf{C}}/_C\left(\begin{smallmatrix} X \\ p\downarrow \\ C \end{smallmatrix}, \begin{smallmatrix} \overline{Y} \\ \bar{q}\downarrow \\ C \end{smallmatrix}\right)$, cf. (2.15).

(c) Commutes since it is the square which *defines* the map $A \pitchfork q$.

(d) Showing that this commutes requires an easy diagram chase, involving the $\underline{\mathcal{A}}$-naturality of the canonical isomorphism displaying the cotensor $A \pitchfork C$, and the definition of $\check{\mathrm{id}}$.

All that remains is to apply the limit preserving functor $(\text{-})^A$ to the pullback defining the "homset" $\underline{\mathbf{C}}/_C\begin{pmatrix} X & Y \\ {\scriptstyle p}\downarrow, & {\scriptstyle q}\downarrow \\ C & C \end{pmatrix}$, and so obtain a pullback

$$\begin{tikzcd} \underline{\mathbf{C}}/_C\begin{pmatrix} X & Y \\ {\scriptstyle p}\downarrow, & {\scriptstyle q}\downarrow \\ C & C \end{pmatrix}^{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathbf{C}}(X,Y)^A \arrow[d, "\underline{\mathbf{C}}(X,q)^A"] \\ 1^A \arrow[r, "p^A"'] & \underline{\mathbf{C}}(X,C)^A \end{tikzcd}$$

which when considered alongside (2.18) induces a unique isomorphism

$$\beta_p\colon \underline{\mathbf{C}}/_C\begin{pmatrix} X & Y \\ {\scriptstyle p}\downarrow, & {\scriptstyle q}\downarrow \\ C & C \end{pmatrix}^{A} \xrightarrow{\ \cong\ } \underline{\mathbf{C}}/_C\begin{pmatrix} X & \overline{Y} \\ {\scriptstyle p}\downarrow, & {\scriptstyle \bar{q}}\downarrow \\ C & C \end{pmatrix}$$

making the evident diagram commute. These isomorphisms are $\underline{\mathcal{A}}$-natural in the variable p and so display $\bar{q}\colon \overline{Y} \longrightarrow C$ as the cotensor $A \pitchfork \begin{pmatrix} Y \\ {\scriptstyle q}\downarrow \\ C \end{pmatrix}$.

(iii) In [30] the concept of the conical $\underline{\mathcal{A}}$-limit in $\underline{\mathbf{C}}$ of a diagram $\Gamma_0\colon \underline{\mathcal{D}} \longrightarrow \underline{\mathbf{C}}_0$ was introduced, where $\underline{\mathcal{D}}$ is a small and unenriched parameterising category. An $\underline{\mathcal{A}}$-limiting cone for this notion may be described concretely as a traditional cone $\varprojlim_{\mathcal{D}} \Gamma_0 \xrightarrow{\ \cdot\ } \Gamma_0(\text{-})$ in $\underline{\mathbf{C}}_0$, with the property that each representable functor

$$\underline{\mathbf{C}}(A,\text{-})\colon \underline{\mathbf{C}}_0 \longrightarrow \underline{\mathcal{A}}_0$$

carries it to a limit cone in $\underline{\mathcal{A}}_0$. The conical $\underline{\mathcal{A}}$-colimit of Γ_0 is defined dually.

Given this description it is a relatively easy matter to generalise the usual proofs, establishing the conical completeness and cocompleteness of slice categories, to the $\underline{\mathcal{A}}$-enriched context. In any case we will not explicitly do this here, since it is well known (for instance see [30]) that if $\underline{\mathbf{C}}$ admits all tensors then a cone over Γ_0 is $\underline{\mathcal{A}}$-limiting iff it is a traditional limiting cone in the unenriched category $\underline{\mathbf{C}}_0$. Of course a dual result holds for conical $\underline{\mathcal{A}}$-colimits and so, since all of the categories we will be considering admit all tensors and cotensors, we are quite happy in stating

- If $\underline{\mathbf{C}}$ admits all conical $\underline{\mathcal{A}}$-colimits then so does $\underline{\mathbf{C}}/_C$ and they are created by Σ_C.
- If $\underline{\mathbf{C}}$ admits all conical $\underline{\mathcal{A}}$-limits then so does $\underline{\mathbf{C}}/_C$.

with no further justification than an appeal to the corresponding unenriched versions.

All that remains is to apply theorem 3.73 of [30], which demonstrates the construction of an arbitrary weighted $\underline{\mathcal{A}}$-limit(colimit) from $\underline{\mathcal{A}}$-cotensors(tensors) and conical $\underline{\mathcal{A}}$-limits(colimits). $\square$

Now consider a pair of right $\mathbb{A}$-actions B and C then a straightforward calculation, using the definition of the enriched slice $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathbb{A}}$ given above, shows that:

$$\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathbb{A}}\left(\begin{array}{c}\mathbb{G}\,(B)\\ {\scriptstyle\bar{b}}\downarrow\\ \mathbb{A}\end{array},\begin{array}{c}\mathbb{G}\,(C)\\ {\scriptstyle\bar{c}}\downarrow\\ \mathbb{A}\end{array}\right)\cong\left\{f\in C^B\;\middle|\;\begin{array}{l}(\forall b\in B)(\bar{c}f(b)=\bar{b}(b))\\ (\forall b\in B,\alpha\in\mathbb{A}_1)((\bar{b}(b)=d_0(\alpha))\Rightarrow\\ \qquad\qquad\qquad\qquad((fb)\cdot\alpha=f(b\cdot\alpha)))\end{array}\right\}\tag{2.19}$$

This is in essence the internalisation to $\underline{\mathcal{A}}$ of the material of section 2.3, up to and including observation 2.3.1. As such it is a simple exercise in the internal language of $\underline{\mathcal{A}}$, and we omit it here, referring the reader instead to that section. From it flows the following theorem:

Theorem 2.5.7 *Under the enrichment of each slice category $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathbb{A}}$ provided by propositions 2.5.4 and 2.5.6 the functors $\mathbb{G}_{\underline{\mathcal{A}}}$ enrich to fully faithful $\underline{\mathcal{A}}$-functors:*

$$\mathbb{G}_{\underline{\mathcal{A}}}\colon\mathcal{P}(\underline{\mathbf{A}})\longrightarrow\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$$

Proof. Let $X,Y\colon\underline{\mathbf{A}}\longrightarrow\!\!\!\!\!\!\!+\;\;\underline{\mathbf{1}}$ be presheaves in $\mathcal{P}(\underline{\mathbf{A}})$ then the "homset" $\mathcal{P}(\underline{\mathbf{A}})(X,Y)$ is given by the right Kan extension:

$$\begin{tikzcd}\underline{\mathbf{A}} \arrow[rr, "X", "|"] \arrow[dr, "Y"', "|"] & \arrow[d, phantom, "\Leftarrow"] & \underline{\mathbf{1}} \arrow[dl, "\mathcal{P}(\underline{\mathbf{A}})(X,Y)", "|"'] \\ & \underline{\mathbf{1}} & \end{tikzcd}$$

Now apply the bicategory morphism $\bar{\mathrm{I}}_\star$ to this, and by proposition 1.7.7(ii) we get a Kan extension

$$\begin{tikzcd}\mathrm{I}_\star\underline{\mathbf{A}} \arrow[rr, "\bar{\mathrm{I}}_\star X", "|"] \arrow[dr, "\bar{\mathrm{I}}_\star Y"', "|"] & \arrow[d, phantom, "\Leftarrow"] & 1 \arrow[dl, "\bar{\mathrm{I}}_\star(\mathcal{P}(\underline{\mathbf{A}})(X,Y))", "|"'] \\ & 1 & \end{tikzcd}$$

wherein $\bar{\mathrm{I}}_\star(\mathcal{P}(\underline{\mathbf{A}})(X,Y))\in\widetilde{\underline{\mathcal{C}}}$ is simply $\mathcal{P}(\underline{\mathbf{A}})(X,Y)\in\underline{\mathcal{A}}$ considered as an object of $\widetilde{\underline{\mathcal{C}}}$. Therefore returning to the description of $\bar{\mathrm{I}}_\star Y \Leftarrow_{\mathrm{I}_\star\underline{\mathbf{A}}} \bar{\mathrm{I}}_\star X$ furnished by (1.30) we see that, on setting $B=\bar{\mathrm{I}}_\star X$ and $C=\bar{\mathrm{I}}_\star Y$, $\mathcal{P}(\underline{\mathbf{A}})(X,Y)$ is isomorphic to the object depicted symbolically on the right hand side of (2.19). It is an easy matter to check that these isomorphisms are the "homset" actions of a fully faithful $\underline{\mathcal{A}}$-functor enriching the functor $\mathbb{G}_{\underline{\mathcal{A}}}$. $\square$

As a result of the last proposition we will, from now on, reserve the symbol $\mathbb{G}_{\underline{\mathcal{A}}}$ for the $\underline{\mathcal{A}}$-enriched functor $\mathcal{P}(\underline{\mathbf{A}}) \longrightarrow \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$, and follow our conventions by using $(\mathbb{G}_{\underline{\mathcal{A}}})_0$ for its action on underlying categories. Observation 2.3.3 identified the objects in the full image of this functor as the discrete fibrations over $\mathrm{I}_\star\underline{\mathbf{A}}$, and from now on for each $\mathbb{B} \in \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ we will use $Dfib(\underline{\mathcal{A}})/_{\mathbb{B}}$ to denote the $\underline{\mathcal{A}}$-enriched full sub-category of $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathbb{B}}$ determined by these. It follows that $\mathbb{G}_{\underline{\mathcal{A}}}$ identifies $\mathcal{P}(\underline{\mathbf{A}})$ with $Dfib(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$ and we exploit this in the corollary to the next proposition in demonstrating the $\underline{\mathcal{A}}$-enrichment of the left adjoint $\mathbf{L}_{\underline{\mathcal{A}}}$:

Proposition 2.5.8 *The full sub-$\underline{\mathcal{A}}$-category $Dfib(\underline{\mathcal{A}})/_{\mathbb{B}}$ is closed in* $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathbb{B}}$ *under all $\underline{\mathcal{A}}$-tensors and $\underline{\mathcal{A}}$-cotensors.*

Proof. First recall from observation 2.3.3 that an internal functor $p\colon \mathbb{E} \longrightarrow \mathbb{B}$ is a discrete fibration iff the diagram

$$\begin{tikzcd} \mathbb{E}_1 \arrow[r, "d_0"] \arrow[d, "p_1"'] & \mathbb{E}_0 \arrow[d, "p_0"] \\ \mathbb{B}_1 \arrow[r, "d_0"'] & \mathbb{B}_0 \end{tikzcd} \tag{2.20}$$

is a pullback. Using this description we may verify that $Dfib(\underline{\mathcal{A}})/_{\mathbb{B}}$ is closed in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathbb{B}}$ under $\underline{\mathcal{A}}$- tensors and cotensors directly from the description of their construction furnished by propositions 2.5.4 and 2.5.6:

<u>Tensors</u>

The tensor $A \otimes \begin{pmatrix} \mathbb{E} \\ {\scriptstyle p}\downarrow \\ \mathbb{B} \end{pmatrix}$ is simply the internal functor $\hat{p}\colon A \times \mathbb{E} \longrightarrow \mathbb{B}$, for which the diagram in (2.20) is the composite square:

$$\begin{tikzcd} A \times \mathbb{E}_1 \arrow[r, "A \times d_0"] \arrow[d, "\pi_{\mathbb{E}_1}"'] \arrow[dr, phantom, "\text{(a)}"] & A \times \mathbb{E}_0 \arrow[d, "\pi_{\mathbb{E}_0}"] \\ \mathbb{E}_1 \arrow[r, "d_0"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\text{(b)}"] & \mathbb{E}_0 \arrow[d, "p_0"] \\ \mathbb{B}_1 \arrow[r, "d_0"'] & \mathbb{B}_0 \end{tikzcd}$$

The square marked (a) is (always) a pullback, as is (b) so long as $p\colon \mathbb{E} \longrightarrow \mathbb{B}$ is a discrete fibration, in which case applying the composition lemma for pullbacks establishes that $\hat{p}$ is also in $Dfib(\underline{\mathcal{A}})/_{\mathbb{B}}$.

<u>Cotensors:</u>

We prove this in two steps:

(a) *if* $p\colon \mathbb{E} \longrightarrow \mathbb{B}$ *is a discrete fibration then so is* $p^A\colon \mathbb{E}^A \longrightarrow \mathbb{B}^A$*.*

In order for p^A to be discrete fibration we need

$$\begin{tikzcd}
\mathbb{E}_1^A \arrow[r, "d_0^A"] \arrow[d, "p_1^A"'] & \mathbb{E}_0^A \arrow[d, "p_0^A"] \\
\mathbb{B}_1^A \arrow[r, "d_0^A"'] & \mathbb{B}_0^A
\end{tikzcd}$$

to be a pullback, but this is just the square obtained by applying the limit preserving functor $(\text{–})^A$ to the pullback in (2.20),

(b) *if* $p\colon \mathbb{E} \longrightarrow \mathbb{B}$ *is a discrete fibration and*

$$\begin{tikzcd}
\mathbb{E}' \arrow[r, "g"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{E} \arrow[d, "p"] \\
\mathbb{B}' \arrow[r, "f"'] & \mathbb{B}
\end{tikzcd} \tag{2.21}$$

is a pullback in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ *then* $p'\colon \mathbb{E}' \longrightarrow \mathbb{B}'$ *is also a discrete fibration.*

Consider the commutative cube

$$\begin{tikzcd}
\mathbb{E}_1' \arrow[rr, "g_1"] \arrow[dr, "d_0"] \arrow[dd, "p_1'"'] & & \mathbb{E}_1 \arrow[dr, "d_0"] \arrow[dd, "p_1"' near start] & \\
& \mathbb{E}_0' \arrow[rr, "g_0" near start] \arrow[dd, "p_0'" near start] & & \mathbb{E}_0 \arrow[dd, "p_0"] \\
\mathbb{B}_1' \arrow[rr, "f_1" near end] \arrow[dr, "d_0"] & & \mathbb{B}_1 \arrow[dr, "d_0"] & \\
& \mathbb{B}_0' \arrow[rr, "f_0"] & & \mathbb{B}_0
\end{tikzcd}$$

of which the front and back faces are pullbacks, because 2.21 is a pullback in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$, as is the right hand face when p is a discrete fibration. Therefore, by the composition and cancellation lemma for pullbacks, the left hand face of this cube must also be a pullback, implying that p' is a discrete fibration.

We combine these two lemmas by observing that $A \pitchfork \begin{pmatrix} \mathbb{E} \\ p\downarrow \\ \mathbb{B} \end{pmatrix}$ is constructed by pulling back p^A along a map into $\mathbb{B}^A$ from which it follows that $Dfib(\underline{\mathcal{A}})/_{\mathbb{B}}$ is closed under cotensors in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathbb{B}}$. $\square$

Corollary 2.5.9 *The $\underline{\mathcal{A}}$-functor $\mathbb{G}_{\underline{\mathcal{A}}}\colon \mathcal{P}(\underline{\mathbf{A}}) \longrightarrow \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$ preserves all $\underline{\mathcal{A}}$-colimits and has a left $\underline{\mathcal{A}}$-adjoint.*

Proof. The (honest) functor underlying $\mathbb{G}_{\underline{\mathcal{A}}}$ preserves all (small) colimits, cf. proposition 2.5.3, and so it itself preserves all (small) conical $\underline{\mathcal{A}}$-colimits, furthermore the closure property of $Dfib(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$ under $\underline{\mathcal{A}}$-tensors, given in the last proposition, implies that $\mathbb{G}_{\underline{\mathcal{A}}}$ also preserves all $\underline{\mathcal{A}}$-tensors. Of course any (small) weighted $\underline{\mathcal{A}}$-colimit may be constructed from $\underline{\mathcal{A}}$-colimits of these two types, and so $\mathbb{G}_{\underline{\mathcal{A}}}$ preserves all $\underline{\mathcal{A}}$-colimits.

In a similar vein, the fact that $Dfib(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$ is closed in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$ under $\underline{\mathcal{A}}$-cotensors implies that $\mathbb{G}_{\underline{\mathcal{A}}}$ preserves $\underline{\mathcal{A}}$-cotensors. So by theorem 4.85 of [30] the left adjoint to $(\mathbb{G}_{\underline{\mathcal{A}}})_0$ enriches to a left $\underline{\mathcal{A}}$-adjoint of $\mathbb{G}_{\underline{\mathcal{A}}}$. Again, following our convention, we will now reserve the symbol $\mathbf{L}_{\underline{\mathcal{A}}}$ for this $\underline{\mathcal{A}}$-adjoint and use $(\mathbf{L}_{\underline{\mathcal{A}}})_0$ for its action on underlying categories. $\square$

We are now in a position to establish the result we promised at the beginning of the section, which was to provide a way of defining closed classes of colimits in terms of (large) sets of categories in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$. First we make precise what it means for a category in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ to have a terminal object:

Definition 2.5.10 *A category $\mathbb{C} \in \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ is said to have a* (global) terminal object *iff there exists a global section $t\colon 1 \longrightarrow \mathbb{C}_0$ and a section $c\colon \mathbb{C}_0 \longrightarrow \mathbb{C}_1$ of $d_1\colon \mathbb{C}_1 \longrightarrow \mathbb{C}_0$ such that $(d_0)^*t \cong c$. In other words we have a pullback*

$$\begin{tikzcd} \mathbb{C}_0 \arrow[r, "\square"] \arrow[d, "c"'] & 1 \arrow[d, "t"] \\ \mathbb{C}_1 \arrow[r, "d_0"'] & \mathbb{C}_0 \end{tikzcd} \tag{2.22}$$

with the property that

$$\begin{tikzcd} \mathbb{C}_0 \arrow[r, "c"] \arrow[dr, "\mathrm{id}_{\mathbb{C}_0}"'] & \mathbb{C}_1 \arrow[d, "d_1"] \\ & \mathbb{C}_0 \end{tikzcd} \tag{2.23}$$

commutes. Notice that in accordance with the conventions we have followed throughout d_0 denotes the <u>codomain</u> map and d_1 the <u>domain</u> one.

In abstract terms all this says is that the extension of the predicate "$x \in \mathbb{C}_0$ is a terminal object of $\mathbb{C}$", as expressed in the internal language of $\underline{\mathcal{A}}$, has a global section. Another way of expressing this concept, which is often quite useful, is to weaken the condition of (2.22), by only insisting that the square commute, and add two extra conditions:

$$
\begin{tikzcd}
\mathbb{C}_1 \arrow[r, "\langle cd_0, \mathrm{id}\rangle"] \arrow[d, "d_1"'] & \mathbb{C}_1 \underset{\mathbb{C}_0}{\times} \mathbb{C}_1 \arrow[d, "\circ"] \\
\mathbb{C}_0 \arrow[r, "c"'] & \mathbb{C}_1
\end{tikzcd}
\tag{2.24}
$$

commutes, which is the naturality condition making c into an internal cone (defined at 1) under the identity diagram $id_{\mathbb{C}}: \mathbb{C} \longrightarrow \mathbb{C}$ (cf. (2.8)), and

$$
\begin{tikzcd}
1 \arrow[r, "t"] & \mathbb{C}_0 \arrow[r, shift left, "c"] \arrow[r, shift right, "i"'] & \mathbb{C}_1
\end{tikzcd}
\tag{2.25}
$$

is a commutative fork, more concretely this says that the component of cone c at t is the identity.

