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ON THE OPERADS OF J.P. MAY

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Author's Note. When this manuscript was submitted in January 1972, the editor asked that it be expanded to study the relation of operads to clubs. The author found this too daunting a task at a busy time and the manuscript was never published.

Reading through the manuscript now, more than thirty years later, elicits two strong impressions. First, the treatment is very complete: the only item not discussed in detail is the *coherence* of the monoidal structure given by the functor $T \circ S$ on $[\mathbf{P}, \mathcal{V}]$. Secondly, it was done—for instance in proving the associativity $(R \circ T) \circ S \cong R \circ (T \circ S)$—with bare hands. Today one could argue as follows, using universal properties; the author learned this approach from Aurelio Carboni.

$\mathbf{P}^{\mathrm{op}}$, which is in fact isomorphic to $\mathbf{P}$, is the free symmetric monoidal category on 1. So to give an object of $[\mathbf{P}, \mathcal{V}]$, or a functor $T : 1 \to [\mathbf{P}, \mathcal{V}]$, is equally to give a strong monoidal functor $\mathbf{P}^{\mathrm{op}} \to [\mathbf{P}, \mathcal{V}]$, where the latter has the convolution monoidal structure $\otimes$; this is the strong monoidal functor sending m to the tensor power $T^m = T \otimes T \otimes \ldots \otimes T$. By Theorem 5.1 of [12], this is equally to give a cocontinuous strong monoidal functor $T' : [\mathbf{P}, \mathcal{V}] \to [\mathbf{P}, \mathcal{V}]$; this is the left Kan extension $- \circ T$, and T is recovered from T' as $T'(J) = J \circ T$. Now the desired associativity $(- \circ T) \circ S \cong - \circ (T \circ S)$ is just the associativity of these cocontinuous strong monoidal functors.

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1. Introduction

In his work on iterated loop spaces [1], J.P. May introduces the concept of an *operad*, and shows that each operad gives rise to a monad on the category $\mathbf{Top}_0$ of pointed hausdorff k-spaces. In particular May produces, for each n with $1 \leq n \leq \infty$, an operad such that the connected algebras for the corresponding monad are the n-fold loop spaces. There is a close formal similarity between operads and the *clubs* introduced in the present author's work on coherence problems in categories [5]; each club gives rise to a monad on the category $\mathbf{Cat}$ of small categories. This similarity led the author to wonder whether the analogue of an operad could be defined with the category $\mathbf{Top}$ of hausdorff k-spaces (it is in this category, and not in $\mathbf{Top}_0$, that May's operads really live) replaced by an arbitrary

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cosmos. (The word *cosmos* has been suggested by Bénabou for "complete and cocomplete symmetric monoidal closed category".)

This is indeed the case. The purpose of the present note is to throw light on operads from the categorical point of view by defining them in this greater generality and by showing abstractly why they give rise to monads.

Write $\mathbf{P}$ for the category whose objects are the integers $n \geq 0$, with no morphisms $n \to m$ for $n \neq m$, and with the morphisms $n \to n$ being the permutations of n. Since $\mathbf{P}$ has an evident monoidal structure, the functor category $[\mathbf{P}, \mathcal{V}]$, where $\mathcal{V}$ is a cosmos, is again a cosmos by the work of Day [1]. However, it turns out that $[\mathbf{P}, \mathcal{V}]$ also admits another, unfamiliar, and this time non-symmetric, monoidal closed structure. If we denote the tensor product for the latter by $\circ$, an operad T is just a $\circ$-monoid in $[\mathbf{P}, \mathcal{V}]$; as such it gives rise to a monad $T \circ$ - on $[\mathbf{P}, \mathcal{V}]$, which restricts to a monad on the full subcategory $\mathcal{V}$ of $[\mathbf{P}, \mathcal{V}]$. In May's case, where $\mathcal{V} = \mathbf{Top}$, the algebras for the monad $T \circ$ - on $\mathbf{Top}$ are also, for a simple reason, the algebras for a certain monad on $\mathbf{Top}_0$. Such is our "explanation" of operads.

Of course only *some* of the monads on $\mathcal{V}$ arise from operads; the value of operads lies in their forming a category that is much easier to handle than that of all monads. The usefulness of the category of operads is sufficiently attested to by the above work of May. We merely point out here that, as the category of $\circ$-monoids in $[\mathbf{P}, \mathcal{V}]$, it is obviously complete; and will be cocomplete under reasonable hypotheses on $\mathcal{V}$. Moreover every morphism $\mathcal{V} \to \mathcal{V}'$ of cosmoi transforms a $\mathcal{V}$-operad into a $\mathcal{V}'$-operad, as we show in §6.

In spite of the above, the author's clubs turn out not to be operads after all; they are $\circ$-monoids not in the functor category $[\mathbf{P},\mathbf{Cat}]$ but in the closely-related category $\mathbf{Cat}/\mathbf{P}$, which again has an "unusual" monoidal closed structure; the $\mathbf{Cat}$-operads are in fact a subset of the clubs. The observed similarity has served its turn in suggesting the above generalization.

