Reprints in Theory and Applications of Categories, No. 11, 2005, pp. 1–35.

AN ELEMENTARY THEORY OF THE CATEGORY OF SETS (LONG VERSION) WITH COMMENTARY

F. WILLIAM LAWVERE

Received by the editors 2005-04-01.
Transmitted by M. Hyland, A. Kock, R. Rosebrugh. Reprint published on 2005-05-23.
2000 Mathematics Subject Classification: 18B05, 00A30, 03A05.
Key words and phrases: Category of sets, Axiom of choice, Mathematical logic and foundations.
This article is an expanded version of 'An elementary theory of the category of sets', *Proceedings of the National Academy of Science of the U.S.A* **52**, 1506–1511. Article and commentary ©F. William Lawvere and Colin McLarty. Permission to copy for private use granted.

ETCS and the Philosophy of Mathematics

Colin McLarty

Philosophers and logicians to this day often contrast "categorical" foundations for mathematics with "set-theoretic" foundations as if the two were opposites. Yet the second categorical foundation ever worked out, and the first in print, was a set theory—Lawvere's axioms for the category of sets, called ETCS, (Lawvere 1964). These axioms were written soon after Lawvere's dissertation sketched the category of categories as a foundation, CCAF, (Lawvere 1963). They appeared in the *PNAS* two years before axioms for CCAF were published (Lawvere 1966). The present longer version was available since April 1965 in the Lecture Notes Series of the University of Chicago Department of Mathematics.[1] It gives the same definitions and theorems, with the same numbering as the 5 page PNAS version, but with fuller proofs and explications.

Lawvere argued that set theory should not be based on membership (as in Zermelo Frankel set theory, ZF), but on "isomorphism-invariant structure, as defined, for example, by universal mapping properties" (p. 1). He later noticed that Cantor and Zermelo differed over this very issue. Cantor gave an isomorphism-invariant account of sets, where elements of sets are "mere units" distinct from one another but not individually identifiable. Zermelo sharply faulted him for this and followed Frege in saying set theory must be founded on a membership relation between sets.[2]

Paul Benacerraf made the question prominent for philosophers one year after (Lawvere 1964). The two were no doubt independent since philosophers would not look in the *PNAS* for this kind of thing. Benacerraf argued that numbers, for example, cannot be sets since numbers should have no properties except arithmetic relations. The set theory familiar to him was ZF, where the elements of sets are sets in turn, and have properties other than arithmetic. So he concluded that numbers cannot be sets and the natural number structure cannot be a set.

Benacerraf wanted numbers to be elements of *abstract structures* which differ from ZF sets this way:

> in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an *abstract structure*—and the distinction lies in the fact that the "elements" of the structure have no properties other than those relating them to other "elements" of the same structure. (Benacerraf 1965, p. 70)

[1] Date verified by Lawvere through the University of Chicago math librarian.

[2] Zermelo's complaint about Cantor and endorsement of Frege are on (Cantor 1932, pp. 351, 440-42). See (Lawvere 1994).

The sets of ETCS are *abstract structures* in exactly this sense. An element $x \in S$ in ETCS has no properties except that it is an element of S and is distinct from any other elements of S. The *natural number structure* in ETCS is a triad of a set N, a selected element $0 \in N$, and a successor function $s{:}N{\to}N$. Then $N, 0, s$ expresses the arithmetic relations as for example $m = s(n)$ says m is the successor of n. But $N, 0, s$ simply have no properties beyond those they share with every isomorphic triad.[3] So in relation to 0 and s the elements of N have arithmetic relations, but they have no other properties.

Lawvere carefully says ETCS "provides a foundation for mathematics ... in the sense that much of number theory, elementary analysis, and algebra can apparently be developed within it." This aspect relies entirely on the elementary axioms. But he did not claim that such formal adequacy makes ETCS a formalist or logicist starting point for mathematics, as if there was no mathematics before it. He says that ETCS condenses and systematizes knowledge we already have of "the category of sets and mappings... denoted by $\mathcal{S}$" (pp. 1-2). This aspect deals with models and metatheorems for the axioms. Gödel's theorem says we will never have a proof-theoretically complete description of this category. Lawvere gives various metatheoretic proofs about categorical set theory based on this and on the Löwenheim-Skolem theorem. Yet he takes it that there are sets, we know much about them apart from any formalization, and we can learn much from seeing how formal axioms describe them. The category exists objectively in mathematical experience as a whole—not in a platonic heaven, nor in merely subjective individual experience. This historical (and dialectical) realism continues through all his work and gets clear recent statements in (Lawvere & Rosebrugh 2003) and (Lawvere 2003).

Of course the category of sets is not the only one that exists. It is not the only one formally adequate as a "foundation" nor the first to be described that way. Lawvere concludes this paper, as he also does the *PNAS* version, by saying that more powerful foundations would be more naturally expressed in the category of categories (p.32, or (Lawvere 1964, p. 1510)).

References.

Benacerraf, Paul (1965), 'What numbers could not be', *Philosophical Review* **74**, 47–73.

Cantor, Georg (1932), *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, Julius Springer, Berlin.

Lawvere, F. William (1963), Functorial Semantics of Algebraic Theories, PhD thesis, Columbia University. Republished in: *Reprints in Theory and Applications of Categories*, No. 5 (2004) pp 1-121, available on-line at http://tac.mta.ca/tac/reprints/articles/5/tr5abs.html.

Lawvere, F. William (1964), 'An elementary theory of the category of sets', *Proceedings of the National Academy of Science of the U.S.A* **52**, 1506–1511.

[3]For a precise statement and proof see (McLarty 1993).

Lawvere, F. William (1966), The category of categories as a foundation for mathematics, *in* S.Eilenberg et al., eds, 'Proceedings of the Conference on Categorical Algebra, La Jolla, 1965', Springer-Verlag, pp. 1–21.

Lawvere, F. William (1994), 'Cohesive toposes and Cantor's "lauter Einsen"', *Philosophia Mathematica* **2**, 5–15.

Lawvere, F. William (2003), 'Foundations and applications: Axiomatization and education', *Bulletin of Symbolic Logic* **9**, 213–24.

Lawvere, F. William & Robert Rosebrugh (2003), *Sets for Mathematics*, Cambridge University Press.

McLarty, Colin (1993), 'Numbers can be just what they have to', *Noûs* **27**, 487–98.

Author Commentary

F. William Lawvere

Saunders Mac Lane wisely insisted on this more complete version of the work that was summarized in the brief article he communicated in October 1964 to the Proceedings of the National Academy of Science. Accordingly, this manuscript, with its relatively complete proofs and extensive remarks on the metamathematical status of the work, was deposited in the University of Chicago mathematics library at the end of April 1965. I hope that the present publication, made possible thanks to Colin McLarty's efforts in TEX and thanks to the editors of *TAC*, will assist those who wish to apply the Elementary Theory of the Category of Sets to philosophy and to teaching.

Of course, the central theory is more fully developed in the joint textbook with Robert Rosebrugh *Sets for Mathematics* (and even bolder metamathematical comments are offered in its appendix). However, the present document may still be of interest, at least historically, giving as it does a glimpse of how these issues looked even as the crucial need for elementary axiomatization of the theory of toposes was crystallizing. (In July 1964 I noted in Chicago that sheaves of sets form categories with intrinsic properties just as sheaves of abelian groups form abelian categories; in January 1965 Benabou told me that Giraud had axiomatized "toposes"; in June 1965 Verdier's lecture on the beach at LaJolla made clear that those axioms were not yet "elementary" though extremely interesting....).

Among mathematical/logical results in this work is the fact that the axiom of choice implies classical logic. That result was later proved by Diaconescu under the strong hypothesis that the ambient category is a topos (i.e. that the notion of arbitrary monomorphism is representable by characteristic functions) but in this work it is shown that the topos property is itself also implied by choice. Specifically, characteristic functions valued in 2 are constructed by forming the union (proved to exist) of all complemented subobjects that miss a given monomorphism, and then using the choice principle to show that there is no excluded middle. (Presumably with modern technology this calculation could be carried out with more general generators than 1).

This elementary theory of the category of sets arose from a purely practical educational need. When I began teaching at Reed College in 1963, I was instructed that first-year analysis should emphasize foundations, with the usual formulas and applications of calculus being filled out in the second year. Since part of the difficulty in learning calculus stems from the rigid refusal of most textbooks to supply clear, explicit, statements of concepts and principles, I was very happy with the opportunity to oppose that unfortunate trend. Part of the summer of 1963 was devoted to designing a course based on the axiomatics of Zermelo-Fraenkel set theory (even though I had already before concluded that the category of categories is the best setting for "advanced" mathematics). But I soon

realized that even an entire semester would not be adequate for explaining all the (for a beginner bizarre) membership-theoretic definitions and results, then translating them into operations usable in algebra and analysis, then using that framework to construct a basis for the material I planned to present in the second semester on metric spaces.

However I found a way out of the ZF impasse and the able Reed students could indeed be led to take advantage of the second semester that I had planned. The way was to present in a couple of months an explicit axiomatic theory of the mathematical operations and concepts (composition, functionals, etc.) as actually needed in the development of the mathematics. Later, at the ETH in Zurich, I was able to further simplify the list of axioms.

April 1, 2005

An Elementary Theory of the Category of Sets (Long Version)

F. William Lawvere [4]

The elementary theory presented in this paper is intended to accomplish two purposes. First, the theory characterizes the category of sets and mappings as an abstract category in the sense that any model for the axioms which satisfies the additional (non-elementary) axiom of completeness (in the usual sense of category theory) can be proved to be equivalent to $\mathcal{S}$. Second, the theory provides a foundation for mathematics which is quite different from the usual set theories in the sense that much of number theory, elementary analysis, and algebra can apparently be developed within it even though no relation with the usual properties of $\in$ can be defined.

Philosophically, it may be said that these developments partially support the thesis that even in set theory and elementary mathematics it is also true as has long been felt in advanced algebra and topology, namely that the substance of mathematics resides not in Substance (as it is made to seem when $\in$ is the irreducible predicate, with the accompanying necessity of defining all concepts in terms of a rigid elementhood relation) but in Form (as is clear when the guiding notion is isomorphism-invariant structure, as defined, for example, by universal mapping properties). As in algebra and topology, here again the concrete technical machinery for the precise expression and efficient handling of these ideas is provided by the Eilenberg-Mac Lane theory of categories, functors, and natural transformations.