Given these two it is an easy matter to show that the square in (2.22) is a pullback, and conversely they follow from that assumption. If the condition of 2.25 is dropped we say that $\mathbb{C}$ possesses a *(global) natural weak terminal object*. We may interpret this as saying that every object $x \in \mathbb{C}$ has associated with it a map $\Box_x : x \longrightarrow t$, which may possibly be chosen from amongst many such, and that the totality of these form a natural cone under the identity diagram $\mathrm{id}_{\mathbb{C}}$.

Let $\mathcal{T}(\underline{\mathcal{A}})$ denote the set of categories in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ with terminal objects and $\mathcal{W}(\underline{\mathcal{A}})$ the larger set of those with *natural weak* terminal objects, which we will often confuse with the corresponding full sub-$\underline{\mathcal{A}}$-categories of $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$. The important proposition concerning these is:

Proposition 2.5.11 *For any given (small) $\underline{\mathcal{A}}$-category $\underline{\mathbf{A}}$, the $\underline{\mathcal{A}}$-adjunction*

$$
\begin{tikzcd}
\mathcal{P}(\underline{\mathbf{A}}) \arrow[r, hook, shift right, "\mathbb{G}_{\underline{\mathcal{A}}}"'] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathrm{I}_\star \underline{\mathbf{A}}} \arrow[l, shift right, "\mathbf{L}_{\underline{\mathcal{A}}}"']
\end{tikzcd}
$$

restricts to

$$
\begin{tikzcd}
\underline{\mathbf{A}} \arrow[r, hook, shift right, "\mathbb{G}_{\underline{\mathcal{A}}}"'] \arrow[r, phantom, "\perp"] & \mathcal{T}(\underline{\mathcal{A}})/_{\mathrm{I}_\star \underline{\mathbf{A}}} \arrow[l, shift right, "\mathbf{L}_{\underline{\mathcal{A}}}"']
\end{tikzcd}
$$

where $\underline{\mathbf{A}}$ is identified with the full subcategory of $\mathcal{P}(\underline{\mathbf{A}})$ on the right representables $a^*: \underline{\mathbf{A}} \longrightarrow\!\!\!\!\!\!\mapsto \underline{\mathbf{1}}$.

Proof. First consider the adjunction

$$
\begin{tikzcd}
\mathrm{Prof}(\widetilde{\underline{\mathcal{C}}})(\mathbb{A}, 1) \arrow[r, shift left, "\mathbf{L}"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}(\widetilde{\underline{\mathcal{C}}})/_{\mathbb{A}} \arrow[l, shift left, "\mathbb{G}"]
\end{tikzcd}
$$

We have the following two lemmas, the proofs of which are practically direct internalisations of the classical results:

(a) *If $X \in r(\mathbb{A})$ is a right representable $\mathbb{A}$-action then the category $\mathbb{G}(X)$ has an initial object.*

First recall that the right representable associated with an internal functor $a\colon 1 \longrightarrow \mathbb{A}$ has underlying object given by the pullback

$$\begin{tikzcd} 1 \underset{a\ d_0}{\times} \mathbb{A}_1 \arrow[r] \arrow[d, "\pi_{\mathbb{A}_1}"'] \arrow[dr, phantom, "\square"] & 1 \arrow[d, "a"] \\ \mathbb{A}_1 \arrow[r, "d_0"'] & \mathbb{A}_0 \end{tikzcd}$$

equipped with projection

$$1 \underset{a\ d_0}{\times} \mathbb{A}_1 \xrightarrow{\pi_{\mathbb{A}_1}} \mathbb{A}_1 \xrightarrow{d_1} \mathbb{A}_0$$

on which we have a right action induced by the composition of $\mathbb{A}$. From this description it follows that the category $\mathbb{G}(a^*)$ looks like

$$1 \underset{a\ d_0}{\times} \mathbb{A}_1 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \rightrightarrows\!\!\!\!\!\rightarrow 1 \underset{a\ d_0}{\times} \mathbb{A}_1 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \leftrightarrows\!\!\!\!\!\rightarrow 1 \underset{a\ d_0}{\times} \mathbb{A}_1$$

and so is just the comma "object" $\mathbb{A} \downarrow a$ in the 2-category $\underline{\underline{\mathrm{Cat}}}(\underline{\mathcal{A}})$. It is now a routine verification to check that the maps

$$1 \xrightarrow{\langle 1, i \circ a \rangle} 1 \underset{a\ d_0}{\times} \mathbb{A}_1$$

$$1 \underset{a\ d_0}{\times} \mathbb{A}_1 \xrightarrow{\langle 1, i \circ a \rangle \underset{\mathbb{A}_0}{\times} \mathbb{A}_1} 1 \underset{a\ d_0}{\times} \mathbb{A}_1 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1$$

satisfy the conditions in (2.22)–(2.25), and so provide a terminal object for $\mathbb{A} \downarrow a = \mathbb{G}(a^*)$.

(b) *If $p\colon \mathbb{E} \longrightarrow \mathbb{A}$ is an object in $\mathrm{Cat}(\widetilde{\underline{\mathcal{C}}})/_{\mathbb{A}}$, and $\mathbb{E}$ has an initial object, then $\mathbf{L}\begin{pmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{A}\end{pmatrix}$ is a right representable.*

Referring to the proof of observation 2.3.2 we see that the underlying object of $\mathbf{L}\begin{pmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{A}\end{pmatrix}$, in $\widetilde{\underline{\mathcal{C}}}/_{\mathbb{A}_0}$, is the coequaliser of a pair

$$\mathbb{E}_1 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \overset{f}{\underset{g}{\rightrightarrows}} \mathbb{E}_0 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \tag{2.26}$$

where
$$\begin{aligned} f\langle \epsilon, \alpha\rangle &= \langle d_0(\epsilon), p_1(\epsilon)\circ\alpha\rangle \\ g\langle \epsilon, \alpha\rangle &= \langle d_1(\epsilon), \alpha\rangle \end{aligned}$$

which we give the right action induced by the canonical one on $\mathbb{E}_0 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1$. Given that $\mathbb{E}$ has a terminal object $t\colon 1 \longrightarrow \mathbb{E}_0$ displayed by a cone $c\colon \mathbb{E}_0 \longrightarrow \mathbb{E}_1$, we may define maps $t' = p_0 \circ t\colon 1 \longrightarrow \mathbb{A}_0$ and

$$\begin{array}{ccc} \mathbb{E}_0 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 & \xrightarrow{\quad h \quad} & 1 \,{}_{t'}\!\times_{d_0} \mathbb{A}_1 \\ \langle e, \alpha\rangle & \longmapsto & \langle *, p_1(c(e))\circ\alpha\rangle \end{array}$$

We will see that the latter of these, when appended to the pair in (2.26), gives a split coequaliser fork. To do this we must introduce the splitting maps

$$1 \,{}_{t'}\!\times_{d_0} \mathbb{A}_1 \xrightarrow{\;t \underset{\mathbb{A}_0}{\times} \mathbb{A}_1\;} \mathbb{E}_0 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1$$
$$\mathbb{E}_0 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \xrightarrow{\;c \underset{\mathbb{A}_1}{\times} \mathbb{A}_1\;} \mathbb{E}_1 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1$$

and fit all of this information together into a diagram

$$\begin{tikzcd}
\mathbb{E}_0 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \arrow[r, "c \underset{\mathbb{A}_0}{\times} \mathbb{A}_1"] \arrow[d, "h"'] & \mathbb{E}_1 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \arrow[r, "d_1 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1"] \arrow[d, "f"] & \mathbb{E}_0 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \arrow[d, "h"] \\
1 \,{}_{t'}\!\times_{d_0} \mathbb{A}_1 \arrow[r, "t' \underset{\mathbb{A}_0}{\times} \mathbb{A}_1"'] & \mathbb{E}_0 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \arrow[r, "h"'] & 1 \,{}_{t'}\!\times_{d_0} \mathbb{A}_1
\end{tikzcd}$$

the important properties of which flow directly from the fact that c presents t as a terminal object in $\mathbb{E}$. We summarise these as follows; the condition in:

(2.22)	$\Rightarrow$	the left hand square commutes.
(2.23)	$\Rightarrow$	the composite of the maps in the upper line is the identity on $\mathbb{E}_0 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1$.
(2.24)	$\Rightarrow$	the right hand square commutes.
(2.25)	$\Rightarrow$	the composite of the maps in the lower line is the identity on $1 \,{}_{t'}\!\times_{d_0} \mathbb{A}_1$.

In fact these results are exactly what we need in order to verify that

$$\begin{tikzcd}
\mathbb{E}_1 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] & \mathbb{E}_0 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \arrow[l, bend right, "c \underset{\mathbb{A}_0}{\times} \mathbb{A}_1"'] \arrow[r, shift right, "h"'] & 1 \,{}_{t'}\!\times_{d_0} \mathbb{A}_1 \arrow[l, shift right, "t \underset{\mathbb{A}_0}{\times} \mathbb{A}_1"']
\end{tikzcd}$$

is a split coequaliser diagram. Furthermore the action induced on $1\,{}_{t'}\!\times_{d_0}\mathbb{A}_1$ is exactly that of the representable t'^*, and so $\mathbf{L}\begin{pmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{A}\end{pmatrix} \cong t'^* = (p_0 \circ t)^*$.

Now establish the truth of the original proposition by first recalling that the action of the $\underline{\mathcal{A}}$-adjunction $\mathbf{L}_{\underline{\mathcal{A}}} \dashv \mathbb{G}_{\underline{\mathcal{A}}}$ on underlying categories is obtained from the composite

$$\begin{tikzcd} \underline{\mathcal{A}}\text{–Prof}(\underline{\mathbf{A}}, \underline{\mathbf{1}}) \arrow[r, shift right, "\bar{\mathrm{I}}_\star"'] \arrow[r, phantom, "\perp"] & \mathrm{Prof}(\underline{\mathcal{A}})(\mathrm{I}_\star\underline{\mathbf{A}}, \mathbb{1}) \arrow[l, shift right, "\mathbf{F}"'] \arrow[r, shift right, "\mathbb{G}"'] \arrow[r, phantom, "\perp"] & \mathrm{Cat}(\widetilde{\underline{\mathcal{C}}})/_{\mathrm{I}_\star\underline{\mathbf{A}}} \arrow[l, shift right, "\mathbf{L}"'] \end{tikzcd}$$

by restriction. The equipment adjunction $\ddot{\mathrm{I}}^\star \dashv \ddot{\mathrm{I}}_\star$ is an inclusion, and so (by definition) $\ddot{\mathrm{I}}_\star$ preserves right representables, its action on profunctors ($\bar{\mathrm{I}}_\star$) is locally fully faithful and the functor $\mathrm{I}_\star$ is fully faithful. It follows that for any object $a \in \underline{\mathbf{A}}$, which we confuse notationally with the functor $a\colon \underline{\mathbf{1}} \longrightarrow \underline{\mathbf{A}}$, we have

$$\mathbb{G}\bar{\mathrm{I}}_\star(a^*) \cong \mathbb{G}((\mathrm{I}_\star a)^*)$$

where, by lemma (a), the internal category on the right has a terminal object. This is enough to establish the restriction of $\mathbb{G}_{\underline{\mathcal{A}}}$ to:

$$\mathbb{G}_{\underline{\mathcal{A}}}\colon \underline{\mathbf{A}} \longrightarrow \mathcal{T}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}}$$

Conversely lemma (b) implies that if $\mathbb{E}$ has a terminal object then $\mathbf{L}\begin{pmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathrm{I}_\star\underline{\mathbf{A}}\end{pmatrix}$ is right representable and therefore, since $\mathrm{I}_\star$ is fully faithful, there exists an object $e\colon \underline{\mathbf{1}} \longrightarrow \underline{\mathbf{A}}$ such that:

$$\bar{\mathrm{I}}_\star(e^*) \cong (\mathrm{I}_\star(e))^* \cong \mathbf{L}\begin{pmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathrm{I}_\star\underline{\mathbf{A}}\end{pmatrix}$$

Finally, because $\bar{\mathrm{I}}_\star$ is locally fully faithful, we get

$$\mathbf{FL}\begin{pmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathrm{I}_\star\underline{\mathbf{A}}\end{pmatrix} \cong \mathbf{F}\bar{\mathrm{I}}_\star(e^*) \cong e^*$$

which establishes the restriction of $\mathbf{L}_{\underline{\mathcal{A}}}$ to:

$$\mathbf{L}_{\underline{\mathcal{A}}}\colon \mathcal{T}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}} \longrightarrow \underline{\mathbf{A}}$$

$\square$

It is worth pointing out that $\mathcal{T}(\underline{\mathcal{A}})$ may not be the largest subset of $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ satisfying the result established in the last proposition for each $\underline{\mathbf{A}} \in \underline{\mathcal{A}}$–Cat. If we were doing straight internal category theory this would in fact hold true, but in order to prove that we need to take a condition which universally quantifies over all categories in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$, and not just the ones with a discrete $\underline{\mathcal{C}}$-set of objects (i.e. those of the form $\mathrm{I}_\star\underline{\mathbf{A}}$). This deficiency may allow categories not in $\mathcal{T}(\underline{\mathcal{A}})$ to have this property, which might seem to be undesirable, but is no problem to us since (as we will see) we are only really interested in having *some* such subset to act as a starting point, $\mathcal{T}(\underline{\mathcal{A}})$ has the advantage of being easy to define and work with.

Now suppose that we were given a set of categories $\mathcal{X} \subset \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ parameterising some $\underline{\mathcal{A}}$-colimits, the calculations in observation 2.5.2 and the work of the last two sections clearly identify the corresponding class of $\underline{\mathcal{A}}$-weights, which is given by:

$$\mathcal{X}(\underline{\mathbf{A}}) = \left\{ X \in \mathcal{P}(\underline{\mathbf{A}}) \,\middle|\, \left(\exists \begin{pmatrix} \mathbb{E} \\ {\scriptstyle p}\downarrow \\ \mathrm{I}_\star\underline{\mathbf{A}} \end{pmatrix} \in \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}) \right) \text{ s.t. } \mathbb{E} \in \mathcal{X} \text{ and } X \cong \mathbf{L}\begin{pmatrix} \mathbb{E} \\ {\scriptstyle p}\downarrow \\ \mathrm{I}_\star\underline{\mathbf{A}} \end{pmatrix} \right\}$$

Often it is less important to know about $\mathcal{X}(\text{–})$ than $\mathcal{X}^*(\text{–})$, its closure as a class of weights, the definition of which we reminded ourselves in section 2.1. It is therefore natural to ask whether we may describe this latter class in terms of some set of categories in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$?

The principal result of [1] gives us a strong clue about how to do this. Identifying sets of objects in an $\underline{\mathcal{A}}$-category with the corresponding full sub-$\underline{\mathcal{A}}$-categories (as we always do), the authors prove that $\mathcal{X}^*(\underline{\mathbf{A}})$ is the $\mathcal{X}$–colimit closure of $\underline{\mathbf{A}}$ in $\mathcal{P}(\underline{\mathbf{A}})$ as defined below. Combining this with the work of this section and in particular propositions 2.5.6, 2.5.9 and 2.5.11 indicate that we should consider the subset $\mathcal{X}^\# \subset \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ which is the $\mathcal{X}$–colimit closure of $\mathcal{T}(\underline{\mathcal{A}})$ in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$. Indeed the next proposition establishes that $\mathcal{X}^*(\text{–})$ admits the simple description

$$\mathcal{X}^*(\underline{\mathbf{A}}) = \left\{ X \in \mathcal{P}(\underline{\mathbf{A}}) \,\middle|\, \mathbb{G}_{\underline{\mathbf{A}}}(X) \in \mathcal{X}^\# \right\} \tag{2.27}$$

with respect to $\mathcal{X}^\#$.

First recall the transfinite construction (in the universe $\underline{\mathrm{SET}}$) of the $\mathcal{X}$–colimit closure $\underline{\mathbf{A}}^\circ$ of a full sub-$\underline{\mathcal{A}}$-category $\underline{\mathbf{A}}$ in $\underline{\mathbf{B}}$, which is a $\mathcal{X}$–cocomplete $\underline{\mathcal{A}}$-category:

$$\begin{aligned} \underline{\mathbf{A}}^\circ_0 &= \underline{\mathbf{A}} \\ \underline{\mathbf{A}}^\circ_{\alpha^+} &= \underline{\mathbf{A}}^\circ_\alpha \cup \left\{ B \in \underline{\mathbf{B}} \,\middle|\, \begin{array}{l} \exists \text{ a diagram } \Gamma\colon \mathbb{C} \longrightarrow \underline{\mathbf{A}}^\circ_\alpha \subset \underline{\mathbf{B}} \text{ with} \\ \mathbb{C} \in \mathcal{X} \text{ s.t. } B \cong \varinjlim_{\mathbb{C}} \Gamma \end{array} \right\} \\ \underline{\mathbf{A}}^\circ_\lambda &= \textstyle\bigcup_{\alpha<\lambda} \underline{\mathbf{A}}^\circ_\alpha \text{ for each limit ordinal } \lambda. \end{aligned}$$

This sequence must eventually converge, since $\underline{\mathbf{B}}$ has only a large set of objects, to a full sub-$\underline{\mathcal{A}}$-category $\underline{\mathbf{A}}^\circ$, which is clearly the closure we are looking for. We may now prove the ultimate theorem of this section:

Theorem 2.5.12 *For any given (small)* $\underline{\mathcal{A}}$*-category* $\underline{\mathbf{A}}$*, the* $\underline{\mathcal{A}}$*-adjunction*

$$\begin{tikzcd} \mathcal{P}(\underline{\mathbf{A}}) \arrow[r, hook, shift right, "\mathbb{G}_{\underline{\mathcal{A}}}"'] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}} \arrow[l, shift right, "\mathbf{L}_{\underline{\mathcal{A}}}"'] \end{tikzcd}$$

restricts to

$$\begin{tikzcd} \mathcal{X}^*(\underline{\mathbf{A}}) \arrow[r, hook, shift right, "\mathbb{G}_{\underline{\mathcal{A}}}"'] \arrow[r, phantom, "\perp"] & \mathcal{X}^\#/_{\mathrm{I}_\star\underline{\mathbf{A}}} \arrow[l, shift right, "\mathbf{L}_{\underline{\mathcal{A}}}"'] \end{tikzcd}$$

and this verifies the description of $\mathcal{X}^*(\text{-})$ *in (2.27).*

Proof. We know that $\mathcal{X}^*(\underline{\mathbf{A}}) = \underline{\mathbf{A}}^\circ$ (in $\mathcal{P}(\underline{\mathbf{A}})$) and $\mathcal{X}^\# = \mathcal{T}(\underline{\mathcal{A}})^\circ$ (in $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$) so we establish the theorem by transfinite induction on the construction of these. To get started notice that proposition 2.5.11 gives the restriction of $\mathbf{L}_{\underline{\mathcal{A}}} \dashv \mathbb{G}_{\underline{\mathcal{A}}}$ to:

$$\begin{tikzcd} \underline{\mathbf{A}}^\circ_0 \arrow[r, hook, shift right, "\mathbb{G}_{\underline{\mathcal{A}}}"'] \arrow[r, phantom, "\perp"] & \mathcal{T}(\underline{\mathcal{A}})^\circ_0/_{\mathrm{I}_\star\underline{\mathbf{A}}} \arrow[l, shift right, "\mathbf{L}_{\underline{\mathcal{A}}}"'] \end{tikzcd}$$