2. The cosmos structure on $[\mathbf{P}, \mathcal{V}]$

Let the cosmos $\mathcal{V}$ have tensor product $\otimes$, identity object I, and internal hom $[\ ,\ ]$. For simplicity of exposition we treat $\otimes$ as strictly associative and I as a strict identity, which is justified by Mac Lane's coherence theorem [8]. Rather than give a special name to the symmetry $A \otimes B \to B \otimes A$ of $\mathcal{V}$, we write $\langle \xi \rangle : A_1 \otimes \cdots \otimes A_n \to A_{\xi 1} \otimes \cdots \otimes A_{\xi n}$ for the natural isomorphism obtained by iterating it; here $\xi \in \mathbf{P}_n = \mathbf{P}(n,n) =$ the set of permutations of n. The functor $V = \mathcal{V}(I, -) : \mathcal{V} \to \mathbf{Sets}$ has a left adjoint F, where FX is the coproduct of X copies of I; it is harmless to suppose that F preserves tensor products strictly, and not just to within isomorphism; so that $F(X \times Y) = FX \otimes FY$ and $F\star = I$, where $\star$ is the distinguished one-point set. For $X \in \mathbf{Sets}$ and $A \in \mathcal{V}$ we write $X \otimes A$ for $FX \otimes A$; it is the coproduct of X copies of A, but defining it as above allows us to write without parentheses $X \otimes A \otimes B$ for $X \in \mathbf{Sets}$ and $A, B \in \mathcal{V}$.

By functor etc., we shall always mean *ordinary* functor etc., not $\mathcal{V}$-functor etc.; in particular the ends and coends below are all relative to $\mathbf{Sets}$, not to $\mathcal{V}$. The reader who is not familiar with the integral notation for ends and coends can find it for instance in [2].

We shall henceforth write $\mathcal{F}$ for the functor category $[\mathbf{P}, \mathcal{V}]$. An object T of $\mathcal{F}$ is a graded $\mathcal{V}$-object Tn ($n \geq 0$), together with for each n a left operation of the symmetric group $\mathbf{P}_n$ on Tn; a morphism of $\mathcal{F}$ is an equivariant graded map of degree 0.

The category $\mathbf{P}$ has a symmetric monoidal structure, with $+$ for its tensor product; $m+n$ is the ordinary sum of integers, and $\xi+\eta : m+n \to m+n$ is the evident permutation. The identity object is of course 0. For the iterated symmetry isomorphism we are going to use the same notation $\langle\xi\rangle$ as we do in $\mathcal{V}$; however it is technically convenient to regard the monoidal structure as one on $\mathbf{P}^{op}$ rather than on $\mathbf{P}$ for this purpose, and therefore we write $\langle\xi\rangle : m_{\xi 1} + \cdots + m_{\xi n} \to m_1 + \cdots + m_n$ (rather than $\langle\xi^{-1}\rangle$) for the appropriate permutation.

It then follows from Day [1] that $\mathcal{F} = [\mathbf{P}, \mathcal{V}]$ is again a cosmos. Day's formulas for the cosmos structure on $[\mathcal{A}, \mathcal{V}]$ refer directly to the case when $\mathcal{A}$ is a $\mathcal{V}$-category; and have to be adapted to the present case by replacing $\mathbf{P}$ by the free $\mathcal{V}$-category on $\mathbf{P}$ and then simplifying. On doing this, we see that the tensor product on $\mathcal{F}$ is given by

$$T \otimes S = \int^{m,n} \mathbf{P}(m+n, -) \otimes Tm \otimes Sn \,. \tag{2.1}$$

If we actually calculate this coend, we get for $T \otimes S$ the explicit formula

$$(T \otimes S)k = \sum_{m+n=k} \mathrm{Sh}(m,n) \otimes Tm \otimes Sn \,, \tag{2.2}$$

where $\mathrm{Sh}(m,n)$ is the set of (m,n)-shuffles; but we do not use this, (2.1) being much easier to handle. Associativity of $\otimes$ is immediate; using the Yoneda lemma, and the fact that the $\otimes$ of $\mathcal{V}$ preserves colimits, we see that both iterated $\otimes$-products are canonically isomorphic to

$$T \otimes S \otimes R \;=\; \int^{m,n,k} \mathbf{P}(m+n+k, -) \otimes Tm \otimes Sn \otimes Rk \,.$$

This formula admits an obvious extension to

$$T_1 \otimes \cdots \otimes T_m = \int^{n_1,\cdots,n_m} \mathbf{P}(n_1 + \cdots + n_m, -) \otimes T_1 n_1 \otimes \cdots T_m n_m \,. \tag{2.3}$$

The identity object for $\otimes$ is

$$\mathbf{P}(0,-) \otimes I \,. \tag{2.4}$$

Again we write as if the associativity and identity were strict; coherence has been formally established by Day. The symmetry for $\mathcal{F}$ comes from those for $\mathbf{P}$ and for $\mathcal{V}$, being given for a multiple $\otimes$-product by the following diagram:

$$\begin{tikzcd}
T_1 \otimes \ldots \otimes T_m \arrow[r, equal] \arrow[d, "\langle\xi\rangle"'] & \int^{n_i} \mathbf{P}(n_1 + \ldots + n_m, -) \otimes T_1 n_1 \otimes \ldots \otimes T_m n_m \arrow[d, "\int^{n_1,\ldots,n_m} \mathbf{P}(\langle\xi\rangle,-)\otimes\langle\xi\rangle"] \\
T_{\xi 1} \otimes \ldots \otimes T_{\xi m} \arrow[r, equal] & \int^{n_i} \mathbf{P}(n_{\xi 1} + \ldots + n_{\xi m}, -) \otimes T_{\xi 1} n_{\xi 1} \otimes \ldots \otimes T_{\xi m} n_{\xi m}
\end{tikzcd} \tag{2.5}$$

