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On σ-Chebyshev Subspaces of Banach Spaces

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Abstract

σ-Chebyshev subspaces of a Banach space X are defined as those in which the set of best approximants of any vector x in X is non-empty and σ-compact. Moreover, it is shown that there exists a proximinal subspace G of X which is σ-Chebyshev, but is not quasi-Chebyshev in X. Also, some other related results are presented.

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1 Introduction

Let X be a (complex or real) Banach space and let G be a linear subspace of X. A point $y_0 \in G$ is said to be a best approximation for $x \in X$ if

$$\|x - y_0\| = d(x, G) = \inf\{\|x - y\| : y \in G\}.$$

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If each $x \in X$ has at least one best approximation in G, then G is called a proximinal subspace of X. If each $x \in X$ has a unique best approximation in G, then G is called a Chebyshev subspace of X (see[7,8]). For $x \in X$, put

$$P_G(x) = \{y \in G : \|x - y\| = d(x, G)\},$$

and

$$\hat{G} = P_G^{-1}(0) = \{x \in X : \|x\| = d(x, G)\}.$$

It is clear that $P_G(x)$ is a closed, bounded and convex subset of X. We say that G is a quasi-Chebyshev subspace of X if $P_G(x)$ is a non-empty and compact set in X for every $x \in X$.

In the following we give a definition which extends the definition of quasi-Chebyshev subspace.

1.1. Definition. Let X be a Banach space. A linear subspace G of X is called σ-Chebyshev if $P_G(x)$ is non-empty and σ-compact set in X for every $x \in X$ (see[3,4,5,6]).

It is clear that every quasi-Chebyshev subspace is σ-Chebyshev. In the following we shall give an example in which the converse is not true.

Let X^* be the dual space of the Banach space X. For $0 \neq f \in X^*$, put

$$M_f = \{x \in X : f(x) = \|f\|, \|x\| = 1\}.$$

We say that a Banach space X is quasi-strictly convex if for every $0 \neq f \in X^*$ the set M_f is compact (see [7]). Every k-strictly convex Banach space is quasi-strictly convex ($k = 1, 2, \ldots$).

1.2. Lemma([1]). Let $C_R([0, 1])$ be class of all continuous real functions on $[0, 1]$ with the supremum norm. Let $c_0, c_1 > 0$ and let S consist of all differentiable functions $f : [0, 1] \longrightarrow R$ such that $\|f\| \leq c_0$ and $\|f'\| \leq c_1$ then S is a compact subset of $C_R([0, 1])$.

2 Introduction

Now, we are ready to state and prove our main results.

2.1. lemma. *Let X be a Banach space and let G is a proximinal subspace of X. If $\hat{G}$ is a σ-compact set, then G is a σ-Chebyshev subspace.*

Proof. Let $x \in X \backslash G$. We have

$$P_G(x) = \{y \in G : \|x - y\| = d(x, G)\}$$
$$= \{y \in G : x - y \in \hat{G}\} = G \bigcap (x - \hat{G})$$

.
Since $\hat{G}$ is a σ-Chebyshev set, then $x - \hat{G}$ and hence $P_G(x)$ is σ-compact set.□
2.2.Example. Let $X = G_R^1([0,1])$ be the Banach space of all real-valued differentiable continuously functions on $[0,1]$, and equipped with the supremum norm. Suppose $G = \{g \in G_R^1([0,1]) : g(0) = 0\}$. Then for every $f \in X$, we have $g = f - f(0) \in G$ is a best approximation for f in X. In fact

$\|f - g\| = |f(0)| = |f(0) - h(0)| \leq \|f - h\|$, for all $h \in G$.

Thus G is proximinal subspace of X. Now it is sufficent that prove $\hat{G}$ is a σ-Chebyshev set.

We have

$$\hat{G} = \{f \in C^1([0,1]) : \|f\| = d(f, G)\}$$
$$= \{f \in C^1([0,1]) : \|f\| = |f(0)|\}$$
$$= \bigcup_{n=1}^{\infty} \{f \in C^1([0,1]) : \|f\| = |f(0)|, \|f^{'}\| \leq n\}$$

.