The argument at limit ordinals is clear, so all that remains is to check what happens at successors. As our induction hypothesis suppose that $\mathbf{L}_{\underline{\mathcal{A}}} \dashv \mathbb{G}_{\underline{\mathcal{A}}}$ restricts to

$$\begin{tikzcd} \underline{\mathbf{A}}^\circ_\alpha \arrow[r, hook, shift right, "\mathbb{G}_{\underline{\mathcal{A}}}"'] \arrow[r, phantom, "\perp"] & \mathcal{T}(\underline{\mathcal{A}})^\circ_\alpha/_{\mathrm{I}_\star\underline{\mathbf{A}}} \arrow[l, shift right, "\mathbf{L}_{\underline{\mathcal{A}}}"'] \end{tikzcd}$$

then we have

(a) given a diagram $\Gamma\colon \mathbb{D} \longrightarrow \underline{\mathbf{A}}^\circ_\alpha \subset \mathcal{P}(\underline{\mathbf{A}})$ with $\mathbb{D} \in \mathcal{X}$, we know that $\mathbb{G}_{\underline{\mathcal{A}}}$ preserves all $\underline{\mathcal{A}}$-colimits (by proposition 2.5.9) so

$$\mathbb{G}_{\underline{\mathcal{A}}}\left(\varinjlim_{\mathbb{D}} \Gamma\right) \cong \varinjlim_{\mathbb{D}} (\mathbb{G}_{\underline{\mathcal{A}}} \circ \Gamma)$$

but $\mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathrm{I}_\star\underline{\mathbf{A}}} \xrightarrow{\Sigma} \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$ creates all $\underline{\mathcal{A}}$-colimits (by proposition 2.5.6) and $(\mathbb{G}_{\underline{\mathcal{A}}} \circ \Gamma)$ is a diagram in $\mathcal{T}(\underline{\mathcal{A}})^\circ_\alpha/_{\mathrm{I}_\star\underline{\mathbf{A}}}$ (by the induction hypothesis). Therefore (by the induction clause defining $\mathcal{T}(\underline{\mathcal{A}})^\circ_{\alpha+}$) we know that $\varinjlim_{\underline{\mathbf{A}}} (\mathbb{G}_{\underline{\mathcal{A}}} \circ \Gamma)$ is in $\mathcal{T}(\underline{\mathcal{A}})^\circ_{\alpha+}/_{\mathrm{I}_\star\underline{\mathbf{A}}}$. But examining the clause defining $\underline{\mathbf{A}}^\circ_{\alpha+}$ we see that this is enough to show that $\mathbb{G}_{\underline{\mathcal{A}}}$ restricts to:

$$\underline{\mathbf{A}}^\circ_{\alpha+} \overset{\mathbb{G}_{\underline{\mathcal{A}}}}{\hookrightarrow} \mathcal{T}(\underline{\mathcal{A}})^\circ_{\alpha+}/_{\mathrm{I}_\star\underline{\mathbf{A}}}$$

(b) given a diagram $\Theta\colon \mathbb{A} \longrightarrow \mathcal{T}(a)^{\circ}_{\alpha}/_{\mathrm{I}_\star \underline{\mathbf{A}}} \subset \mathrm{Cat}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})/_{\mathrm{I}_\star \underline{\mathbf{A}}}$ with $\mathbb{D} \in \mathcal{X}$, we know that $\mathbf{L}_{\underline{\mathcal{A}}}$ preserves $\underline{\mathcal{A}}$-colimits (since it is a left $\underline{\mathcal{A}}$-adjoint) so:

$$\mathbf{L}_{\underline{\mathcal{A}}}\left(\varinjlim_{\mathbb{D}} \Theta\right) \cong \varinjlim_{\mathbb{D}} (\mathbf{L}_{\underline{\mathcal{A}}} \circ \Theta)$$

By the induction hypothesis $(\mathbf{L}_{\underline{\mathcal{A}}} \circ \Theta)$ is a diagram in $\underline{\mathbf{A}}^{\circ}_{\alpha}$, and so $\varinjlim_{\mathbb{D}} (\mathbf{L}_{\underline{\mathcal{A}}} \circ \Theta)$ is in $\underline{\mathbf{A}}^{\circ}_{\alpha^+}$ (by the induction clause which defines it). Finally examining the clause which defines $\mathcal{T}(a)^{\circ}_{\alpha^+}/_{\mathrm{I}_\star \underline{\mathbf{A}}}$ we see that this is enough to show that $\mathbf{L}_{\underline{\mathcal{A}}}$ restricts to:

$$\mathcal{T}(\underline{\mathcal{A}})^{\circ}_{\alpha^+}/_{\mathrm{I}_\star \underline{\mathbf{A}}} \xrightarrow{\ \mathbf{L}_{\underline{\mathcal{A}}}\ } \underline{\mathbf{A}}^{\circ}_{\alpha^+}$$

It is now established that the adjunction $\mathbf{L}_{\underline{\mathcal{A}}} \dashv \mathbb{G}_{\underline{\mathcal{A}}}$ restricts at each stage of the construction of these closures, so it must also do so for the categories to which the sequences converge. The principle restriction result of the theorem follows, with the remainder a simple corollary. $\square$

2.6 Persistent 2-limits.

In this section we describe and characterise a class of limits, in the theory of 2-categories, defined by a stability property with respect to equivalences. This was first introduced by Paré in [38] and dubbed by him the class of *persistent* limits, although his definition was not quite correct as stated; we fix it here. He also gave there a characterisation of this class which we study here in more detail. In the next section we will use the work of the last section to show that not only is this class a closed one, but furthermore it is identical to the class of *flexible* limits, a detailed account of which may be found in [7].

For the remainder of this chapter we will identify any 2-category $\underline{\mathbf{C}}$ with the double category $\mathrm{I}_\star\underline{\mathbf{C}}$, but continue to refer to its horizontal and double cells as 1- and 2-cells, its vertical cells are all identities, and we say it is *vertically discrete*. Before proceeding we should first clarify the structure of $\mathrm{Cat}(\underline{\mathrm{Cat}})$ as an $\mathrm{Cat}(\underline{\mathrm{Cat}})$-enriched category. If $\mathbb{A}$, $\mathbb{B}$ are categories in $\mathrm{Cat}(\underline{\mathrm{Cat}})$ we may describe the double category $\mathrm{Cat}(\underline{\mathrm{Cat}})(\mathbb{A},\mathbb{B})$ concretely as follows

Objects: Known as *double functors*, map the double cells of $\mathbb{A}$ to those of $\mathbb{B}$ while preserving horizontal and vertical identities, so we may picture the action of some such $\Gamma\colon \mathbb{A} \longrightarrow \mathbb{B}$ by:

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, dotted, "v"'] \arrow[dr, phantom, "\Downarrow\alpha"] & a' \arrow[d, dotted, "v'"] \\ \bar{a} \arrow[r, "\bar{h}"'] & \bar{a}' \end{tikzcd} \longmapsto \begin{tikzcd} \Gamma(a) \arrow[r, "\Gamma(h)"] \arrow[d, dotted, "\Gamma(v)"'] \arrow[dr, phantom, "\Downarrow\Gamma(\alpha)"] & \Gamma(a') \arrow[d, dotted, "\Gamma(v')"] \\ \Gamma(\bar{a}) \arrow[r, "\Gamma(\bar{h})"'] & \Gamma(\bar{a}') \end{tikzcd}$$

Furthermore this must be functorial with respect to both the horizontal and vertical composition of double cells.

Horizontal cells: Known as *horizontal (natural) transformations.* An example of such a transformation $\tau\colon \Gamma \longrightarrow \Gamma'$ is given by the following data:

For each object $a \in \mathbb{A}$ a horizontal cell

$$\Gamma(a) \xrightarrow{\tau_a} \Gamma'(a)$$

in $\mathbb{B}$, collectively obeying the naturality rule with respect horizontal cells in

$\mathbb{A}$, and for each vertical cell $v\colon a \cdots\!> \bar{a}$ a double cell

$$\begin{tikzcd}
\Gamma(a) \arrow[r, "\tau_a"] \arrow[d, dotted, "\Gamma(v)"'] \arrow[dr, phantom, "\Downarrow \tau_v"] & \Gamma'(a) \arrow[d, dotted, "\Gamma'(v)"] \\
\Gamma(\bar{a}) \arrow[r, "\tau_{\bar{a}}"'] & \Gamma'(\bar{a})
\end{tikzcd}$$

in $\mathbb{B}$. These collectively satisfy the obvious horizontal naturality condition w.r.t. the double cells of $\mathbb{A}$ and are compatible with vertical composition and identities. We may clarify these rules diagrammatically as follows:

$$\begin{tikzcd}
\Gamma(a) \arrow[r, "\tau_a"] \arrow[d, dotted, "\Gamma(v)"'] \arrow[dr, phantom, "\Downarrow^{\tau_v}"] & \Gamma'(a) \arrow[r, "\Gamma'(h)"] \arrow[d, dotted, "\Gamma'(v)"'] \arrow[dr, phantom, "\Downarrow^{\Gamma'(\alpha)}"] & \Gamma'(a') \arrow[d, dotted, "\Gamma'(v')"] \\
\Gamma(\bar{a}) \arrow[r, "\tau_{\bar{a}}"'] & \Gamma'(\bar{a}) \arrow[r, "\Gamma'(\bar{h})"'] & \Gamma'(\bar{a}')
\end{tikzcd}
=
\begin{tikzcd}
\Gamma(a) \arrow[r, "\Gamma(h)"] \arrow[d, dotted, "\Gamma(v)"'] \arrow[dr, phantom, "\Downarrow^{\Gamma(\alpha)}"] & \Gamma(a') \arrow[r, "\tau_{a'}"] \arrow[d, dotted, "\Gamma(v')"] \arrow[dr, phantom, "\Downarrow^{\tau_{v'}}"] & \Gamma'(a') \arrow[d, dotted, "\Gamma'(v')"] \\
\Gamma(\bar{a}) \arrow[r, "\Gamma(\bar{h})"'] & \Gamma(\bar{a}') \arrow[r, "\tau_{\bar{a}'}"'] & \Gamma'(\bar{a}')
\end{tikzcd}$$

for each double cell α in $\mathbb{A}$ and

$$\begin{tikzcd}
\Gamma(a) \arrow[r, "\tau_a"] \arrow[d, dotted, "\Gamma(v)"'] \arrow[dr, phantom, "\Downarrow^{\tau_v}"] & \Gamma'(a) \arrow[d, dotted, "\Gamma'(v)"] \\
\Gamma(\bar{a}) \arrow[r, "\tau_{\bar{a}}"] \arrow[d, dotted, "\Gamma(w)"'] \arrow[dr, phantom, "\Downarrow^{\tau_w}"] & \Gamma'(\bar{a}) \arrow[d, dotted, "\Gamma'(w)"] \\
\Gamma(\tilde{a}) \arrow[r, "\tau_{\tilde{a}}"] & \Gamma'(\tilde{a})
\end{tikzcd}
=
\begin{tikzcd}
\Gamma(a) \arrow[r, "\tau_a"] \arrow[d, dotted, "\Gamma(w\bullet v)"'] \arrow[dr, phantom, "\Downarrow^{\tau_w \bullet \tau_v}"] & \Gamma'(a) \arrow[d, dotted, "\Gamma'(w\bullet v)"] \\
\Gamma(\tilde{a}) \arrow[r, "\tau_{\tilde{a}}"] & \Gamma'(\tilde{a})
\end{tikzcd}$$

for each pair of vertical cells $v, w \in \mathbb{A}$. Compatibility with identities simply means that $\tau_{j_a} = j_{\tau_a}$ for each object $a \in \mathbb{A}$, where j_h denotes the vertical identity on the horizontal cell h.

Vertical cells: Known as *vertical (natural) transformations.* These are dual to horizontal transformations, we swap the rôles of the horizontal and vertical cells of $\mathbb{A}$ and $\mathbb{B}$ in the last definition.

Double cells: Known as *double (natural) transformations.* Such a transformation

$$\begin{tikzcd} \Gamma \arrow[r, "\tau"] \arrow[d, dotted, "\mu"'] \arrow[dr, phantom, "\Phi \Downarrow"] & \Gamma' \arrow[d, dotted, "\mu'"] \\ \overline{\Gamma} \arrow[r, "\bar{\tau}"'] & \overline{\Gamma}' \end{tikzcd} \quad \text{consists of double cells} \quad \begin{tikzcd} \Gamma(a) \arrow[r, "\tau_a"] \arrow[d, dotted, "\mu_a"'] \arrow[dr, phantom, "\Phi_a \Downarrow"] & \Gamma'(a) \arrow[d, dotted, "\mu'_a"] \\ \overline{\Gamma}(a) \arrow[r, "\bar{\tau}_a"'] & \overline{\Gamma}'(a) \end{tikzcd}$$

in $\mathbb{B}$, one for each object $a \in \mathbb{A}$, collectively obeying

$$\begin{tikzcd} \Gamma(a) \arrow[r, "\tau_a"] \arrow[d, dotted, "\mu_a"'] \arrow[dr, phantom, "\Phi_a \Downarrow"] & \Gamma'(a) \arrow[r, "\Gamma'(h)"] \arrow[d, dotted, "\mu'_a"] \arrow[dr, phantom, "\mu'_h \Downarrow"] & \Gamma'(a') \arrow[d, dotted, "\mu'_{a'}"] \\ \overline{\Gamma}(a) \arrow[r, "\bar{\tau}_a"'] & \overline{\Gamma}'(a) \arrow[r, "\overline{\Gamma}'(h)"'] & \overline{\Gamma}'(a') \end{tikzcd} = \begin{tikzcd} \Gamma(a) \arrow[r, "\Gamma(h)"] \arrow[d, dotted, "\mu_a"'] \arrow[dr, phantom, "\Downarrow \mu_h"] & \Gamma(a') \arrow[r, "\tau_{a'}"] \arrow[d, dotted, "\mu_{a'}"] \arrow[dr, phantom, "\Downarrow \Phi_{a'}"] & \Gamma'(a') \arrow[d, dotted, "\mu'_{a'}"] \\ \overline{\Gamma}(a) \arrow[r, "\overline{\Gamma}(h)"'] & \overline{\Gamma}(a') \arrow[r, "\bar{\tau}_{a'}"'] & \overline{\Gamma}'(a') \end{tikzcd}$$

for each horizontal cell $h\colon a \longrightarrow a'$ in $\mathbb{A}$, and the dual rule, involving the structure of τ and $\bar{\tau}$, for each vertical cell $v\colon a \cdots\!> \bar{a}$.

The various compositions and identities making the totality of these structures into a Cat(Cat)-enriched category are largely apparent and we leave the details to the reader. Observe that the 2-category we get by forgetting the horizontal and double transformations is the one described in proposition 2.5.4. A horizontal transformation $\tau\colon \Gamma \longrightarrow \Gamma'$ corresponds to an equivariant map $\Gamma'_* \longrightarrow \Gamma_*$ in $\mathrm{Prof}([\mathbf{\Delta}^{\mathrm{op}}, \underline{\mathrm{Set}}])$, and so if Γ, Γ' are diagrams in a 2-category $\underline{\mathbf{C}}$, then τ induces a unique 1-cell $(\varinjlim_{\mathbb{D}} \tau)\colon \varinjlim_{\mathbb{D}} \Gamma \longrightarrow \varinjlim_{\mathbb{D}} \Gamma'$ (when these double colimits exist in $\underline{\mathbf{C}}$). It also means that the 2-category obtained by dropping the vertical and double transformations is the obvious full sub-2-category of the canonical one obtained by applying proposition 1.2.10 to the equipment $\mathrm{Equip}([\mathbf{\Delta}^{\mathrm{op}}, \underline{\mathrm{Set}}])$.

Examples of horizontal and double transformations are the cones under a double diagram, defined at 1 and 2 respectively, as described in section 2.4. When we consider a 2-category $\underline{\mathbf{C}}$ as a vertically discrete double category, a double functor $\Gamma\colon \mathbb{D} \longrightarrow \underline{\mathbf{C}}$ takes all vertical cells to identities, so these merely serve to force their domain and codomain to map to the same object in $\underline{\mathbf{C}}$. Of course the vertical cells are far from being redundant since it is they that allow us to insert 2-cells into the horizontal and double cones over a double diagram.

Now for the principle definition of this section:

Definition 2.6.1 *(Paré) A (small) double category $\mathbb{D}$ parameterises a persistent limit iff for any (possibly large) 2-category $\underline{\mathbf{C}}$ and pair of diagrams $\Gamma, \Gamma'\colon \mathbb{D} \longrightarrow \underline{\mathbf{C}}$ related by a horizontal transformation $\tau\colon \Gamma \longrightarrow \Gamma'$ satisfying* *(i) for each object $d \in \mathbb{D}$, the 1-cell τ_d is an equivalence and*

(ii) for each vertical cell $(v\colon d \dashrightarrow \bar{d}) \in \mathbb{D}$, the 2-cell τ_v is an isomorphism,

the unique induced 1-cell $(\varprojlim_{\mathbb{D}} \tau)\colon \varprojlim_{\mathbb{D}} \Gamma \longrightarrow \varprojlim_{\mathbb{D}} \Gamma'$ (when these double limits exist in $\underline{\mathbf{C}}$) is an equivalence as well.