Finally the internal hom of $\mathcal{F}$, which we do not explicitly use, is given by [[S, R]] where

$$[[S,R]]k = \int_n [Sn, R(n+k)] \ . \tag{2.6}$$

$\mathcal{F}$ is actually a $\mathcal{V}$-category, and we do need its $\mathcal{V}$-valued hom, which is given by

$$[S,R] = \int_n [Sn, Rn] \ . \tag{2.7}$$

We could write this as ϕ [[S, R]], where $\phi : \mathcal{F} \to \mathcal{V}$ is the functor given by

$$\phi T = T0 \ . \tag{2.8}$$

This functor has the left adjoint $\psi : \mathcal{V} \to \mathcal{F}$ where

$$\psi A = \mathbf{P}(0,-) \otimes A \ ; \tag{2.9}$$

that is to say, $(\psi A)0 = A$ and, for $n \neq 0, (\psi A)n = 0$, the initial object of $\mathcal{V}$. Clearly $\phi\psi = 1$, so that ψ embeds $\mathcal{V}$ as a full coreflective subcategory of $\mathcal{F}$. It is easily verified that $\psi(A \otimes B) = \psi A \otimes \psi B$; that ψI is the identity object (2.4) of $\mathcal{F}$; and that $[\psi A, \psi B] = [A, B]$. So no confusion arises if we write A for ψA, and regard $\mathcal{V}$ as a subcategory of $\mathcal{F}$; note that the identity object (2.4) in $\mathcal{F}$ is then just I. (Observe that $[[A, B]]$, which is easily seen to be given by $[[A, B]]k = [A, P(0, k) \otimes B]$ when $A, B \in \mathcal{V}$, differs from $[A, B]$ whenever $[A, 0]$, as in **Sets** or **Top**, is different from 0; they coincide if $\mathcal{V}$ is pointed.)

It is also immediate from the Yoneda lemma that, for $A \in \mathcal{V}$ and $S \in \mathcal{F}$, we have

$$(A \otimes S)k = A \otimes Sk \ . \tag{2.10}$$

For $A \in \mathcal{V}$ and $S, R \in \mathcal{F}$ we clearly have:

$$\mathcal{F}(A \otimes S, R) \ \cong \ \mathcal{V}(A, [S, R]) \ . \tag{2.11}$$

3. The non-symmetric monoidal closed structure on $[\mathbf{P}, \mathcal{V}]$

For $T \in \mathcal{F} = [\mathbf{P}, \mathcal{V}]$, write T^m for the m-fold tensor product $T \otimes T \otimes \cdots \otimes T$. This is contravariantly functorial in $m \in \mathbf{P}$ if we define $T^\xi : T^m \to T^m$, for $\xi \in \mathbf{P}(m, m)$, to be $\langle \xi \rangle : T \otimes \cdots \otimes T \to T \otimes \cdots \otimes T$. Thus $(m, T) \mapsto T^m$ is a functor $\mathbf{P}^{op} \times \mathcal{F} \to \mathcal{F}$; it restricts to a functor $\mathbf{P}^{op} \times \mathcal{V} \to \mathcal{V}$, since $A^m \in \mathcal{V}$ if $A \in \mathcal{V}$.

For fixed T, the functor $m \mapsto T^m : \mathbf{P}^{op} \to \mathcal{F}$ is strict monoidal; for $T^{m+n} = T^m \otimes T^n$ and $T^{\xi+\eta} = T^\xi \otimes T^\eta$, while $T^0 = I$. It also respects the symmetries; since we defined the $\langle \xi \rangle$ in $\mathbf{P}$ as the symmetry appropriate to $\mathbf{P}^{op}$, this says that we have commutativity in

$$\begin{tikzcd} T^{k_1} \otimes \ldots \otimes T^{k_m} \arrow[r, equal] \arrow[d, "\langle \xi \rangle"'] & T^{k_1 + \ldots + k_m} \arrow[d, "T^{\langle \xi \rangle}"] \\ T^{k_{\xi 1}} \otimes \ldots \otimes T^{k_{\xi m}} \arrow[r, equal] & T^{k_{\xi 1} + \ldots + k_{\xi m}} \end{tikzcd} \ , \tag{3.1}$$

verification of which is immediate.

We define a new tensor product $\circ$ on $\mathcal{F}$ by

$$T \circ S = \int^m Tm \otimes S^m \ . \tag{3.2}$$

If $S \in \mathcal{V}$, then $S^m \in \mathcal{V}$ and $Tm \otimes S^m \in \mathcal{V}$; since the inclusion $\mathcal{V} \to \mathcal{F}$, having the right adjoint ϕ, preserves colimits, it follows that $T \circ S \in \mathcal{V}$. Thus $\circ$ is a functor

$$\circ : \mathcal{F} \times \mathcal{F}, \mathcal{F} \times \mathcal{V} \to \mathcal{F}, \mathcal{V} \ . \tag{3.3}$$

To prove the associativity of $\circ$, we first establish the following lemma. We use freely in the proof the fact that $A \otimes -$ and $- \otimes A : \mathcal{V} \to \mathcal{V}$, having right adjoints, preserve colimits and in particular coends.

3.1. Lemma. *$(S \circ R)^m \cong S^m \circ R$, naturally in S, R and m.*

Proof. We have

$$\begin{aligned}
(S \circ R)^m &= \int^{n_1, \cdots, n_m} \mathbf{P}(n_1 + \cdots + n_m, -) \otimes (S \circ R)n_1 \otimes \cdots \otimes (S \circ R)n_m \ \text{ by (2.3)} \\
&\cong \int^{n_i, k_i} \mathbf{P}(n_1 + \cdots + n_m, -) \otimes (Sk_1 \otimes (R^{k_1})n_1) \otimes \cdots \otimes (Sk_m \otimes (R^{k_m})n_m) \ \text{ by (3.2)} \\
&\cong \int^{n_i, k_i} (Sk_1 \otimes \cdots \otimes Sk_m) \otimes \mathbf{P}(n_1 + \cdots + n_m, -) \otimes (R^{k_1})n_1 \otimes \cdots \otimes (R^{k_m})n_m \\
&\cong \int^{k_i} (Sk_1 \otimes \cdots \otimes Sk_m) \otimes (R^{k_1} \otimes \cdots \otimes R^{k_m}) \ \text{ by (2.3)} \\
&\cong \int^{k_i} Sk_1 \otimes \cdots \otimes Sk_m \otimes R^{k_1 + \cdots + k_m} \\
&\cong \int^{k_i, t} \mathbf{P}(k_1 + \cdots + k_m, t) \otimes Sk_1 \otimes \cdots \otimes Sk_m \otimes R^t \ \text{ by Yoneda} \\
&\cong \int^t (S^m)t \otimes R^t \ \text{ by (2.3)} \\
&= (S^m) \circ R \ \text{ by (3.2)} \ .
\end{aligned}$$