The undefined terms of our theory are *mappings*, *domain*, *co-domain*, and *composition*. The first of these is merely a convenient name for the elements of the universe over which all quantifiers range, the second two are binary relations in that universe, and the last is a ternary relation. The heuristic intent of these notions may be briefly explained as follows: a *mapping* f consists of three parts, a set S, a set S', and a "rule" which assigns to every element of S exactly one element of S'. The identity rule on S to S determines the *domain* of f and the identity rule on S' to S' determines the *codomain* of f. Thus every mapping has a unique domain and codomain, and these are also mappings. Mappings which appear as domains or codomains are also called *objects* and are frequently denoted by capital letters to distinguish them from more general mappings. If A, f, B are mappings, and if A is the domain of f while B is the codomain of f, we denote this situation by $A \xrightarrow{f} B$.

[4]Work partially supported by AF Contract No. AFOSR-520-64 at the University of Chicago and partially by a NATO Postdoctoral Fellowship at the ETH, Zurich.

A binary operation called *composition* is defined in the obvious way for precisely those pairs of mappings f, g such that the codomain of f is the domain of g. The result of such composition is denoted by juxtaposition fg, and we have

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad \Rightarrow \qquad A \xrightarrow{fg} C \ .$$

It is clear that composition is associative insofar as it is defined, and that objects behave as neutral elements with respect to composition.

The heuristic class of all mappings, with the above described domain, codomain, and composition structure, will be denoted by $\mathcal{S}$. The axioms of our theory may be regarded as a listing of some of the basic and mathematically useful properties of $\mathcal{S}$. There are three groups of axioms and two special axioms.

The first group of axioms (to which we do not assign numbers) consists of the axioms for an abstract *category* in the sense of (Eilenberg and Mac Lane 1945). We take these axioms as known but note that these axioms are obviously elementary (first-order) and that they have been stated informally in the third paragraph. We remark that a map $A \xrightarrow{f} B$ in a category is called an *isomorphism* if there exists a map $B \xrightarrow{g} A$ in the category such that $fg = A$ and $gf = B$; g is then a uniquely determined isomorphism called the *inverse* of f.

The second group of axioms calls for the existence of solutions to certain "universal mapping problems". As usual in such situations, it will be clear that the objects and operations thus asserted to exist are actually unique up to an isomorphism which is itself uniquely determined by a "naturalness" condition suggested by the structure of the particular axiom. This naturalness condition is simply that the isomorphism in question commutes with all the "structural maps" occurring in the statement of the universal mapping problem.

Axiom 1. All finite roots exist. Explicitly, this is guaranteed by assuming, in the sense explained below, that a terminal object 1 and an initial object 0 exist; that the product $A \times B$ and the coproduct (sum) $A + B$ of any pair of objects exists; and that the equalizer $E \xrightarrow{k} A$ and the coequalizer $B \xrightarrow{q} E^*$ of any pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ of maps exist.

Axiom 2. The exponential B^A of any pair of objects exists. The defining universal mapping property of exponentiation, discussed below, is closely related to the concept of λ-conversion and requires, in essence, that for each object A, the functor $(\ )^A$ is co-adjoint to the functor $A \times (\ )$.

Axiom 3. There exists a Dedekind-Pierce object N. This plays the role of our axiom of infinity; the defining property of the natural numbers is here taken to be the existence and uniqueness of sequences defined by a very simple sort of recursion, as explained below.

We now explain in more detail the universal properties associated with the terminology occurring in the existential axioms above. A *terminal* object is an object 1 such that for any object A there is a unique mapping $A \to 1$. An *initial* object is an object 0 such that for any object B there is a unique mapping $0 \to B$. It is clear that in our heuristic picture of $\mathcal{S}$,

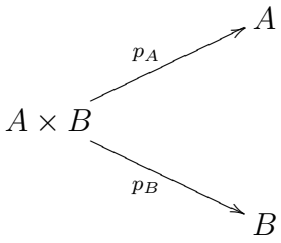
any singleton set is a terminal object and the null set is an initial object. (More precisely, we refer here to the identity mappings on these sets.) Because nothing of substance that we wish to do in our theory depends on the size of the isomorphism classes of objects, we find it a notational convenience to assume that there is exactly one terminal object 1 and exactly one initial object 0; however, we do not make this convention a formal axiom as it can be avoided by complicating the notation somewhat.

DEFINITION 1. x is an *element* of A, denoted $x \in A$, iff $1 \xrightarrow{x} A$.

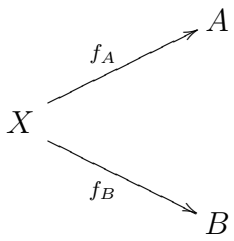
REMARK 1: Elementhood, as defined here, is a special case of membership, to be defined presently. Neither of these notions has many formal properties in common with the basic relation of the usual set theories; $x \in y \in z$, for example, never holds except in trivial cases. However, we are able (noting that the evaluation of a mapping at an element of its domain may be viewed as a special case of composition) to deduce the basic property required by our heuristic definition of mapping, namely that if $A \xrightarrow{f} B$ then

$$\forall x \; [x \in A \;\Rightarrow\; \exists! y \; [y \in B \;\&\; xf = y]]$$

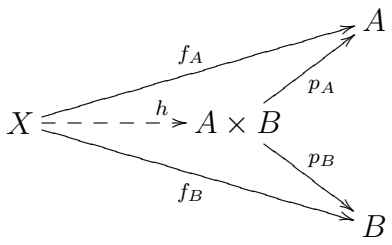
We must still describe the universal mapping properties of product, coproduct, equalizer and coequalizer in order to explicate our first axiom. Given two objects A and B their *product* is an object $A \times B$, together with a pair of mappings



such that for any object X and pair of mappings



there is a unique mapping $X \xrightarrow{h} A \times B$ such that $hp_A = f_A$ and $hp_B = f_B$, i.e. such that the following diagram commutes:



We use the notation $h = \langle f_A, f_B \rangle$. Thus taking $X = 1$, we deduce that the elements x of $A \times B$ are uniquely expressible in the form $x = \langle x_A, x_B \rangle$ where $x_A \in A$ and $x_B \in B$. Returning to consideration of an arbitrary X and pair of maps f_A, f_B it then follows that the "rule" of $\langle f_A, f_B \rangle$ is given by

$$x\langle f_A, f_B \rangle \;=\; \langle x f_A, x f_B \rangle \qquad \text{for all } x \in X.$$

The "structural maps" p_A and p_B are called *projections*. There are unique natural (commuting with projections) isomorphisms

$$A \times 1 \;\cong\; A \;\cong 1 \times A$$
$$(A \times B) \times (C \times D) \;\cong\; (A \times C) \times (B \times D)\ .$$

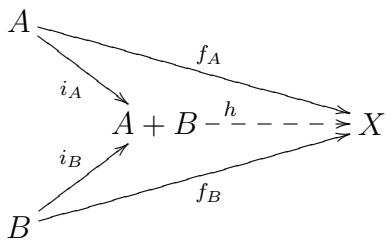
Given mappings $A \xrightarrow{f} A'$, $B \xrightarrow{g} B'$, there is a unique natural mapping

$$A \times B \xrightarrow{\quad f \times g \quad} A' \times B'$$

and this product operation on mappings is functorial in the sense that given further mappings $A' \xrightarrow{\overline{f}} A''$, $B' \xrightarrow{\overline{g}} B''$, one has

$$(f\,\overline{f}) \times (g\,\overline{g}) \;=\; (f \times g)(\overline{f} \times \overline{g})$$

The *coproduct* $A + B$ of two objects is characterized by the dual universal mapping property

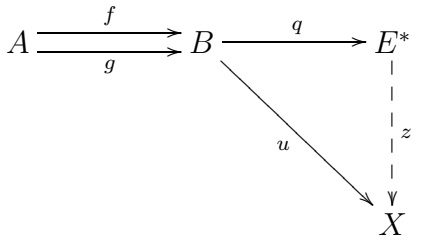


$$\forall X\ \forall f_A\ \forall f_B\ \exists! h\ [i_A h \;=\; f_A\ \&\ i_B h \;=\; f_B]$$

(where the quantifiers range only over diagrams of the above form). Here the structural maps i_A and i_B are called *injections*. In the usual discussions of $\mathcal{S}$ and hence in our theory, the coproduct is called the *sum*. The functorial extension of this operation from objects to all mappings follows just as in the case of the product operation, but the deduction of the nature of the elements of $A + B$ is not so easy; in fact we will find it necessary to introduce further axioms to describe the elements of a sum. This is not surprising in view of the great variation from category to category in the nature of coproducts. Suffice it for the moment to note that the existence of sums in $\mathcal{S}$ can be verified by one of the usual "disjoint union" constructions; the universal mapping property corresponds to the heuristic principle that a mapping is well defined by specifying its rule separately on each piece of a finite partition of its domain.

The *equalizer* of a pair of mappings $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is a mapping $E \overset{k}{\to} A$ such that $kf = kg$ and which satisfies the universal mapping property: for any $X \overset{u}{\to} A$ such that $uf = ug$ there is a unique $X \overset{z}{\to} E$ such that $u = zk$. It follows immediately that the elements a of A such that $af = ag$ are precisely those which factor through k, and the latter are in one-to-one correspondence with the elements of E. It also follows that k is, in a sense to be defined presently, a subset of A; clearly it is just the subset on which f and g agree. The equalizer construction, like the coequalizer to be described next, has certain functorial properties which we do not bother to state explicitly.

The *coequalizer* of a pair of mappings $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is a mapping $B \overset{q}{\to} E^*$ such that $fq = gq$ and which satisfies the dual universal mapping property:



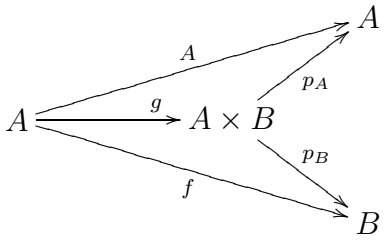
$$\forall X\ \forall u\ [fu = gu\ \Rightarrow\ \exists!\ z\ [u = qz]\ ]$$

(where the quantified variables range only over diagrams of the sort pictured). As in the case of sums, an elementwise investigation of coequalizers is not immediate; the discussion of the elements of E^* can be reduced to the consideration of the pairs of elements of B which are identified by q, but the latter is non-trivial. However, using our full list of axioms we will be able to prove a theorem to the effect that q is the quotient map obtained by partitioning B in the finest way which, for each $a \in A$, puts af and ag in the same cell of the partition.

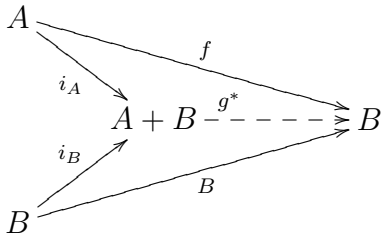
This completes the explication of our axiom "finite roots exist" except for the remark that in category theory "finite roots" refers to a more general class of operations including inverse images and intersections and their duals as well as more complicated constructions.

