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Algorithms for Dynamic Negotiation

Aimeé Bella, Ferenc Szidarovszky, and Lucien Duckstein

Abstract

Most conflict situations are dynamic processes where negotiation is carried out and the payoff not only depends on the final state but on the entire process leading to that final state. In this paper, we present three techniques by generalizing the concepts of conflict and solution method so as to include dynamic features into the negotiation methodology. The three techniques are: a graph concept in a dynamic framework, a utility gain model and a negotiation algorithm. In an upper Rio Grande basin case study, we use the negotiation algorithms to resolve conflicts in water resources management, especially water allocation where sources of conflicts appear to involve:

1. satisfaction of water demand,
2. preservation of water quality, and
3. the non-quantification of Native American rights.

Results of the case study are analyzed and interpreted showing how negotiations can lead to satisfactory solutions.

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INTRODUCTION

In this paper, we provide three appproaches for conflict resolution, namely a graph concept in a dynamic framework, a utility gained model, and a negotiation algorithm. We are mainly concerned with conflict resolution and negotiation considered as evolutionary processes involving dynamic changes. Conflict resolution and negotiation are virtually inevitable whenever humans interact,

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either individually or in groups. A conflict is a situation in which there is a "condition of opposition" and parties with opposing goals affect one another. In other words, in a conflict, two or more decision makers or decision making groups are in dispute over some issue and the resolution depends on the choices of all participants. Conflicts which may be either simple, two-player, complex, or multi-player ones (Bella and Szidarovszky, 1992), can range from a small scale dispute between individuals, to a large military confrontation where huge armies are involved. However, the underlying philosophy and mathematical structure that describe conflict situations have many similar features regardless of the size of the conflicting parties. Fraser and Hipel (1984) have shown that it is often very convenient to model real-world conflicts by describing the possible states, the players' preferences over these states, the inter-state transitions and the transition control by the players. Preferences among the feasible outcomes can be realistically modeled using a minimum amount of ordinal information. As an example, in conflict negotiation, only relative preference information is required when a given player decides that one outcome is preferred over another and it is not necessary to know by how much a player prefers a particular outcome. Therefore, real numbers or utilities do not have to be always assigned to each outcome for representing the preferences of a player and for modeling his/her behavior in a conflict situation. The concepts of conflicts and conflict resolution techniques are also discussed for example by Fang et al. (1993) who give a comprehensive summary of solution concepts based on stability conditions, and introduce a general solution algorithm in graph form. In addition, the authors compared the different solution concepts in order to give a guideline to practitioners for selecting the most appropriate solution method in modeling and solving real world problesm. In a dynamic framework, which is the focus of this paper, there are two main approaches to model and analyze the situation. The first approach is to develop a new model each time period that a model parameter changes (Rapoport, 1969). The second approach taken here incorporates formally the time as a variable in a conflict model by using a graph concept in which at each time period, players make moves by taking into account:

(i) past history of previous moves,

(ii) a utility gain model based on utility theory and

(iii) a negotiation algorithm in which case moves at each time period are done by the weakest player in some sense.

Therefore, the paper is intended to be another contribution in the direction of enriching the application of game theoretic-methods to real-world conflict situations. Specifically, players can make choices (moves) which would interact with those of other players in determining the state of the system in future time periods. We have to point out that to the best of our knowledge, the possibility that the payoff matrix and the relationships between the system states (such as the state transition relations) during the negotiation process could also change over time have not been recognized, or at least no formal methods for coping with such changes have been introduced in the literature. We contend that many water resources conflicts are best regarded as non-stationary processes, in which the system state-payoff matrix relationship is itself subject to change over time. Thus, we generalize the conflict resolution concepts and solution algorithms in a graph theoretical framework, utility gain model and a negotiation process in order to include the above mentioned dynamic features into the methodology.

The paper is organized as follows. In section 1, we present the graph concept in dynamic framework where players' moves are decided according to past history of previous moves. Section 2 develops the utility gain model in which the conflict is carried out by computing a utility gain for each decision maker when he/she decides to change his/her strategy. In section 3, we provide a negotiation scheme showing when and how the decision makers make their moves. Finally, in section 4, using the negotiation algorithm of the previous section, the upper Rio Grande Basin conflict is modeled and solved using water allocation conflict model that involves fifteen (15) decision makers.

1 GRAPH CONCEPT IN DYNAMIC FRAMEWORK

In applying any conflict analysis methodology, the available information on the dispute has first to be organized, and then possible solutions of the dispute have to be constructed. Based on the results, a decision maker can then select actions which would be most beneficial for his/her purposes. The general scheme can be formulated as follows. Let N denote the total number of players, and let $k = 1, 2, \ldots, N$ be the notation of each player. Let $\varphi_k, k = 1, 2, \ldots, N$, denote the payoff function of player k, where

$$\varphi_k : S_1 \times S_2 \times \ldots \times S_N \to R.$$

Here $S_1, \ldots, S_N$ denote the sets of feasible actions of the players. Assume furthermore that the players form G groups, and let $I_j, j = 1, 2, \ldots, N$, denote the set of players in group j. It is also assumed that the cooperation of player k with all other players is expressed as a utility function $\psi_k(\varphi_1, \varphi_2, \ldots, \varphi_N)$, which contains the payoffs of all players. For example, if ψ_k is linear, then it can be written as

$$\psi_k = \varphi_k + \sum_{l \neq k} c_{kl} \varphi_l \tag{1}$$

The payoff of each group j is a function of ϕ_j of the utilities ψ_k of the players belonging to that group. That is, a new game is defined when G is the number of players and the set of strategies of each group is given as Cartesian product of the action sets of the players from that group. In addition, the payoff function of each group is represented as:

$$\phi_j = \phi_j(\psi_{j_i}, \psi_{j_2}, \ldots, \psi_{j_{M(j)}}) \tag{2}$$

where we assume that $\left\{I_j = j_1, j_2, \ldots\ldots, j_{M(j)}\right\}$. Here $M(j)$ denotes the number of players of group j, which might depend on j.

Considering the individual moves of the players, let now the state space be denoted by S, which is usually (but not necessarily) a finite set. The initial state of the conflict is s_o and at each time period $t \geq 0, s_t$ denotes the state at t. Introduce the notation $\mathbf{N} = \{1, 2, \ldots, N\}$. We say that a combination $(i_1, \ldots, i_l; s_1, \ldots, s_l) \in N^l \times S^l$ is feasible, when at $t = 1$, player i_1 can move from s_o to s_1. After this move at $t = 2$, player i_2 can move from s_1 to s_2, etc. Finally, at $t = l$, after all previous moves, player i_l is able to move from S_{l-1} to s_l. Let D_l denote the set of all feasible combinations of length l. At each time period $t \geq 1$, the conflict is given by the combination $(i_1, \ldots, i_t; s_1, \ldots, s_t)$, which shows all moves leading to the final state s_t.

The consequence of any combination $(i_1, \ldots, i_l; s_1, \ldots, s_l)$ to player k is an element of the consequence space Q_k, which is assumed to be partially ordered with binary relation $\Im(k, t)$. For example, "$c\Im(k,t)c^*$" means that player k prefers c^* to c at time period t.

Hence the partial order is time dependent. Introducing the notation

$$D = \bigcup_{l=1}^{\infty} D_l, \tag{3}$$

the payoff function of player k is given as $\beta_k : D \to Q_k$. Notice that for different values of l, β_k has a different number of variables. Therefore β_k depends on time, even if it is not explicitly shown in the notation. Here $\beta_k(i_1, \ldots, i_l; s_1, \ldots, s_l)$ gives the consequence of the entire combination set resulting in the final state s_l to player k. That is, β_k depends on the entire sequence of moves, not only on the final state s_l. We also assume that if $(i_1, \ldots, i_l; s_1, \ldots, s_l) \in D_l$, then for all players $n, (i_1, \ldots, i_l, k; s_1, \ldots, s_l, s_l) \in D_{l+1}$. That is, no change is assumed to be a feasible move to all players. For each time period a directed graph can be defined, where the possible states are the nodes and the possible moves are the arcs. The drawback of this approach is that players need to know the entire feasible combination or the entire history of past moves.

2 UTILITY GAIN MODEL

Let now denote the set of states of $S = \{s_o, s_1, \ldots, S_M\}$, where s_o is the "status quo" and M is the total number of states. Let also $u_i(s_j)$ be the utility of decision maker i if state s_j is selected and $V_{ik}(s_h, s_j)$ be the largest utility value that player k is willing to pay to player i for agreeing in changing state s_h to s_j. Let finally $T_{ik}(s_h, s_j)$ be the maximal threat that k poses on i if he agrees in changing s_h to $s_j, C_{ik}(T)$, the cost of threat T, which is posed on i by k and $W_i(T)$, the utility loss by threat T for i. In the negotation process, decision maker i will agree to change s_h to s_j is he incurs an overall utility gain which has the following components: $\{u_i(s_j) - u_i(s_h)\}$ as the utility gained, $\left\{\sum_{k\neq i} V_{ik} - \sum_{k\neq i} V_{ki}\right\}$ as the received utility minus the lost utility subject to $0 \leq v_{ik} \leq V_{ik}(s_h, s_j)$, $\left\{\sum_{k\neq i} W_{ki}(t_{ik})\right\}$ as total threat received, subject to $0 \leq t_{ik} \leq T_{ik}(s_h, s_j)$ and $\{\sum_{k\neq i} C_{ki}(t_{ki})\}$ as total cost paid for threatening others, subject to $0 \leq t_{ki} \leq T_{ki}(s_h, s_j)$. Hence the total gain is given by:

$$G_i(s_h, s_j) = u_i(s_j) - u_i(s_h) + \sum_{k\neq i} \{v_{ik} - v_{ki} - W_i(t_{ik}) - C_{ki}(t_{ki})\} \tag{4}$$

Therefore, i will agree on changing s_h to s_j if and only if there are feasible values of v_{ik}, v_{ki}, t_{ik}, and t_{ki} such that $G_i(s_h, s_j) \geq 0$.