The above isomorphisms are clearly natural in S and R. We have to prove naturality in m. For $\xi \in \mathbf{P}_m$, consider $(S \circ R)^\xi = \langle \xi \rangle$. The following display exactly imitates the display above, but this time at the level of maps:

$$
\begin{aligned}
(S \circ R)^\xi = \langle \xi \rangle &= \int^{n_i} \mathbf{P}(\langle \xi \rangle, -) \otimes \langle \xi \rangle \ \text{ by (2.5)} \\
&= \int^{n_i k_i} \mathbf{P}(\langle \xi \rangle, -) \otimes \langle \xi \rangle \\
&= \int^{n_i, k_i} \langle \xi \rangle \otimes \mathbf{P}(\langle \xi \rangle, -) \otimes \langle \xi \rangle \\
&= \int^{k_i} \langle \xi \rangle \otimes \langle \xi \rangle \ \text{ by (2.5)} \\
&= \int^{k_i} \langle \xi \rangle \otimes R^{\langle \xi \rangle} \ \text{ by (3.1)} \\
&= \int^{k_i, t} \mathbf{P}(\langle \xi \rangle, t) \otimes \langle \xi \rangle \otimes R^t \\
&= \int^{t} \langle \xi \rangle t \otimes R^t \ \text{ by (2.5)} \\
&= S^\xi \circ R \, .
\end{aligned}
$$

So we have naturality in m and the lemma is proved.

The associativity of $\circ$ now follows at once. We have

$$
\begin{aligned}
T \circ (S \circ R) &= \int^m Tm \otimes (S \circ R)^m \ \text{ by (3.2)} \\
&\cong \int^m Tm \otimes (S^m \circ R) \ \text{ by Lemma 3.1} \\
&\cong \int^{m,k} Tm \otimes (S^m)k \otimes R^k \ \text{ by (3.2)} \\
&\cong \int^{m,k} (Tm \otimes S^m)k \otimes R^k \ \text{ by (2.10)} \\
&\cong \int^{k} (T \circ S)k \otimes R^k \ \text{ by (3.2)} \\
&= (T \circ S) \circ R \ \text{ by (3.2)} \ ;
\end{aligned}
$$

in the penultimate line we have used the fact that colimits in a functor category are computed evaluation-wise.

An identity object for $\circ$ is

$$J = \mathbf{P}(1, -) \otimes I \ ; \tag{3.4}$$

thus $J1 = I$ and $Jn = 0$ for $n \neq 1$; note that $J \neq I$. To see that J is a left identity, observe that $J \circ S = \int^m Jm \otimes S^m = \int^m \mathbf{P}(1, m) \otimes I \otimes S^m \cong S^1$ by Yoneda. To see that

J is a right identity, observe that

$$\begin{aligned} J^m &= \int^{n_1,\cdots,n_m} \mathbf{P}(n_1+\cdots+n_m,-)\otimes(\mathbf{P}(1,n_1)\otimes I)\otimes\cdots\otimes(\mathbf{P}(1,n_m)\otimes I)\\ &\cong \mathbf{P}(1+1+\cdots+1,-)\otimes I \ \text{ by Yoneda}\\ &= \mathbf{P}(m,-)\otimes I\ . \end{aligned} \tag{3.5}$$

This is easily verified to be natural in m, so that

$$T\circ J \cong \int^m Tm\otimes J^m \cong \int^m Tm\otimes \mathbf{P}(m,-)\otimes I = T \ \text{ by Yoneda}\ .$$

.

Since all the isomorphisms involved are the canonical ones, it is clear that the monoidal structure given by $\circ$ on $\mathcal{F}$ will be coherent; the details, which we omit, would require a series of lemmas along the lines of §2 of [1].

This monoidal structure is closed; that is; $-\circ S$ has a right adjoint. For

$$\begin{aligned} \mathcal{F}(T\circ S,R) \ &= \ \mathcal{F}(\int^m T_m\otimes S^m,R)\\ &\cong \int_m \mathcal{F}(T_m\otimes S^m,R)\\ &\cong \int_m \mathcal{V}(T_m,[S^m,R]) \qquad \text{by (2.11)}\\ &= \mathcal{F}(T,\{S,R\})\ , \end{aligned}$$

if we define $\{S,R\}\in\mathcal{F}$ by

$$\{S,R\}m = [S^m,R]\ . \tag{3.6}$$

We have already remarked that $T\circ A\in\mathcal{V}$ for $A\in\mathcal{V}$; note that, if $T\in\mathcal{F}$ and $A,B\in\mathcal{V}$, the isomorphism $\mathcal{F}(T\circ S,R)\cong\mathcal{F}(T,\{S,R\})$ has the special case

$$\mathcal{V}(T\circ A,B)\cong\mathcal{F}(T,\{A,B\})\ . \tag{3.7}$$

This monoidal structure is not symmetric; in fact, it is not even biclosed - that is, $T\circ -$ does not have a right adjoint. For if it did, it would have to preserve the initial object 0 of $\mathcal{F}$; which is the initial object 0 of $\mathcal{V}$, since $\psi:\mathcal{V}\to\mathcal{F}$ preserves colimits. But $0^0=I$ and $0^m=0$ for $m\neq 0$; so $T\circ 0 = T0\otimes I = T0$, which is $\neq 0$ in general.