However, all these can be proved to exist (within the first order language) once the existence of the six special cases we have assumed is affirmed.

Before proceeding to the discussion of the exponentiation axiom, let us note that the usual formal definition of mapping can be dualized. The usual notion of a mapping f from A to B describes f as a subset of $A \times B$, namely the *graph* g of f in the diagram below.



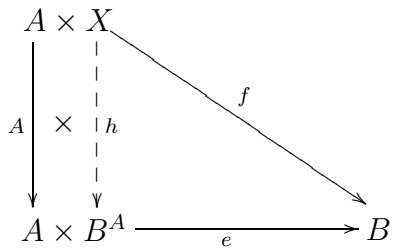
Dualizing the diagram, we obtain the *cograph* g^* of f defined to make the following diagram commutative.



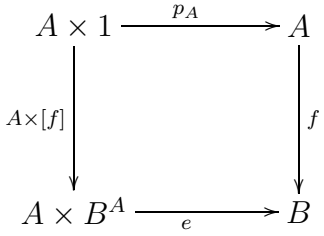
This suggests the alternative concept of a mapping from A to B as a partition of the disjoint union with the property that each cell in the partition contains exactly one member of B. The concept of cograph corresponds to the intuitive diagrams sometimes used to represent mappings in which both A and B are displayed (disjointly!) and a line is drawn from each member of A to its image in B; two members of $A + B$ can then be thought of as identified if they can be connected by following these lines.

Whatever formal definition is taken, it is intuitively clear that for each pair of sets A, B there exists a set whose elements are precisely the mappings from A to B or at least "names" for these mappings in some sense. This is one of the consequences of our axiom that "exponentiation exists".

The meaning of the exponentiation axiom is that the operation of forming the product has a co-adjoint operation in the sense that given any pair of objects A, B there exists an object B^A and a mapping $A \times B^A \xrightarrow{e} B$ with the property that for any object X and any mapping $A \times X \xrightarrow{f} B$ there is a unique mapping $X \xrightarrow{h} B^A$ such that $(A \times h)e = f$. This universal mapping property is partly expressed by the following diagram.



Here the structural mapping e is called the *evaluation* map. By taking $X = 1$, and noting that the projection $A \times 1 \to A$ is an isomorphism, it follows immediately that the elements of B^A are in one-to-one correspondence with the mappings from A to B. Explicitly, a mapping $A \xrightarrow{f} B$ and its "name" $[f] \in B^A$ are connected by the commutativity of the following diagram.



It follows immediately that for $a \in A$, $A \xrightarrow{f} B$, one has

$$\langle a, [f] \rangle e \;=\; af \;,$$

i.e., the rule of the evaluation map is evaluation.

When extended to a functor in the natural (i.e., coherent with evaluation) fashion, exponentiation is actually contravariant in the exponent. That is, given $A' \xrightarrow{f} A$, $B \xrightarrow{g} B'$, the induced map goes in the direction opposite of f:

$$B^A \xrightarrow{\;g^f\;} B'^{A'}$$

Functoriality in this case just means that if $A'' \xrightarrow{\overline{f}} A'$, $B' \xrightarrow{\overline{g}} B''$ are further mappings then $(g\overline{g})^{(\overline{f}f)} = (g^f)(\overline{g}^{\overline{f}})$. The rule of g^f is of course that for $A \xrightarrow{u} B$,

$$[u](g^f) \;=\; [fug] \;.$$

A simple example of such an induced map is the "diagonal" $B \to B^A$ which is induced by $A \to 1$ and $B \xrightarrow{B} B$ and which assigns to each $b \in B$ the "name" of the corresponding constant mapping $A \to B$.

Also deducible from the exponentiation axiom is the existence, for any three objects A, B, C, of a mapping

$$B^A \times C^B \xrightarrow{\;\gamma\;} C^A$$

such that for any $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ one has

$$\langle [f], [g] \rangle \gamma \;=\; [fg]$$

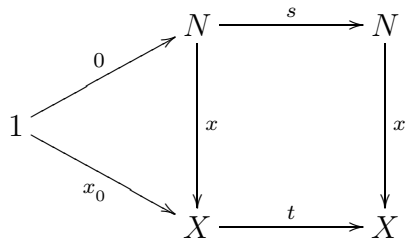
(Of course the usual "laws of exponents" can be proved as well.)

Another consequence of the exponentiation axiom is the distributivity relation

$$A \times B \;+\; A \times C \;\cong\; A \times (B + C)$$

Actually a canonical mapping from the left hand side to the right hand side can be constructed in any category in which products and coproducts exist; in a category with exponentiation the inverse mapping can also be constructed. A related conclusion is that if q is the coequalizer of f with g, then $A \times q$ is the coequalizer of $A \times f$ with $A \times g$. Both these conclusions are special cases of the fact that for each A the functor $A \times (\ )$, like any functor with a co-adjoint, must preserve all right hand roots. We remark that in the usual categories of groups, rings, or modules, exponentiation does not exist, which follows immediately from the fact that distributivity fails.

The third axiom asserts the existence of a special object N equipped with structural maps $1 \xrightarrow{0} N \xrightarrow{s} N$ with the property that given any object X and any $x_0 \in X$, $X \xrightarrow{t} X$ there is a unique $N \xrightarrow{x} X$ such that the following diagram commutes:



As usual, the universal mapping property determines the triple $N, 0, s$ up to a unique isomorphism. A mapping $N \xrightarrow{x} X$ is called a *sequence* in X, and the sequence asserted to exist in the axiom is said to be defined by simple recursion with the starting value x_0 and the transition rule t. Most of Peano's postulates could now be proved to hold for $N, 0, s$ but we delay this until all our axioms have been stated because, for example, the existence of an object with more than one element is needed to conclude that N is infinite. However we can now prove the

THEOREM 1 (Primitive Recursion). Given a pair of mappings

$$A \xrightarrow{f_0} B$$
$$N \times A \times B \xrightarrow{u} B$$

there is a mapping

$$N \times A \xrightarrow{f} B$$

such that for any $n \in N$, $a \in A$ one has

$$\begin{aligned} \langle 0, a\rangle f &= a f_0 \\ \langle ns, a\rangle f &= \langle n, a, \langle n, a\rangle f\rangle u \end{aligned}$$

(It will follow from the next axiom that f is uniquely determined by these conditions.)

PROOF. In order to understand the idea of the proof, notice that primitive recursion is more complicated than simple recursion in essentially two ways. First, the values of f depend not only on $n \in N$ but also on members of the parameter object A. However, the exponentiation axiom enables us to reduce this problem to a simpler one involving a sequence whose values are functions. That is, the assertion of the theorem is equivalent to the existence of a sequence

$$N \xrightarrow{y} B^A$$

such that $0y = [f_0]$ and such that for any $n \in N,\ a \in A$, one has

$$\langle a, (ns)y\rangle e = \langle n, a, \langle a, ny\rangle e\rangle u$$

where e is the evaluation map for B^A.

The second way in which primitive recursion (and the equivalent problem just stated) is more complicated than the simple recursion of our axiom is that the transition rule u depends not only on the value calculated at the previous step of the recursion but also on the number of steps that have been taken. However, this can be accomplished by first constructing by simple recursion a sequence whose values are ordered pairs in which the first coordinate is used just to keep track of the number of steps. Thus the proof of this theorem consists of constructing a certain sequence

$$N \xrightarrow{x} N \times B^A$$

and then defining y as required above to be simply x followed by projection onto the second factor. That is, we define by simple recursion the *graph* of y.

Explicitly, the conditions on the sequence x are that

$$0x = \langle 0, [f_0]\rangle 0$$

and that for each $n \in N$ and $a \in A$

$$\begin{aligned} (ns)xp_N &= ns \\ \langle a, (ns)xp_{B^A}\rangle e &= \langle n, a, \langle a, nxp_{B^A}\rangle e\rangle u \end{aligned}$$

By our axiom, the existence of such x is guaranteed by the existence of a mapping

$$N \times B^A \xrightarrow{t} N \times B^A$$

such that for any $n \in N$, $h \in B^A$, $a \in A$

$$\begin{aligned} \langle n, h\rangle t p_N &= ns \\ \langle a, \langle n, h\rangle t p_{B^A}\rangle e &= \langle n, a, \langle a, h\rangle e\rangle u \end{aligned}$$

But such a t can be constructed by defining

$$t = \langle p_N s, t_2\rangle$$

where $N \times B^A \xrightarrow{t_2} B^A$ is the mapping corresponding by exponential adjointness to the composite

$$A \times N \times B^A \longrightarrow N \times A \times A \times B^A \longrightarrow N \times A \times B \xrightarrow{u} B$$

where the first is induced by the diagonal map $A \to A \times A$ and where the second is induced by the evaluation map $A \times B^A \to B$. (Here we have suppressed mention of certain commutativity and associativity isomorphisms between products.) This completes the proof of the Primitive Recursion theorem. ∎

Our first three axioms, requiring the existence of finite roots, exponentiation, and the Dedekind-Pierce object N, actually hold in certain fairly common categories other than $\mathcal{S}$, such as the category $\mathcal{C}$ of small categories and functors or any functor category $\mathcal{S}^{\mathbb{C}}$ for a fixed small category $\mathbb{C}$ (examples of the latter include the category $\mathcal{S}^{\mathbb{G}}$ whose objects are all permutation representations of a given group $\mathbb{G}$ and whose maps are equivariant maps as well as the category $\mathcal{S}^2$ whose objects correspond to the mappings in $\mathcal{S}$ and whose maps are commutative squares of mappings in $\mathcal{S}$). However, these categories do not satisfy our remaining axioms.

We state now our two special axioms

Axiom 4. 1 is a *generator*. That is, if $A \underset{g}{\overset{f}{\rightrightarrows}} B$ then

$$f \neq g \;\Rightarrow\; \exists a\, [a \in A \;\&\; af \neq ag]$$

In other words, two mappings are equal if they have the same domain and codomain and if they have the same value at each element of their domain. We mention the immediate consequence that if A has exactly one element then $A = 1$.

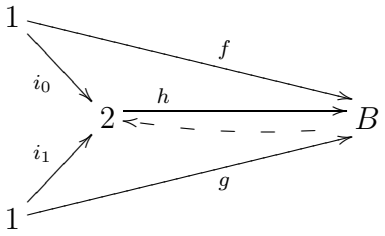
Axiom 5(Axiom of Choice). If the domain of f has elements, then there exists g such that $fgf = f$.