Assume finally that the current state is s_h. Compute all utility gains $u_i(s_j) - u_i(s_h)$, for player i which are denoted as $u_i(s_j, s_h)$. Let $u_{i^*}(s_{j^*}, s_h)$ be equal to $\max\{u_i(s_j, s_h)\}$. Then player i^* tries to make the move to s_{j^*}. First, he checks on the possibility of convincing all other parties or sufficiently many other parties to agree with this change. Then he suggests a move and tries to convince as many participants as possible by negotiating with them on possible side payments and threats. If successful, then s_{j^*} will become the new state. Otherwise, this move is dropped, the second largest of the numbers $u_i(s_j, s_h)$ is selected, and a new move is examined. This process terminates when no more move is possible. An alternative approach is based on maximizing the relative utility gain

$$\frac{u_i(s_j) - u_i(s_h)}{|u_i(s_h)|}, \tag{5}$$

when relative gain is more important to the players than simple utility increase.

3 NEGOTIATION ALGORITHM

This section describes a negotiation algorithm which can especially be used in resource allocation conflicts. It determines which decision makers should make the concession at each time period. To start, let N denote again the number of players in a conflict. Let x_k denote the demand of player k, and let $u_i(x)$ denote the utility function of player k if simultaneous demand vector $x = (x_1, \ldots, x_N)$ is selected. Assume that not enough resources are available to satisfy all demands, then at least one player will get less than the demand according to the regulations and agreement among players. Let $x^* = (x_1^*, \ldots, x_N^*)$ denote the "status quo" or the amount each player gets if no agreement is achieved. The negotiation procedure is modeled in the following step. At time period $t = 0$, each player selects a demand. If the corresponding simultaneous strategy vector is feasible, then agreement is reached. Otherwise,at least one player changes his/her demand. If the new simultaneous strategy vector becomes feasible, then the negotiation procedure terminates. Otherwise, a new player makes the next move. This procedure continues until a feasible solution is reached. If no agreement is reached, that is, no further moves are made, then the players receive the corresponding components of the disagreement vector. Assume now that at a certain time period of the negotiating process, the current demand vector $x = (x_1, \ldots, x_N)$ is not feasible. Then the players must continue the bargaining process, otherwise, the "status quo" point will be automatically selected, which is unfavorable to all decision makers. The question that arises now is the following: who makes the next move and which offer is made by this player? It is natural to assume that the player who will have the largest relative loss in switching to the "status quo" point should make the next concession, and he has to continue relaxing his position until the new state becomes feasible or someone else becomes the weakest player. After his offer is made, a new negotiation state with a new demand vector will be in effect, and the weakest player at this new stage makes the next move. This procedure continues until a feasible state is found. This concept can be formulated mathematically in the following way. Define for all x, i, and $t \in [0, 1]$,

$$Q_i(x,t) = \frac{U_i(x_1, \ldots, x_{i-1}, x_i - t(x_i - x_i^*), x_{i+1}, \ldots, x_N) - U_i(x^*)}{U_i(x_1, \ldots, x_{i-1}, x_i - t(x_i - x_i^*), x_{i+1}, \ldots, x_N)} \tag{6}$$

which gives the relative loss of player i by switching the state vector to the "status quo" point. We say that at state x, player i is the weakest, if for some small $\epsilon > 0$,

$$Q_i(x,t) > Q_j(x,t) \tag{7}$$

for all $j \neq i$ and $0 < t \leq \epsilon$.

In this case, player i has to relax his demand from state x_i to $x_i - t(x_i - x_i^*)$, where t is the smallest positive number such that player i is not the weakest anymore. It at a certain state x, there are two or more weakest players, then all of them should relax their demands in this way simultaneously. If the resulting new state is feasible, then the negotiation ends. If the new state is nondominated, then this is considered to be the outcome of the negotiation process. If is is not, then the final outcome is defined as follows. Let x be the state before the last stage. Then define:

$$t^* = \min \{t | x + t(x - x^*) \text{ is nondominated}\},$$

and select the final outcome as $x + t^*(x - x^*)$. Notice that an "equity" type adjustment is made is this final step.

4 NUMERICAL APPLICATION

The upper Rio Grande Basin conflict (Figure 1) which involves multiple objectives – multiple decision makers, is modeled using the negotiation algorithm of the previous section. The conflict is over water demand satisfaction scenarios. In other words, we model how available water supplies should be shared among all users. The conflict involves 15 major decision makers (Table 1) and 30 plausible option strategies (Bella et al., 1995), reduced to 15 major strategies (Table 1) that are hypothetically defined by the authors based on available reports (US Army Corps, 1989 and US Bureau of Reclamation, 1992). Historically, the division of surface waters has been carried out amiably by mutual agreement between the three groups of users: the states of Colorado, New Mexico, and Texas. Specifically, the water balance along the river is monitored and allocated by the US Bureau of Reclamation, the US Army Corps of Engineers, and the International Boundary and Water Commission (IBWC) according to the Rio Grande compact of 1938 and the two US-Mexico agreements of 1906 and 1944. Water consumption in the Rio Grande Basin is primarily for urban, agricultural, and recreational uses, with some water used for hydroelectric power generation and cooling. The largest water-use category is irrigation. The bulk of the irrigation water comes from the river. A significant fraction of the total water use is pumped ground water. Most cities along the valley depend completely on groundwater for domestic use.

The lack of surface water in the past and present management schemes have been made up through interbasin diversions and ground water mining, but extensive ground water withdrawal from stream-aquifer systems has resulted in a reduction of available surface water supplies of the Rio Grande. The current management of water resources is complicated by a multiplicity of governmental units on both sides of the border. On the Mexican side, the Federal government exercises the dominant role through a variety of agencies. On the US side, primary responsibility is placed in three different states, each having its separate, independent and different water laws. Economic activities and population for the region are expected to increase, especially with the recent signing of the North American Free Trade Agreement (NAFTA) with the Republic of Mexico and Canada. These developments, which will place greater demands on water resources, may require changes in public policies presumably through negotiation. Thus, water management in the basin is complex because many of the water needs are incompatible. Allocation changes designed to meet a diminishing water supply would be difficult to implement because of the complex web of international treaties, interstate compacts and local, state and federal laws (Waterstone, 1994). The high degree of complexity makes it difficult for a decision maker to comprehend the problem fully and eventually reach a reasonable decision.

Furthermore, climatologists predict that the increase in greenhouse gases may result in a warmer and drier climate in many parts of the world in the next century. The Upper Rio Grande is one such area and it will be extremely vulnerable to any change that would decrease available supply. Revell and Waggoner (1983) predicted that a 2-degree temperature increase and a 10 percent decrease in annual precipitation would reduce runoff in the Rio Grande river basin by 76 percent. This represents the largest percentage reduction for any basin in the United States (Thomson and Stephen, 1990), and this is the main motivation for having selected this region as a case study.

Sources of water allocation conflicts appear to be as follows (Bella et al., 1995): (1) The absence of agreements on ground water usage and water supply, especially since increased demands and possibly climate change may lead to increased pumpage and a possible lowering of the water table; (2) Water quality problems; (3) The non-quantification of Indian rights, which as a necessary requirement will require an adjudication. The set of selected alternatives or strategy options include the following (Duckstein et al., 1995):

1. (1) increasing water supply and irrigation efficiency,
2. (2) meeting water quality and protecting the public interest,
3. (3) constructing hydropower and enhancing flood control,
4. (4) reducing sediment by pool stabilization, and
5. (5) establishing groundwater regulation.

Combining these actions, we have indentified the set of strategies given in Table 2. If we let Table 3 represent the players' utilities at a certain time period in the negotiation process, then we can compute values of $Q_i(x,t)$ using equation (6). The results are given in Table 4. For example, the values of $Q_1(A_1,x_1)$ and $U_1(A_3,x_1)$ are computed as follows:

$$Q_1(A_1,x_1) = \frac{U_1(A_1,x_1) - U_1(x^*)}{U_1(A_1,x_1)} = \frac{0.5-0.1}{0.5} = 0.8$$

$$Q_1(A_3,x_1) = \frac{U_1(A_3,x_1) - U_1(x^*)}{U_1(A_3,x_1)} = \frac{0.2-0.1}{0.2} = 0.5$$

Based on these values, the weakest players are identified and underlined in Table 4. For example, since $Q_1(A_4,x_1)$ corresponds to the larger relative loss for strategy x_1, A_4 (or player 4) has to make the concession by relaxing his water demand. For strategy x_5, or "Reducing sediment by pool stabilization", player A_7 or "Bureau of Reclamation" makes the move because he appears to be the weakest player. For strategy x_2, three players, namely player A_4 (Irrigation Districts), player A_{10} (New Mexico Compact Commissioners), and A_{14} (Middle Rio Grande Conservation District) make the move. Zero values in Table 4 indicate the status quo for the player.