4. Operads

By an *operad* we mean a monoid for the tensor product $\circ$ in $\mathcal{F}$, that is, an object T of $\mathcal{F}$ together with morphisms $\mu: T\circ T\to T$ and $\eta: J\to T$ satisfying the associative and identity laws. Operads form a category, a morphism of operads being a map $T\to T'$ respecting μ and η. An example of an operad is the *endomorphism operad* $\{S,S\}$ of S;

the internal endomorphism object in a closed category is of course always a monoid, $\mu : \{S,S\} \circ \{S,S\} \to \{S,S\}$ corresponding by adjunction to

$$\{S,S\} \circ \{S,S\} \circ S \xrightarrow{1 \circ e} \{S,S\} \circ S \xrightarrow{e} S$$

where e is the evaluation, and $\eta : J \to \{S,S\}$ corresponding by adjunction to $J \circ S \cong S$.

If T is an operad, $T \circ -$ is obviously a monad on $\mathcal{F}$; it restricts to a monad on $\mathcal{V}$, since $T \circ A \in \mathcal{V}$ for $A \in \mathcal{V}$. An algebra for this monad on $\mathcal{V}$ is an $A \in \mathcal{V}$ together with an *action* $T \circ A \to A$ satisfying the usual laws; however it is at once seen that these laws express precisely that the corresponding map $T \to \{A, A\}$ is a morphism of operads. We call such an algebra A a T-*algebra*, and write T-Alg for the category of T-algebras.

To show that this generalizes May's definition, we must give the data for an operad in more primitive terms. To give $\eta : J \to T$ is just to give a map $\eta_1 : I \to T1$, since $J1 = I$ and $Jn = 0$ for $n \neq 1$. To give $\mu : T \circ T \to T$ is to give $\sigma : Tm \otimes T^m \to T$, natural in m; the naturality requirements says $\sigma(T\xi \otimes 1) = \sigma(1 \otimes T^\xi)$, that is, $\sigma(T\xi \otimes 1) = \sigma(1 \otimes \langle\xi\rangle)$. By (2.3) and (2.5), this is to give maps $\tau : Tm \otimes \mathbf{P}(n_1 + \cdots + n_m, k) \otimes Tn_1 \otimes \cdots \otimes Tn_m \to Tk$, natural in k and the n_i, and such that $\tau(T\xi \otimes 1 \otimes 1) = \tau(1 \otimes \mathbf{P}(\langle\xi\rangle, 1) \otimes \langle\xi\rangle)$. By the Yoneda lemma, this is finally to give maps $\theta : Tm \otimes Tn_1 \otimes \cdots \otimes Tn_m \to T(n_1 + \cdots + n_m)$, natural in the n_i, and such that the following diagram commutes:

$$\begin{tikzcd}
Tm \otimes Tn_1 \otimes \ldots \otimes Tn_m \arrow[r, "\theta"] & T(n_1 + \ldots + n_m) \\
Tm \otimes Tn_1 \otimes \ldots \otimes Tn_m \arrow[u, "T\xi\otimes 1"] \arrow[d, "1\otimes\langle\xi\rangle"'] & \\
Tm \otimes Tn_{\xi 1} \otimes \ldots \otimes Tn_{\xi m} \arrow[r, "\theta"'] & T(n_{\xi 1} + \ldots + n_{\xi m}) \arrow[uu, "T\langle\xi\rangle"']
\end{tikzcd} \tag{4.1}$$

In a case such as $\mathcal{V} = \mathbf{Top}$, where we have elements, we can write the data still more simply. To give η is now just to give an element 1 of $T1$. Write the image under θ of $(a, b_1, \cdots, b_n)$ as $a[b_1, \cdots, b_n]$. For $a \in Tm$, write ξa for $(T\xi)a$. Denote the permutation

$$n_{\xi 1} + \ldots + n_{\xi m} \xrightarrow{\langle\xi\rangle} n_1 + \ldots + n_m \xrightarrow{\eta_1 + \ldots + \eta_m} n_1 + \ldots + n_m \tag{4.2}$$

by $\xi[\eta_1, \cdots, \eta_m]$. Then the commutativity of (4.1), *together with* the naturality of θ in the n_i, is expressed by

$$(\xi a)[\eta_1 b_1, \cdots, \eta_n b_n] = (\xi[\eta_1, \cdots, \eta_n])a[b_{\xi 1}, \cdots, b_{\xi n}] \, . \tag{4.3}$$

Finally the associative and identity laws for μ and η are expressed by

$$(a[b_1, \cdots, b_m])[c_1, \cdots, c_k] = a[b_1[c_1, \cdots, c_{k_1}], b_2[c_{k_1+1}, \cdots, c_{k_2}], \cdots, b_m[\cdots, c_k]] \, , \tag{4.4}$$

$$a[1, \cdots, 1] = a \, , \tag{4.5}$$

$$1[b] = b\ . \tag{4.6}$$

We now have May's definition of operad, except that he uses right actions of $\mathbf{P}_n$ rather than left ones, and that he requires $T0$ to reduce a single point.

To give a T-algebra in **Top** is to give an action $T \circ A \to A$; so we must give maps $Tm \otimes A^m \to A$, natural in m; if these maps send $(a, x_1, \cdots, x_m)$ to $a\{x_1, \cdots, x_m\}$, the naturality in m is expressed by

$$(\xi a)\{x_1, \cdots, x_m\} = a\{x_{\xi 1}, \cdots, x_{\xi m}\}\ , \tag{4.7}$$

and the conditions for an action are

$$(a[b_1, \cdots, b_m])\{x_1, \cdots, x_k\} = a\{b_1\{x_1, \cdots, x_{k_1}\}, \cdots, b_m\{\cdots, x_k\}\}\ , \tag{4.8}$$

$$1\{x\} = x\ . \tag{4.9}$$

5. May's monad on $\mathbf{Top}_0$

As we said above, May imposes on his operads the extra condition that $T0$ be the one-point set $\star$. It is then the case that every T-algebra is canonically pointed, and that T-Alg $= T_0$-Alg for a certain monad T_0 on $\mathbf{Top}_0$.