Remark 2: This axiom has many uses in our theory even at the elementary level, as will soon become apparent. That, in relation to the other axioms, the axiom of choice is

stronger in our system than in the usual systems is indicated by the fact that the axiom of choice (as stated above) is obviously independent. Namely, the category $\mathcal{O}$ of partially ordered sets and order-preserving maps is a model for all our axioms (even including the final group of axioms not yet described) with the exception that the axiom of choice is false in $\mathcal{O}$. To see the latter we need only take f to be the "identity" map from a set with trivial partial ordering to the same set with a stronger ordering. Most of the basic theorems that we will prove fail in $\mathcal{O}$.

PROPOSITION 1. 2 is a cogenerator.

PROOF. Here we define $2 = 1+1$, and by the statement that 2 is a *cogenerator* we mean that if $A \underset{g}{\overset{f}{\rightrightarrows}} B$ are different then there is $B \xrightarrow{u} 2$ such that $fu \neq gu$. Since 1 is a generator we are immediately reduced to the case where $A = 1$ and f, g are elements of B. By the universal property of sums there is an induced map $2 \xrightarrow{h} B$ which by the axiom of choice has a "quasi-inverse" u



$$huh = h$$

But this u clearly separates f, g if they are different, since

$$fu = gu \quad \Rightarrow \quad fuh = guh \quad \Rightarrow \quad i_0huh = i_1huh$$
$$\Rightarrow \quad i_0h = i_1h \quad \Rightarrow \quad f = g$$

■

In order to state our next proposition, we must define subset, inclusion, and member.

DEFINITION 2. a is a *subset* of A iff a is a monomorphism with codomain A.

Here a monomorphism a is a mapping such that

$$\forall b\ \forall b'\ [ba = b'a \ \Rightarrow\ b = b']$$

By the axiom of choice it is clear that in $\mathcal{S}$ (indeed in any model of our theory) every monomorphism, except for trivial cases, is a retract, i.e. has a right inverse. Also, since 1 is a generator, it suffices in verifying that a is a monomorphism to consider b, b' which are elements of the domain of a. Note that the subset is a itself, not just the domain of a. The domain of a is an object, and an object in $\mathcal{S}$ is essentially just a cardinal number, whereas the notion of subset contains more structure than just number. Namely, a subset of A

involves a specific way of identifying its members as elements of A, hence the mapping a.

REMARK 3: A mapping $A \xrightarrow{f} B$ is an epimorphism iff

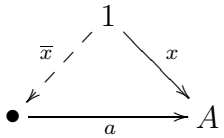
$$\forall y \in B \; \exists x \in A \; [xf = y] \; ,$$

a monomorphism iff

$$\forall x \in A \; \forall x' \in A \; [xf = x'f \;\Rightarrow\; x = x'] \; ,$$

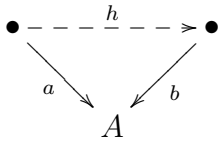
an isomorphism iff it is both an epimorphism and a monomorphism. The proof (using the axiom of choice) is left to the reader. (The concept of epimorphism is defined dually to that of monomorphism.)

DEFINITION 3. x is a member of a [notation $x \in a$] iff for some A (which will be unique) x is an element of A, a is a subset of A, and there exists $\overline{x}$ such that $\overline{x}a = x$.



Note that $\overline{x}$ is an element of the domain of a and is uniquely determined. Note also that x is an element of a iff x is a member of a and a is an object; the use of the same notation for the more general notion causes no ambiguity.

DEFINITION 4. $a \subseteq b$ iff for some A, a and b are both subsets of A and there exists h such that $a = hb$.



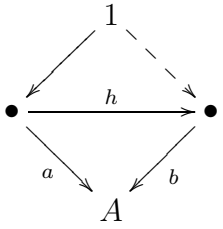
Note that h is a uniquely determined monomorphism if it exists. The inclusion relation thus defined for subsets of A is clearly reflexive and transitive. Also $x \in a$ iff $x \subseteq a$ and the domain of x is 1, and a is a subset of A iff $a \subseteq A$ and A is an object.

PROPOSITION 2. Let a, b be subsets of A. Then $a \subseteq b$ iff

$$\forall x \in A \; [x \in a \;\Rightarrow x \in b]$$

(For the proof we assume that the domain of b has elements; however the next axiom makes this restriction unnecessary.)

PROOF. If $a \subseteq b$ and $1 \xrightarrow{x} A$, then clearly $x \in a \;\Rightarrow\; x \in B$.



Conversely, suppose $\forall x \in A\ [x \in a \ \Rightarrow\ x \in B]$. By the axiom of choice there is $A \xrightarrow{g} dom\ b$ such that $bgb = b$. Since b is a monomorphism, $bg = dom\ b$ (the identity mapping). We define $h = ag$, and attempt to show $a = hb$. It suffices to show that $\overline{x}a = \overline{x}agb$ for every element $\overline{x}$ of the domain of a. But given $\overline{x}$, then $x = \overline{x}a$ is an element of A for which $x \in a$, so by hypothesis $x \in b$, i.e., $\exists y\ [x = yb]$. Then

$$\overline{x}hb \ = \ \overline{x}agb \ = \ xgb \ = \ ybgb \ = \ yb \ = \ x \ = \ \overline{x}a$$

Since $\overline{x}$ was arbitrary, $hb = a$ because 1 is a generator. This shows $a \subseteq b$. ∎

Our final group of axioms is the following:

AXIOM 6. Each object other than 0 (i.e. other than initial objects) has elements.

AXIOM 7. Each element of a sum is a member of one of the injections.

AXIOM 8. There is an object with more than one element.

We remark that Axiom 8 is independent of all the remaining axioms, and that, in fact, the full set of axioms with 8 excluded is easily proved consistent, for all these axioms are verified in the finite category

$$0 \longrightarrow 1$$

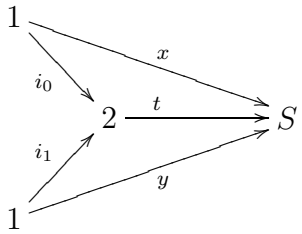
(In this case $N = 1$.) However, the question of independence for Axioms 4,6,7 is still unsettled.

PROPOSITION 3. 0 has no elements.

PROOF. If $1 \to 0$, then $1 = 0$, so that every object has exactly one element, contradicting axiom 8. ∎

PROPOSITION 4. The two injections $1 \underset{i_1}{\overset{i_0}{\rightrightarrows}} 2$ are different and are the only elements of 2.

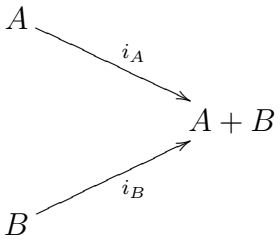
PROOF. The second assertion is immediate by Axiom 7. Suppose $i_0 = i_1$ and let S be an object with two distinct elements x, y. Then there is t such that



commutes by the sum axiom, so $x = i_0 t = i_1 t = y$, a contradiction. Thus $i_0 \neq i_1$. ∎

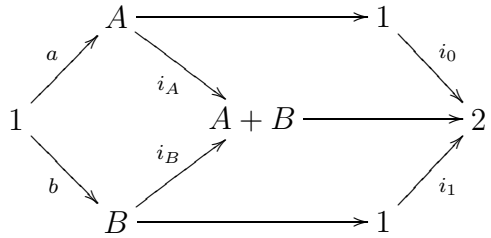
It is also easy to establish, by using the axiom of choice, that if A has exactly two elements, then $A \cong 2$.

PROPOSITION 5. To any two objects A, B, the two injections



have no members in common.

PROOF. Consider $A + B \to 2$ induced by the unique maps $A \to 1$, $B \to 1$. Suppose there are a, b such that $ai_A = bi_B$ are equal elements of $A + B$. Then the diagram



is commutative. But the composite maps $1 \xrightarrow{a} A \to 1$ and $1 \xrightarrow{b} B \to 1$ must both be the identity, yielding $i_0 = i_1$, a contradiction. ∎

We now prove five theorems which, together with Theorem 1, are basic to the development of mathematics within our theory.

THEOREM 2. Peano's postulates hold for N.

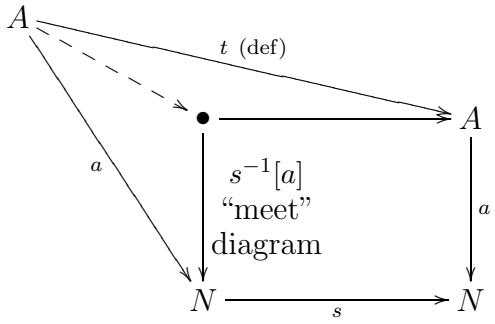
PROOF. That the successor mapping $N \xrightarrow{s} N$ is well-defined is already implicit. To show that s is injective, note that the predecessor mapping can be defined by primitive recursion; one of the defining equations of the predecessor is that it be right inverse to s. The other defining equation for the predecessor p is that $0p = 0$, so that if 0 were a successor, say $ns = 0$, then $n = n(sp) = (ns)p = 0p = 0$. But if $0s = 0$, then *every* object has only the identity endomorphism. For let $X \xrightarrow{t} X$ and $x_0 \in X$; then there is $N \xrightarrow{x} X$ such that $0x = x_0$ and $sx = xt$, hence $x_0 t = 0xt = 0sx = 0x = x_0$. But x_0 being arbitrary, we can conclude $t = X$ since 1 is a generator. However, Axiom 8 implies that 2, for example,

has non-identity endomorphisms; this contradiction shows that 0 is not the successor of any $n \in N$.

Finally we prove Peano's induction postulate. Let $A \xrightarrow{a} N$ be a subset such that $0 \in A$ and

$$\forall n \in N\ [n \in a\ \Rightarrow\ ns \in a]$$

The last means that a is included in its own inverse image under s, so there is $A \xrightarrow{t} A$ such that $as = ta$.



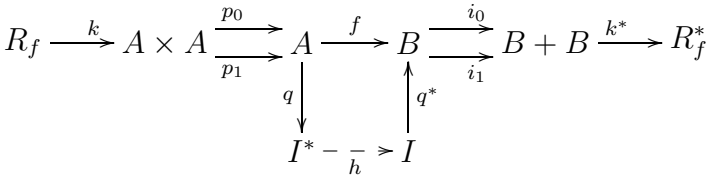
The first means there is $\overline{0} \in A$ such that $\overline{0}a = 0$. By simple recursion $\overline{0}, t$ determine $N \xrightarrow{x} A$ such that $0x = \overline{0}$ and $sx = xt$. This implies

$$\begin{aligned} 0(xa) &= 0 \\ s(xa) &= (xa)s \end{aligned}$$

so that by the uniqueness of mappings defined by simple recursion, $xa = N$. Thus for any $n \in N$, $n = (nx)a$, so that $n \in a$. ∎

The following theorem has the consequence that every mapping can be factored into an epimorphism followed by a monomorphism. However, from the viewpoint of general categories, the result is stronger. (It does not hold in the category of commutative rings with unit, for example.)