5 CONCLUSIONS

In this paper, we have focused on dynamic negotiation algorithms for a conflict situation by presenting three conflict resolution techniques, namely: the graph concept, the utility gain model and a negotiation algorithm. Then, using a case study, the upper Rio Grande Basin conflict, a real world application is given as an example to show how players make moves during the negotiation process. This methodology is highly recommended in practical cases when the relevant data are available.

References

[1] Bella A. A. and Szidarovszky F. Water Allocation Conflict under Climatic Uncertainty: Case of Upper Rio Grande Basin, USA. Pure Math. and Appl. Ser. C., Vol. 3, No. 1-4, pp. 35-58 (1992).

[2] Bella A. A., Duckstein L. and Szidarovszky, F. A Multicriterion Analysis of Water Resources in the Upper Rio Grande Basin. Invited paper, Western Regional Science Association Meeting, San Diego, Feb. 1995.

[3] Brams S. J. Theory of Moves. Cambridge University Press, Cambridge, U.K., 1993

[4] Duckstein L., Bella A., Stakhiv E. Z. and Waterstone M. Conflict Analysis Using Multiple Criterion Decision Making Under Uncertainty In: Proceedings NATO ASI Transboundary Water Resources: Management and Environmental Issues Conference, Skopelos, Greece, May 15-25 (to appear 1995).

[5] Fang L., Hipel K. W. , and Kilgour D. M. Interactive Decision Making: The Graph Model for Conflict Resolution. John Wiley & Sons, Inc., New York, 1993.

[6] Fraser N. M. and Hipel K. W. Conflict Analysis: Models and Resolutions. North-Holland, New York, 1984

[7] Rasmusen E. Games and Information: An Introduction to Game Theory. Basil Blackwell, Oxford, 1989

[8] Rapoport, A. Effects of Payoff Information in Multi-Stage Mixed Motive Games. Behavioral Sci., Vol. 14, 1969, pp. 204-215.

[9] Revell R. R. and Waggoner P. E. Efforts of the Carbon Dioxide Induced Climatic Change on Water Supplies in the Western United States in Changing Climate. Report of the Carbon Dioxide Assessment Committee. National Academy Press, Washington, D.C. 1983.

[10] U.S. Army Corps of Engineers Reevaluation of the Operating Plan. Report: Albuquerque Project Office. Albuquerque, New Mexico. 1989

[11] U.S. Bureau of Reclamation Annual Operating Plan. Report: Albuquerque Projects Office. Albuquerque, New Mexico. 1992.

[12] Thomson B. and Stephen A. Rio Grande Laboratory/ Consortium Concepts. Proceedings of the Rio Grande Basin - Global Climate Change Scenarios, WRRI Rep. M24, 1990, pp. 33-36.

[13] Waterstone M. Institutional Analysis of Water Resources. In: Engineering Risk in Natural Resources Management with Special References to Hydrosystems under changes of Physical or Climatic Environment. L. Duckstein and Parent E. (eds). Kluwer, Amsterdam. Series E: Applied Sciences Vol. 275, 1994 pp. 59-75.

Table 1. Identified decision makers in the Upper Rio Grande

Notation	Decision Makers
A_1	US Forest Service and Land Management
A_2	Pueblo Indian Tribes
A_3	Recreation Groups
A_4	Irrigation Districts
A_5	User Groups
A_6	Army Corps of Engineers
A_7	Bureau of Reclamation
A_8	International Boundary and Water Commission
A_9	Colorado Compact Commissioners
A_{10}	New Mexico Compact Commissioners
A_{11}	Texas Compact Commissioners
A_{12}	Fish and Wildlife Service
A_{13}	Department of Interior
A_{14}	Middle Rio Grande Conservation District
A_{15}	Mexican Government

Table 2. Selected strategies in the Upper Rio Grande Notation & Strategies

Notation	Strategies
X_1	Do nothing
X_2	Increasing water supply and irrigation efficiency
X_3	Meeting water quality and protecting the public interest
X_4	Constructing hydropower and enhancing flood control
X_5	Reducing sediment by pool stabilization
X_6	Establishing groundwater regulation
X_7	Increasing water supply and irrigation efficiency, meeting water quality and protecting the public interest
X_8	Increasing water supply and irrigation efficiency, constructing hydropower and enhancing flood control
X_9	Increasing water supply, irrigation efficiency and reducing sediment by pool stabilization
X_{10}	Increasing water supply and irrigation efficiency and establishing groundwater regulation
X_{11}	Meeting water quality, protecting the public interest, constructing hydropower and enhancing flood control
X_{12}	Meeting water quality, protecting the public interest and reducing sediment by pool stabilization
X_{13}	Meeting water quality, protecting the public interest establishing groundwater regulation
X_{14}	Constructing hydropower, enhancing flood control and reducing sediment by pool stabilization
X_{15}	Constructing hydropower, enhancing flood control and establishing groundwater regulation

Table 3. Utility function of strategies and player

$U_i(x)$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}
X_1	0.5	0.1	0.2	0.7	0.8	0.5	0.5	0.3	0.6	0.1	0.12	0.1	0.7	0.1	0.1
X_2	0.2	0.6	0.2	0.95	0.9	0.7	0.75	0.5	0.8	0.9	0.75	0.4	0.3	0.9	0.5
X_3	0.85	0.65	0.9	0.4	0.77	0.5	0.55	0.35	0.67	0.6	0.55	0.95	0.33	0.43	0.5
X_4	0.3	0.2	0.4	0.25	0.45	0.9	0.95	0.7	0.6	0.75	0.55	0.4	0.6	0.5	0.3
X_5	0.2	0.4	0.6	0.5	0.65	0.8	0.75	0.3	0.4	0.5	0.45	0.3	0.55	0.7	0.6
X_6	0.1	0.15	0.25	0.3	0.32	0.5	0.45	0.6	0.9	0.85	0.97	0.23	0.7	0.82	0.5
X_7	0.2	0.65	0.6	0.95	0.6	0.5	0.7	0.43	0.8	0.6	0.82	0.9	0.7	0.3	0.5
X_8	0.6	0.5	0.53	0.27	0.4	0.74	0.8	0.3	0.5	0.5	0.6	0.4	0.3	0.5	0.1
X_9	0.1	0.3	0.4	0.7	0.21	0.4	0.35	0.5	0.6	0.7	0.65	0.41	0.5	0.6	0.12
X_{10}	0.83	0.9	0.7	0.1	0.15	0.2	0.3	0.9	0.1	0.3	0.2	0.7	0.12	0.3	0.8
X_{11}	0.2	0.5	0.5	0.8	0.43	0.9	0.1	0.21	0.7	0.6	0.3	0.44	0.6	0.22	0.35
X_{12}	0.11	0.82	0.6	0.22	0.92	0.5	0.5	0.42	0.6	0.61	0.53	0.74	0.3	0.47	0.4
X_{13}	0.12	0.6	0.52	0.46	0.6	0.7	0.4	0.31	0.7	0.6	0.44	0.53	0.4	0.36	0.39
X_{14}	0.3	0.3	0.1	0.5	0.4	0.97	0.91	0.12	0.22	0.4	0.61	0.33	0.94	0.8	0.94
X_{15}	0.45	0.21	0.2	0.3	0.57	0.92	0.87	0.3	0.7	0.5	0.7	0.5	0.5	0.63	0.81

Table 4. Relative loss function of strategies and players

$Q_i(A,X)$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}
X_1	.8	0	.5	$\underline{.86}$	.81	.6	.8	.6	.83	0	0	0	.83	0	0
X_2	.5	.83	.5	$\underline{.89}$	.83	.71	.87	.76	.87	$\underline{.89}$	.87	.75	.6	$\underline{.89}$	.8
X_3	.88	.85	$\underline{.89}$	.75	.81	.6	.82	.66	.85	.83	.82	$\underline{.89}$	.64	.77	.8
X_4	.67	.5	.75	.6	.67	.78	$\underline{.89}$	.83	.83	.87	.82	.75	.8	.8	.67
X_5	.5	.75	.83	.8	.77	.75	$\underline{.87}$	.6	.75	.8	.78	.67	.78	.86	.83
X_6	0	.33	.6	.67	.53	.6	.78	.8	.89	.88	$\underline{.9}$	.57	.83	.88	.8
X_7	.5	.85	.83	$\underline{.89}$	.78	.6	.86	.72	.87	.83	.88	$\underline{.89}$	.83	.67	.8
X_8	.83	.8	.81	.63	.62	.73	$\underline{.87}$	.6	.8	.8	.83	.75	.6	.8	0
X_9	0	.67	.75	$\underline{.86}$	.29	.5	.75	.76	.83	$\underline{.86}$	.85	.76	.76	.83	.17
X_{10}	.88	$\underline{.89}$	.86	0	0	0	.67	.87	0	.67	.5	.86	0	.67	.88
X_{11}	.5	.8	.8	$\underline{.87}$	.65	.78	0	.43	.86	.83	.67	.77	.8	.55	.71
X_{12}	.09	$\underline{.88}$	.83	.55	.84	.6	.8	.71	.83	.84	.81	.86	.6	.79	.75
X_{13}	.17	.83	.81	.78	.75	.71	.75	.61	$\underline{.86}$	.83	.77	.81	.7	.72	.74
X_{14}	.67	.67	0	.8	.62	.79	$\underline{.89}$	0	.55	.75	.84	.7	.87	.88	$\underline{.89}$
X_{15}	.78	.52	.5	.67	.74	.78	$\underline{.89}$	.6	.86	.8	.86	.8	.76	.84	.88

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Polynomial–Time Partition of a Graph into Cliques

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Abstract

The properties of finite partially ordered set are used to find the solution of the partition finding problem of arbitrary undirected graph into the minimum number of cliques. The solving algorithm has a partical orientation. The run time of the solving algorithm is equal to $O(n^5)$, where n is the number of vertices of the graph.