In fact, in his notes [38], Paré leaves out condition (ii) on those horizontal transformations that we require to be equivalence inducing. The following example demonstrates that the class defined strictly in this way does not even include cotensors, and is therefore not the one he was interested in:

Consider the 2-category $\underline{\underline{\mathrm{Cat}}}$, which has all weighted (and therefore double) limits, and define a category $\mathbf{B}$ in here with two objects $\perp$, $\top$ and homsets

$$\mathbf{B}(\top,\top) = \mathbf{B}(\perp,\perp) = \mathbb{N}$$
$$\mathbf{B}(\top,\perp) = \mathbf{B}(\perp,\top) = \mathbb{N} \setminus \{0\}$$

with composition

$$\begin{tikzcd} \mathbf{B}(y,z) \times \mathbf{B}(x,y) \arrow[r, "\circ"] & \mathbf{B}(x,z) \\ \langle n,m \rangle \arrow[r, mapsto] & n+m \end{tikzcd}$$

which is associative with identities $0 \in \mathbf{B}(\top,\top), \mathbf{B}(\perp,\perp)$. Notice that the objects of this category are non-isomorphic, since $0 \notin \mathbf{B}(\top,\perp), \mathbf{B}(\perp,\top)$. On this we have an involutive endo-functor

$$\begin{tikzcd} \mathbf{B} \arrow[r, "\neg"] & \mathbf{B} \\ \top,\perp \arrow[r, mapsto] & \perp,\top \\ x \arrow[d, "n"'] & \neg x \arrow[d, "n"'] \\ y \arrow[u, phantom, ""] \arrow[r, mapsto, shift left=4] & \neg y \end{tikzcd}$$

and a natural transformation

$$\begin{tikzcd} \mathbf{B} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\neg"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathbf{B} \end{tikzcd} \quad \text{with components} \quad \begin{array}{l} \tau_\top = 1\colon \top \longrightarrow \perp, \\ \tau_\perp = 1\colon \perp \longrightarrow \top, \end{array}$$

all of which, when taken together, forms an endomorphic horizontal transformation, τ, on the constant diagram at $\mathbf{B}$ (in $\underline{\underline{\mathrm{Cat}}}$) of the double category:

$$\mathbb{T} = \begin{tikzcd} \bullet \arrow[d, dotted] \\ \bullet \end{tikzcd}$$

Of course any diagram of $\mathbb{T}$ in a 2-category must simply pick out a single 0-cell and, by examining (and dualising) the definition of double colimit given in section 2.4, we conclude that the double limit of such a diagram is simply the cotensor of the 0-cell it picks out by $\mathbb{2}$. The functor induced by our horizontal transformation takes an object $(n\colon x \longrightarrow y) \in \mathbb{2} \pitchfork \mathbf{C}$ to one represented by the horizontal composite

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow n"] & \mathbf{B} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\neg"'] \arrow[r, phantom, "\Downarrow \tau"] & \mathbf{B} \end{tikzcd}$$

which, more explicitly, is the object $(n+1\colon x \longrightarrow \neg y) \in \mathbb{2} \pitchfork \mathbf{B}$. But $\top, \bot$ are non-isomorphic in $\mathbf{B}$, so if ${}^{x}_{y}\!\!\downarrow^{n} \cong {}^{x'}_{y'}\!\!\downarrow^{n'}$ in $\mathbb{2} \pitchfork \mathbf{B}$ then $x = x'$,$y = y'$ and $n = n'$ which implies that ${}^{\top}_{\top}\!\!\downarrow^{0}$ is not in the full image of $(\varprojlim_{\mathbb{T}} \tau)$, since 0 is not a successor. Hence this induced functor is not an equivalence, whereas both the identity and $\neg$ are isomorphisms.

Before the next theorem we introduce a little bit of notation. If $\mathbb{D}$ is a double category let $\mathbb{D}_H$ denote the category obtained by "dropping" the vertical and double cells of $\mathbb{D}$ with $\mathbb{D}_O$ being the set of its objects. We start to get a grip on the nature of persistent limits with:

Proposition 2.6.2 *If the double category $\mathbb{D}$ parameterises a persistent limit then each connected component of $\mathbb{D}_H$ has a natural weak initial object.*

Proof. Given a (small) double category $\mathbb{D}$ we will prove the proposition by constructing a suitable diagram $\Gamma\colon \mathbb{D} \longrightarrow \underline{\underline{\mathrm{Cat}}}$ with the property that each category $\Gamma(d)$ is the chaotic groupoid on some (non empty) set. This implies that the unique horizontal transformation from Γ to the constant diagram at $\mathbb{1}$ (call this $\Delta_{\mathbb{1}}$) satisfies the conditions of definition 2.6.1. Condition (ii) is satisfied vacuously since every natural transformation between functors into $\mathbb{1}$ is an identity, and condition (i) is a result of the fact that the chaotic groupoid on any (non empty) set is equivalent to $\mathbb{1}$. Now, assuming that $\mathbb{D}$ parameterises a persistent limit, these would imply that $\varprojlim_{\mathbb{D}} \Gamma \simeq \varprojlim_{\mathbb{D}} \Delta_{\mathbb{1}} \cong \mathbb{1}$, in particular $\varprojlim_{\mathbb{D}} \Gamma$ is non-empty and so there exists a horizontal cone $\tau\colon \mathbb{1} \overset{\cdot}{\longrightarrow} \Gamma$. With this in mind we would want to construct Γ with the hope that this cone would pick a weak initial object in each component of $\mathbb{D}_H$, along with the canonical natural maps from these to any other object.

As an example of how we might do this consider the simple case when $\mathbb{D}$ is vertically and doubly discrete (in other words the only vertical or double cells are the vertical identities), which implies that we may identify it with the category $\mathbb{D}_H$. Define a functor

$$\begin{array}{ccc} \mathbb{D}_H & \xrightarrow{\quad\Psi\quad} & \underline{\mathrm{Set}} \\ d & \longmapsto & \{h \in \mathbb{D}_H \,|\mathrm{cod}(h) = d\} \\ f\big\downarrow & \longmapsto & \big\downarrow\Psi(f) \qquad \text{g.b. } h \longmapsto f \circ h \\ d' & \longmapsto & \{h' \in \mathbb{D}_H \,|\mathrm{cod}(h') = d'\} \end{array}$$

and let

$$\underline{\mathrm{Set}} \xrightarrow{\text{chaotic}} \underline{\mathrm{Cat}}$$

denote the functor taking a set to the corresponding chaotic groupoid, then the composite (chaotic) $\circ\, \Psi$ is, in truth, a diagram in $\underline{\underline{\mathrm{Cat}}}$ on $\mathbb{D}$, since this is vertically and doubly discrete. A horizontal cone from $\mathtt{1}$ to this diagram gives rise to a classical cone from the one point set $\{*\}$ to Ψ in $\underline{\mathrm{Set}}$, picking out a horizontal cell $h_d\colon e_d \longrightarrow d$ for each object $d \in \mathbb{D}$ which collectively satisfy the naturality condition $f \circ h_d = h_{d'}$ for each horizontal cell $f\colon d \longrightarrow d'$. It follows directly that if d, d' are in the same connected component $\mathbf{C}$ of $\mathbb{D}_H$ then $\mathrm{dom}(h_d) = \mathrm{dom}(h_{d'})$ and this object along with attendant maps $\{h_d | d \in \mathbf{C}\}$ form a natural weak initial object for $\mathbf{C}$

In the general case we have the vertical and double cells of $\mathbb{D}$ to contend with, and so (chaotic) $\circ\, \Psi$ may not enrich to a double diagram, but we may modify this special case while preserving it's essence. Consider $\prod_{e \in \mathbb{D}_O} (\Psi(e) \times \Psi(e))$ and for each $f\colon d \longrightarrow d' \in \mathbb{D}_H$ define as function

$$\begin{array}{ccc} \displaystyle\prod_{e \in \mathbb{D}_O} (\Psi(e) \times \Psi(e)) & \xrightarrow{\Gamma(f)} & \displaystyle\prod_{e \in \mathbb{D}_O} (\Psi(e) \times \Psi(e)) \\ \underset{\sim}{h} = \{(h_{e,0}, h_{e,1})\}_{e \in \mathbb{D}_O} & \longmapsto & \underset{\sim}{k} = \{(k_{e,0}, k_{e,1})\}_{e \in \mathbb{D}_O} \end{array}$$

$$\text{g.b. } k_{e,0} = h_{e,0}$$
$$k_{e,1} = \begin{cases} h_{e,0} & \text{if } e \neq d' \\ f \circ h_{d,1} & \text{if } e = d' \end{cases}$$

It is clear that $\Gamma(g \circ f) = \Gamma(g) \circ \Gamma(f)$ and that $\Gamma(i_d)$ is the identity so we have defined a functor:

$$\mathbb{D}_H \xrightarrow{\quad\Gamma\quad} \underline{\mathrm{Set}}$$

$$\begin{array}{ccc} d & \longmapsto & \displaystyle\prod_{e \in \mathbb{D}_O} (\Psi(e) \times \Psi(e)) \\ f \big\downarrow & \longmapsto & \big\downarrow \Gamma(f) \\ d' & \longmapsto & \displaystyle\prod_{e \in \mathbb{D}_O} (\Psi(e) \times \Psi(e)) \end{array}$$

Observe that from any cone over Γ, with vertex $\{*\}$, we may derive a cone with the same vertex over Ψ, and that we may extend (chaotic) $\circ\, \Gamma$ to a double diagram:

$$\mathbb{D} \xrightarrow{\quad\overline{\Gamma}\quad} \underline{\underline{\mathrm{Cat}}}$$

Each object of $\mathbb{D}$ maps to the same thing under Γ so the action of $\overline{\Gamma}$ on vertical cells takes care of itself, as does the action on double cells, simply because between any pair of functors

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathrm{chaotic}(X) \end{tikzcd}$$

there exists a unique natural transformation $\alpha\colon f \longrightarrow g$. Now apply the same reasoning as before using $\overline{\Gamma}$ to obtain cones over Γ and therefore Ψ, with vertex $\{*\}$, thus establishing the proposition. $\square$

In order to establish the converse we simplify matters with:

Lemma 2.6.3 *$\mathbb{D}$ parameterises a persistent limit iff it satisfies the property of definition 2.6.1 for all diagrams in* $\underline{\underline{\mathrm{CAT}}}$

Proof. For clarification of bicategorical terminology and the Yoneda lemma for bicategories see [48] and [3].

Let $\underline{\mathcal{B}}$, $\underline{\mathcal{C}}$ be bicategories and consider a *strong* transformation

$$\begin{tikzcd} \underline{\mathcal{B}} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \underline{\mathcal{C}} \end{tikzcd}$$

with the property that, for each 0-cell $b \in \underline{\mathcal{B}}$, the 1-cell $\theta_b\colon F(b) \longrightarrow G(b)$ is an equivalence. Of course we may choose equivalence inverses θ_b^{-1} and 2-cells $\eta_b\colon i_{G(b)} \xRightarrow{\simeq} \theta_b \circ \theta_b^{-1}$, $\epsilon_b\colon \theta_b^{-1} \circ \theta_b \xRightarrow{\simeq} i_{F(b)}$ making each quadruple $(\theta_b, \theta_b^{-1}, \eta_b, \epsilon_b)$ into an *adjoint* equivalence. By taking mates of the structure 2-cells of θ under these adjunctions, we provide the 1-cells θ_b^{-1} with the 2-cellular structure of a strong transformation:

$$\begin{tikzcd} \underline{\mathcal{B}} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Uparrow\theta^{-1}"] & \underline{\mathcal{C}} \end{tikzcd}$$

Furthermore the construction of θ^{-1} is motivated by the fact that the 2-cells η_b, ϵ_b then satisfy the rules making them into modifications:

$$I_G \xrightarrow[\simeq]{\ \eta\ } \theta \circ \theta^{-1} \quad \text{and} \quad \theta^{-1} \circ \theta \xrightarrow[\simeq]{\ \epsilon\ } I_F$$

In other words θ is an equivalence in $\mathcal{H}om_S(\underline{\mathcal{B}}, \underline{\mathcal{C}})$, the bicategory of homomorphisms, strong transformations and modifications from $\underline{\mathcal{B}}$ to $\underline{\mathcal{C}}$.

Assume now that a double category $\mathbb{D}$ has the persistency property with respect to diagrams in $\underline{\underline{\mathrm{CAT}}}$ and consider a pair of diagrams $\Gamma, \Gamma'\colon \mathbb{D} \longrightarrow \underline{\mathbf{C}}$, both of which we will assume to have double limits in $\underline{\mathbf{C}}$. Connect them by a horizontal transformation $\tau\colon \Gamma \longrightarrow \Gamma'$ satisfying the conditions of definition 2.6.1, and examine what happens when we apply the 2-functor $\underline{\mathbf{C}}(c, \text{-})\colon \underline{\mathbf{C}} \longrightarrow \underline{\underline{\mathrm{CAT}}}$ (for an arbitrary 0-cell $c \in \underline{\mathbf{C}}$) to this data.

This functor carries the double limiting cones over Γ and Γ' to double limiting cones over the composite diagrams $\underline{\mathbf{C}}(c, \Gamma(\text{-}))$ and $\underline{\mathbf{C}}(c, \Gamma'(\text{-}))$ in $\underline{\underline{\mathrm{CAT}}}$, therefore the functor $\underline{\mathbf{C}}(c, \varprojlim_{\mathbb{D}} \tau)\colon \underline{\mathbf{C}}(c, \varprojlim_{\mathbb{D}} \Gamma) \longrightarrow \underline{\mathbf{C}}(c, \varprojlim_{\mathbb{D}} \Gamma')$ is the unique one induced by the horizontal transformation $\underline{\mathbf{C}}(c, \tau)\colon \underline{\mathbf{C}}(c, \Gamma(\text{-})) \longrightarrow \underline{\mathbf{C}}(c, \Gamma'(\text{-}))$. This satisfies the two conditions of definition 2.6.1, and so since $\mathbb{D}$ is persistent for diagrams in $\underline{\underline{\mathrm{CAT}}}$ each $\underline{\mathbf{C}}(c, \varprojlim_{\mathbb{D}} \tau)$ is an equivalence.

These functors constitute the 1-cellular components of the strong transformation obtained by applying the Yoneda homomorphism $\mathcal{Y}\colon \underline{\mathbf{C}} \longrightarrow \mathcal{H}om_S(\underline{\mathbf{C}}^{\mathrm{op}}, \underline{\underline{\mathrm{CAT}}})$ to the 1-cell $(\varprojlim_{\mathbb{D}} \tau)$. Hence, by the "coherence" result at the beginning of this proof, $\mathcal{Y}(\varprojlim_{\mathbb{D}} \tau)$ is an equivalence in $\mathcal{H}om_S(\underline{\mathbf{C}}^{\mathrm{op}}, \underline{\underline{\mathrm{CAT}}})$ but the bicategorical Yoneda lemma informs us that $\mathcal{Y}$ reflects equivalences, and so $(\varprojlim_{\mathbb{D}} \tau)$ is an equivalence. This establishes that $\mathbb{D}$ parameterises a persistent limit. $\square$

In $\underline{\underline{\mathrm{CAT}}}$ it is quite easy to give a description of $(\varprojlim_{\mathbb{D}} \Gamma)$, using the usual trick of identifying objects of a category $\mathbf{C}$ with functors

$$\mathtt{1} \xrightarrow{\quad c \quad} \mathbf{C}$$

and its morphisms with natural transformations:

$$\begin{tikzcd} \mathtt{1} \arrow[r, bend left, "c"] \arrow[r, bend right, "c'"'] \arrow[r, phantom, "\Downarrow f"] & \mathbf{C} \end{tikzcd}$$

Under this identification $(\varprojlim_{\mathbb{D}} \Gamma)$ has objects (morphisms) corresponding to horizontal (double) cones over Γ with vertex $\mathtt{1}$, furthermore some notation and calculations become simplified, for instance we may now talk of the horizontal composition of a morphism in a category with a natural transformation. To explain what we mean consider the simple example of a morphism $f\colon b \longrightarrow b' \in \mathbf{B}$ and a natural transformation

$$\begin{tikzcd} \mathbf{B} \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathbf{C} \end{tikzcd}$$

then the horizontal composite $\alpha \circ f$ is simply that of α with the natural transformation identified with f, in other words it is the leading diagonal of the commutative square:

$$\begin{tikzcd} g(b) \arrow[r, "\alpha_b"] \arrow[d, "g(f)"'] \arrow[dr] & g'(b) \arrow[d, "g'(f)"] \\ g(b') \arrow[r, "\alpha_{b'}"'] & g'(b') \end{tikzcd}$$

This convention also implies that we should write the composite of f with another morphism $g\colon b' \longrightarrow \bar{b} \in \mathbf{B}$ as $g \bullet f$, in other words it is the vertical composite of the corresponding 2-cells. Often we will follow tradition by dropping explicit use of $\circ$ to denote horizontal composition, but we will always retain instances of $\bullet$ for vertical composition in order to avoid confusion. Let $\mathbb{D}_V$ denote the category of vertical cells in $\mathbb{D}$ under vertical composition, and we have the following description of $(\varprojlim_{\mathbb{D}} \Gamma)$:

Objects:

$$\underset{\sim}{x} = \left\langle \{x_d \in \Gamma(d)_{\mathrm{O}}\}_{d\in\mathbb{D}_{\mathrm{O}}}, \left\{x_v : x_d \longrightarrow x_{\bar{d}} \in \Gamma(d) = \Gamma(\bar{d})\right\}_{(v:d\to\bar{d})\in\mathbb{D}_V} \right\rangle$$

satisfying

(a) $x_{w\bullet v} = x_w \bullet x_v$ for all compatible cells $v, w \in \mathbb{D}_V$,
$x_{j_d} = i_{x_d}$ for all objects $d \in \mathbb{D}_{\mathrm{O}}$.

(b) $\Gamma(h)x_d = x_{d'}$ for each $h: d \longrightarrow d' \in \mathbb{D}_H$.

(c) $\Gamma(\alpha)x_v = x_{v'}$ for each double cell

$$\begin{tikzcd} d \arrow[r, "h"] \arrow[d, dotted, "v"'] \arrow[dr, phantom, "\Downarrow\alpha"] & d' \arrow[d, dotted, "v'"] \\ \bar{d} \arrow[r, "\bar{h}"'] & \bar{d}' \end{tikzcd} \quad \text{in } \mathbb{D}.$$

Morphisms: An example $\underset{\sim}{f}: \underset{\sim}{x} \longrightarrow \underset{\sim}{y}$ consists of a family of morphisms $\{f_d : x_d \longrightarrow y_d \in \Gamma(d)\}_{d\in\mathbb{D}_{\mathrm{O}}}$ such that:

(a) $\Gamma(h)f_d = f_{d'}$ for each $h: d \longrightarrow d' \in \mathbb{D}_H$.

(b) The square

$$\begin{tikzcd} x_d \arrow[r, "x_v"] \arrow[d, "f_d"'] & x_{\bar{d}} \arrow[d, "f_{\bar{d}}"] \\ y_d \arrow[r, "y_v"'] & y_{\bar{d}} \end{tikzcd}$$

commutes for each vertical cell $v: d \cdots\!\!> \bar{d} \in \mathbb{D}$.

The action of the functor $(\underleftarrow{\lim}_{\mathbb{D}}\,\tau)$, induced by a horizontal transformation $\tau: \Gamma \longrightarrow \Gamma'$, is also of interest and is given by:

$$\begin{aligned} (\underleftarrow{\lim}_{\mathbb{D}}\,\tau)\underset{\sim}{x} &= \left\langle \{\tau_d x_d\}_{d\in\mathbb{D}_{\mathrm{O}}}, \{\tau_v x_v : \tau_d x_d \longrightarrow \tau_{\bar{d}} x_{\bar{d}}\}_{(v:d\to\bar{d})\in\mathbb{D}_V} \right\rangle \\ (\underleftarrow{\lim}_{\mathbb{D}}\,\tau)\underset{\sim}{f} &= \{\tau_d f_d : \tau_d x_d \longrightarrow \tau_d y_d\}_{d\in\mathbb{D}_{\mathrm{O}}} \end{aligned}$$

We have an easy lemma:

Lemma 2.6.4 *For any (small) double category $\mathbb{D}$ and horizontal transformation $\tau: \Gamma \longrightarrow \Gamma'$ between two diagrams of $\mathbb{D}$ in $\underline{\underline{\mathrm{CAT}}}$ satisfying*

(i) for each object $d \in \mathbb{D}$ the functor τ_d is fully faithful.