THEOREM 3. For any mapping f, the regular co-image of f is canonically isomorphic to the regular image of f. More precisely, the unique h rendering the following diagram commutative is an isomorphism, where k is the equalizer of $p_0 f, p_1 f$ and q is the coequalizer of kp_0, kp_1, while k^* is the coequalizer of fi_0, fi_1 and q^* is the equalizer of $i_0 k^*, i_1 k^*$.



PROOF. We first show that hq^* is a monomorphism. This is clear if $A = 0$, so we assume A has elements. Then by the axiom of choice, q has a left inverse t: $tq = I^*$. Suppose $uhq^* = u'hq^*$. Then

$$\langle ut, u't\rangle p_0 f \;=\; utf \;=\; uhq^* \;=\; u'hq^* \;=\; u'tf \;=\; \langle ut, u't\rangle p_1 f$$

Hence, k being the equalizer of $p_0 f, p_1 f$, there is w such that $\langle ut, u't\rangle = wk$. But then, q being a coequalizer,

$$u \;=\; utq \;=\; \langle ut, u't\rangle p_0 q \;=\; wkp_0 q \;=\; wkp_1 q$$
$$= \langle ut, u't\rangle p_1 q \;=\; u'tq \;=\; u'$$

The above shows hq^* is a monomorphism, from which it follows that h is a monomorphism. A dual argument shows that h is an epimorphism and hence an isomorphism by Remark 3. ∎

The above theorem implies that any reasonable definition of image gives the same result, so we may drop the word "regular". There is no loss in assuming that $I* = h = I$; we then refer to the equation $f = qq*$ as the factorization of f through its image.

For the discussion of the remaining theorems we assume that a fixed labelling of the two maps $1 \underset{i_1}{\overset{i_0}{\rightrightarrows}} 2$ has been chosen. It is clear heuristically that the (equivalence classes of) subsets of any given set X are in one-to-one correspondence with the functions $X \to 2$. It is one of our goals to prove this. More exactly, we wish to show that every subset of X has a *characteristic function* $X \to 2$ in the sense of

DEFINITION 5. The mapping $X \xrightarrow{\varphi} 2$ is the *characteristic function* of the subset $A \xrightarrow{a} X$ iff

$$\forall x \in X \; [x \in a \;\Leftrightarrow\; x\varphi = i_1]$$

It is easy to see (using equalizers) that every function $X \to 2$ is the characteristic function of some subset. We will refer to those subsets which have characteristic functions as *special* subsets, until we have shown that every subset is special. This assertion will be made to follow from the construction of a "complement" $A' \xrightarrow{a'} X$ for every subset $A \xrightarrow{a} X$.

Essentially, the complement of a will be constructed as the union of all special subsets which do not intersect a. We first prove a theorem to the effect that such unions exist.

THEOREM 4. Given any "indexed family of special subsets of X"

$$I \xrightarrow{\quad\alpha\quad} 2^X$$

there exists a subset

$$\bigcup_\alpha \xrightarrow{\quad a\quad} X$$

which is the union of the α_j, $j \in I$, in the sense that for any $x \in X$,

$$x \in a \;\Leftrightarrow\; \exists j \in I \; [\langle x, j\alpha\rangle e_{2^X} = i_1]$$

where e_{2^X} is the evaluation $X \times 2^X \to 2$.

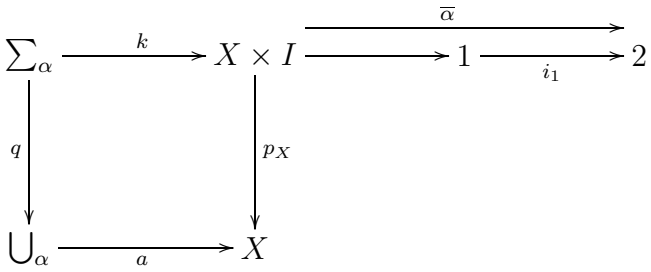
PROOF. By exponential adjointness, α is equivalent to a mapping

$$X \times I \xrightarrow{\overline{\alpha}} 2$$

and the desired property of $\bigcup_\alpha$ is equivalent to

$$x \in a \;\Leftrightarrow\; \exists j \in I\ [\langle x, j\rangle \overline{\alpha} = i_1]$$

We construct $\bigcup_\alpha$ as follows:



Here k is the equalizer of $\overline{\alpha}$ with the composite $X \times I \to 1 \xrightarrow{i_1} 2$, and $kp_X = qa$ is the factorization of kp_X through its image. Suppose $x \in a$. Then since q is an epimorphism there exists $\overline{x} \in \sum_\alpha$ such that $\overline{x}kp_X = x$. Define $j = \overline{x}kp_I$. Then

$$\langle x, j\rangle\overline{\alpha} \;=\; \overline{x}k\overline{\alpha} \;=\; \overline{x}k(X \times I \to 1 \xrightarrow{i_1} 2) \;=\; i_1$$

since k is an equalizer and since $\overline{x}k(X \times I \to 1) = 1$. The converse is even easier: if there is $j \in I$ such that $\langle x, j\rangle\overline{\alpha} = i_j$, then by the universal mapping property of the equalizer k, $\langle x, j\rangle$ must come from $\overline{x} \in \sum_\alpha$, so that on applying q to $\overline{x}$ one finds that $x \in a$. ∎

REMARK 4: Using the above construction of $\sum_\alpha$ we can prove a perhaps more familiar form of the axiom of choice, namely, given

$$I \xrightarrow{\alpha} 2^X$$

if every α_j is non-empty ($\alpha_j \neq 0$), there is $I \xrightarrow{f} X$ such that $jf \in \alpha_j$ for each $j \in I$. (Here by α_j we mean the subset of X obtained by equalizing with $X \to 1 \xrightarrow{i_1} 2$ the map $X \to 2$ whose "name" is $j\alpha$.) We leave the details to the reader, but note that the essential step is the application of the axiom of choice to the composite mapping

$$\sum_\alpha \longrightarrow X \times I \xrightarrow{p_I} I$$

We also leave to the reader the construction of the "product" $\prod_\alpha$ which is the subset of $(\sum_\alpha)^I$ consisting of all choice functions f as above. Notice that, although in our heuristic model $\mathcal{S}$, $\sum_\alpha$ and $\prod_\alpha$ are actual (infinite) coproduct and product in the categorical sense,

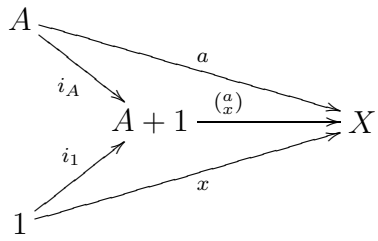
there is no way to guarantee by first order axioms that this will be true in every first order model of our theory. (See also Remark 11.)

THEOREM 5. Given any subset $A \xrightarrow{a} X$, there is a subset $A' \xrightarrow{a'} X$ such that $X \cong A + A'$, with a, a' as the injections. Thus $x \in a'$ iff $x \in X$ and $x \notin a$, and a has a characteristic function $X \to 2$.

We will need for the proof the following

LEMMA 1. Given a subset $A \xrightarrow{a} X$ and an element $x \in X$ such that $x \notin a$, then there is $X \xrightarrow{\varphi} 2$ such that $x\varphi = i_1$ and $a\varphi \equiv i_0$; that is, there is a special subset containing x but intersecting a vacuously.

PROOF OF LEMMA 1. Define $\binom{a}{x}$ by the universal mapping property of sums so that



and let u be a "quasi-inverse" for $\binom{a}{x}$ as guaranteed by the axiom of choice. Then define φ to be the composition

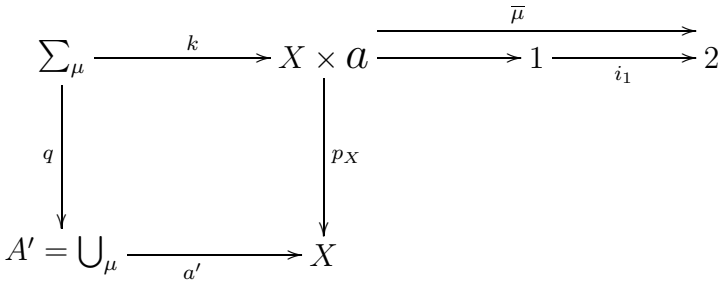
$$X \xrightarrow{u} A + 1 \xrightarrow{(A \to 1)+1} 1 + 1 \;=\; 2$$

By axiom 7, $\binom{a}{x}$ is a monomorphism since a is and since $x \notin a$, so we actually have that $\binom{a}{x}\, u = A + 1$. Therefore (setting $t_A = (A \to 1) + 1$)

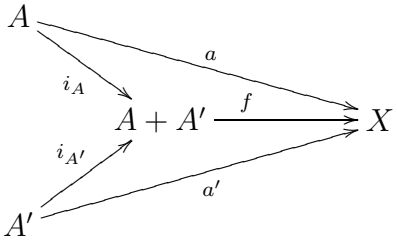
$$\begin{aligned} x\varphi \;&=\; xut_A \;=\; i_1\tbinom{a}{x}ut_A \;=\; i_1t_A \;=\; i_1 \\ a\varphi \;&=\; aut_A \;=\; i_A\tbinom{a}{x}ut_A \;=\; i_At_A \;=\; i_0 \end{aligned}$$

■

PROOF OF THEOREM 5. Let $j_0 \in 2^A$ be the element corresponding by exponential adjointness to the composition $A \times 1 \xrightarrow{p_1} 1 \xrightarrow{i_0} 2$, and then define $\mathcal{a} \xrightarrow{\mu} 2^X$ to be the equalizer of 2^a with the composition $2^X \to 1 \xrightarrow{j_0} 2^A$. Thus intuitively μ is the subset of 2^X whose members are just the special subsets of X which intersect a vacuously. Define a' to be the union of μ. Thus



where $\overline{\mu}$ corresponds to μ and k is the equalizer of $\overline{\mu}$ with the function constantly i_1, and where q is an epimorphism while a' is a monomorphism. There is, of course a mapping f such that the diagram



commutes. We must show that f is an isomorphism. For this it suffices to show that f is a monomorphism and an epimorphism. That f is a monomorphism follows from axiom 7 and the fact that, by construction, a and a' have no members in common, since $x \in a$ implies that $\langle x, t\rangle e_{2^X} = i_0$ for every $t \in \mu$, whereas $x \in a'$ implies that $\exists t \in \mu\ [\langle x, t\rangle e_{2^X} = i_1]$. On the other hand, to show that f is an epimorphism, it suffices to show that f is surjective, so consider an arbitrary $x \in X$. If $x \notin A$ then by the lemma there is $[\varphi] \in 2^X$ such that $x\varphi = i_1$ while $a\varphi \equiv i_0$. Since $a\varphi \equiv i_0$, we actually have $[\varphi] \in \mu$ by construction of $\mathfrak{a} \overset{\mu}{\rightarrow} 2^X$. Because $\langle x, [\varphi]\rangle e_{2^X} = i_1$, $\langle x, [\varphi]\rangle$ is actually a member of $k(X \times \mu)$ where k is the equalizer of $\overline{\mu}$ with the map constantly i_1, so finally applying $\sum_\mu \overset{q}{\rightarrow} A'$ to the element of $\sum_\mu$ which corresponds to $\langle x, [\varphi]\rangle$, we find that $x \in a'$. Thus, since we have shown that x comes either from A or A', f is surjective. ∎

The final "mathematical" theorem we shall prove concerns a more "internal" description of the nature of coequalizers. Actually the theorem only refers to an important special case, but we will remark that the general case can also be handled, leaving details to the reader. We consider a pair

$$R \underset{f_1}{\overset{f_0}{\rightrightarrows}} A$$

of mappings, and will sometimes denote by $f = \langle f_0, f_1\rangle$ the corresponding single mapping $R \rightarrow A \times A$, from which f_0 and f_1 can be recovered.