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1 Statement of the problem.

We will consider the class L_n of undirected n-graphs without loops and multiple edges.

Let $G = (X, \Gamma) \in L_n$, where $X = \{x_1, \ldots, x_n\}$ is the set of vertices of the graph G, Γ is mapping of X into X.

The set $N \subset X$ creates *a clique* of the graph G if the correlation $x_i \in \Gamma x_j$ takes place for any $x_i, x_j \in N$ $(i \neq j; \forall i, j \in \{1, 2, \ldots, n\})$. The clique of graph G, defined by the set of vertices N, we will denote: $Q(N)$. In the special case, when $Card(N) = 1$, we will call it a single-vertex clique.

The graph G is partitioned into cliques $Q(N_1), \ldots, Q(N_m)$ if

$$\bigcup_{i=1}^{m} N_i = X,$$

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where $N_i \cap N_k = \oslash\ \forall i, k \in \{1, \ldots, m\}\ (i \neq k)$.

This evident that there exists a set of partitions of the graph G into cliques. The partition $\mathbf{F} = \{Q(N_1), \ldots, Q(N_m)\}$ of the graph G into cliques will be called *minimum (MCP)* if it has the minimum number of elements in comparison with other partitions of the graph into cliques.

It is necessary to find the partition $\mathbf{F}$ of the graph $G \in L_n$ with the minimum number of cliques.

The formulated problem on the partitioning of the graph $G \in L_n$ with the minimum number of the cliques is one of the NP-complete problems on graphs as is defined in [1], [6]. This problem has numerous applications.

2 Vertex-saturated digraph

Let there be a graph $G = (X, \Gamma) \in L_n$.

A set $U \subset X$, satisfying the condition $U \cap \Gamma U = \oslash$, is called *an independent set of the vertices* of the graph G. An independent set of vertices U is called *maximal (MIS)*, if $U \cup \Gamma U = X$. Finally, MIS $\hat{U}$ is called *maximum (MMIS)*, if $Card(\hat{U}) > Card(U)$ for all MIS $U \subset X$ of the graph $G \in L_n$.

In the graph G we will find the MIS X^0 and we will construct the acyclic digraph $\vec{G}(X^0) = (X, \vec{\Gamma})$, by way of orientation of the edges of the graph G so that the set of vertices X^0 will have only outgoing arcs. We will call the set X^0 *initiating*, and digraph $\vec{G}(X^0)$ – *a directed mapping* of the graph $G = (X, \Gamma) \in L_n$. There is a set $D(G)$ a directed mapping of the graph $G \in L_n$.

We will further consider only digraphs, belonging to the set $D(G)$.

The length $\rho(x)$ of the maximum directed path, connecting the initiating set X^0 with the vertex x, we will call the *rank* of the vertex $x \in X$. We will call the set of all vertices of rank $\rho(x) = k$ *k-th layer* of the vertices of the digraph $\vec{G}(X^0)$ and will designate it as: X^k $(k = \overline{0, m})$, where m is the maximum rank of the vertices from $X \setminus X^0$.

We will construct the graph $\vec{G}_t(X^0) = (X, \vec{\Gamma}_t)$ of the transitive closure digraph $\vec{G}(X^0)$. Obviously, the graph $\vec{G}_t(X^0)$ determines a finite partially ordered set $(X, >)$.

It is not worth while to distinguish between the graph of transitive closure $\vec{G}_t(X^0)$ and the partially ordered set $(X, >)$. Therefore in the graph $\vec{G}_t(X^0)$ we will also consider antipaths.

We will say, that the antipath U_1 *precedes* the antipath U_2 in the graph $\vec{G}_t(X^0)$ and designate this as: $U_1 \prec U_2$, if for all vertices $x \in U_1 \setminus U_2$ there is a vertex $y \in U_2 \setminus U_1$ such that $x < y$. The precedence relation, obviously, is transitive.

In the digraph $\vec{G}(X^0)$ we will consider the set of all vertices having ranks greater than or equal to k: $X^k \cup \cdots \cup X^m$. An oriented subgraph of digraph $\vec{G}(X^0)$, defined for the set of vertices $X^k \cup \cdots \cup X^m$, will be called *a directed subgraph of digraph $\vec{G}(X^0)$, generated by layer X^k*, and designate as: $\vec{G}(X^k)$.

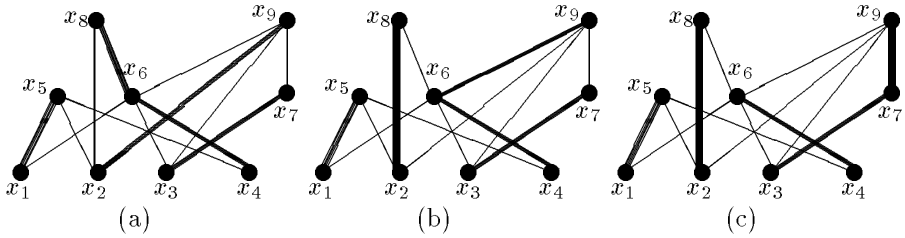

Figure 1: Some MPP of the graph

The acyclic digraph $\vec{G}(X^0)$ will be called *correctly constructed (CCD)*, if in each directed subgraph $\vec{G}(X^k)$ $(k = \overline{0,m})$ the initiating set X^k is a MIS of this subgraph.

We will consider only correctly constructed digraphs from $D(G)$.

Let there be CCD $\vec{G}(X^0)$ and its graphs of transitive closure $\vec{G}_t(X^0)$.

A search of the maximum antipath with a simultaneous partitioning of the vertices of the graph $\vec{G}_t(X^0)$ into a minimum number of paths is possible to execute by any algorithm of a finding of the maximum matching in bipartite graph [2], [5], [4]. Using the methodology of Ford and Fulkerson [2] in the graph $\vec{G}_t(X^0)$ one such the maximum antipath $\mathcal{U}$ can be found, for which any other maximum antipath U_1 precedes, i.e. $\mathcal{U}_1 \prec \mathcal{U}$ for any antipath $\mathcal{U}_1$ of the graph $\vec{G}_t(X^0)$. Such an antipath of the graph $\vec{G}_t(X^0)$ we will call *general.*

In addition to the general antipath, if it possible to find other antipaths of the graph $\vec{G}_t(X^0)$ if they exist. So for any vertex $x \in X$ of the graph $\vec{G}_t(X^0)$ it is possible to find a maximum antipath $\mathcal{U}(x)$ such that $x \in \mathcal{U}(x)$. Technically to search for antipath $\mathcal{U}(x)$ it is sufficient in the Ford-Fulkerson working table in which the maximum number of units in allowable cells [2] is found, to mark a row, corresponding to vertex x, and to execute a cycle of appointment of marks. Clearly, the antipath $\mathcal{U}(x)$ will be general for the vertex x, i. e. any other antipath, containing vertex x, will precede this antipath.

We will notice, that in the general case it is not true that for any vertex $x \in X$ of the graph $\vec{G}_t(X^0)$ there exists the maximum antipath $\mathcal{U}(x)$.

In fig. 1 a digraph is shown for the graph of transitive closure for which there is a general antipath $\mathcal{U} = \mathcal{U}(1) = \mathcal{U}(2) = \mathcal{U}(4) = \mathcal{U}(7) = \{x_1, x_2, x_4, x_7\}$ but there is not an antipath $\mathcal{U}(8)$. (The orientation of arcs of a digraph on a drawing is from below upwards).

The directed subgraph $\vec{G}(X^k)$ $(k = \overline{0,m})$ of the digraph $\vec{G}(X^0)$ we will call *vertex-saturated relative to the initiating set* X^k, if in its graph of transitive closure $\vec{G}_t(X^k)$ any maximum antipath $\mathcal{U}(x) \subset X$, when it exists, is a MIS of subgraph $\vec{G}(X^k)$ and satisfies the relation: $Card(\mathcal{U}(x)) = Card(X^k)$.

We will call CCD $\vec{G}(X^0)$ *vertex-saturated*, if any of its directed subgraphs $\vec{G}(X^k)$ $(k = \overline{0,m})$, generated by the layer X^k, is vertex-saturated relative to the initiating set X^k.

Obviously, for any graph $G = (X, \Gamma) \in L_n$ it is easy to construct a vertex-saturated CCD $\vec{G}(X^0)$. For this purpose consider any subgraph $\vec{G}(X^k)$ ($k = \overline{0, m}$) and construct its graph of transitive closure $\vec{G}_t(X^k)$ which may be considered as the graph of a partial ordering. For each vertex $x \in X$ find the maximum antipath $\mathcal{U}(x) \subset X$, if it exists.