That it must be so is best seen abstractly by defining as follows an operad S in **Top** : $S0 = S1 = \star$, Sn is empty for other n. The operad structure on S is then unique, and it is immediate that S-Alg $= \mathbf{Top}_0$, with $S \circ A$ being the free pointed space $\star + A$ on A. If T is an operad with $T0 = \star$, there is a unique operad map $S \to T$, inducing a monad map $S \circ - \to T \circ -$ and hence an algebraic functor T-Alg $\to S$-Alg. By the general theory of monads this has a left adjoint and is monadic, so that T-Alg $= T_0$-Alg for some monad T_0 on S-Alg $= \mathbf{Top}_0$.

Added in 2005 for the TAC Reprint. *I must have been thinking of the adjoint triangle theorem, whereby* T-Alg $\to S$-Alg *has a left adjoint if* S-Alg $\to \mathbf{Top}_0$ *is conservative with a left adjoint and the composite* T-Alg $\to S$-Alg $\to \mathbf{Top}_0$ *(here the forgetful* T-Alg $\to \mathbf{Top}_0$*) has a left adjoint; but this requires coequalizers in* T-Alg*. Here we do have these since, by Section 8 below,* T-Alg *is the category of models in* $\mathbf{Top}_0$ *of a* $\mathbf{Top}_0$*-enriched finitary Lawvere theory* $\mathcal{T}$*, and is therefore a reflective subcategory of* $[\mathcal{T}, \mathbf{Top}_0]$ *by Theorem 6.11 of [11], the cartesian closed category* $\mathbf{Top}_0$ *being locally bounded by Section 6.1 of [11].*

It is easy to give T_0 explicitly. First observe that the forgetful functor T-Alg $\to$ S-Alg $= \mathbf{Top}_0$ sends A to the space A with base-point $\dagger = \star\{\}$. The operad $\{A, A\}$ has a sub-operad $\{A, A\}_0$ given by $\{A, A\}_0 n = [A^n, A]_0 =$ the object of *pointed* maps $A^n \to A$. The operad map $T \to \{A, A\}$ always factorizes through $\{A, A\}_0$; that is,

$$a\{\dagger, \cdots, \dagger\} = \dagger\ . \tag{5.1}$$

To see this, observe that $a[\star, \cdots, \star] = \star$, since $\star$ is the only element of $T0$. Using (4.8), we have

$$a\{\dagger, \cdots, \dagger\} = a\{\star\{\}, \cdots, \star\{\}\} = (a[\star, \cdots, \star])\{\} = \star\{\} = \dagger\ .$$

It follows that a T-algebra A may equally be defined as a *pointed* space A together with an operad map $T \rightarrow \{A, A\}_0$; if, with May, we call *this* an action, then an action is a map $a(x_1, \cdots, x_m) \longmapsto a\{x_1, \cdots, x_m\}$ satisfying (5.1) as well as (4.7) - (4.9). Using the special case

$$\star\{\} = \dagger \tag{5.2}$$

of (5.1), together with (4.8) and (4.9), we get

$$(a[1, 1, \cdots, \star, \cdots, 1])\{x_1, \cdots, x_{i-1}, x_{i+1} \cdots, x_m\} = a\{x_1, \cdots, x_{i-1}, \dagger, x_{i+1}, \cdots, x_m\}\ ; \tag{5.3}$$

conversely, (5.2) and (5.3) imply (5.1) – put each $x_i = \dagger$ in (5.3) and use induction.

Define therefore T_0A for $A \in \mathbf{Top}_0$ as $(\sum_m Tm \times A^m)/\mathbf{q}$, where $\mathbf{q}$ is the equivalence relation needed to force (4.7) and (5.3); then an action is given by a map $T_0A \rightarrow A$, pointed to guarantee (5.2), and satisfying (4.8) and (4.9); but T_0 is clearly a monad on $\mathbf{Top}_0$, and this is just an action for T_0.

6. Change of $\mathcal{V}$

If a functor $\Phi : \mathcal{V} \rightarrow \mathcal{V}'$ admits enrichment to a symmetric monoidal functor, it is easy to see that $[1, \Phi] : [\mathbf{P}, \mathcal{V}] \rightarrow [\mathbf{P}, \mathcal{V}']$ does so too, for the $\otimes$-structure; and that as a consequence $[1, \Phi]$ admits enrichment to a monoidal functor for the $\circ$-structure. It follows that a $\mathcal{V}$-operad T then gives rise to a $\mathcal{V}'$-operad, that we may call ΦT. If Φ preserves $\otimes$ and colimits, so does $[1, \Phi]$, and then $[1, \Phi]$ preserves $\circ$; so in this case the operad structure on ΦT is especially simple.

There are two evident operads in **Sets**; one is N, given by $Nn = \star$ for all n, with the unique operad structure; the other is P, given by $Pn = \mathbf{P}_n$, with $\xi[\eta_1, \cdots, \eta_n]$ defined as in (4.2). The functor F of the first paragraph of §2 now gives operads FN and FP in any $\mathcal{V}$. It is easy to see that $FP \circ A = \sum_m A^m =$ the free monoid on A, and that an FP-algebra is a monoid in $\mathcal{V}$; while $FN \circ A$ is the free abelian monoid on A, and an FN-algebra is an abelian monoid in $\mathcal{V}$. In the case $\mathcal{V} = \mathbf{Top}$, the corresponding monads $(FP)_0$ and $(FN)_0$ on $\mathbf{Top}_0$ are respectively the James reduced product construction and the infinite symmetric product construction.