DEFINITION 6. We say that f is *reflexive* iff

$$\exists d\ [A \xrightarrow{d} R \quad \& \quad df_0 = A = df_1]$$

symmetric iff

$$\exists t\ [R \xrightarrow{t} R \quad \& \quad tf_0 = f_1 \quad \& \quad tf_1 = f_0]$$

transitive iff

$$\forall h_0\ \forall h_1\ \{X \underset{h_1}{\overset{h_0}{\rightrightarrows}} R \quad \& \quad h_0 f_1 = h_1 f_0 \Rightarrow \exists u\ [uf_0 = h_0 f_0 \quad \& \quad uf_1 = h_1 f_1]\}$$

It has previously been pointed out (Lawvere 1963a) that the following theorem holds in every "algebraic" category and is in a sense characteristic of such categories. Since $\mathcal{S}$ is the basic algebraic category, it is essential that we should be able to prove the theorem from our axioms for $\mathcal{S}$. Notice that although our proof depends on the axiom of choice, and although the validity of the theorem for algebraic categories depends on its validity for $\mathcal{S}$, nevertheless the axiom of choice does not hold in most algebraic categories.

THEOREM 6. If $R \underset{f_1}{\overset{f_0}{\rightrightarrows}} A$ is such that $f = \langle f_0, f_1 \rangle$ is reflexive, symmetric, transitive, and a subset of $A \times A$, then f is the equalizer of $p_0 q$ with $p_1 q$, where $A \xrightarrow{q} Q$ is the coequalizer of f_0 with f_1.

PROOF. We actually show that if f is reflexive, symmetric, and transitive, and if $a_0 \in A,\ a_1 \in A$, then

$$a_0 q = a_1 q \ \Leftrightarrow\ \exists r \in R\ [r f_0 = a_0\ \&\ r f_1 = a_1]$$

which clearly implies the assertion in the theorem. We accomplish this by constructing a mapping $A \xrightarrow{g} 2^A$ which is intuitively just the classification of elements of A into f-equivalence classes. For the construction of g we need two lemmas:

LEMMA 2. for any A there is a mapping $A \xrightarrow{\{\}} 2^A$ such that for any $x \in A,\ y \in A$, y is a member of the subset whose characteristic function has the name $\{x\}$ iff $y = x$.

PROOF. Let $\{\}$ be the mapping corresponding by exponential adjointness to the characteristic function of the diagonal subset

$$A \xrightarrow{\langle A, A \rangle} A \times A$$

■

LEMMA 3. Given any mapping $R \xrightarrow{h} A$ there is a mapping

$$2^R \xrightarrow{\overline{h}} 2^A$$

with the property that given any subset of R with characteristic function ψ, then $[\psi]\overline{h}$ is the name of the characteristic function of the subset a of A having the property that for any $x \in A$,

$$x \in a \;\Leftrightarrow\; \exists r\, [r\psi = i_1 \quad \& \quad rh = x]$$

Briefly, $\overline{h}$ is the direct image mapping on subsets determined by the mapping h on the elements.

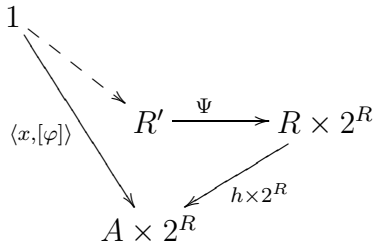
PROOF. Consider the equalizer Ψ of e_{r^R} with i_1:

$$R' \xrightarrow{\Psi} R \times 2^R \underset{\xrightarrow[i_1]{}}{\overset{e_{2^R}}{\rightrightarrows}} 1 \xrightarrow{} 2$$

Thus Ψ is a subset whose members are just the pairs $\langle r, [\psi]\rangle$ where ψ is the characteristic function of a subset of R of which r is a member. Form the composition of Ψ with

$$R \times 2^R \xrightarrow{h \times 2^R} A \times 2^R$$

Then the image of this composition is a subset of $A \times 2^R$ and hence has a characteristic function $A \times 2^R \xrightarrow{\Phi} 2$. Let $\overline{h}$ be the mapping $2^R \to 2^A$ corresponding by exponential adjointness to Φ. To show that $\overline{h}$ has the desired property, let $R \xrightarrow{\psi} 2$ and let a be the subset of A whose characteristic function φ has the name $[\psi]\overline{h}$, i.e $[\varphi] = [\psi]\overline{h}$ and φ is the characteristic function of a. Suppose $r\psi = i_1$ and $rh = x$; then we must show that $x \in a$, i.e. that $x\varphi = i_1$. However, since $\langle r, [\psi]\rangle \in \Psi$, one has also $\langle r, [\psi]\rangle(h \times 2^R) \in \Psi(h \times 2^R)$, which is the same as to say $rh, [\psi]\rangle\Phi = i_1$, which since $x = rh$, implies by definition of $\overline{h}$ that x is a member of the subset whose characteristic function is $[\psi]\overline{h} = [\varphi]$; but the latter subset is a, as required. Conversely, assume $x \in a$. Then $x\varphi = i_1$, so that $\langle x, [\varphi]\rangle\Phi = i_1$ and hence $\langle x, [\varphi]\rangle \in \Psi(h \times 2^R)$.



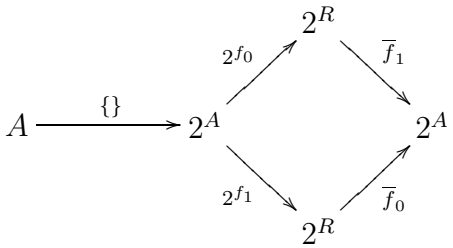
Therefore there is $r \in R \;\ni\; rh = x$ and $\langle r, [\psi]\rangle \in \Psi$. But this means $r\psi = i_1$. ∎

REMARK 5: It is easy to verify that in these lemmas we have actually constructed a covariant functor and a natural transformation $\{\}$ from the identity functor into this functor. There are actually three different "power-set" functors: the contravariant one $f \rightsquigarrow 2^f$, the covariant "direct image" functor $h \rightsquigarrow \overline{h}$ just constructed, and a dual covariant functor related to universal quantification in the same way that the direct image functor is related to existential quantification.

We now return to the proof of the theorem. Given the two mappings

$$R \underset{f_1}{\overset{f_0}{\rightrightarrows}} A$$

we can by the lemmas construct two mappings $A \to 2^A$ as follows



We claim that if $f = \langle f_0, f_1 \rangle$ is symmetric, then the two composites $A \to 2^A$ are actually equal (to a map which we will call g), that if f is symmetric and transitive, then $f_0 g = f_1 g$, and that if f is reflexive, symmetric and transitive, then the induced map $Q \to 2^A$ (where $A \xrightarrow{q} Q$ is the coequalizer of f_0 with f_1) is actually the image of g (so in particular a subset of 2^A). When reflexivity does not hold, the difference between q and the co-image of g is that q keeps separate all elements of A which are not related to any element, whereas g maps all such elements to the empty subset of A.

First we show that symmetry implies $\{\}2^{f_0}\overline{f}_1 = \{\}2^{f_1}\overline{f}_0$ and hence a unique definition of g. Let $a, a' \in A$. Then a' is a member of the subset of A whose characteristic function has the name $\{a\}2^{f_0}\overline{f}_1$ (briefly $a' \in \{a\}2^{f_0}\overline{f}_1$) iff there is $r \in R$ such that $rf_0 = a$ and $rf_1 = a'$. But this condition is clearly symmetrical in f_0, f_1. (And of course "members" (in the sense of $\in$) determine elements of 2^A.)

Next we show that $f_0 g = f_1 g$, where $g = 2^{f_0}\overline{f}_1 = 2^{f_1}\overline{f}_0$, if symmetry and transitivity hold. Let $r \in R$. Then the desired relation $rf_0 g = rf_1 g$ holds iff

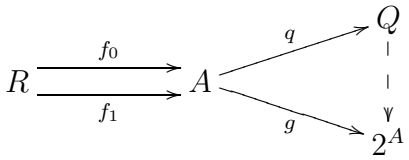
$$\forall a' \in A \; [a' \in rf_0 g \;\Leftrightarrow\; a' \in rf_1 g]$$

So let $a' \in A$. Then what we must show is that

$$\exists \overline{r} \in R \; [\overline{r} f_0 = a' \;\&\; \overline{r} f_1 = rf_0] \quad \Leftrightarrow \quad \exists \overline{\overline{r}} \in R \; [\overline{\overline{r}} f_0 = a' \;\&\; \overline{\overline{r}} f_1 = rf_1]$$

That the left hand side implies the right hand side follows immediately from transitivity ($a' \equiv rf_0 \;\&\; rf_0 \equiv rf_1 \;\Rightarrow\; a' \equiv rf_1$, where $\equiv$ is defined as the image of f) and the

converse follows in the same way after first applying symmetry. Thus $f_0 g = f_1 g$ and so there is a unique induced map



The third claim made above was that under the additional hypothesis of reflexivity, q is actually the co-image of g. This is essentially just the assertion that $a_0 g = a_1 g \ \Rightarrow \ a_0 q = a_1 q$. But $a_0 g = a_1 g$ means just that

$$\forall a' \in A \ [a' \equiv a_0 \ \Leftrightarrow \ a' \equiv a_1]$$

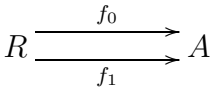
(where the relation $\equiv$ is just the image of f as above). By reflexivity there is at least one a' such that $a' \equiv a_0$ (namely a_0 itself) and by symmetry $a_0 \equiv a'$; but since $a_0 g = a_1 g$, it also follows from $a' \equiv a_0$ that $a' \equiv a_1$, so by transitivity $a_0 \equiv a_1$. This means $\exists r \in R \ [r f_0 = a_0 \ \& \ r f_1 = a_1]$, and thus $a_0 q = r f_0 q = r f_1 q = a_1 q$ because $f_0 q = f_1 q$, q being the coequalizer of f_0 with f_1.