Vertex-saturated digraph will have constructed if each found the maximum antipath $\mathcal{U}(x) \subset X$ of the graph $\vec{G}_t(X^k)$ satisfies the relation $Card(\mathcal{U}(x)) = Card(X^k)$ and is a MIS of the subgraph $\vec{G}(X^k)$. Consequently, any subgraph $\vec{G}(X^k)$ also must be vertex-saturated relative to the inducing layer X^k ($k = \overline{0, m}$). Otherwise we form MIS $S \subset X$ ($U(x) \subset S$) of the digraph $\vec{G}(X^k)$ and all arcs, coming in the vertices of the set S, are reoriented.

For the new digraph $\vec{G}(S)$ the procedure described above is repeated.

Thus, the construction of a vertex-saturated digraph $\vec{G}(X^0)$ is a recursive procedure, i. e. after construction of the subgraph $\vec{G}(X^m)$, which is vertex-saturated relative to the initiating set X^m, for all k ($k = \overline{0, m}$) vertex-saturating of each subgraph $\vec{G}(X^k)$ relative to the set X^k must be checked again.

Clearly, one of the main properties of a vertex-saturated digraph is that each path of a partition $\mathcal{P}$ of the vertices of the graph $\vec{G}_t(X^0)$ into a minimum number of paths begins at a vertex of the initiating set X^0.

Let $U_1, U_2 \subset X$ are antipaths, maximal by inclusion, of the graph $\vec{G}_t(X^0)$ such that $U_1 \prec U_2$.

We will consider the bipartite digraph $\widetilde{G} = (U_1, U_2, \widetilde{\Gamma})$. This digraph is a subgraph of $\vec{G}_t(X^0)$, which we will say is *the induced* by maximal antipaths U_1, U_2. The number

$$\nabla(\widetilde{G}) = max\{Card(S) - Card(U_1 \cap \widetilde{\Gamma}^{-1} S)\} \qquad \forall S \subset U_2$$

will call *the deficit* of the bipartite digraph $\widetilde{G} = (U_1, U_2, \widetilde{\Gamma})$ [5].

Theorem 1. *Let there be a vertex-saturated digraph $\vec{G}(X^0)$. Then the deficit $\nabla(\widetilde{G})$ of any bipartite subgraph $\widetilde{G}) \subset \vec{G}_t(X^0)$, induced by the maximal antipaths $U_1, U_2 \subset X$ such, that $U_1 \prec U_2$, is equal to zero and $Card(U_1) > Card(U_2)$.*

This is a direct result of the König theorem [2], [5] and the definition of a vertex-saturated digraph.□

3 Recursive sequence of digraphs.

Let there be a vertex-saturated CCD $\vec{G}(X^0)$ and $\vec{G}_t(X^0)$ be its graph of transitive closure.

Partition the set of the vertices X of the graph $\vec{G}_t(X^0)$ into the minimum number of paths $\mathcal{P} = \{S_1, \dots, S_p\}$, where $p = Card(X^0)$. We will call the partition $\mathcal{P}$ *the minimum path partition (MPP)* and which in the general case is not single valued.

As usual we will call an arc of the graph $\vec{G}_t(X^0)$ *thick*, or *engaged*, if its belongs to some path of MPP $\mathcal{P}$ of this graph. Otherwise, an arc of the graph $\vec{G}_t(X^0)$ will be called *thin*, or *free* (see the fig. 1).

We will call the arc (x_i, x_j) of the graph $\vec{G}_t(X^0)$ *essential* and we will use the notation: $x_i > x_j$, if arc (x_i, x_j) of digraph $\vec{G}(X^0)$ corresponds to it. Otherwise, we will call the arc (x_i, x_j) of the graph $\vec{G}_t(X^0)$ *fictitious* and we will use the notation: $x_i \gg x_j$. We will call the terminal vertex x_j of a fictitious arc $x_i \gg x_j$, both ends of which belong to the same path $S_q \in \mathcal{P}$ $(q = \overline{1,p})$, *marked* (in the given MPP $\mathcal{P}$), and for the set of all marked vertices of the graph $\vec{G}_t(X^0)$ we will use the notation: $\mathcal{B}(\mathcal{P})$.

We will call a fictitious arc $x_i \gg x_j$ of the graph $\vec{G}_t(X^0)$ *elementary (EFA)*, if in the digraph $\vec{G}(X^0)$ the vertices x_i and x_j are connected by the shortest directed path of length 2. We will call corresponding essential arcs $x_i > x_k$ and $x_k > x_j$, belonging to paths of the graph $\vec{G}_t(X^0)$, generators for EFA. Otherwise, we will call the fictitious arc $x_i \gg x_j$ *compound.*

Obviously, any compound fictitious arc can be represented as union of elementary fictitious arcs.

Lemma 2. *Let $\vec{G}(X^0)$ be some digraph and $\vec{G}_t(X^0)$ be its graph of transitive closure. If $\mathcal{B}(\mathcal{P}) = \oslash$, where $\mathcal{P}$ is MPP of the graph $\vec{G}_t(X^0)$, its maximum antipath $\mathcal{U}$ is MMIS of the digraph $\vec{G}(X^0)$, and MPP $\mathcal{P}$ is its minimum cliques partition.*

For the proof, we observe that if conditions of Lemma 2 are satisfied, each path $S_q \in \mathcal{P}$ $(q = \overline{1,p})$ is a clique of the digraph $\vec{G}(X^0)$ and, hence, MPP $\mathcal{P}$ is a minimum cliques partition of the digraph. On the other hand, the vertices of the maximum antipath $\mathcal{U}$ belong to the set of its cliques, i.e. the number of vertices in the MMIS U does not exceed number of vertices in the set $\mathcal{U}$.□

Notice, if the graph of the transitive closure $\vec{G}_t(X^0)$ satisfies the conditions of Lemma 2, then the initial graph $G \in L_n$ certainly is not transitively directed, and the digraph $\vec{G}(X^0)$ is not correctly constructed or vertex-saturated.

Theorem 3. *Let CCD $\vec{G}(X^0)$ be vertex-saturated. Then in the graph $\vec{G}_t(X^0)$ the MPP $\mathcal{P}$ exists and its paths contain only essential arcs.*

The validity of the assertion follows from the definition of a correctly constructed digraph and Theorem 1.□

We will call the MPP $\mathcal{P}$ of the graph $\vec{G}_t(X^0)$ *correct* relative to the marked vertex $x_j \in \mathcal{B}(\mathcal{P})$, if the vertex x_k preceding it in the path $S_q \in \mathcal{P}$ $(x_j, x_k \in X(S_q))$ is connected to the vertex x_j by an essential arc and there does not exist another path connecting these vertices. We will say the arc $x_k > x_j$ of the graph $\vec{G}_t(X^0)$, satisfying the indicated conditions, *the leading* arc for the vertex $x_j \in \mathcal{B}(\mathcal{P})$.

It is easy to see, that the leading arc always determines the oriented path of maximum length, connecting the vertices x_k and x_j in digraph $\vec{G}(X^0)$. Obviously, for any marked vertex $x_j \in \mathcal{B}(\mathcal{P})$ there always exists MPP of the graph $\vec{G}_t(X^0)$ which is correct relative to x_j.

To find a leading arc in the digraph $\vec{G}(X^0)$ we will construct a procedure (similar breadth-first search), which is started at the vertex x_k, which precedes the vertex $x_j \in \mathcal{B}(\mathcal{P})$ in the path $S_q \in \mathcal{P}$ $(x_k, x_j \in X(S_q))$ of the MPP $\mathcal{P}$ of the graph $\vec{G}_t(X^0)$. As we proceed we will store all arcs, ending at the vertex x_j. The last arc found, obviously, determines the required leading arc in the correct MPP of the graph $\vec{G}_t(X^0)$.

Let there be a vertex-saturated CCD $\vec{G}(X^0)$.

The sequence of digraphs

$$\mathcal{T} = (\vec{G}^{(1)}(X^0), \vec{G}^{(2)}(X^0), \ldots, \vec{G}^{(h)}(X^0)),$$

induced by the digraph $\vec{G}(X^0)$ we will call *recursive (RSD)*, if for each of digraphs $\vec{G}^{(z)}(X^0)$ $(z = \overline{1, h-1})$ within this sequence there is a vertex-saturated CCD, $\vec{G}^{(1)}(X^0) = \vec{G}(X^0)$ where the digraph $\vec{G}^{(z+1)}(X^0)$ was obtained from the digraph $\vec{G}^{(z)}(X^0)$ by reorientation of one of its arcs.

The construction of RSD $\mathcal{T}$ is made as follows (in an initial digraph all arcs are unmarked).

For an initial vertex-saturated CCD $\vec{G}^{(1)}(X^0) = \vec{G}(X^0)$ find its graph of transitive closure $\vec{G}_t^{(1)}(X^0)$. In the graph $\vec{G}_t^{(1)}(X^0)$ we will construct some of its MPP $\mathcal{P}$.

Further, in the graph $\vec{G}_t^{(1)}(X^0)$ the marked vertex of minimum rank can be found. If there are some such vertices, then a marked vertex, which located first by the order path of MPP $\mathcal{P}$, is chosen. The order of a path in $\mathcal{P}$ is defined by the numbering of the vertex of initiating set X^0, from which the considered path of MPP is going.

After a finding the marked vertex the search for its leading arc is executed. The leading arc found in the digraph $\vec{G}^{(1)}(X^0)$ is reoriented. This is yielded a transition to the consideration of a new digraph $\vec{G}^{(2)}(X^0)$ RSD $\mathcal{T}$. In the new digraph the reoriented arc is marked as *"considered"*.