7. Relation to props

Slightly altering Mac Lane's definition on p.97 of [9], define a ($\mathcal{V}$-) *prop* to be a $\mathcal{V}$-category $\mathcal{T}$ with the same objects as $\mathbf{P}$, with a strict symmetric monoidal structure over $\mathcal{V}$, and with a strict symmetric monoidal functor $\Phi : \mathbf{P}^{\mathrm{op}} \to \mathcal{T}$ which is the identity on objects. Then a $\mathcal{T}$-*algebra* is a strict symmetric monoidal $\mathcal{V}$-functor $\Psi : \mathcal{T} \to \mathcal{V}$. If $\Psi 1 = A$, then Ψn must be A^n, and the $\mathcal{T}$-algebra is given by maps $\mathcal{T}(m,n) \to [A^m, A^n]$ satisfying whatever is necessary to make this a $\mathcal{V}$-functor and strict symmetric monoidal.

Every operad T determines a prop $\mathcal{T} = \hat{T}$. Let $\mathcal{T}(m,n) = (T^n)m$. The map $\mu : T \circ T \to T$ gives $\mu^n : (T \circ T)^n \to T^n$, or $T^n \circ T \to T^n$ by Lemma 3.1; in view of the definition of $\circ$ this gives maps $(T^n)m \otimes (T^m)k \to (T^n)k$, or $\mathcal{T}(m,n) \otimes \mathcal{T}(k,m) \to \mathcal{T}(k,n)$, defining composition in $\mathcal{T}$. Similarly $\eta^n : J^n \to T^n$ gives by (3.5) maps $\mathbf{P}(n,m) \otimes I \to \mathcal{T}(m,n)$, or $\mathbf{P}(n,m) \to V\mathcal{T}(m,n)$, where V is the underlying-set functor $\mathcal{V}(I,-)$ of §2; this gives identities for $\mathcal{T}$ and gives the functor $\Phi : \mathbf{P}^{\mathrm{op}} \to \mathcal{T}$. A T-algebra A with action $\nu : T \circ A \to A$ gives a $\mathcal{T}$-algebra, via $\nu^n : (T \circ A)^n = T^n \circ A \to A^n$, which gives $T^n \to \{A, A^n\}$ and hence $\mathcal{T}(m,n) = (T^n)m \to \{A, A^n\}m = [A^m, A^n]$. Moreover it may be verified that every $\mathcal{T}$-algebra arises thus from a unique T-algebra.

Conversely every prop $\mathcal{T}$ determines an operad T by setting $Tn = \mathcal{T}(n,1)$. However the prop $\hat{T}$ constructed in the last paragraph will not in general be $\mathcal{T}$; props of the form $\hat{T}$ are only those in which

$$\mathcal{T}(m,n) = ((\mathcal{T}(-,1))^n)m \; . \tag{7.1}$$

For instance, the prop whose algebras are Hopf algebras is not of this kind, containing an element in $\mathcal{T}(1,2)$ not describable in term of elements of the various $\mathcal{T}(n,1)$. For a general prop $\mathcal{T}$, a $\mathcal{T}$-algebra structure on $A \in \mathcal{V}$ gives a T-algebra structure, but the converse is no longer true. The fact is that the $\mathcal{T}$-algebras, unlike the T-algebras, are not monadic over $\mathcal{V}$.

We conclude that operads may be identified with props of a very special kind; it is not clear than any advantage would follow from so considering them.

8. Other domain categories

Write $\mathbf{N}$ for the discrete category whose objects are the integers $n \geq 0$. The monoidal structure on $\mathbf{P}$ restricts to one on $\mathbf{N}$, and we can clearly repeat everything we have said above with $\mathbf{N}$ replacing $\mathbf{P}$ throughout. The only thing that changes is the *explicit* formula (2.2) for the $\otimes$-product in the functor category; for $[\mathbf{N}, \mathcal{V}]$ it becomes $(T \otimes S)k = \sum_{m+n=k} Tm \otimes Sn$; but we never used this formula. Of course, with no permutations to worry about, everything is now simpler; the formula (3.2) may now be written as $T \circ S = \sum_m Tm \otimes S^m$.

The new "operads" we get, or $\mathbf{N}$-*operads*, are of course quite different things from the old ones, or $\mathbf{P}$-*operads*; in the case of **Top** they are the "non- $\sum$ operads" of May ([10], §§3.12-3.14). Of course each such gives a monad on $\mathcal{V}$; but in fact we get no new monads in this way; every $\mathbf{N}$-operad T determines a $\mathbf{P}$-operad ΓT giving the same monad on $\mathcal{V}$.

For take $\Gamma : [\mathbf{N}, \mathcal{V}] \to [\mathbf{P}, \mathcal{V}]$ as the left Kan adjoint of the functor $[\mathbf{P}, \mathcal{V}] \to [\mathbf{N}, \mathcal{V}]$ induced by the inclusion $\mathbf{N} \to \mathbf{P}$. The usual integral formula for Γ simplifies because $\mathbf{N}$ is discrete to

$$\Gamma T = \sum_m \mathbf{P}(m,-) \otimes Tm \; ; \tag{8.1}$$

so that $(\Gamma T)n = \mathbf{P}_n \otimes Tn$. It is easy to see that Γ preserves $\otimes$-products (or we can observe that its right adjoint is clearly a *normal* closed functor and appeal to §5.2 of [4]). Since it also preserves colimits, it takes an $\mathbf{N}$-operad T to a $\mathbf{P}$-operad ΓT. For $A \in \mathcal{V}$, we have

$$\begin{aligned} (\Gamma T) \circ A &= \int^{n \in \mathbf{P}} (\Gamma T)n \otimes A^n \\ &= \int^{n \in \mathbf{P}} \sum_m \mathbf{P}(m,n) \otimes Tm \otimes A^n \\ &= \sum_m Tm \otimes A^m \;\; \text{by Yoneda} \\ &= T \circ A \; ; \end{aligned}$$

it easily follows that T-algebras coincide with ΓT-algebras.