Finally we can assert our theorem

$$a_0 q = a_1 g \ \Leftrightarrow \ a_0 \equiv a_1$$

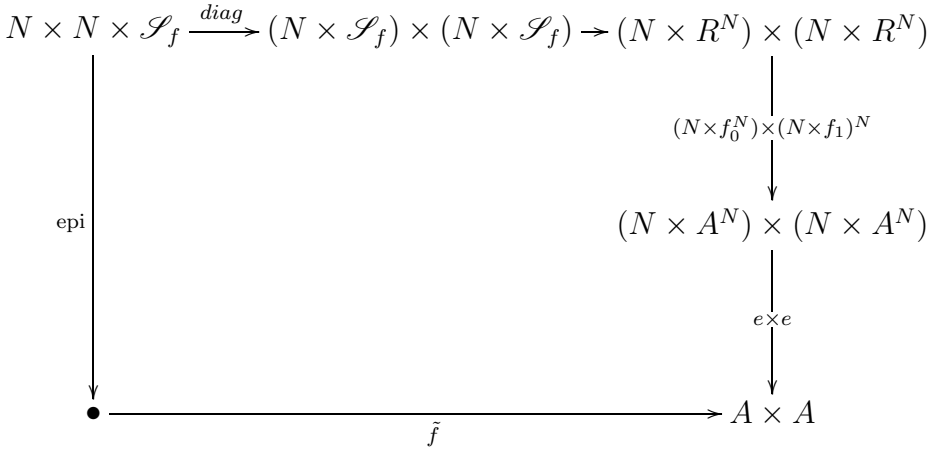
The implication $a_0 \equiv a_1 \ \Rightarrow \ a_0 q = a_1 q$ is trivial (and we just used it), while the converse was proved in the preceding paragraph. ∎

REMARK 6: It follows easily now that for an *arbitrary* pair of mappings



the equalizer $\tilde{f}$ of $p_0 q$ with $p_1 q$, where q is the coequalizer, is the smallest RST subset of $A \times A$ containing the image of f, and that the coequalizer of $\tilde{f}$ is the same as the coequalizer of f. Thus an "internal" construction of an arbitrary coequalizer will be obtained if one knows a way to construct the RST hull of an arbitrary relation. The construction below involves the natural numbers (see Remark 8 in connection with our metatheorem below). Since the symmetric hull is easy to construct we may assume that f is symmetric and reflexive. The reader may verify that the following then describes a construction of the RST hull $\tilde{f}$.

Let $\mathscr{S}_f \rightarrow R^N$ be the subset whose members are just the sequences $N \overset{r}{\rightarrow} R$ such that $s r f_0 = r f_1$, s being the successor mapping (that is, form the equalizer of f_0^s with f_1^N). Then define $\tilde{f}$ to be the image of the following composite mapping $N \times N \times \mathscr{S}_f \rightarrow A \times A$:



(Here e is the evaluation mapping for sequences $N \to A$.) That $\tilde{f}$ is the RST hull of the reflexive and symmetric f then follows from the fact that $\langle a_0, a_1 \rangle \in \tilde{f}$ iff there exists a sequence r in R such that $srf_0 = rf_1$ and there exist n_0, n_1 such that $a_0 = n_0 x$ and $a_1 = n_1 x$ where $x = rf_0$.

Remark 7: For the development of mathematics in our theory it is also convenient to make use of a "Restricted Separation Schema" the best form of which the author has not yet determined. This theorem schema is to the effect that given any formula involving sets and maps as well as a free variable $x \in A$, if the quantifiers in the formula are suitably restricted (say to maps between sets which result from certain numbers of applications of the operations of Axioms 1-3 to the given sets and maps), then a subset of A exists whose members are precisely the elements of A which satisfy the formula. (Note that A itself could be a product set $B^3 \times C$, etc.) The Restricted Separation Schema is equivalent to a schema which asserts that there exists a mapping having any given (suitably restricted) rule. This theorem schema is surely strong enough to guarantee the existence of all the constructions commonly encountered in analysis. Without using the theorem schema, one can of course derive particular constructions using equalizers, exponentiation, images, etc, and this can often be illuminating. For example, the ring of Cauchy sequences of rational numbers and the ideal of sequences converging to 0 can be constructed and calculus developed.

The reader should also be able now to construct proofs of Tarski's fixed point theorem, the Cantor-Schroeder-Bernstein theorem, and Zorn's lemma.

Finally, we wish to prove a metatheorem which clarifies the extent to which our theory characterizes $\mathcal{S}$. Though our proof is informal it could easily be formalized within a sufficiently strong set theory of the traditional type or within a suitable theory of the category of categories. [For a preliminary description of the latter, see the author's doctoral dissertation (Lawvere 1963b).]

Let $\mathbb{C}$ be any category which satisfies the eight axioms of our theory. (To be more precise, we also assume that for each pair of objects C, C' in $\mathbb{C}$, the class (C, C') of maps $C \to C'$ is "small", i.e. is a "set".) There is then a canonical functor

$$\mathbb{C} \xrightarrow{\mathrm{H}^1} \mathcal{S}$$

which assigns to each object C in $\mathbb{C}$ the (identity map of the) set $(1, C)$ of all maps $1 \to C$ in $\mathbb{C}$.

The functor H^1 is clearly left exact (i.e. preserves 1, products, and equalizers) and faithful (i.e. if $C \rightrightarrows C'$ are identified by H^1 then they are equal). Our metatheorem states that under the additional (non-first-order) axiom of completeness, H^1 is an equivalence of categories, i.e. there is a functor $\mathcal{S} \to \mathbb{C}$ which, up to natural equivalence, is inverse to H^1. Thus our axioms serve to separate the structure of $\mathcal{S}$ from the structure of all other complete categories, such as those of topological spaces, vector spaces, partially ordered sets, groups, rings, lattices, etc.

REMARK 8: For any model $\mathbb{C}$, H^1 also preserves sums and coequalizers of RST pairs of maps. However it seems unlikely that it is necessary that H^1 be right exact, since as remarked above, the construction of the RST hull involves N and by Gödel's theorems the nature of the object N may vary from one model $\mathbb{C}$ of any theory to another model. That is, H^1 need not preserve N. Also H^1 will be *full at* 1 in the sense that the induced mapping

$$(1, A) \longrightarrow (1, A\mathrm{H}^1)$$

must be surjective for each object A in $\mathbb{C}$. However, it is certainly not necessary that H^1 be full, because there are countable models by the Skolem-Lowenheim theorem, yet $(1, N)$ is always infinite; thus (N, N) can be countable but $(N\mathrm{H}^1, N\mathrm{H}^1)$ must be uncountable, so that e.g. the induced map

$$(N, N) \longrightarrow (N\mathrm{H}^1, N\mathrm{H}^1)$$

need not be surjective. Of course the countability of $(N, N) = (1, N^N)$ could not be demonstrated by maps $N \to N^N$ in $\mathbb{C}$. For the same reason, H^1 need not preserve exponentiation.

Actually, we will derive our result (that our axioms together with completeness characterize $\mathcal{S}$) from the following more general metatheorem concerning the category of models for our theory. We will use some of the basic facts about adjoint functors.

METATHEOREM. Let $\mathbb{C} \xrightarrow{\mathrm{T}} \mathbb{C}'$ be a functor such that:

1. Both $\mathbb{C}$ and $\mathbb{C}'$ satisfy axioms 1-8.
2. Both $\mathbb{C}$ and $\mathbb{C}'$ have the property that for each object A the lattice of subobjects of A is complete.
3. T has an adjoint $\check{\mathrm{T}}$.

4. T is full at 1, i.e. for each object A in $\mathbb{C}$ the induced mapping

$$(1, A)_{\mathbb{C}} \longrightarrow (1\mathrm{T}, A\mathrm{T})_{\mathbb{C}'}$$

is surjective.

Then T is an equivalence of categories, i.e. $\mathrm{T}\check{\mathrm{T}}$ and $\check{\mathrm{T}}\mathrm{T}$ are naturally equivalent to the identity functors of $\mathbb{C}$ and $\mathbb{C}'$, respectively.

Remark 9: Completeness of the lattice of subobjects of A in $\mathbb{C}$ means that every family $A_\alpha \overset{a_\alpha}{\to} A$, $\alpha \in \mathcal{F}$ of subobjects of A has an intersection $D \to A$ and a union $V \to A$. Here *every* family is to be understood in an absolute sense, i.e., relative to the universe in which we are discussing model theory; neither $\mathcal{F}$ nor the mapping which takes $\alpha \rightsquigarrow a_\alpha$ need be an object or map in $\mathbb{C}$. We can show of course (Theorems 4 & 5) that in any model $\mathbb{C}$, any family $I \overset{\alpha}{\to} 2^A$ in $\mathbb{C}$ has a uniquely determined union and intersection in $\mathbb{C}$. However, as with any first-order set theory, we cannot guarantee that, in a given model $\mathbb{C}$ every absolute family of subsets of A is represented by a single mapping in $\mathbb{C}$ with codomain 2^A. Note that, in particular, lattice-completeness of $\mathbb{C}$ implies that to every (absolute) family of elements of A there is a subset of A with precisely those elements as members.

Note that we clearly need to assume that T is full at 1, since for any model $\mathbb{C}$ and any object I in $\mathbb{C}$, $A \rightsquigarrow A^I$ is a functor $\mathbb{C} \to \mathbb{C}$ which has an adjoint; but which is surely not an equivalence if, e.g., I is infinite.

Proof of Metatheorem. Since T has an adjoint it must preserve products and equalizers and in particular $1\mathrm{T} = 1$. Since T is full at 1, $0\mathrm{T}$ can have no elements, and so $0\mathrm{T} = 0$ by Axiom 6.