Thus, any arc, reoriented in the digraph $G(z)(X^0)$ $(z = \overline{1, h-1})$, in each subsequent digraph $\vec{G}^{(z+1)}(X^0)$ $(z+1 < r < h)$ of the RSD $\mathcal{T}$ will be marked as "considered". Further such an arc of the digraph $\vec{G}^{(z)}(X^0) \in \mathcal{T}$ is reoriented, which is not marked as "considered" and in its corresponding graph of transitive closure $\vec{G}_t^{(z)}(X^0)$ one arc is a leading arc for some marked vertex.

After construction of each new digraph $\vec{G}^{(z+1)}(X^0) \in \mathcal{T}$ $(z = \overline{1, h-1})$ a check is executed: is it vertex-saturated? If so, then the obtained digraph is vertex-saturated and its corresponding graph of transitive closure $\vec{G}_t^{(z+1)}(X^0)$

has been constructed. In this case all thick arcs of the graph $\vec{G}_t^{(z)}(X^0)$ (except the leading arc) are chosen as the thick arcs of an initial (not optimum) partition of the new graph $\vec{G}_t^{(z+1)}(X^0)$ into a paths.

After reorientation of an arc (x, y) of the digraph $\vec{G}^{(z)}(X^0)$ a new vertex-saturated digraph $\vec{G}^{(z+1)}(X^0) \in \mathcal{T}$ has been constructed. Then, it is first determined whether the vertex x is marked in the graph $\vec{G}_t^{(z+1)}(X^0)$. If so, a leading arc for the vertex x is found. If this arc is marked as "considered", the construction of a RSD is finished and all marks of the arcs in the obtained digraph $\vec{G}^{(z+1)}(X^0) = \vec{G}^{(h)}(X^0)$ are removed.

If the vertex x is marked in the graph $\vec{G}_t^{(z+1)}(X^0)$ but its leading arc is not marked as "considered" then in the corresponding path of MPP of the graph $\vec{G}_t^{(z+1)}(X^0)$ the vertex of the minimum rank is found.

In all other cases for the graph $Gt(z+1)(X^0)$ a marked vertex of the minimum rank is found which is first by order of the path of MPP of this graph.

Thus, RSD T is finite and turns, if:

(1) Digraph $\vec{G}^{(h)}(X^0)$ is not vertex-saturated;

(2) The digraph $\vec{G}^{(h)}(X^0)$ is vertex-saturated and its corresponding graph of transitive closure does not have the marked vertices ;

(3) In the vertex-saturated digraph $\vec{G}^{(h)}(X^0)$ the arc marked earlier is leading.

In case (2) we will call the digraph $\vec{G}^{(h)}(X^0)$ *deadlocked.*

Lemma 4. *In the graph $G = (X, \Gamma) \in L_n$ let $\mathbf{Q}(\mathcal{U})$ be the set of all of its cliques to each of which belongs one of vertices of the maximum antipath $\mathcal{U}$ of the corresponding vertex-saturated digraph $\vec{G}(X^0) \in D(G)$. Further, let there be an odd cycle C_{2k+1} $(k \geq 2)$ of graph G such that k of the vertices belong to the maximum antipath $\mathcal{U}$. Then such an edge of cycle C_{2k+1} exists, which does not belong to any one of the cliques in the set $\mathbf{Q}(\mathcal{U})$.*

Let conditions of a Lemma 4 be satisfied.

Since k of the vertices of cycle C_{2k+1} $(k \geq 2)$, belonging to the maximum antipath $\mathcal{U}$, are independent, there are $2k$ of edges of this cycle, each of which is incident to any of k of the vertices. Hence, there will be such an edge of cycle C_{2k+1}, which does not belong to any of cliques of set $\mathbf{Q}(\mathcal{U})$.$\Box$

Theorem 5. *In the construction of a RSD of a vertex-saturated digraph $\vec{G}(X^0)$, the leading arc will be found to be marked "considered" iff this arc belongs to the odd (undirected) cycle C_{2k+1} $(k \geq 2)$ of this digraph.*

The **necessity** conditions of Theorem 5 follow from the Lemma 4.

Sufficiency. Let the designated RSD $\mathcal{T}$ of the vertex-saturated digraph $\vec{G}(X^0)$ be completed as the result of the detection in a graph of transitive closure $\vec{G}_t^{(h)}(X^0)$, a leading arc marked as "considered". Let it be the arc $x_i > x_j$.

Then, according to the definition of RSD $\mathcal{T}$, the vertex-saturated digraph $\vec{G}^{(h)}(X^0)$ is obtained from the vertex-saturated digraph $\vec{G}^{(h-1)}(X^0)$ by a reorientation of an arc $x_j > x_i$ in it. We tighten the arc $x_j > x_i$ in the digraph $\vec{G}^{(h-1)}(X^0)$ and arc $x_i > x_j$ in the digraph $\vec{G}^{(h)}(X^0)$ at one vertex $x_{i,j}$. We obtain as a result same digraph $\vec{H}(X^0)$ in both cases. Since the MPPs of graphs $\vec{G}_t^{(h-1)}(X^0)$ and $\vec{G}_t^{(h)}(X^0)$ are different, it follows, that in the graph $\vec{H}_t(X^0)$ there are two distinct MPPs, non-coincident (thick) arcs which will form an even cycle C_{2k} $(k \geq 2)$, containing vertex $x_{i,j}$. Therefore, the arc $x_i > x_j$, marked as "considered" in the digraph $\vec{G}_t^{(h)}(X^0)$, belongs to the odd cycle C_{2k+1} $(k \geq 2$ in this digraph.

We will specify a way to search for this odd cycle in the digraph $\vec{G}(X^0)$.

Let $\mathcal{P}^{(h-1)}$ and $\mathcal{P}^{(h)}$ be the MPP of graphs $\vec{G}_t^{(h-1)}(X^0)$ and $\vec{G}_t^{(h)}(X^0)$ respectively. We will combine the paths of MPP of the graphs $\vec{G}_t^{(h-1)}(X^0)$ and $\vec{G}_t^{(h)}(X^0)$ and will remove coinciding arcs (except the leading arc of the graph $\vec{G}_t^{(h)}(X^0)$). It is easy to see, that part of the remaining arcs of MPP $\mathcal{P}^{(h-1)}$ and $\mathcal{P}^{(h)}$ forms the odd cycle C_{2k+1} $(k \geq 2)$. It follows that such cycle includes (possibly not all) arcs of alternating cycle, which was constructed during the transition from the initial (not optimum) partition of the graph $\vec{G}_t^{(h)}(X^0)$ into a paths of the required MPP $\mathcal{P}^{(h)}$.□

Theorem 6. *If the digraph $\vec{G}(X^0) \in D(G)$ does not have odd cycles, then its deadlocked digraph can be found by construction of the sequence $\mathcal{T}_1, \ldots, \mathcal{T}_m$ of a RSD of this digraph.*

This follows from Theorem 5.□

4 Co-vertex.

Let $\vec{G}(X^0)$ be a vertex-saturated digraph, for which RSD $\mathcal{T}$ has been constructed. Further, assume that for the digraph $\vec{G}^{(z)}(X^0) \in \mathcal{T}$ a leading arc (x_i, x_j), marked as "considered", has been is found. According to Theorem 5 this arc will belong to some odd cycle C_{2k+1} $(k \geq 2)$, the arcs of which $x_l > x_i$ and $x_i > x_j$ are generators for the elementary fictitious arc $x_l \gg x_j$ of graph $\vec{G}_t^{(z)}(X^0)$.

In this case in the digraph $\vec{G}^{(z)}(X^0) = \vec{G}(X^0)$ we will introduce the additional vertex x', which we will call *the co-vertex* of the disclosed cycle C_{2k+1} $(k \geq 2)$. With the co-vertex x' we will connect the $(k+1)$ vertices of cycle such that the introduced co-vertex and remaining vertices of cycle C_{2k+1} form an independent set. We assert that the vertex x' is joined by arcs with all such vertices of the cycle C_{2k+1}, which belong to the initiating set X^0, as well as with x_i, which is initial vertex for the leading arc $x_i > x_j$ of the graph $\vec{G}_t^{(z)}(X^0) = \vec{G}_t(X^0)$. The vertex x' will be terminal for such arcs.

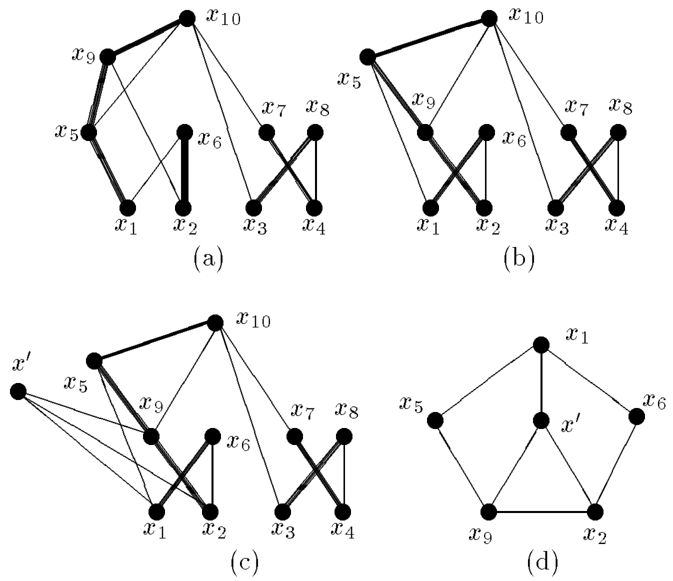

Figure 2: Introducing of co-vertex

The process of introducing a co-vertex in a digraph is illustrated in fig. 2 (a), (b) and (c). In fig. 2 (a) vertex-saturated digraph is shown in a graph of transitive closure in which the vertex x_9 is found to be marked. In fig. 2 (b) is shown a vertex-saturated digraph, obtained from the previous digraph by reorientation of the arc (x_5, x_9), in a graph of transitive closure for which the vertex x_5 was found to be marked.

