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On Cantor's singular moments

H. Prodinger

Abstract

We evaluate a constant explicitly, thereby answering a question raised in [1].

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In problem **10621** *of the American Mathematical Monthly, Cantor's singular moments* J_n *were to be computed. In the published answers* [1] *they come out as*

$$J_n = \frac{2}{3(n+1)} \sum_{j=0}^{n} \binom{n+1}{j} \frac{B_j}{3 \cdot 2^{j-1} - 1} \quad \textit{for} \quad n \geq 1$$

and $J_0 = 1$, *with Bernoulli numbers* B_n.

The editor asked, whether it is possible to compute

$$J_{-1} = \sum_{n \geq 0} J_n$$

exactly.

The purpose of this note is to do that. In [5] *we considered a similar problem, and the gentle reader is invited to consult this paper for more background about the technique, as well as* [6, 4] *for more information about the Cantor distribution.*

Centre for Applicable Analysis and Number Theory Department of Mathematics
University of the Witwatersrand, 2050 Johannesburg, South Africa
E-mail: helmut@gauss.cam.wits.ac.za
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Following the method described in [3], *we can write* J_n *as a contour integral viz.*

$$\begin{aligned} J_n &= \frac{2}{3(n+1)} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(n+2)\Gamma(1-s)}{\Gamma(n+2-s)} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds \\ &= \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(n+1)\Gamma(1-s)}{\Gamma(n+2-s)} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n=0}^{N} J_n = J_0 &+ \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(N+2)\Gamma(1-s)}{\Gamma(N+2-s)s} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds \\ &- \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(2)\Gamma(1-s)}{\Gamma(2-s)s} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds. \end{aligned}$$

From this form, one could even compute the asympotics as $N \to \infty$. *However, here, we only have to note that the first integral is of order* $N^{1-\log_2 3}$, *which means that it goes to zero. Consequently*

$$\begin{aligned} \sum_{n\geq 0} J_n &= 1 - \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{1}{(1-s)s} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds \\ &= \frac{4}{3} + \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{s(s-1)} \frac{\zeta(s)}{3\cdot 2^{-s}-1}\, ds \\ &= \frac{4}{3} + \frac{2}{3} \sum_{k,m\geq 1} 3^{-k} \cdot \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{s(s-1)} \Big(\frac{2^k}{m}\Big)^s\, ds. \end{aligned}$$

The last step was by using the Dirichlet series for $\zeta(s)$ *and the geometric series, both valid for* $\Re s = \frac{3}{2}$. *A simple application of residue calculus, as it is often used in the context of the Mellin–Perron summation formula (see* [7, 2]*) evaluates the integrals inside the summation:*

$$\frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{s(s-1)} t^s ds = t - 1 for\ t \geq 1 0 for\ t < 1\ .$$

Therefore

$$\begin{aligned} \sum_{n\geq 0} J_n &= 1 + \frac{2}{3} \sum_{k\geq 1} \sum_{1\leq m\leq 2^k} 3^{-k} \Big(\frac{2^k}{m} - 1\Big) \\ &= 1 + \frac{2}{3} \sum_{k\geq 1} \Big(\frac{2}{3}\Big)^k H_{2^k} - \frac{2}{3} \sum_{k\geq 1} \Big(\frac{2}{3}\Big)^k \\ &= -\frac{1}{3} + \frac{2}{3} \sum_{k\geq 1} \Big(\frac{2}{3}\Big)^k H_{2^k} = 3.36465\,07281\,00925\,16083\,89349\,6289\ldots, \end{aligned}$$

with harmonic numbers $H_n = \sum_{1\leq k\leq n} \frac{1}{k}$.

References

[1] H. G. Diamond and B. Reznick. Problems and solutions. *American Mathematical Monthly*, 106:175–176, 1999.

[2] P. Flajolet, P. Grabner, P. Kirschenhofer, H. Prodinger, and R.F. Tichy. Mellin transforms and asymptotics: Digital sums. *Theoretical Computer Science*, 123:291–314, 1994.

[3] P. Flajolet and R. Sedgewick. Mellin transforms and asymptotics: Finite differences and Rice's integrals. *Theoretical Computer Science*, 144:101–124, 1995.

[4] P. Grabner and H. Prodinger. Asymptotic analysis of the moments of the Cantor distribution. *Statistics and Probability Letters*, 26:243, 1996.

[5] P. Grabner and H. Prodinger. An asymptotic study of a recursion occurring in the analysis of an algorithm on broadcast communication. *Information Processing Letters*, 65:89–93, 1998.

[6] A. Knopfmacher and H. Prodinger. Exact and asymptotic formulæ for average values of order statistics of the Cantor distribution. *Statistics and Probability Letters*, 27:189, 1996.

[7] G. Tenenbaum. *Introduction to analytic and probabilistic number theory*. Cambridge University Press, 1995.