We claim that T is faithful, i.e., that each induced mapping

$$(A, B) \longrightarrow (A\mathrm{T}, B\mathrm{T})$$

is injective. Suppose $A \underset{g}{\overset{f}{\rightrightarrows}} B$ are such that $f\mathrm{T} = g\mathrm{T}$ and consider $1 \overset{x}{\to} A$. We must show $xf = xg$, so consider the equalizer $E \to 1$ of xf with xg. We must have $E = 1$ or $E = 0$. If $E = 0$, then $E\mathrm{T} = 0$ is the equalizer of $(xf)\mathrm{T}$ with $(xg)\mathrm{T}$ which implies $(x\mathrm{T})(f\mathrm{T}) \neq (x\mathrm{T})(g\mathrm{T})$, a contradiction. Hence $E = 1$ which implies $xf = xg$ and so $f = g$. Thus T is faithful.

Also, T must preserve sums, because for each A and B in $\mathbb{C}$, the canonical map

$$A\mathrm{T} + B\mathrm{T} \longrightarrow (A + B)\mathrm{T}$$

is surjective by Axiom 7 and the fact that T is full at 1, and is injective by Axiom 7 and the fact that any functor whose domain satisfies our axiom of choice must preserve monomorphisms.

Now the lattice-completeness of $\mathbb{C}$ and $\mathbb{C}'$ together with the fact that T is full at 1 implies that every subset of $A\mathrm{T}$ is the value of T at some subset of A, for any object A in $\mathbb{C}$. For

$$(1, 2^A) \longrightarrow (1, 2^{A\mathrm{T}})$$

is a homomorphism of complete atomic Boolean algebras which induces an isomorphism

$$(1, A) \xrightarrow{\approx} (1, A\mathrm{T})$$

on the respective sets of atoms, and hence must be itself an isomorphism. But this implies that T must be full, since arbitrary maps are determined by their graphs, which are subsets.

In general, if a functor T is full and faithful and has an adjoint $\check{\mathrm{T}}$, then for each A, the canonical map $A\mathrm{T}\check{\mathrm{T}} \to A$ is an isomorphism. Hence this must hold in our case.

To complete the proof of the metatheorem we need only show that for each I in $\mathbb{C}'$, the canonical map

$$I \xrightarrow{\varphi_I} I\check{\mathrm{T}}\mathrm{T}$$

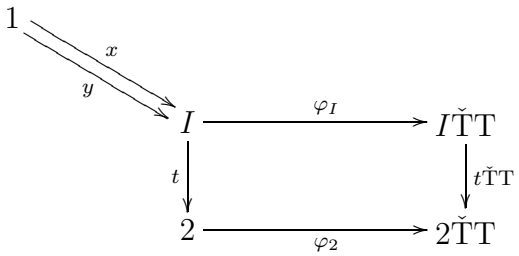
is injective, for it follows then by the previous remark that φ_I, being a subset of a value of T, must actually have its domain I isomorphic to a value of T, which immediately implies that φ_I is an isomorphism.

Because $A\mathrm{T}\check{\mathrm{T}} \cong A$, T is full and hence $1\mathrm{T} = 1$. Also, $\check{\mathrm{T}}$, being an adjoint functor, must preserve sums. Since we have already seen that T preserves 1 and sums, it follows that the particular canonical map

$$2 \xrightarrow{\varphi_2} 2\check{\mathrm{T}}\mathrm{T}$$

is an isomorphism. This, together with the fact that 2 is a cogenerator for $\mathbb{C}'$ (Proposition 1), will imply that every φ_I is injective.

For suppose $1 \overset{x}{\underset{y}{\rightrightarrows}} I$ are different. Then there is $I \xrightarrow{t} 2$ such that $xt \neq yt$. In the commutative diagram



φ_2 is an isomorphism and hence $t\check{\mathrm{T}}\mathrm{T}$ must separate $x\varphi_I$ from $y\varphi_I$. ∎

Metacorollary. If $\mathbb{C}$ is a complete category which satisfies our axioms then the canonical functor

$$\mathbb{C} \xrightarrow{\mathrm{H}^1} \mathcal{S}$$

is an equivalence.

PROOF. Completeness means that the categorical sum and product of any family of objects exists. However, this can easily be shown to imply lattice completeness. If we define

$$\mathcal{S} \xrightarrow{\check{\mathrm{T}}} \mathbb{C}$$

so that for each $I \in \mathcal{S}$, $I\check{\mathrm{T}}$ is the I-fold repeated sum of 1 with itself, then $\check{\mathrm{T}}$ is adjoint to $\mathrm{T} = \mathrm{H}^1$. By definition of $\mathrm{H}^1 = \mathrm{T}$, it is full at 1. ∎

REMARK 10: To give some indication of the extent to which a model of our axioms may fail to be complete, note that the set of all mappings between sets of rank less than $\omega + \omega$ is a model.

REMARK 11: Even countable repeated sums of an object with itself may fail to exist, for the equation

$$(\sum_N A, X) \cong (A, X)^N$$

shows that any non-trivial category in which such sums exist must be uncountable, whereas any first-order theory has countable models. Of course, if such sums do exist in a model for our theory, then

$$\sum_N A \cong N \times A$$

and the latter operation *can* always be performed.

REMARK 12: Of course, it is important to investigate first-order strengthenings of our axiom system in the direction of completeness, although cardinals larger than $\aleph_\omega$ are usually not needed, e.g., for the development of analysis. The existence of this and many more cardinals would be guaranteed, and hence the model of Remark 10 excluded, if we take the following as an axiom schema

$$\begin{array}{l} \forall A \forall B \forall B' \,[\, \varphi(A,B) \;\&\; \varphi(A,B') \quad \Rightarrow \quad B \cong B' \,] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall X \exists Y \,[\, X < Y \;\&\; \forall A \forall B \,[\, A < Y \;\&\; \varphi(A,B) \quad \Rightarrow \quad B < Y]] \end{array}$$

where φ is any formula and $X < Y$ refers to cardinality.

However, it is the author's feeling that when one wishes to go substantially beyond what can be done in the theory presented here, a much more satisfactory foundation for practical purposes will be provided by a theory of the category of categories.

REFERENCES.

Eilenberg, Samuel and Saunders Mac Lane (1945), 'General Theory of Natural Equivalences', *Transactions of the American Mathematical Society* **58**, 231–94.

Lawvere, F. William (1963a), 'Functorial Semantics of Algebraic Theories', *Proceedings of the National Academy of Sciences of the U.S.A.* **50**, 869–872.

Lawvere, F. William (1963b), Functorial Semantics of Algebraic Theories, PhD thesis, Columbia University. Republished in: *Reprints in Theory and Applications of Categories*, No. 5 (2004) pp 1-121, available on-line at http://tac.mta.ca/tac/reprints/articles/5/tr5abs.html.

This article may be accessed from `http://www.tac.mta.ca/tac/reprints` or by anonymous ftp at
`ftp://ftp.tac.mta.ca/pub/tac/html/tac/reprints/articles/11/tr11.{dvi,ps}`

REPRINTS IN THEORY AND APPLICATIONS OF CATEGORIES will disseminate articles from the body of important literature in Category Theory and closely related subjects which have never been published in journal form, or which have been published in journals whose narrow circulation makes access very difficult. Publication in 'Reprints in Theory and Applications of Categories' will permit free and full dissemination of such documents over the Internet. Articles appearing have been critically reviewed by the Editorial Board of *Theory and Applications of Categories*. Only articles of lasting significance are considered for publication. Distribution is via the Internet tools `WWW/ftp`.

SUBSCRIPTION INFORMATION. Individual subscribers receive (by e-mail) abstracts of articles as they are published. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor.

SELECTION OF REPRINTS. After obtaining written permission from any copyright holder, any three TAC Editors may propose a work for TAC Reprints to the Managing Editor. The proposal will be reported to all Editors. The Managing Editor may either accept the proposal or require that the Editors vote on it. Before such a vote, the author, title and original publication data will be circulated to Editors. If a 2/3 majority of those TAC Editors responding within one month agrees, the work will be accepted for TAC Reprints. After a work is accepted, the author or proposer must provide to TAC either a usable TeX source or a PDF document acceptable to the Managing Editor that reproduces a typeset version. Up to five pages of corrections, commentary and forward pointers may be appended by the author. When submitting commentary, authors should read and follow the Format for submission of *Theory and Applications of Categories* at `http://www.tac.mta.ca/tac/`.

Managing editor. Robert Rosebrugh, Mount Allison University: `rrosebrugh@mta.ca`

TEXnical editor. Michael Barr, McGill University: `mbarr@barrs.org`

Transmitting editors.
Richard Blute, Université d' Ottawa: `rblute@mathstat.uottawa.ca`
Lawrence Breen, Paris 13: `breen@math.univ-paris13.fr`
Ronald Brown, University of North Wales: `r.brown@bangor.ac.uk`
Jean-Luc Brylinski, Pennsylvania State University: `jlb@math.psu.edu`
Aurelio Carboni, Università dell Insubria: `aurelio.carboni@uninsubria.it`
Valeria de Paiva, Xerox Palo Alto Research Center: `paiva@parc.xerox.com`
Ezra Getzler, Northwestern University: `getzler(at)math(dot)northwestern(dot)edu`
Martin Hyland, University of Cambridge: `M.Hyland@dpmms.cam.ac.uk`
P. T. Johnstone, University of Cambridge: `ptj@dpmms.cam.ac.uk`
G. Max Kelly, University of Sydney: `maxk@maths.usyd.edu.au`
Anders Kock, University of Aarhus: `kock@imf.au.dk`
Stephen Lack, University of Western Sydney: `s.lack@uws.edu.au`
F. William Lawvere, State University of New York at Buffalo: `wlawvere@acsu.buffalo.edu`
Jean-Louis Loday, Université de Strasbourg: `loday@math.u-strasbg.fr`
Ieke Moerdijk, University of Utrecht: `moerdijk@math.uu.nl`
Susan Niefield, Union College: `niefiels@union.edu`
Robert Paré, Dalhousie University: `pare@mathstat.dal.ca`
Jiri Rosicky, Masaryk University: `rosicky@math.muni.cz`
Brooke Shipley, University of Illinois at Chicago: `bshipley@math.uic.edu`
James Stasheff, University of North Carolina: `jds@math.unc.edu`
Ross Street, Macquarie University: `street@math.mq.edu.au`
Walter Tholen, York University: `tholen@mathstat.yorku.ca`
Myles Tierney, Rutgers University: `tierney@math.rutgers.edu`
Robert F. C. Walters, University of Insubria: `robert.walters@uninsubria.it`
R. J. Wood, Dalhousie University: `rjwood@mathstat.dal.ca`