Now the leading arc (x_9, x_5) is marked as "considered", for the odd cycle C_5, determined by the vertices x_1, x_2, x_5, x_6 and x_9, the co-vertex x' (see fig. 2 (c)) is introduced. We see that the required odd cycle is easily identified by imposing the MPPs of the graphs of transitive closure for the digraphs, shown in fig. 2 (a) and 2 (b). The undirected cycle C_5 and the introduced co-vertex are separately represented in fig. 2 (d). It is not difficult to see that the undirected cycle C_{2k+1} ($k \geq 2$) of the digraph $\mathbf{G}(X^0)$ to which arc (x_i, x_j) belongs, is enclosed in the cycle C_{2k+2}, formed by vertices of cycle C_{2k+1} and the introduced co-vertex in the extended digraph $\mathbf{G}''(X^0)$, and the digraph $\mathbf{G}''(X^0)$ will never be vertex-saturated.

Let $X_1 \subset X(C_{2k+1})$ be the set of all vertices of cycle C_{2k+1}, adjacent to the introduced co-vertex.

Further, we will connect co-vertex x' by arcs to all vertices of the digraph $\vec{G}(X^0)$, with which one vertex from the set X_1 is connected.

As a result of these constructions, we obtain the extended digraph $\vec{G}'(X^0) = (X', \vec{\Gamma}')$, where $X' = X \cup \{x'\}$.

Clearly, in the extended digraph $\vec{G}'(X^0)$, the undirected cycle C_{2k+1} $(k \geq 2)$ of the digraph $\vec{G}(X^0)$, to which arc (x_i, x_j) belongs, is inserted in the cycle C_{2k+2}, formed by the vertices of the cycle C_{2k+1} and introduced co-vertex, and the digraph $\vec{G}'(X^0)$ will never be vertex-saturated.

Lemma 7. *In the undirected graph $G' = (X', \Gamma') \in L_{n'}$, which is obtained from the graph $G = (X, \Gamma) \in Ln$ by the introduction in the latter some co-vertex x' of the cycle C_{2k+1} $(k \geq 2)$, for any vertex x of this cycle, adjacent to the co-vertex, the following correlation exists:*

$$\Gamma' x \backslash X(C_{2k+1}) \subset \Gamma' x' \backslash X(C_{2k+1}).$$

The result follows directly from the definition of a co-vertex.□

Theorem 8. *Let $G = (X, \Gamma) \in L_n$. Further, let the graph $G' = (X', \Gamma')$ be obtained from the graph G by the introduction of some co-vertices. Then the MCP $\mathbf{F}'$ of the graph G' with $Card(\mathbf{F}) = Card(\mathbf{F}')$ can be found for any MCP $\mathbf{F}$ of the graph G.*

Let the graph $G = (X, \Gamma) \in L_n$ and the MCP $\mathbf{F}$ of the graph G be given. Further, let the graph $G' = (X', \Gamma')$ be obtained from the graph G by the introduction of some co-vertices.

Clearly, the relation $Card(\mathbf{F}') > Card(\mathbf{F})$ is true for any MCP $\mathbf{F}'$ of the graph G'.

We will now show that any MCP of the graph G' can be obtained from any MCP $\mathbf{F}$ of the graph G by the connection of cliques from $\mathbf{F}$ to existing co-vertices.

In fact, by the definition, the co-vertex x' is introduced in the graph G, when the odd cycle C_{2k+1} $(k \geq 2)$ is found. The co-vertex x' is connected by edges with $(k+1)$ vertices of cycle C_{2k+1} so that it and the remaining k vertices of this cycle form the independent set. Then in MCP $\mathbf{F}'$ there will be at least one vertex x of cycle C_{2k+1} adjacent to the co-vertex x'. Therefore, by Lemma 7, if the co-vertex x' is connected to the clique MCP $\mathbf{F}'$, to which the vertex x of cycle C_{2k+1} belongs, then we obtain a new clique of the extended graph G'.

After executing similar connections of all co-vertices of the graph G' to cliques MCP $\mathbf{F}$, we obtain MCP $\mathbf{F}'$ of the graph G' such that $Card(\mathbf{F}') = Card(\mathbf{F}')$.□

Theorem 9. *A deadlocked digraph $\vec{G}^{(h)}(X^0)$ can be constructed for any vertex-saturated digraph $\vec{G}(X^0)$ as result of a sequential construction of RSD $\mathcal{T}_1, \ldots, \mathcal{T}_t$ and, if necessary, by introduction of a co-vertex into given the digraph.*

As a result of a introduction of a co-vertex the resulting odd cycle is enclosed in the even cycle C_{2k+1} $(k \geq 2)$, the validity of the assertion follows from Theorem 6.□

5 Algorithm CLIPAT

Let $\vec{G}^{(h)}(X^0)$ be a deadlocked digraph.

Because the deadlocked digraph $\vec{G}^{(h)}(X^0)$ is a directed mapping of some extended graph $G' = (X', \Gamma') \in L_{n'}$, where $n' > n$, then the MPP $\mathcal{P}^{(h)}$ of graph $\vec{G}_t^{(h)}(X^0)$ determines, according to Lemma 3, a MCP F' of graph G'.

The following statement defines main property of a deadlocked digraph.

Theorem 10. *Let $\vec{G}^{(h)}(X^0)$ be a deadlocked digraph. Then for any independent set $U \subset X$ of the graph $G \in L_n$, in the digraph $\vec{G}^{(h)}(X^0)$ there exist $Card(U)$ non-intersecting cliques of graph G, each of which contains one of the vertices of the set U and belongs to one of the paths of MPP $\mathbf{A}^{(h)}$ of the graph $\vec{G}_t^{(h)}(X^0)$.*

Let $\vec{G}(h)(X^0)$ be a deadlocked digraph.

Clearly, if $U \subset X$ is some independent set of vertices of the graph $G = (X, \Gamma) \in L_n$, according to the construction of a deadlocked digraph, the vertices of set U belong to various paths of a MPP $\mathcal{P}(h)$ of the graph $\vec{G}_t^{(h)}(X^0)$. We take into account, that if in each path of a MPP $\mathcal{P}^{(h)}$ of the graph $\vec{G}_t^{(h)}(X^0)$, passing through a vertex of set U, we delete a co-vertex, then we will obtain set with $Card(U)$ disjoint from the cliques of the graph G, each of which contains some vertex of the set U.$\square$

Theorem 11. *Let there be graph $G \in L_n$ and $\vec{G}^{(h)}(X^0)$ is a deadlocked digraph, obtained as result of construction of some system of a RSD. Then in the graph $\vec{G}_t^{(h)}(X^0)$ there exists such MPP $\mathcal{P}$, in which each clique of a MCP of graph G belongs to different its paths and $\mathcal{B}(\mathcal{P}) = \oslash$.*

Let $\mathbf{F}$ be some MCP of the graph $G \in L_n$.

Further, let $\vec{G}_t^{(h)}(X^0)$ be a graph of transitive closure of the deadlocked digraph $\vec{G}(h)(X^0)$, in which some MPP $\mathcal{P}^{(h)}$ has vertices not marked.

In the graph $\vec{G}_t^{(h)}(X^0)$ we will designate vertices, belonging to different cliques from MCP $\mathbf{F}$. Obviously, the designated vertices will form a set P of non-intersecting oriented paths of the graph $\vec{G}_t^{(h)}(X^0)$, which contains all vertices of the graph G.

If $\vec{G}^{(h)}(X^0) \in D(G)$, then all paths from P form the required MPP $\mathcal{P}$ of graph $\vec{G}_t^{(h)}(X^0)$.

Otherwise we have $\vec{G}^{(h)}(X^0) \in D(\vec{G}')$, where $\vec{G} \subset \vec{G}'$ i. e. the deadlocked digraph is an extended mapping of the graph $\vec{G}'$.

We will construct a new MPP $\mathcal{P}$ of the graph $\vec{G}_t^{(h)}(X^0)$ in the following way.

We will assign to $\mathcal{P}$ paths from P, i.e. vertices of each path of P we believe belong to the same path of the constructed MPP $\mathcal{P}$. The vertices of the assigned paths form some subgraph $\vec{G}'_t$ of the graph $\vec{G}_t^{(h)}(X^0)$.

Further, in the obtained subgraph $\vec{G}'_t$ and subsequent subgraphs of the graph $\vec{G}_t^{(h)}(X^0)$ we will introduce its co-vertices one by one. For example, in the order in which they were introduced during design of the deadlocked digraph.

At any time we will attach an introduced co-vertex as expansion of such path of a constructed MPP $\mathcal{P}$, with which this co-vertex will form a clique. By Theorem 8 it is always possible.