(ii) for each vertical cell $v: d \cdots\!\!> \bar{d} \in \mathbb{D}$ the 2-cell τ_v is an isomorphism.

the induced functor $(\varprojlim_{\mathbb{D}} \tau)$ *is fully faithful.*

Proof. Suppose that τ satisfies the conditions given, then from the description of $(\varprojlim_{\mathbb{D}} \tau)$ its faithfulness is clearly evident, and fullness remains to be proved. So suppose that we have $\underset{\sim}{g}: (\varprojlim_{\mathbb{D}} \tau)\underset{\sim}{x} \longrightarrow (\varprojlim_{\mathbb{D}} \tau)\underset{\sim}{y}$ then each τ_d is fully faithful and it follows that there exists a unique $f_d: x_d \longrightarrow y_d$ such that $\tau_d f_d = g_d$. Observe that we can establish the two conditions that the family $\{f_d\}_{d \in \mathbb{D}_O}$ must satisfy to be a morphism in $\varprojlim_{\mathbb{D}} \Gamma$ as follows:

(i) For an arbitrary horizontal cell $(h: d \longrightarrow d') \in \mathbb{D}$ the naturality of τ dictates that $\tau_{d'} \circ \Gamma(h) = \Gamma'(h) \circ \tau_d$ therefore

$$\begin{array}{rcll} \tau_{d'}\Gamma(h)f_d & = & \Gamma'(h)\tau_d f_d & \\ & = & \Gamma'(h)g_d & \text{by the definition of } f_d \\ & = & g_{d'} & \text{rule (a) for } \underset{\sim}{g} \\ & = & \tau_{d'}f_{d'} & \text{by the definition of } f_{d'} \end{array}$$

and hence $\Gamma(h)f_d = f_{d'}$ because $\tau_{d'}$ is faithful.

(ii) For an arbitrary vertical cell $v: d \cdots\!> d' \in \mathbb{D}$ the morphism $\underset{\sim}{g}$ satisfies rule (b) which we combine with the definition of each f_d to get:

$$\begin{array}{rcll} \tau_v(y_v \bullet f_d) & = & (\tau_v y_v) \bullet (\tau_d f_d) & \text{middle four interchange} \\ & = & (\tau_{d'} f_{d'}) \bullet (\tau_v x_v) & \text{rule (ii) for } \underset{\sim}{g} \\ & = & \tau_v(f_{d'} \bullet x_v) & \text{middle four interchange} \end{array}$$

But suppose we had $(k: x \longrightarrow y) \in \Gamma(d)$ then $\tau_v k = (\tau_v x) \bullet (\tau_{\bar{d}} k)$ holds, $\tau_{\bar{d}}$ is an isomorphism and τ_v is faithful so if $\tau_v k = \tau_v k'$ then $k = k'$. Applying this to the calculation above we find that $y_v \bullet f_d = f_{d'} \bullet x_v$, thus concluding the verification of rule (ii) for $\underset{\sim}{f}$ and demonstrating that this is the unique morphism in $(\varprojlim_{\mathbb{D}} \Gamma)$ with $(\varprojlim_{\mathbb{D}} \tau)\underset{\sim}{f} = \underset{\sim}{g}$.

$\square$

We might naïvely expect the proof that $(\varprojlim_{\mathbb{D}} \tau)$ is essentially surjective on objects (when τ satisfies the conditions of definition 2.6.1) to go along similar lines, but are immediately presented with a complicating factor. Starting with an object $\underset{\sim}{x}' \in (\varprojlim_{\mathbb{D}} \Gamma')$, it is certainly true that for each $d \in \mathbb{D}_O$ we may pick an object $x_d \in \Gamma(d)$ with $\tau_d x_d \cong x'_d$, but for these to form an object in $(\varprojlim_{\mathbb{D}} \Gamma)$ they must (at least) satisfy the horizontal naturality rule $\Gamma(h)x_d = x_{d'}$, for each horizontal cell h. The problem is that we are allowed a degree of choice in selecting each x_d and therefore cannot rely on this rule holding merely because it does so for $\underset{\sim}{x}'$. In fact if the horizontal structure of $\mathbb{D}$ imposes any *essentially* non-trivial relations on the x_ds we may be unable to choose them in any coherent way at all and it is here that our condition on $\mathbb{D}_H$ comes into play.

Let $\{\mathbb{C}_l\}_{l\in I}$ denote the indexed set of connected components of $\mathbb{D}_H$, and $\{w_l\}_{l\in I}$ a set of objects such that w_l is a natural weak initial object for $\mathbb{C}_l$. We will adopt the convention that if x is an object of $\mathbb{C}_l$ then $\square_x\colon w_l \longrightarrow x$ is the leg (at x) of the cone which displays the natural weak initiality of w_l. The idea of the next proposition is that we only need make one arbitrary choice in each component $\mathbb{C}_l$, at the object w_l:

Proposition 2.6.5 *If the (small) double category $\mathbb{D}$ is such that each connected component of $\mathbb{D}_H$ has a natural weak initial object, and $\tau\colon\Gamma \longrightarrow \Gamma'$ is a horizontal transformation between two diagrams of $\mathbb{D}$ in $\underline{\underline{\mathrm{CAT}}}$ satisfying the conditions of definition 2.6.1 then $(\varprojlim_{\mathbb{C}}\tau)$ is essentially surjective.*

Proof. Suppose $\underset{\sim}{x}'$ is an arbitrary object of $(\varprojlim_{\mathbb{D}}\Gamma')$ and construct an object $\underset{\sim}{x}$ of $(\varprojlim_{\mathbb{D}}\Gamma)$ by first picking an $\bar{x}_l \in \Gamma(w_l)$ and an isomorphism $\bar{g}_l\colon \tau_{w_l}\bar{x}_l \xrightarrow{\cong} x'_{w_l}$ for each w_l, which we are at liberty to do this since each τ_{w_l} is essentially surjective. Now define $x_d = \Gamma(\square_d)\bar{x}_l$ and $g_d = \Gamma'(\square_d)g_{w_l}\colon x_d \xrightarrow{\cong} \Gamma'(\square_d)x'_l = x'_d$ when d is an object in component $\mathbb{C}_l$, and notice that these represent a coherent choice of x_ds satisfying the horizontal naturality rule, simply because $h \circ \square_d = \square_{d'}$ for any horizontal cell h (the naturality property of the $\square_d$s) and therefore:

$$\begin{aligned} \Gamma(h)x_d = \Gamma(h)\Gamma(\square_d)\bar{x}_l = \Gamma(\square_{d'})\bar{x}_l = x_{d'} \quad &\text{and} \\ \Gamma'(h)g_d = \Gamma'(h)\Gamma'(\square_d)\bar{g}_l = \Gamma'(\square_{d'})\bar{g}_l = g_{d'}. & \end{aligned}$$

Notice that it is not necessarily true that $x_{w_l} = \bar{x}_l$, but this is of no consequence since the "$\bar{x}_l$" and "$\bar{g}_l$" have done their job in allowing us to define the "x_d" and "g_d" so are no longer needed.

For any vertical cell $v\colon d \longrightarrow \bar{d} \in \mathbb{D}$ define $x_v\colon x_d \longrightarrow x_{\bar{d}}$ to be the unique map making the diagram

$$\begin{tikzcd} \tau_d x_d \arrow[r, "g_d", "\cong"'] \arrow[d, "\tau_v x_v"'] & x'_d \arrow[d, "x'_v"] \\ \tau_{\bar{d}} x_{\bar{d}} \arrow[r, "\cong", "g_{\bar{d}}"'] & x'_{\bar{d}} \end{tikzcd}$$

commute. This exists and is unique since $\tau_v k = (\tau_v x) \bullet (\tau_{\bar{d}} k)$ for any $(k\colon x \longrightarrow y) \in \Gamma(d)$, $g_{\bar{d}}$ and $\tau_v x$ are isomorphisms and $\tau_{\bar{d}}$ is fully faithful. If we can demonstrate that the collection $\underset{\sim}{x}$ is an object of $(\varprojlim_{\mathbb{D}}\Gamma)$ then the x_vs are defined in order to ensure that $\underset{\sim}{g}$ would then become an isomorphism in $(\varprojlim_{\mathbb{D}}\Gamma')$ between $(\varprojlim_{\mathbb{D}}\tau)\underset{\sim}{x}$ and $\underset{\sim}{x}'$.

First off we check the compatibility of the collection $\underset{\sim}{x}$ with vertical composition in $\mathbb{D}$, consider the calculation

$$\begin{array}{rcll} \tau_{w\bullet v}(x_w \bullet x_v) & = & (\tau_w \bullet \tau_v)(x_w \bullet x_v) & \text{compatibility of } \tau \text{ with } \bullet \\ & = & (\tau_w x_w) \bullet (\tau_v x_v) & \text{middle four interchange} \\ & = & g_{\tilde{d}}^{-1} \bullet x'_w \bullet x'_v \bullet g_d & \text{by the definition of } x_v, x_w \\ & = & g_{\tilde{d}}^{-1} \bullet x'_{w\bullet v} \bullet g_d & \text{compatibility of } \underset{\sim}{x}' \text{ with } \bullet \\ & = & \tau_{w\bullet v}(x_{w\bullet v}) & \text{definition of } x_{w\bullet v} \end{array}$$

from which it follows that $x_{w\bullet v} = x_w \bullet x_v$ by the uniqueness clause in the definition of the x_vs. A similar calculation establishes compatibility with vertical identities.

All that remains is to demonstrate the horizontal naturality rule with respect to any double cell α of $\mathbb{D}$, via the calculation

$$\begin{array}{rcll} g_{\bar{d}'} \bullet (\tau_{v'}\Gamma(\alpha)x_v) & = & (\Gamma'(\bar{h})g_{\bar{d}}) \bullet (\Gamma'(\alpha)\tau_v x_v) & \\ & = & \Gamma'(\alpha)(g_{\bar{d}} \bullet (\tau_v x_v)) & \text{middle four interchange} \\ & = & \Gamma'(\alpha)(x'_v \circ g_d) & \text{definition of } x_v \\ & = & (\Gamma'(\alpha)x'_v) \bullet (\Gamma'(h)g_d) & \text{middle four interchange} \\ & = & x'_{v'} \bullet g_{d'} & \text{by horizontal naturality for } \underset{\sim}{x}' \end{array}$$

entailing that $\Gamma(\alpha)x_v = x_{v'}$ by the uniqueness clause of the definition of $x_{v'}$. $\square$

The ultimate result of this section is:

Theorem 2.6.6 *A (small) double category $\mathbb{D}$ parameterises a persistent limit iff each connected component of $\mathbb{D}_H$ has a natural weak initial object.*

Proof.
"$\Rightarrow$" This is just proposition 2.6.2.
"$\Leftarrow$" If $\tau\colon \Gamma \longrightarrow \Gamma'$ is a horizontal transformation between diagrams on $\mathbb{D}$ in $\underline{\underline{\mathrm{CAT}}}$ satisfying the conditions of definition 2.6.1 then lemma 2.6.4 and proposition 2.6.5 establish that $(\varprojlim_{\mathbb{D}} \tau)$ is an equivalence. Lemma 2.6.3 extends this result to diagrams in all 2-categories and hence $\mathbb{D}$ parameterises a persistent limit. $\square$

Of course the we have a dual result for colimits, characterising persistent colimits as those parameterised by double categories with a natural weak *terminal* object for each horizontal component.

2.7 Flexible Limits.

Naturally we might ask whether the class of persistent limits has been studied in other contexts. Early in the development of 2-category theory it was noticed that certain 2-limits, like for instance 2-pullbacks or 2-equalisers, were quite badly behaved and as a result did not even exist in many of the natural 2-categories of interest. It was soon realised that this behaviour was due to a lack of "flexibility" in the 2-limit concerned. For instance Persistent limits are defined by a natural "flexibility" property, and the work we did demonstrates quite clearly that "rigidity" of a 2-limit seems to stem from the presence of essentially non-trivial relations imposed on objects in its construction.

Indeed one particularly well studied class of 2-limits has been dubbed the class of *flexible* limits, the rather technical definition of which we will spare the reader from here, referring him or her to [7] instead. In that paper the authors introduce this class and demonstrate that every flexible limit may be constructed from products, inserters, equifiers and splitting of (strict) idempotents, each of these being kinds of flexible limit themselves. For us their most important result establishes that flexible limits form a closed class and so, for our purposes, it is sufficient to think of it as the closure of the class consisting of those 2-limit types mentioned in the last sentence, and they use the acronym (PIES)*

From the last section two important questions remain to be answered:

- Is the class of persistent limits closed?
- How does it relate to the class of flexible ones?

It will become clear that these questions may be answered in one go but first, as a warm up to doing that, it is worth thinking a little more about the limits from which we construct flexible ones. In fact, in view of what we discussed in section 2.5, we are more interested in colimits and in particular how to calculate them in the 2-category $\mathrm{Cat}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Cat}})$:

coproducts These are those colimits parameterised by double categories consisting of a discrete set of objects, so they are persistent (by the characterisation of the previous section) and are calculated in $\mathrm{Cat}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Cat}})$ as coproducts in the underlying category $\mathrm{Cat}_0(\underline{\mathrm{Cat}})$.

(co)splitting of idempotents Parameterised by the category with one object and one non-identity arrow

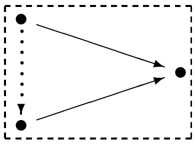
$$\mathbb{J} = \boxed{\begin{tikzcd} \bullet \arrow[loop above, "j"] \end{tikzcd} \quad \text{s.t. } j \circ j = j}$$

which we consider to be a vertically and doubly discrete double category. A double diagram on $\mathbb{J}$ in some 2-category $\underline{\mathbf{A}}$ corresponds to a (strict) idempotent $j\colon A \longrightarrow A$ in $\underline{\mathbf{A}}_0$ the double colimit of which, when it exists, is simply a splitting

$$\begin{tikzcd} A_j \arrow[r, tail, shift right, "m_j"'] & A \arrow[l, shift right, "e_j"'] \end{tikzcd}$$

Idempotent splitting is a persistent colimit since $j\colon \bullet \longrightarrow \bullet$ is a natural cone displaying $\bullet$ as a natural weak terminal object of $\mathbb{J}$.

coinserter Parameterised by the double category:

$\mathbb{I} =$ 

The category $\mathbb{I}_H$ has a terminal object so coinserters are persistent 2-colimits. A diagram of this in a 2-category $\underline{\mathbf{A}}$ consists of a parallel pair of 1-cells

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd} \tag{2.28}$$

and a horizontal cone with that codomain is a pair (l, θ) where $l\colon B \longrightarrow C$ is a 1-cell and θ is a 2-cell:

$$\begin{tikzcd} & B \arrow[dr, "l"] & \\ A \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow\theta"] & & C \\ & B \arrow[ur, "l"'] & \end{tikzcd}$$

Leaving the determination of the double cones up to the reader we go on to talk about the computation of these in $\mathrm{Cat}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Cat}})$. Given a pair of double functors $f, g\colon \mathbb{A} \longrightarrow \mathbb{B}$ we construct their coinserter $\mathrm{coins}(f, g)$ by starting with $\mathbb{B}$ and freely adjoining:

vertical cells $\lceil a \rceil\colon fa \cdots\!> ga$ one for each object $a \in \mathbb{A}$ and

double cells $\begin{tikzcd} fa \arrow[r, "fh"] \arrow[d, dotted, "\lceil a \rceil"'] \arrow[dr, phantom, "\Downarrow \lceil h \rceil"] & fa' \arrow[d, dotted, "\lceil a' \rceil"] \\ ga \arrow[r, "gh"'] & ga' \end{tikzcd}$ one for each horizontal $(h\colon a \longrightarrow a') \in \mathbb{A}$

subject to the extra relations:

$$\lceil \bar{a} \rceil \bullet (fv) = (gv) \bullet \lceil a \rceil \quad \text{for each vertical cell} \quad (v\colon a \cdots\!\!> \bar{a}) \in \mathbb{A}$$

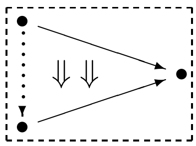
$$\lceil \bar{h} \rceil \bullet (f\alpha) = (g\alpha) \bullet \lceil h \rceil \quad \text{for each double cell} \quad \begin{tikzcd} a \arrow[r, "h"] \arrow[d, dotted, "v"'] \arrow[dr, phantom, "\Downarrow\alpha"] & a' \arrow[d, dotted, "v'"] \\ \bar{a} \arrow[r, "\bar{h}"'] & \bar{a}' \end{tikzcd} \in \mathbb{A}$$

$$\lceil i_a \rceil = i_{\lceil a \rceil} \quad \text{for each object} \quad a \in \mathbb{A}$$

$$\lceil h' \rceil \circ \lceil h \rceil = \lceil h' \circ h \rceil \quad \text{for each pair of horizontal cells} \quad \begin{matrix} (h\colon a \longrightarrow a'), \\ (h'\colon a' \longrightarrow a'') \end{matrix} \in \mathbb{A}$$

The cone displaying this as a colimit is a pair (i, ϕ) where $i\colon \mathbb{B} \longrightarrow \operatorname{coins}(f, g)$ is the canonical inclusion and $\phi\colon i \circ f \cdots\!\!> i \circ g$ has components $\phi_a = \lceil a \rceil (\forall a \in \mathbb{A}_O)$ and $\phi_h = \lceil h \rceil (\forall h \in \mathbb{A}_H)$. The relations imposed above are precisely chosen to ensure that these cells satisfy the conditions necessary for ϕ to be a vertical transformation.

coequifier Parameterised by the double category:

$\mathbb{E} =$ 

The category $\mathbb{E}_H$ has a terminal object so coequifiers are persistent 2-colimits. A diagram of this in a 2-category consists of a parallel pair of 1-cells with a pair of 2-cells between them

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha \Downarrow \Downarrow \beta"] & B \end{tikzcd} \tag{2.29}$$

and a horizontal cone with that codomain consists of a 1-cell $l\colon B \longrightarrow C$ satisfying the equation $l \circ \alpha = l \circ \beta$, as before we leave an explicit description double cones up to the reader. Calculating the coequifier $\operatorname{coeqf}(\alpha, \beta)$ of vertical transformations $\alpha, \beta\colon f \Rightarrow g\colon \mathbb{A} \longrightarrow \mathbb{B}$ in $\mathrm{Cat}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Cat}})$ is easy, we simply start with $\mathbb{B}$, to which we add new relations:

$$\alpha_a = \beta_a \quad \text{for each object} \quad a \in \mathbb{A}$$

$$\alpha_h = \beta_h \quad \text{for each horizontal cell} \quad (h\colon a \longrightarrow a') \in \mathbb{A}$$

All these extra relations do is to force corresponding components of α and β to become equal under the canonical quotient map $q\colon \mathbb{B} \longrightarrow \operatorname{coeqf}(\alpha, \beta)$ therefore $q \circ \alpha = q \circ \beta$ and so q constitutes a cone under our diagram. It is clear that this must be the universal such cone.