Now let $\mathbf{S}$ be the category whose objects are the integers $n \geq 0$, and whose morphisms $n \to m$ are the set-maps $\{1, 2, \cdots, n\} \to \{1, 2, \cdots, m\}$. Then $\mathbf{S}$ has a symmetric monoidal structure extending that of $\mathbf{P}$, the tensor product $m + n$ now being the coproduct. This being so, we have $\mathbf{S}(m+n,-) = \mathbf{S}(m,-) \times \mathbf{S}(n,-)$, and therefore if we replace $\mathbf{P}$ by $\mathbf{S}$ in (2.1) and use the Yoneda lemma, we get in place of (2.2) the explicit formula $(T \otimes S)k = Tk \otimes Sk$; the cosmos structure on $[\mathbf{S}, \mathcal{V}]$ has the *pointwise* monoidal structure.

If we wish to imitate all that we have done, but with $\mathbf{S}$ in place of $\mathbf{P}$, we must suppose that $\mathcal{V}$ is *cartesian* closed. Then we can extend the definition of $\langle \xi \rangle$ in $\mathcal{V}$ to $\langle \alpha \rangle : A_1 \times \cdots \times A_n \to A_{\alpha 1} \times \cdots \times A_{\alpha m}$ where $\alpha \in \mathbf{S}(m,n)$. Since $\mathbf{S}^{\mathrm{op}}$ too has the cartesian monoidal structure, we have similarly $\langle \alpha \rangle : k_{\alpha 1} + \cdots + k_{\alpha m} \to k_1 + \cdots + k_n$ in $\mathbf{S}$. The closed structure on $[\mathbf{S}, \mathcal{V}]$, being pointwise, is again cartesian; and (2.5) holds with $\langle \xi \rangle$ replaced by $\langle \alpha \rangle$.

For $T \in [\mathbf{S}, \mathcal{V}]$, note that T^m is contravariantly functorial in $m \in \mathbf{S}$, with $T^\alpha = \langle \alpha \rangle$; and (3.1) holds with ξ replaced by α (and $\otimes$ by the $\times$ that is more usual in the cartesian case). So everything carries over.

When we come to §7 in the $\mathbf{S}$-case, we insist that the monoidal structure on $\mathcal{T}$ be cartesian; the more usual name for the $\mathbf{S}$-analogue of $\mathcal{V}$-prop is $\mathcal{V}$*-theory*; these are not the most general kind of $\mathcal{V}$-theory as defined by Dubuc [3], but bear to the latter the same relation as do the finitary **Sets**-theories of Lawvere [6] to the more general ones of Linton [7]. However in the $\mathbf{S}$-case (7.1) always holds, because $\mathcal{T}(-,1)^n = \mathcal{T}(-,n)$, since $+$ is the cartesian product in $\mathcal{T}$. Hence an $\mathbf{S}$-operad in a cartesian closed $\mathcal{V}$ is exactly the same thing as a finitary $\mathcal{V}$-theory; an example would be the theory in **Top** of topological modules over a topological ring. Every $\mathbf{P}$-operad T determines an $\mathbf{S}$-operad ΓT, exactly as in the transition from $\mathbf{N}$-operads to $\mathbf{P}$-operads; and the T-algebras are the ΓT-algebras. This shows that in the cartesian closed case the monads that arise from $\mathbf{P}$-operads are among those of finite rank, and certainly do not constitute all monads.

References

[1] Day, B.J. On closed categories of functors, Reports of the Midwest Category Seminar IV, *Lecture Notes in Mathematics* **137** (1970), 1–38.

[2] Day, B.J. and Kelly, G.M. Enriched functor categories, Reports of the Midwest Category Seminar III, *Lecture Notes in Mathematics* **106** (1969), 178-191.

[3] Dubuc, E.J. Enriched semantics-structure (meta) adjointness, *Rev. Union Math. Argentina* **25** (1970/71), 5–26.

[4] Kelly, G.M. Adjunction for enriched categories, Reports of the Midwest Category Seminar III, *Lecture Notes in Mathematics* **106** (1969), 166–177.

[5] Kelly, G.M. An abstract approach to coherence, Coherence in categories, *Lecture Notes Mathematics* **281** (1972), 106–147.

[6] Lawvere, F.W. Functorial semantics of algebraic theories, *Proc. Nat. Acad. Sci. U.S.A.* **50** (1963), 869–872.

[7] Linton, F.E.J. Some aspects of equational categories, *Proc. Conf. on Categorical Algebra*, (La Jolla 1965), Springer-Verlag 1966, 84–94.

[8] Mac Lane, S. Natural associativity and commutativity, *Rice University Studies* **49** (1963), 28–46.

[9] Mac Lane, S. Categorical algebra, *Bull. Amer. Math. Soc.* **71** (1965), 40–106.

[10] May, J.P. The geometry of iterated loop spaces, *Lectures Notes in Mathematics* **271** (1972).

[11] Kelly, G.M. *Basic Concepts of Enriched Category Theory*, London Mathematical Society Lecture Notes Series, **64**, Cambridge University Press, 1982. Reprinted as *Reprints in Theory and Applications of Categories*, 10 (2005).

[12] Im, G.B. and Kelly, G.M., A universal property of the convolution monoidal structure, *J. Pure Appl. Alg.* **43** (1986), 75–88.

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