By lemma 1.3, the set $\{f \in C^1([0,1]) : \|f\| = |f(0)|, \|f^{'}\| \leq n\}$ is a compact subset of $C^1([0,1])$. Therefore $\hat{G}$ is σ-Chebyshev subspace of X. Now, we show that G is not quasi-Chebyshev subspace of X.

For that, let $f = 1$ and $g_n(x) = 1 - cosnx (n = 1, 2, ..., x \in [0,1])$. It is clear that $g_n \in P_G(f)$ and $g_n(1/n) = 1 - cos1$ for every $n = 1, 2, ...,$. Since $g_n(0) = 1 (n = 1, 2, ...)$ and $g(0) = 1$ for all $g \in G$, it follows that $\{g_n\}_{n \geq 1}$ has not any uniformly convergent subsequence in $G_G(f)$. Hence G is not a

quasi-Chebyshev subspace of $C_R^1([0,1])$.

2.3. Theorem. *Let X be a Banach space and let G is a proximinal subspace of X. If M_f is σ-compact for every $0 \neq f \in G^{\perp}$, then G is σ-Chebyshev in X.*

Proof. Let M_f is σ-compact for every $0 \neq f \in G^{\perp}$. It is clear that for each $0 \neq f \in G^{\perp}$ and each $x_0 \in M_f$

$$x_0 - P_G(x_0) = M_f \bigcap (x_0 + G)$$

.

Now, suppose that $x_0 \in X - G$, put $y_0 = x_0/\|x_0\|$. Then by an consequence of the Hahn-Banach theorem, there exists $0 \neq f \in G^{\perp}$ such that $\|f_0\| = f_0(y_0)$, therefore $y_0 \in M_f$. Since M_{f_0} is σ-compact, then $y_0 - P_G(y_0)$ is σ-compact and hence $P_G(x_0) = \|x_0\|P_G(y_0)$ is σ-compact. □

The following example show that the inverse of Theorem 2.3 does not hold in general.

Example 2.4. Let ℓ^1 with the basis $\{e_n\}_{n\geq 1}$. Put $X = \ell^1 \bigoplus_1 \Re$and $G = \{0\} \times \Re$. First, we show that G is a σ-Chebyshev subspace of X. Let $z = (x, y) \in X$ be arbitrary. let $g = (0, y)$. Then $g \in G$and we have

$$\|z - g\|_1 = \|(x, y) - (0, y)\|_1$$

$$= \|(x, 0)\|_1 = \|x\|_1 \leq \|x\|_1 + |y - u|$$

$$\leq \|(x, y) - (0, u)\|_1$$

for all $(o, u) \in G$. It follows that $g \in P_G(z)$ and hence $P_G(z)$ is non-empty for every $z \in X$. Therefore, G is a proximinal subspace of X.

It is not difficult to show that for every $z = (x, y) \in X$ we have $P_G(z) = \{(o, y)\}$, hence $P_G(z)$ is σ-compact for each $(x, y) \in X$.

Finally, define $f_0 : X \longrightarrow \mathcal{C}$ by $f_0(x,y) = \sum_{n=1}^{\infty} \mathcal{C}_n$ for all $(x,y) \in X$, where $x = \sum_{n=1}^{\infty} e_n$ and $\mathcal{C}$ is the field of complex numbers. Then we have

$$|f_0(x,y)| = |\sum_{n=1}^{\infty} \mathcal{C}_n| \leq \sum_{n=1}^{\infty} |\mathcal{C}_n| = \|x\|_1 \leq \|x\|_1 + |y| = \|(x,y)\|_1$$

for all $(x,y) \in X$. It follows that $f_0 \in X^*$, $f_0|_G = 0$ and $\|f_0\| = 1$. But, we have

$$\begin{aligned} M_{f_0} &= \{(x,y) \in X : f_0(x,y) = \|f_0\|, \|(x,y)\|_1 = 1\} \\ &= \{(x,y) \in X : \sum_{n=1}^{\infty} \mathcal{C}_n = 1, \|x\|_1 + |y| = 1\} \\ &= \{(x,y) \in X : \|x\|_1 = 1, y = 0\} = \{(x,0) \in X : \|x\|_1 = 1\} \\ &= S_{\ell^1} \times \{0\} \end{aligned}$$

where $S_{\ell^1} = \{x \in \ell^1 : \|x\|_1 = 1\}$.

Hence M_{f_0} is not σ-compact.

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