Our description of coinserters and coequifiers indicates strongly their importance. We may use them to build up the vertical and double structure of any double category $\mathbb{D}$ starting from its underlying category of horizontal cells $\mathbb{D}_H$. This is achieved by freely adjoining the double and vertical cells to $\mathbb{D}_H$ using coinserters while imposing the required relations between them using coequifiers. This observation leads directly to the important theorem of this section:

Theorem 2.7.1 *The class $\mathcal{P}^{\mathrm{co}} \subset \mathrm{Cat}(\underline{\mathrm{Cat}})$, of those double categories parameterising persistent 2-colimits, is the closure of $\mathcal{T}(\underline{\mathrm{Cat}})$ in $\mathrm{Cat}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Cat}})$ under coproducts, coinserters, coequifiers and splitting of idempotents (PIES-colimits).*

Proof. Recall first, from the last section, that $\mathbb{D} \in \mathcal{P}^{\mathrm{co}}$ iff its underlying category of horizontal cells $\mathbb{D}_H$ has a natural weak terminal object for each of its connected components. Examining the calculations of $\mathrm{coins}(f, g)$ and $\mathrm{coeqf}(\alpha, \beta)$ in $\mathrm{Cat}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Cat}})$ given above we see that they do not affect the horizontal cell structure of $\mathbb{B}$. In other words $\mathrm{coins}(f, g)_H \cong \mathrm{coeqf}(\alpha, \beta)_H \cong \mathbb{B}_H$ and so if $\mathbb{B} \in \mathcal{P}^{\mathrm{co}}$ then both of these colimits are also in there. Similarly $(\coprod_{i\in I} \mathbb{D}_i)_H \cong \coprod_{i\in I}(\mathbb{D}_i)_H$ from which it follows that $\mathcal{P}^{\mathrm{co}}$ is closed under coproducts. It remains to check the splitting of an idempotent $j\colon \mathbb{D} \longrightarrow \mathbb{D}$, its action $j\colon \mathbb{D}_H \longrightarrow \mathbb{D}_H$ on horizontal cells is also an idempotent and $(\mathbb{D}_j)_H \cong (\mathbb{D}_H)_j$. The idempotent on $\mathbb{D}_H$ restricts to one on each of its connected components $\mathbf{C}$, and we obtain $(\mathbb{D}_H)_j$ by splitting each of these and taking the disjoint union of the results. But if $\mathbb{D} \in \mathcal{P}^{\mathrm{co}}$ then each $\mathbf{C}$ has a natural weak terminal object w presented by some cone

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\Delta_w"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathbf{C} \end{tikzcd}$$

which we post-compose with $e_j\colon \mathbf{C} \longrightarrow \mathbf{C}_j$ and pre-compose with $m_j\colon \mathbf{C}_j \longrightarrow \mathbf{C}$ to obtain a cone displaying $e(w)$ as a natural weak initial object of $\mathbf{C}_j$, it follows that $\mathbb{D}_j \in \mathcal{P}^{\mathrm{co}}$.

If a double category has a terminal object then this is certainly terminal in the underlying category of horizontal cells therefore $\mathcal{T}(\underline{\mathrm{Cat}}) \subset \mathcal{P}^{\mathrm{co}}$. So having already demonstrated that $\mathcal{P}^{\mathrm{co}}$ is closed under PIES-colimits, it remains to show that we may construct all double categories in $\mathcal{P}^{\mathrm{co}}$ by starting with those in $\mathcal{T}(\underline{\mathrm{Cat}})$ and successively applying PIES-colimits.

Given a double category $\mathbb{D} \in \mathcal{P}^{\mathrm{co}}$ we proceed by first constructing its horizontal part $\mathbb{D}_H$ (considered as a vertically and doubly discrete double category). If $\mathbb{C}$ is a connected component of $\mathbb{D}_H$ consider $\mathbb{C}_\perp$, the category constructed by freely adjoining a terminal object to that component. Of course we always have a canonical inclusion $m\colon \mathbb{C} \longrightarrow \mathbb{C}_\perp$ but, since $\mathbb{C}$ has a natural weak terminal object w, we may also define a functor $e\colon \mathbb{C}_\perp \longrightarrow \mathbb{C}$, such that $e \circ m = \mathrm{id}_{\mathbb{C}}$, by mapping the adjoined terminal object $\bot$ to w. So $\mathbb{C}$ can be constructed by splitting the idempotent $m \circ e$ on $\mathbb{C}_\bot \in \mathcal{T}(\underline{\mathrm{Cat}})$ therefore, having constructed each of its components in this way, we may form $\mathbb{D}_H$ as their coproduct.

The remainder of our construction concerns the use of coinserters and coequifiers in appending the vertical and double structure of $\mathbb{D}$ to $\mathbb{D}_H$. This is essentially a two step process. First we freely adjoin generating cells

$$d \overset{\lceil v \rceil}{\cdots\triangleright} \bar{d} \qquad \text{one for each vertical cell} \qquad (d \overset{v}{\cdots\triangleright} \bar{d}) \in \mathbb{D}$$

$$\begin{tikzcd} d \arrow[r, "h"] \arrow[d, dotted, "\lceil v \rceil"'] \arrow[dr, phantom, "\Downarrow \lceil\alpha\rceil"] & d' \arrow[d, dotted, "\lceil v' \rceil"] \\ \bar{d} \arrow[r, "\bar{h}"'] & \bar{d}' \end{tikzcd} \qquad \text{one for each double cell} \qquad \begin{tikzcd} d \arrow[r, "h"] \arrow[d, dotted, "v"'] \arrow[dr, phantom, "\Downarrow \alpha"] & d' \arrow[d, dotted, "v'"] \\ \bar{d} \arrow[r, "\bar{h}"'] & \bar{d}' \end{tikzcd} \in \mathbb{D}$$

to obtain a double category which we will call $\mathbb{D}_F$. The construction of this must be performed in two sub-steps since coinserters only allow us to add generating cells with specified vertical domain and codomain; we must also make sure that the horizontal domain and codomain of our double generators are correct, which we will do with a coequifier. In giving an explicit description of this regard $\mathtt{1}$ and $\mathtt{2}$ as vertically and doubly discrete double categories, they then become elements of $\mathcal{T}(\underline{\mathrm{Cat}})$. Also adopt the notations $\mathrm{O}(\mathbb{D})$, $\mathrm{H}(\mathbb{D})$, $\mathrm{V}(\mathbb{D})$ and $\mathrm{D}(\mathbb{D})$ for the sets of objects, horizontal, vertical and double cells of $\mathbb{D}$ respectively. Now define a double category

$$\mathbb{D}_{\mathrm{cell}} = \left(\coprod_{v \in \mathrm{V}(\mathbb{D})} \mathtt{1}\right) \amalg \left(\coprod_{\alpha \in \mathrm{D}(\mathbb{D})} \mathtt{2}\right)$$

and two double functors

$$\begin{tikzcd} \mathbb{D}_{\mathrm{cell}} \arrow[r, shift left, "\underline{\mathrm{dom}}"] \arrow[r, shift right, "\underline{\mathrm{cod}}"'] & \mathbb{D}_H \end{tikzcd}$$

defined so that the copy of $\mathtt{1}$ [$\mathtt{2}$] corresponding to the vertical [double] cell v [α] maps to $\mathrm{dom}_V(v)$ and $\mathrm{cod}_V(v)$ [$\mathrm{dom}_V(\alpha)$ and $\mathrm{cod}_V(\alpha)$] under $\underline{\mathrm{dom}}$ and $\underline{\mathrm{cod}}$ respectively. If we refer back to the construction of coinserters in $\mathrm{Cat}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Cat}})$ given at the beginning of this section it is clear that taking the coinserter of this pair is tantamount to adjoining vertical and double cells with the correct vertical domains and codomains, but the horizontal domains and codomains of our new double cells are surplus and freely adjoined. To rectify this we need to add relations

$$\begin{aligned} \mathrm{dom}_H(\lceil\alpha\rceil) &= \lceil \mathrm{dom}_H(\alpha) \rceil \\ \mathrm{cod}_H(\lceil\alpha\rceil) &= \lceil \mathrm{cod}_H(\alpha) \rceil \end{aligned} \tag{2.30}$$

for each $\alpha \in \mathrm{D}(\mathbb{D})$, which we achieve with the aid of a double category

$$\mathbb{D}_{\text{dc-rel}} = \coprod_{\alpha \in \mathrm{D}(\mathbb{D})} (\mathtt{1} \amalg \mathtt{1})$$

and two vertical transformations:

$$\begin{tikzcd} \mathbb{D}_{\text{dc-rel}} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\mu \Downarrow \Downarrow \mu'"] & \operatorname{coins}(\underline{\operatorname{dom}}, \underline{\operatorname{cod}}) \end{tikzcd}$$

In the definition of $\mathbb{D}_{\text{dc-rel}}$ there is one copy of $\mathtt{1}$ for each relation of (2.30), and the component of μ $[\mu']$ at a given object is the vertical cell on the left [right] of the corresponding relation. So taking the coequifier of this pair imposes our relations and we have succeeded in constructing $\mathbb{D}_F$.

The second step is to construct $\mathbb{D}$ from $\mathbb{D}_F$ by imposing the relations:

$$\begin{array}{l} \lceil j_h \rceil = j_h \text{ for each } h \in \mathrm{H}(\mathbb{D}) \\ \lceil i_v \rceil = i_{\lceil v \rceil} \text{ for each } v \in \mathrm{V}(\mathbb{D}) \\ \lceil \beta \circ \alpha \rceil = \lceil \beta \rceil \circ \lceil \alpha \rceil \text{ for each pair } \alpha, \beta \in \mathrm{D}(\mathbb{D}) \text{ with } \operatorname{dom}_H(\beta) = \operatorname{cod}_H(\alpha) \\ \lceil \gamma \bullet \alpha \rceil = \lceil \gamma \rceil \bullet \lceil \alpha \rceil \text{ for each pair } \alpha, \gamma \in \mathrm{D}(\mathbb{D}) \text{ with } \operatorname{dom}_V(\gamma) = \operatorname{cod}_V(\alpha) \end{array} \tag{2.31}$$

Again define $\mathbb{D}_{\text{rel}}$ to be a coproduct of copies of $\mathtt{2}$, one for each relation in (2.31), and vertical transformations

$$\begin{tikzcd} \mathbb{D}_{\text{rel}} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\nu \Downarrow \Downarrow \nu'"] & \mathbb{D}_F \end{tikzcd}$$

with the component of ν $[\nu']$ at a given copy of $\mathtt{2}$ being the double cell on the left [right] of the corresponding relation. Then the coequifier of this pair is $\mathbb{D}$ itself.

To recap: $\mathcal{P}^{\text{co}}$ is closed under PIES-colimits, contains $\mathcal{T}(\underline{\mathrm{Cat}})$ and any of its objects may be constructed from those in that subset using PIES-colimits. Therefore $\mathcal{P}^{\text{co}}$ must be the closure of $\mathcal{T}(\underline{\mathrm{Cat}})$ under PIES-colimits in $\mathrm{Cat}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Cat}})$. □

Corollary 2.7.2 *The corresponding class of colimits $\mathcal{P}^{\text{co}}(-)$ is in fact just the class of flexible colimits, and a weight $X \in \mathcal{P}(\underline{\mathbf{A}})$ parameterises a flexible colimit iff the double category $\mathbb{G}_{\underline{\mathrm{Cat}}}(X)$ is in $\mathcal{P}^{\text{co}}$.*

Proof. Follows directly from the last theorem and theorem 2.5.12 □

We might simply phrase the dual result about classes of limits by:

The classes of Flexible and Persistent limits are identical.

Notice that our construction in theorem 2.7.1 is motivated by exactly the same considerations as that of the principle result in [43]. In other words get the 1-dimensional structure right first and then use coinserters and coequifiers to get the 2-dimensional structure back. Our hope is that calculating directly with double categories rather than weights has succeeded in clarifying this sort of argument. Dropping splitting of idempotents from the proof of theorem 2.7.1 gives us exactly their result:

Corollary 2.7.3 *A weight $X \in \mathcal{P}(\underline{\mathbf{A}})$ is in the class* (co–PIE)* *iff each of the horizontal components of $\mathbb{G}_{\underline{\mathrm{Cat}}}(X)$ has a terminal object.* □

We round off this section with a short discussion of the notion of *finite* flexible limit. Consider $\mathrm{Cat}(\underline{\mathrm{Cat}})$ in the context of [31], not only is it an LFP category but we may modify the argument in example (5.9) of loc. cit. to demonstrate that it is LFP as a cartesian closed category. Furthermore comment (8.12) of that paper provides a prototype for the proof that the finitely presentable objects of $\mathrm{Cat}(\underline{\mathrm{Cat}})$ are exactly those which we may describe in terms of a finite number of generators and relations (leaving the precise definition of that concept up to the reader).

Let $\mathrm{Cat}(\underline{\mathrm{Cat}})_f$ be the set of finitely presented double categories and define $\mathcal{P}^{\mathrm{co}}_f = \mathcal{P}^{\mathrm{co}} \cap \mathrm{Cat}(\underline{\mathrm{Cat}})_f$, we say that these parameterise finite flexible colimits since:

Proposition 2.7.4 *The set $\mathcal{P}^{\mathrm{co}}_f$ is the closure of $\mathcal{T}(\underline{\mathrm{Cat}})_f = \mathcal{T}(\underline{\mathrm{Cat}}) \cap \mathrm{Cat}(\underline{\mathrm{Cat}})_f$ in $\mathrm{Cat}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Cat}})$ under $(\mathrm{P}_f\mathrm{IES})$-colimits (i.e. finite coproducts, coinserters, coequifiers and splitting of idempotents).*

Proof. This proposition is an easy re-working of the of theorem 2.7.1. The fact that $\mathrm{Cat}(\underline{\mathrm{Cat}})$ is LFP as a cartesian closed category implies that $\mathrm{Cat}(\underline{\mathrm{Cat}})_f$ is closed in there under $\mathrm{Cat}(\underline{\mathrm{Cat}})$-colimits with finite weights (for a definition of which see [31] (4.1)). This in turn implies that the f.p. double categories are closed in $\mathrm{Cat}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Cat}})$ under finite 2-colimits, which include the $(\mathrm{P}_f\mathrm{IES})$-colimits. already know that $\mathcal{P}^{\mathrm{co}}$ is closed under $(\mathrm{P}_f\mathrm{IES})$-colimits and so it follows that $\mathcal{P}^{\mathrm{co}}_f$ is as well.

It remains to re-work the principle construction of theorem 2.7.1. If $\mathbb{D} \in \mathcal{P}^{\mathrm{co}}_f$ the $\mathbb{D}_H$ is finitely presentable and so therefore so is each of its components $\mathbb{C}$. We may append a terminal object to $\mathbb{C}$ by taking a 2-colimit in $\mathrm{Cat}(\underline{\mathrm{Cat}})_f$ so $\mathbb{C}_\perp \in \mathcal{T}(\underline{\mathrm{Cat}})_f$ and it follows by the same proof as before that we may build $\mathbb{D}_H$ from $\mathcal{T}(\underline{\mathrm{Cat}})_f$ using finite coproducts and splitting of idempotents.

The part of our construction which re-builds the double and vertical structure of $\mathbb{D}$ only uses infinite colimits when putting together $\mathbb{D}_{\mathrm{cell}}$, $\mathbb{D}_{\mathrm{dc\text{-}rel}}$ and $\mathbb{D}_{\mathrm{rel}}$ from $1, 2 \in \mathcal{T}(\underline{\mathrm{Cat}})_f$. We can rectify this situation by replacing them by similar categories $(\mathbb{D}_{\mathrm{cell}})_f$, $(\mathbb{D}_{\mathrm{dc\text{-}rel}})_f$ and $(\mathbb{D}_{\mathrm{rel}})_f$ where the first two only add the (finite number of) vertical and double generators of $\mathbb{D}$, and the last only imposes the (finite number of) specified relations. $\square$

There seems to be no simple finiteness property characterising the double categories in $(\mathrm{P}_f\mathrm{IES})^{\#}$, the closure of $\mathcal{T}(\underline{\mathrm{Cat}})$ in $\mathrm{Cat}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Cat}})$ under $(\mathrm{P}_f\mathrm{IES})$-colimits. In essence this is because at any stage in the construction of some $\mathbb{D} \in (\mathrm{P_f IES})^{\#}$ we are able to add an infinite number of generators or relations. The set $\mathcal{P}^{\mathrm{co}}_f$ is simply the fraction of $(\mathrm{P}_f\mathrm{IES})^{\#}$ which is the easiest to get to grips with.

How does $\mathcal{P}^{\mathrm{co}}_f$ relate to the class of weights for flexible limits which are also finite in the sense of [31]? It is certainly not true that if $\mathbb{D}$ is in $\mathcal{P}^{\mathrm{co}}_f$ then the associated weight $\bar{\mathrm{I}}^{\star}(\square_{*}\colon \mathbb{D} \longrightarrow \mathbb{1})$ is finite. A counter example is the double category:

$$\mathbb{D}_c = \begin{tikzcd} \bullet \arrow[d, shift right] \arrow[d, dotted, shift left] \\ \bullet \end{tikzcd}$$

When we apply $\mathrm{I}^{\star}$ to this its two objects become identified and as a consequence the "homset" of the resulting 2-category is discrete & infinite and therefore not a finitely presentable category. What is in fact true is that if $\underline{\mathbf{A}}$ is a 2-category with a finite number objects and finitely presentable "homsets" then a weight $X \in \mathcal{P}(\underline{\mathbf{A}})$ parameterises a flexible limit and is finite iff $\mathbb{G}\,(X)$ is in $\mathcal{P}_f^{\mathrm{co}}$.

Appendix A

Pasting in Bicategories.

The notion of *pasting* has become fundamental to the development of 2-category theory, since it mirrors and formalises our natural inclination to express more complex composites of 2-cells in terms of diagrams. For some time this concept had no more than an intuitive foundation but has now been given firm footing, and applied to the far more complex case of pasting in n-categories, most notably by Johnson (in [26]), Power (in [40] and [42]) and Street (in [52]). Since we have used pasting throughout our work on bicategories it seems appropriate that we should give some idea of how the theory of 2-categorical pasting extends to the bicategorical case. For our purposes the approach of [40] seems most appropriate and we assume that the reader is familiar with that paper. Most important for us in re-phrasing that work is the following consequence of MacLane's famous coherence theorem for bicategories:

Theorem A.0.5 *Every bicategory* $\underline{\underline{\mathcal{B}}}$ *is biequivalent to some 2-category* $\underline{\underline{\mathbf{B}}}$.

Proof. Recall that a biequivalence of bicategories $\underline{\underline{\mathcal{B}}}$, $\underline{\underline{\mathcal{C}}}$ is a homomorphism

$$\underline{\underline{\mathcal{B}}} \xrightarrow{\quad \mathrm{H} \quad} \underline{\underline{\mathcal{C}}}$$

with the properties:

- local equivalence, i.e. each $\mathrm{H}_{bb'}\colon \underline{\underline{\mathcal{B}}}(b, b') \longrightarrow \underline{\underline{\mathcal{C}}}(\mathrm{H}b, \mathrm{H}b')$ is an equivalence of categories.
- essential surjectivity, i.e. for each 0-cell $c \in \underline{\underline{\mathcal{C}}}$ there exists a 0-cell $b \in \underline{\underline{\mathcal{B}}}$ and an equivalence $\mathrm{H}b \simeq c$ in $\underline{\underline{\mathcal{C}}}$.

Following a similar argument to that for categorical equivalences we may show (in the presence of the axiom of choice) that every biequivalence has a pseudo inverse, that is a homomorphism $\mathrm{H}'\colon \underline{\underline{\mathcal{C}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ admitting equivalences $\mathrm{H}' \circ \mathrm{H} \simeq \mathrm{I}_{\underline{\underline{\mathcal{B}}}}$ in $\mathcal{H}om_S(\underline{\underline{\mathcal{B}}}, \underline{\underline{\mathcal{B}}})$ and $\mathrm{I}_{\underline{\underline{\mathcal{C}}}} \simeq \mathrm{H} \circ \mathrm{H}'$ in $\mathcal{H}om_S(\underline{\underline{\mathcal{C}}}, \underline{\underline{\mathcal{C}}})$, where $\circ$ denotes the usual (strictly associative) composition of homomorphisms (see [3]).

In constructing a 2-category biequivalent to $\underline{\underline{\mathcal{B}}}$ we use the obvious "syntactic" method rather than, for instance, exploiting the bicategorical Yoneda lemma as Power does in [41]. This decision is motivated by a purely technical consideration which dictates that for our specific purpose it would be useful to construct a biequivalence $\mathrm{H}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathbf{B}}}$ which is a *retract* biequivalence, in the sense that it has a biequivalence inverse $\mathrm{H}'\colon \underline{\underline{\mathbf{B}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ with $\mathrm{H}' \circ \mathrm{H} = \mathrm{I}_{\underline{\underline{\mathcal{B}}}}$.

Now notice that we may factor any homomorphism $\mathrm{F}\colon \underline{\underline{\mathcal{A}}} \longrightarrow \underline{\underline{\mathcal{B}}}$ as

$$\underline{\underline{\mathcal{A}}} \xrightarrow{\ \mathrm{F_e}\ } \underline{\underline{\mathcal{A}}}_{\mathrm{F}} \xrightarrow{\ \mathrm{F_m}\ } \underline{\underline{\mathcal{B}}}$$

where $\mathrm{F_e}$ is essentially surjective on both the 0- and 1-cells of $\underline{\underline{\mathcal{A}}}$ and $\mathrm{F_m}$ is locally fully faithful. This is an analogue of traditional epi-mono factorisation, and $\underline{\underline{\mathcal{A}}}_{\mathrm{F}}$ is constructed to have:

0- and 1-cells those of $\underline{\underline{\mathcal{A}}}$.

2-cells $\tilde{\alpha}\colon p \Rightarrow q$ correspond to 2-cells $\alpha\colon \mathrm{F}(p) \Rightarrow \mathrm{F}(q)$, which form "hom" categories $\underline{\underline{\mathcal{A}}}_{\mathrm{F}}(a, a')$ under vertical composition in $\underline{\underline{\mathcal{B}}}$.

composition of 1-cells is that of $\underline{\underline{\mathcal{A}}}$. The horizontal composite of 2-cells $\tilde{\alpha}\colon p \Rightarrow q$ and $\tilde{\alpha}'\colon p' \Rightarrow q'$ corresponds to the composite

$$\mathrm{F}(p' \circledast p) \xrightarrow[\cong]{\text{can}} \mathrm{F}(p') \otimes \mathrm{F}(p) \xrightarrow{\alpha' \otimes \alpha} \mathrm{F}(q') \otimes \mathrm{F}(q) \xrightarrow[\cong]{\text{can}} \mathrm{F}(q' \circledast q)$$

which is clearly functorial. Notice the use of $\circledast$ and $\otimes$ to distinguish the compositions in the two bicategories. We verify easily that the maps (corresponding to)

$$\mathrm{F}(p'' \circledast (p' \circledast p)) \xrightarrow[\cong]{\mathrm{F}(\text{can})} \mathrm{F}((p'' \circledast p') \circledast p)$$

satisfy the naturality and coherence conditions required of the associativity isomorphisms of a bicategory.

identities those of $\underline{\underline{\mathcal{A}}}$ with canonical isomorphisms (corresponding to):

$$\mathrm{F}(\mathrm{i}_{a'} \circledast p) \xrightarrow[\cong]{\mathrm{F}(\text{can})} \mathrm{F}(p)$$
$$\mathrm{F}(p \circledast \mathrm{i}_{a}) \xrightarrow[\cong]{\mathrm{F}(\text{can})} \mathrm{F}(p)$$

The importance of this bicategory lies in:

(a) If $\underline{\underline{\mathcal{A}}}$ is a 2-category then so is $\underline{\underline{\mathcal{A}}}_{\mathrm{F}}$.

(b) There is a canonical *strict* homomorphism $\mathrm{F_e}\colon \underline{\underline{\mathcal{A}}} \longrightarrow \underline{\underline{\mathcal{A}}}_{\mathrm{F}}$ which acts as the identity on 0- and 1-cells and takes a 2-cell $\beta\colon p \Rightarrow q$ to a 2-cell (corresponding to) $\mathrm{F}(\beta)\colon \mathrm{F}(p) \Rightarrow \mathrm{F}(q)$.

(c) There is a naturally defined homomorphism $\mathrm{F_m}\colon \underline{\underline{\mathcal{A}}}_{\mathrm{F}} \longrightarrow \underline{\underline{\mathcal{B}}}$ which acts as F on 0- and 1-cells and takes a 2-cell $\tilde{\alpha}\colon p \Rightarrow q$ to the corresponding 2-cell $\alpha\colon \mathrm{F}(p) \Rightarrow \mathrm{F}(q)$. Clearly this is locally fully faithful with the extra properties:

- If F is essentially surjective and locally essentially surjective then so is $\mathrm{F_m}$, which is therefore a biequivalence.
- If F is surjective on 0-cells and locally surjective on 1-cells then $\mathrm{F_m}$ is a biequivalence (as in the last comment) with the property that the naturally constructed biequivalence inverse $\mathrm{F}'_{\mathrm{m}}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{A}}}_{\mathrm{F}}$ is a retract biequivalence. Its actions on 0- and 1-cells are given by splitting the postulated surjections and therefore $\mathrm{F_m} \circ \mathrm{F}'_{\mathrm{m}} = \mathrm{I}_{\underline{\underline{\mathcal{B}}}}$.

(d) F factors as $\mathrm{F_m} \circ \mathrm{F_e}$.

To apply this result to the question in hand let $\mathbb{B}$ be the free category generated by the graph of 0- and 1-cells of $\underline{\underline{\mathcal{B}}}$. The 0-cells of this coincide with those of $\underline{\underline{\mathcal{B}}}$ and its 1-cells are (possibly empty) compatible sequences $[p_n, \cdots, p_1]$ of 1-cells, with composition given by sequence concatenation. Now define a homomorphism

$$\mathbb{B} \xrightarrow{\quad \mathrm{E} \quad} \underline{\underline{\mathcal{B}}}$$

which is the identity on 0-cells, takes the empty (identity) sequence on a 0-cell $b \in \underline{\underline{\mathcal{B}}}$ to the identity $\mathrm{i}_b \in \underline{\underline{\mathcal{B}}}$ and a non-empty sequence $[p_n, \cdots, p_1]$ to the composite $(p_n \otimes (p_{n-1} \otimes \cdots \otimes (p_2 \otimes p_1) \cdots))$. It really doesn't matter which particular bracketing of these composites we choose, the main thing is that MacLane's coherence theorem provides us with canonical "re-bracketing" isomorphisms:

$$\mathrm{E}([p_n, \cdots, p_1]) \otimes \mathrm{E}([q_m, \cdots, p_1]) \cong \mathrm{E}([p_n, \cdots, p_1, q_m, \cdots, q_1])$$

The uniqueness clause of that theorem also ensures that these isomorphisms satisfy the coherence conditions required of a homomorphism E, which is clearly both surjective on 0-cells and locally surjective on 1-cells. So applying the factorisation above to E we get a homomorphism $\mathrm{E_m}\colon \mathbb{B}_{\mathrm{E}} \longrightarrow \underline{\underline{\mathcal{B}}}$, which we know to have a biequivalence inverse E'_{m} with $\mathrm{E_m} \circ \mathrm{E}'_{\mathrm{m}} = \mathrm{I}_{\underline{\underline{\mathcal{B}}}}$. All that remains is to remark that $\mathbb{B}_{\mathrm{E}}$ is a 2-category since $\mathbb{B}$ is a category. □

Armed with this coherence theorem we proceed to give an account of pasting in bicategories. Firstly the definition of *pasting scheme* which in essence remains the same as in [40], that is to say they are planar directed graphs with source and sink satisfying an acyclicity condition, the edges of which are to represent 1-cells of a pasting diagram and the internal faces its 2-cells. Since we are working in bicategories it is formally quite important not only to specify a sequence of 1-cells for composition but also the order in which they are to be composed, we will also see that it is convenient to distinguish edges which are to be treated as identities from the rest. So the definition in our context is:

Definition A.0.6 A *bicategorical pasting scheme* consists of:

(i) A pasting scheme G in the sense of [40],

(ii) A distinguished set of edges, known as the *identities* of the pasting scheme.

(iii) For each interior face F of G, a *bracketing* of the directed paths of edges which comprise its domain and codomain.

(iv) A bracketing of the directed paths of edges which comprise the domain and codomain of the whole pasting scheme. □

We should make precise the notion of *bracketing* a directed path in a bicategorical pasting scheme. Given a directed path of edges $\vec{e} = \langle e_n, \cdots, e_1\rangle$ in G let the sequence $\sigma(1) < \sigma(2) < \cdots < \sigma(m)$ enumerate precisely the non-identity edges of $\vec{e}$. A bracketing of $\vec{e}$ is then simply any expression obtained by meaningfully inserting $(m-1)$ pairs of brackets into the word $e_{\sigma(m)} \cdots e_{\sigma(2)} e_{\sigma(1)}$, in fact it instructs us about the order in which to form the composite of a realisation of this path in a bicategory using the dyadic operator $\otimes$. It will become apparent that ignoring the identity edges of a path in this way mirrors how we manipulate identities in bicategories themselves, they are generally forgotten about unless we need to explicitly introduce one to act as domain or codomain of a 2-cell. Of course the sequence σ may be empty, if all edges of the path are identities, in which case there is only one bracketing, the empty one $\emptyset$. Given a face F (or the pasting scheme G itself) let $s(F)$ denote its domain, which consists of both the sequence of edges and its given bracketing, and we use $t(F)$ similarly for its codomain.

A realisation of one of these pasting schemes in a bicategory is called a *labelling* which we define:

Definition A.0.7 A labelling $l\colon G \longrightarrow \underline{\underline{\mathcal{B}}}$ of a (bicategorical) pasting scheme G in the bicategory $\underline{\underline{\mathcal{B}}}$ consists of:

(i) An assignment of a 0-cell $l(v) \in \underline{\underline{\mathcal{B}}}$ to each vertex $v \in G$.

(ii) For each edge $e = uv \in G$ a 1-cell $l(e)\colon l(u) \longrightarrow l(v)$ subject to the condition that any identity edge must map to one of the designated identity 2-cells i_b of $\underline{\underline{\mathcal{B}}}$. This condition clearly also imposes a restriction on our choice of labels for vertices, ie. if e is an identity then $l(u) = l(v)$.

Once we have labelled the edges of G we may derive, for each path $\vec{e} = \langle e_1, \cdots, e_n\rangle$ (where $e_i = v_i v_{i+1}$) and bracketing $\tilde{b}$, a unique 1-cell

$$l(\vec{e}, \tilde{b})\colon l(v_1) \longrightarrow l(v_{n+1})$$

given by the following three cases:

- The sequence σ is empty, this means the $\vec{e}$ consists simply of identities implying that each of its edges must be labelled with the same identity 1-cell $\mathrm{i}_{l(v_1)} \in \underline{\underline{\mathcal{B}}}$, so define $l(\vec{e},\tilde{b}) = \mathrm{i}_{l(v_1)}$.
- The sequence σ has length 1, so $\vec{e}$ contains only one non-identity edge $e_{\sigma(1)}$ which we use to define $l(\vec{e},\tilde{b}) = l(e_{\sigma(1)})$.
- Otherwise $l(\vec{e},\tilde{b})$ is formed by composing the (necessarily compatible) 1-cells $l(e_{\sigma(m)}), \cdots, l(e_{\sigma(1)})$ using the dyadic operator $\otimes$ in the order specified in the bracketing $\tilde{b}$.

(iii) For each interior face F a 2-cell $l(F)\colon l(s(F)) \Rightarrow l(t(F))$.

We should point out that our treatment of bracketings and identities is redundant when $\underline{\underline{\mathcal{B}}}$ is a mere 2-category, in that case this notion of labelling reduces to that of [40]. $\square$

We will actually define the pasting of such a labelling in terms of the biequivalence derived in Theorem A.0.5, so to that end we define what it means to apply a homomorphism to a labelling:

Definition A.0.8 Let $l\colon G \longrightarrow \underline{\underline{\mathcal{B}}}$ be a labelling of the pasting scheme G in a bicategory $\underline{\underline{\mathcal{B}}}$, and suppose that $\mathrm{H}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$ is a homomorphism then we may define a labelling $[\mathrm{H}\cdot l]\colon G \longrightarrow \underline{\underline{\mathcal{C}}}$ as follows:

- For each vertex $v \in G$, $[\mathrm{H}\cdot l](v) = \mathrm{H}(l(v))$,
- For each non-identity edge $e \in G$, $[\mathrm{H}\cdot l](e) = \mathrm{H}(l(e))$,
- For each identity edge $e \in G$, if $l(e) = \mathrm{i}_b$ then let $[\mathrm{H}\cdot l](e) = \mathrm{i}_{\mathrm{H}b}$. Notice that from the canonical isomorphisms $\mathrm{H}(p') \otimes \mathrm{H}(p) \cong \mathrm{H}(p' \otimes p)$ and $\mathrm{i}_{\mathrm{H}(b)} \cong \mathrm{H}(\mathrm{i}_b)$ we may construct a unique isomorphism $[\mathrm{H}\cdot l](\vec{e},\tilde{b}) \cong \mathrm{H}(l(\vec{e},\tilde{b}))$ for each bracketed path $(\vec{e},\tilde{b})$. When we say these isomorphisms are "unique" we of course mean so in the formal sense of MacLane's theorem, and this is a consequence of the coherence conditions on the structure of a homomorphism.
- For each face F of G define $[\mathrm{H}\cdot l](F)$ to be the 2-cell given by the composite:

$$
\begin{aligned}
[\mathrm{H}\cdot l](s(F)) \xrightarrow[\cong]{\text{can}} \mathrm{H}(l(s(F))) &\xrightarrow{\mathrm{H}(l(F))} \mathrm{H}(l(t(F))) \\
&\xrightarrow[\cong]{\text{can}} [\mathrm{H}\cdot l](t(F))
\end{aligned}
\tag{A.1}
$$

When given a labelled face

$$\begin{tikzcd}
& \bullet \arrow[r, "p_2"] & \bullet & \cdots & \bullet \arrow[r, "\mathrm{i}_b"] & \bullet & \cdots & \bullet \arrow[r, "p_{n-1}"] & \bullet \arrow[dr, "p_n"] & \\
\bullet \arrow[ur, "p_1"] \arrow[dr, "q_1"'] & & & & \alpha \Downarrow & & & & & \bullet \\
& \bullet \arrow[r, "q_2"'] & \bullet & \cdots & \bullet \arrow[r] & \bullet & \cdots & \bullet \arrow[r, "q_{m-1}"'] & \bullet \arrow[ur, "q_m"'] &
\end{tikzcd}$$

in $\underline{\underline{\mathcal{B}}}$ we will display the result of applying H as:

$$\begin{tikzcd}
& \bullet \arrow[r, "\mathrm{H}(p_2)"] & \bullet & \cdots & \bullet \arrow[r, "\mathrm{i}_{\mathrm{H}(b)}"] & \bullet & \cdots & \bullet \arrow[r, "\mathrm{H}(p_{n-1})"] & \bullet \arrow[dr, "\mathrm{H}(p_n)"] & \\
\bullet \arrow[ur, "\mathrm{H}(p_1)"] \arrow[dr, "\mathrm{H}(q_1)"'] & & & & \mathrm{H}(\alpha) \Downarrow & & & & & \bullet \\
& \bullet \arrow[r, "\mathrm{H}(q_2)"'] & \bullet & \cdots & \bullet \arrow[r] & \bullet & \cdots & \bullet \arrow[r, "\mathrm{H}(q_{m-1})"'] & \bullet \arrow[ur, "\mathrm{H}(q_m)"'] &
\end{tikzcd}$$

Strictly speaking the use of $\mathrm{H}(\alpha)$ to label this 2-cell is misleading, but we rely on the context in the diagram to show that it is not simply the result of applying H to α but rather the composite of diagram (A.1). The convention concerning the application of a homomorphism to identities follows common practice, for instance applying H to the unit of an adjunction $\mathrm{i}_b \Rightarrow u \otimes f$ should give a unit $\mathrm{i}_{\mathrm{H}b} \Rightarrow \mathrm{H}(u) \otimes \mathrm{H}(f)$ not simply a 2-cell $\mathrm{H}(\mathrm{i}_b) \Rightarrow \mathrm{H}(u) \otimes \mathrm{H}(f)$.

Definition A.0.9 (pasting composition) For each bicategory $\underline{\underline{\mathcal{B}}}$ pick a biequivalence $\mathrm{H}_{\underline{\underline{\mathcal{B}}}} \colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathbf{B}}}$ where $\underline{\underline{\mathbf{B}}}$ is a 2-category, in which theorem 3.3 of [40] provides us with a unique *(2-categorical)* pasting composite $k(G) \colon k(s(G)) \Rightarrow k(t(G))$ for any labelling $k \colon G \longrightarrow \underline{\underline{\mathbf{B}}}$. We extend this to labellings $l \colon G \longrightarrow \underline{\underline{\mathcal{B}}}$ by defining $l(G) \colon l(s(G)) \Rightarrow l(t(G))$ to be the unique 2-cell making the diagram

$$\begin{tikzcd}
\mathrm{H}_{\underline{\underline{\mathcal{B}}}}(l(s(G))) \arrow[r, "\mathrm{H}_{\underline{\underline{\mathcal{B}}}}(l(G))"] \arrow[d, "\mathrm{can}"', "\wr"] & \mathrm{H}_{\underline{\underline{\mathcal{B}}}}(l(t(G))) \arrow[d, "\wr\,\mathrm{can}"] \\
{[\mathrm{H}_{\underline{\underline{\mathcal{B}}}} \cdot l](s(G))} \arrow[r, "{[\mathrm{H}_{\underline{\underline{\mathcal{B}}}} \cdot l](G)}"] & {[\mathrm{H}_{\underline{\underline{\mathcal{B}}}} \cdot l](t(G))}
\end{tikzcd} \tag{A.2}$$

commute. Existence and uniqueness follows from the fact that as a biequivalence $\mathrm{H}_{\underline{\underline{\mathcal{B}}}}$ is locally fully faithful. If $\underline{\underline{\mathcal{B}}}$ is a 2-category already then we elect to take the identity homomorphism $\mathrm{I}_{\underline{\underline{\mathcal{B}}}}$ for $\mathrm{H}_{\underline{\underline{\mathcal{B}}}}$, this simply ensures that the notion presented in this definition and that of 2-categorical pasting given in [40] coincide on 2-categories. $\square$

The next lemma provides us with a proof that this definition is independent of the choice of homomorphism $\mathrm{H}_{\underline{\underline{\mathcal{B}}}}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathbf{B}}}$ as well as a result concerning the preservation of pasting composites under homomorphisms of bicategories:

Lemma A.0.10 *Let* $\mathrm{H}\colon \underline{\underline{\mathbf{B}}} \longrightarrow \underline{\underline{\mathbf{C}}}$ *be a homomorphism (pseudo functor) of 2-categories then if* $l\colon G \longrightarrow \underline{\underline{\mathbf{B}}}$ *is any labelling we have a commutative diagram:*

$$\begin{tikzcd} \mathrm{H}(l(s(G))) \arrow[r, "\mathrm{H}(l(G))"] \arrow[d, "\mathrm{can}"', "\wr"] & \mathrm{H}(l(t(G))) \arrow[d, "\mathrm{can}", "\wr"'] \\ {[\mathrm{H}\cdot l](s(G))} \arrow[r, "{[\mathrm{H}\cdot l](G)}"] & {[\mathrm{H}\cdot l](t(G))} \end{tikzcd} \tag{A.3}$$

We might re-phrase this by saying that homomorphisms between 2-categories preserve pastings. Of course in this lemma when we talk about schemes and pasting we are working in 2-categories and we mean the notion presented in [40].

Proof. Recall from proposition 2.10 and theorem 3.3 of [40] that we obtain a pasting composite by induction on the number of faces of G and we establish our result in the same way:

(i) G has one face, so has no internal faces and consists of a single path of edges. In this case the pasting of any labelling is simply the identity on the composite of the 1-cells labelling this path and the result is clear.

(ii) G has two faces, in other words it has a single internal face and we can visualise it as:

$$\begin{tikzcd}[column sep=small] & & & & \bullet \arrow[r, "e_2"] & \bullet & \cdots & \bullet \arrow[dr, "e_m"] & & & & \\ \bullet \arrow[r, "f_1"] & \bullet & \cdots & \bullet \arrow[ur, "e_1"] \arrow[dr, "e'_1"'] & & F\Downarrow & & & \bullet & \cdots & \bullet \arrow[r, "g_l"] & \bullet \\ & & & & \bullet & \cdots & \bullet \arrow[r, "e'_{m'-1}"'] & \bullet \arrow[ur, "e'_{m'}"'] & & & & \end{tikzcd} \tag{A.4}$$

Now consider a labelling $l\colon G \longrightarrow \underline{\underline{\mathcal{B}}}$, the pasting of this is simply the horizontal composite $l(\vec{g}) \circ l(F) \circ l(\vec{f})$ and observing the rules on application of a homomorphism to a labelling we see that the pasting of $[\mathrm{H}\cdot l]$ is given by the composite:

$$\begin{aligned} {[\mathrm{H}\cdot l](\vec{g}) \circ [\mathrm{H}\cdot l](\vec{e}) \circ [\mathrm{H}\cdot l](\vec{f})} &\xrightarrow[\cong]{- \circ \mathrm{can} \circ -} [\mathrm{H}\cdot l](\vec{g}) \circ \mathrm{H}(l(\vec{e})) \circ [\mathrm{H}\cdot l](\vec{f}) \\ &\xrightarrow{- \circ \mathrm{H}(l(F)) \circ -} [\mathrm{H}\cdot l](\vec{g}) \circ \mathrm{H}(l(\vec{e}\ ')) \circ [\mathrm{H}\cdot l](\vec{f}) \\ &\xrightarrow[\cong]{- \circ \mathrm{can} \circ -} [\mathrm{H}\cdot l](\vec{g}) \circ [\mathrm{H}\cdot l](\vec{e}\ ') \circ [\mathrm{H}\cdot l](\vec{f}) \end{aligned}$$

It remains to point out that we have commutative diagrams

$$\begin{tikzcd}
{[\mathrm{H}\cdot l](\vec{g}) \circ [\mathrm{H}\cdot l](\vec{e}) \circ [\mathrm{H}\cdot l](\vec{f})} \arrow[rr, "\mathrm{can}\circ\mathrm{can}\circ\mathrm{can}", "\cong"'] \arrow[d, equal] & & \mathrm{H}(l(\vec{g})) \circ \mathrm{H}(l(\vec{e})) \circ \mathrm{H}(l(\vec{f})) \arrow[d, "\wr\,\mathrm{can}"] \\
{[\mathrm{H}\cdot l](\vec{g}\vec{e}\vec{f})} \arrow[r, "\cong", "\mathrm{can}"'] & \mathrm{H}(l(\vec{g}\vec{e}\vec{f})) \arrow[r, equal] & \mathrm{H}(l(\vec{g}) \circ l(\vec{e}) \circ l(\vec{f}))
\end{tikzcd}$$

as a result of the "uniqueness" of the canonical isomorphisms $[\mathrm{H}\cdot l](\vec{e}) \cong \mathrm{H}(l(\vec{e}))$. On substituting these equalities and expressions for the pastings of l and $[\mathrm{H}\cdot l]$ into (A.3) we get a clearly commutative diagram.

(iii) G has $n > 2$ faces and the result holds for fewer than n faces. The acyclicity requirement on G ensures that we may pick an internal face F with domain entirely immersed that of G itself. This face, the edges that border it and the remaining edges in the domain of G form a sub-pasting scheme G_0 which looks like a *whisker* as in diagram (A.4). We get a second sub-pasting scheme by removing F along with the attendant edges and vertices of its boundary which are *not* in its codomain, call this G_1, together these satisfy $G_0 \cup G_1 = G$, $t(G_0) = s(G_1) = G_0 \cap G_1$. Given a labelling $l\colon G \longrightarrow \underline{\underline{\mathbf{B}}}$ we have restrictions to $l_i\colon G_i \longrightarrow \underline{\underline{\mathbf{B}}}$ $(i = 0, 1)$ and the pasting composite of l is given by $l(G) \overset{\mathrm{def}}{=} l_1(G_1) \bullet l_0(G_0)$, where $\bullet$ denotes vertical composition of 2-cells.

Certainly applying H to a restricted labelling l_i is the same as restricting $[\mathrm{H}\cdot l]$ to G_i therefore $[\mathrm{H}\cdot l](G) = [\mathrm{H}\cdot l_1](G_1) \bullet [\mathrm{H}\cdot l_0](G_0)$ and by the inductive hypothesis we know that both (non trivial) squares in

$$\begin{tikzcd}
\mathrm{H}(l_0(s(G_0))) \arrow[r, "\mathrm{H}(l_0(G_0))"] \arrow[d, "\mathrm{can}\,\wr"'] & \mathrm{H}(l_0(t(G_0)))=\mathrm{H}(l_1(s(G_1))) \arrow[r, "\mathrm{H}(l_1(G_1))"] \arrow[d, "\wr\,\mathrm{can}"] & \mathrm{H}(l_1(t(G_1))) \arrow[d, "\wr\,\mathrm{can}"] \\
{[\mathrm{H}\cdot l_0](s(G_0))} \arrow[r, "{[\mathrm{H}\cdot l_0](G_0)}"] & {[\mathrm{H}\cdot l_0](t(G_0))=[\mathrm{H}\cdot l_1](s(G_1))} \arrow[r, "{[\mathrm{H}\cdot l_1](G_1)}"] & {[\mathrm{H}\cdot l_1](t(G_1))}
\end{tikzcd}$$

commute. Of course this diagram only decomposes the square in (A.3), and so establishes the lemma. □

Now for the promised corollaries:

Corollary A.0.11 *The pasting composite of a labelling $l\colon G \longrightarrow \underline{\underline{\mathcal{B}}}$ in a bicategory $\underline{\underline{\mathcal{B}}}$ (as defined in A.0.9) is preserved by all homomorphisms $\mathrm{K}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathbf{C}}}$ where $\underline{\underline{\mathbf{C}}}$ is a 2-category.*

Proof. We know by the proof of theorem A.0.5 that it may be assumed that we have chosen each $\mathrm{H}_{\underline{\underline{\mathcal{B}}}}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathbf{B}}}$ so that they are retract biequivalences with inverse

$\mathrm{H}'_{\underline{\underline{\mathcal{B}}}}$. The pasting $l(G)\colon l(s(G)) \Rightarrow l(t(G))$ is defined to be the unique map related to the 2-categorical pasting of $[\mathrm{H}_{\underline{\underline{\mathcal{B}}}}{\cdot}l]$ by the commutativity of diagram (A.2), to which we apply the homomorphism $\mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}}$ to obtain the commutative square marked (a) in:

$$\begin{tikzcd}
\mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}} \circ \mathrm{H}_{\underline{\underline{\mathcal{B}}}}(l(s(G))) \arrow[r, "\mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}} \circ \mathrm{H}_{\underline{\underline{\mathcal{B}}}}(l(G))"] \arrow[d, "\mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}}(\mathrm{can})"', "\wr"] \arrow[dr, phantom, "\text{(a)}"] & \mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}} \circ \mathrm{H}_{\underline{\underline{\mathcal{B}}}}(l(t(G))) \arrow[d, "\wr"', "\mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}}(\mathrm{can})"] \\
\mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}}([\mathrm{H}_{\underline{\underline{\mathcal{B}}}}{\cdot}l](s(G))) \arrow[r, "\mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}}([\mathrm{H}_{\underline{\underline{\mathcal{B}}}}\cdot l](G))"] \arrow[d, "\mathrm{can}"', "\wr"] \arrow[dr, phantom, "\text{(b)}"] & \mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}}([\mathrm{H}_{\underline{\underline{\mathcal{B}}}}{\cdot}l](t(G))) \arrow[d, "\wr"', "\mathrm{can}"] \\
[(\mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}} \circ \mathrm{H}_{\underline{\underline{\mathcal{B}}}}){\cdot}l](s(G)) \arrow[r, "{[(\mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}} \circ \mathrm{H}_{\underline{\underline{\mathcal{B}}}})\cdot l](G)}"] & [(\mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}} \circ \mathrm{H}_{\underline{\underline{\mathcal{B}}}}){\cdot}l](t(G))
\end{tikzcd}$$

The square marked (b) commutes by lemma A.0.10, $\mathrm{K} \circ \mathrm{H}'_{\underline{\underline{\mathcal{B}}}} \circ \mathrm{H}_{\underline{\underline{\mathcal{B}}}} = \mathrm{K}$ since $\mathrm{H}_{\underline{\underline{\mathcal{B}}}}$ is a retract biequivalence, and the vertical sides of the two squares compose to the canonical isomorphisms associated with K giving a commutative diagram

$$\begin{tikzcd}
\mathrm{K}(l(s(G))) \arrow[r, "\mathrm{K}(l(G))"] \arrow[d, "\mathrm{can}"', "\wr"] & \mathrm{K}(l(t(G))) \arrow[d, "\wr"', "\mathrm{can}"] \\
{[\mathrm{K}{\cdot}l](s(G))} \arrow[r, "{[\mathrm{K}\cdot l](G)}"] & {[\mathrm{K}{\cdot}l](t(G))}
\end{tikzcd} \tag{A.5}$$

which is, of course, exactly what we mean when we say that the pasting of l is preserved. ☐

Corollary A.0.12 *The definition of pasting composition in a bicategory is independent of the choice of biequivalence* $\mathrm{H}_{\underline{\underline{\mathcal{B}}}}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\mathbf{B}}$*, furthermore pastings are preserved by homomorphisms between bicategories.*

Proof. Given another biequivalence $\mathrm{K}\colon \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\mathbf{C}}$ simply apply the last corollary to it. The resulting commutative square (A.5) exactly demonstrates that the pasting of labelling $l\colon G \longrightarrow \underline{\underline{\mathcal{B}}}$, as defined with respect to $\mathrm{H}_{\underline{\underline{\mathcal{B}}}}$, also satisfies the property required by definition (A.0.9 for a pasting defined with respect to K. Therefore the choice of homomorphisms $\mathrm{H}_{\underline{\underline{\mathcal{B}}}}$ is irrelevant to the definition of bicategorical pasting.

In order to prove that all homomorphisms $\mathrm{H}: \underline{\underline{\mathcal{B}}} \longrightarrow \underline{\underline{\mathcal{C}}}$ preserve pasting composition consider the diagram

$$\begin{tikzcd}
\mathrm{H}_{\underline{\underline{\mathcal{C}}}} \circ \mathrm{H}(l(s(G))) \arrow[r, "\mathrm{H}_{\underline{\underline{\mathcal{C}}}} \circ \mathrm{H}(l(G))"] \arrow[d, "\mathrm{H}_{\underline{\underline{\mathcal{C}}}}(\mathrm{can})"', "\wr"] & \mathrm{H}_{\underline{\underline{\mathcal{C}}}} \circ \mathrm{H}(l(t(G))) \arrow[d, "\wr"', "\mathrm{H}_{\underline{\underline{\mathcal{C}}}}(\mathrm{can})"] \\
\mathrm{H}_{\underline{\underline{\mathcal{C}}}}([\mathrm{H}\cdot l](s(G))) \arrow[r, "\mathrm{H}_{\underline{\underline{\mathcal{C}}}}([\mathrm{H}\cdot l](G))"] \arrow[d, "\mathrm{can}"', "\wr"] & \mathrm{H}_{\underline{\underline{\mathcal{C}}}}([\mathrm{H}\cdot l](t(G))) \arrow[d, "\wr"', "\mathrm{can}"] \\
[(\mathrm{H}_{\underline{\underline{\mathcal{C}}}} \circ \mathrm{H})\cdot l](s(G)) \arrow[r, "{[(\mathrm{H}_{\underline{\underline{\mathcal{C}}}} \circ \mathrm{H})\cdot l](G)}"] & [(\mathrm{H}_{\underline{\underline{\mathcal{C}}}} \circ \mathrm{H})\cdot l](t(G))
\end{tikzcd}$$

in which the lower square commutes, since it is exactly the defining property of the pasting of $[\mathrm{H}\cdot l]$ in $\underline{\underline{\mathcal{C}}}$. That the outer square commutes may be checked by applying the previous corollary to the homomorphism $\mathrm{H}_{\underline{\underline{\mathcal{C}}}} \circ \mathrm{H}$ thus implying, in turn, that the upper square must also commute. This square is obtained by applying the locally fully faithful homomorphism $\mathrm{H}_{\underline{\underline{\mathcal{C}}}}$ to

$$\begin{tikzcd}
\mathrm{H}(l(s(G))) \arrow[r, "\mathrm{H}(l(G))"] \arrow[d, "\mathrm{can}"', "\wr"] & \mathrm{H}(l(t(G))) \arrow[d, "\wr"', "\mathrm{can}"] \\
{[\mathrm{H}\cdot l](s(G))} \arrow[r, "{[\mathrm{H}\cdot l](G)}"] & {[\mathrm{H}\cdot l](t(G))}
\end{tikzcd}$$

which must therefore commute, establishing that H preserves the pasting of l. $\square$

How should we interpret this notion of bicategorical pasting? The inductive proof of lemma A.0.10 has already demonstrated that we may decompose any pasting scheme into a succession of whiskers, and that the pasting composite of a labelling in a 2-category is formed by first taking the horizontal composite of each whisker and then composing these vertically. It is easy to show that this description of pasting extends to the bicategorical context for which we elaborate on the modifications necessary, leaving detailed verification up to the reader. First pick any succession of whiskers $W_1, \cdots, W_n$ with $s(W_1) = s(G)$, $s(W_{i+1}) = t(W_i)$, $t(G) = t(W_n)$ and $\bigcup_{i=1}^{n} W_i = G$, furthermore select arbitrary bracketings $\tilde{b}_i$ of $t(W_i) = s(W_{i+1})$ for $i = 1, \cdots, n-1$ and let $\tilde{b}_0$, $\tilde{b}_n$ denote the bracketings of the domain and codomain of G given as part of its structure. Equipped with the bracketings $\tilde{b}_{i-1}$ and $\tilde{b}_i$ the subgraph W_i forms a pasting scheme.

Given a labelling $l: G \longrightarrow \underline{\underline{\mathcal{B}}}$ restrict it to each whisker to get $l_i: W_i \longrightarrow \underline{\underline{\mathcal{B}}}$ and form the composite of each of these by horizontal composition (as we did in the 2

-categorical case) without worrying about the precise order in which we apply the bifunctor $\otimes$ in doing this. Sadly this is not the true pasting of l_i as it takes no account of the bracketings of domain and codomain of W_i, but this fault is easily fixed by "re-bracketing" these using the canonical associativity isomorphisms of $\underline{\underline{\mathcal{B}}}$. We are now left with a sequence of n abutting 2-cells which we may compose vertically to form the pasting of l.

This is of course exactly the way we might intuitively think of pasting in a bicategory, simply go about things as we might in a 2-category whilst liberally sprinkling our calculations with associativity isomorphisms, so why didn't we couch our definition in these terms? It is exactly this sprinkling of canonical isomorphisms that would make that sort of definition hard to prove anything about directly, in particular how do we establish unicity of pasting composition. Things would get even more messy if we were to start thinking about the preservation of pasting by homomorphisms, forcing us to talk explicitly about the interaction of associativity isomorphisms and the structural 2-cells of the homomorphism.

In essence the coherence lemma A.0.5 gives us a neat way of coding up all of the coherence information involved in process of pasting into a neat parcel. Notice also that it implies that, for most purposes, we may forget about the bracketing of horizontal composites and allow the associativity and identity isomorphisms to take care of themselves. We have only introduced pasting schemes equipped with bracketing in order to make precise the work presented above, in practical situations we will never need to make this information explicit. In fact to all intents and purposes this section may be summed up by saying that pasting diagrams and their composites may be formally manipulated in bicategories in exactly the ways we are used to in 2-category theory.

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