The process of introduction of co-vertices is finished when the list of co-vertices are exhausted, and the next constructed subgraph $\vec{G}''_t$ coincides with the graph $\vec{G}_t^{(h)}(X^0)$. Clearly, the resulting MPP $\mathcal{P}$ has no marked vertices.□

Theorem 12. *The general number of co-vertices of the graph $\vec{G}_t^{(h)}(X^0)$ is equal to $O(n)$.*

This result follows from Theorem 11.□

The obtained results make it possible to formulate an algorithm CLIPAT (CLIque PArTition) for finding the minimal partition of the graph $G \in L_n$ into cliques.

We will formulate the main steps of the algorithm CLIPAT.

Let there be a graph $G = (X, \Gamma) \in L_n$, for example, given by adjacency matrix.

Step 1. Orient the edges of the graph $G = (X, \Gamma) \in L_n$ so as to obtain the acyclic digraph $\vec{G}(X^0)$, where X^0 is the MIS of graph G.

The result of step 1 is an acyclic digraph $\vec{G}(X^0) = (X, \vec{\Gamma})$.

Step 2. Construct a vertex-saturated digraph $\vec{G}(X^0)$ from the given acyclic digraph.

Step 3. Find the marked vertex $x_j \in \mathcal{B}(\mathcal{P})$ of minimum rank in the graph of transitive closure $\vec{G}_t(X^0)$ of the vertex-saturated digraph $\vec{G}(X^0)$ for the constructed MPP $\mathcal{P}$. If $B(\mathcal{P}) = \oslash$ then go to Step 8. Otherwise, we have the marked vertex $x_j \in \mathcal{B}(\mathcal{P})$ of minimum rank.

Step 4. Find a leading arc (x_i, x_j) in the graph $\vec{G}_t(X^0)$ by the given marked vertex $x_j \in \mathcal{B}(\mathcal{P})$.

Step 5. If the leading arc (x_i, x_j) is marked as "considered" then go to Step 7. Otherwise, reorient arc (x_i, x_j) and mark arc (x_j, x_i) as "considered" in the new digraph $\vec{G}(X^0)$. Go to Step 6.

Step 6. If the new digraph $\vec{G}(X^0)$ is vertex-saturated then return to Step 3. Otherwise, remove all marks of arcs in the digraph and go to Step 2.

Step 7. Find the undirected odd cycle ${}_{2k+1}$ $(k \geq 2)$ of digraph $\vec{G}(X^0)$ and introduce a co-vertex x' for it. As the result we obtain the new digraph $\vec{G}(X^0)$. Go to Step 2.

Step 8. If the digraph $\vec{G}(X^0)$ is deadlocked and does not have co-vertices, then stop the calculations, as the required partition **F** of graph G into the minimum number of cliques has been found. Each element of MCP $\mathbf{F}$ is formed by the vertices of a corresponding path of MPP $\mathcal{P}$ of graph $\vec{G}_t(X^0)$. If the deadlocked digraph has even one co-vertex, go to Step 9.

Step 9. Remove all co-vertices in the deadlocked digraph $\vec{G}(X^0)$. Paths of MPP $\mathcal{P}$ of the graph $\vec{G}_t(X^0)$, intersecting remaining vertices, determine the required MCP of the graph $G = (X, \Gamma) \in L_n$.

Theorem 13. *The algorithm CLIPAT finds a partition of the graph $G \in L_n$ into the minimum number of cliques within time $O(n^5)$.*

First of all we notice that the algorithm CLIPAT constructs some deadlocked digraph and consequently, according to Theorem 11, finds some MCP $\mathbf{F}$ of the graph $G = (X, \Gamma) \in L_n$.

Obviously the operating time of the algorithm is determined by the time of design of deadlocked digraph.

We will evaluate an operating time for each step of the algorithm CLIPAT.

Step 1 is executed once and requires $O(n^2)$ time units, where n is number of vertices of the graph $G \in L_n$, since it is necessary to orient each edge of the graph G.

For reception of a vertex-saturated digraph it is necessary, that each of its subgraphs, induced by the layer X^k $(k = \overline{0, m})$, be vertex-saturated with respect to this layer. Therefore it is necessary for each subgraph $\vec{G}(X^k)$:

(a) Find the graph of transitive closure $\vec{G}_t(X^k)$. It is necessary for this purpose to expend $O(n_k^3)$ units of the time [5], where n_k is the number of vertices in the graph $\vec{G}_t(X^k)$.

(b) To design a MPP of the graph $\vec{G}_t(X^k)$, using an algorithm of search of the maximum matching in bipartite graph with $2n_k$ vertices, it is necessary to expend $O(n_k^{5/2})$ units of the time [4].

(c) Find the maximum antipath for each vertex $x \in X^k \cup \cdots \cup X^m$. The time of search of one antipath is equal to $O(n_k^2)$ – the time of design of an alternating tree. Therefore a total time to complete this step is equal to $O(n_k^3)$.

(d) Check, whether each of found the maximum antipaths $\mathcal{U}(x)$ a graph $\vec{G}_t(X^k)$ will be a MIS of a digraph $\vec{G}(X^k)$. If it so, to finish design of a digraph $\vec{G}(X^k)$, vertex-saturated with respect to initiating set X^k. Otherwise to fill a found antipath to a MIS and to construct a new acyclic digraph $\vec{G}(X^k)$ and to return to item (a).

Items (a) – (d) can be repeated. One completion of these steps requires $O(n_k^3)$ time units. Assuming, that for each completion of the steps an increase of the maximum antipath at one vertex occurs, we obtain a time of design of a digraph $\vec{G}(X^k)$, vertex-saturated with respect to initiating set, is equal to $O(n_k^4$.

Hence the time of design of a correctly constructed vertex-saturated digraph $\vec{G}(X^0)$ is equal to:

$$\sum_{\forall n_k} O(n_k^4) = h,$$

where $n_0 + \cdots + n_m = n$.

The size h is maximal, if $n_0 = \cdots = n_m$. Hence we have:

$$h = O((m+1)(\frac{n}{m+1})^4) = O(n^5)$$

.

The time for completion Step 3 is equal to, obviously, $O(n^2)$.

The time for completion of Step 4 is equal to the time of the breadth-first search of the graph and is equal to $O(n^2)$.

Step 5 requires one unit of the time.

The time for completion of Step 6 is equal to the time of required for completion of steps (a) – (d) during the design of a vertex-saturated digraph. It is possible this time to believe is equal to $O(n^3)$.

The time of search of an odd cycle is equal to the time of depth-first search in a graph and is equal to $O(n^2)$. Assuming the time to input a co-vertex is equal to $O(n^3)$, we conclude, that the time for completion of Step 7 is equal to $O(n^3)$.

Obviously steps 3 – 7 can be repeated not more then n^2 times. Therefore the time of design of deadlocked digraph will be equal to $O(n^5)$.

Steps 8 – 9 will not change the estimation.□

References

[1] Garey M.R., Johnson D. S. Computers and Intractability (W.H.Freeman and Company, San Francisco, 1979).

[2] Ford L.R., Fulkerson D.R. Flows in Networks (Princeton University press, Princeton, New Jersey, 1962).

[3] Christofides N. Graph theory (Academic press, New York, London, San Francisco, 1975).

[4] Papadimitriou C.H., Steiglitz K. Combinatorial optimization: Algorithms and Complexity (Prentice-hall Inc., Engiewood Cliffs, New Jersey, 1982).

[5] Swamy M.N.S., Thulasiraman K. Graphs, Networks and Algorithms (A Wiley Interscience Publication, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1981).

[6] Tarjan R.E. Complexity of combinatorial algorithms, SIAM Rew., **20**, N 3, (1978) 457–491.

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SET-VALUED MAPS WITH FIXED AND COINCIDENCE POINTS

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Abstract. In this paper, we prove the existence of solutions of functional equations $f_ix \in Sx \cap Tx$ and $x = f_ix \in Sx \cap Tx$ under certain nonlinear hybrid contraction and asymptotic regularity conditions, generalize and improve a recent result due to V. Popa concerning common fixed points of multi-valued mappings weakly commuting with a single-valued mapping and satisfying a generalized contraction type condition. In the sequel, we derive several independent results.

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I. Introduction

In [16], Nadler proved a fixed point theorem for multi-valued contraction, commonly known as Nadler's contraction principle. Subsequently, a large number of generalizations of Nadler's contraction principle appeared in Čirič [1], Khan [10], Kubiak [12], [13], Kaneko [8], [9], Sessa [22]–[24], Meade-Singh [14], Smithon [27], Wegrzyk [28] and in many others.

Recently, non-linear hybrid contractions, that is, contraction types involving single-valued and multi-valued mappings have been studied by Khan et al. [11], Mukherjee [15], Naimpally et al. [17], Popa [18], Rhoades et al. [19] and Sessa et al. [24], [25].

In this note, we show the existence of solutions of functional equations $f_ix \in Sx \cap Tx$ and $x = f_ix \in Sx \cap Tx$ under certain independent non-linear hybrid contraction and asymptotic regularity conditions where $\{f_i\}$ is a family of single-valued mappings on a metric space, S and T are multi-valued mappings on a metric space. Our results are generalizations and improvement of some results of O. Hadžić, H. Kaneko, M.S. Khan et al., T. Kubiak, R.N. Mukherjee, S.A. Naimpally et al., B.E. Rhoades et al. and many others. In the sequel, we also obtain other related results by using the proximinality of sets.

II. Preliminaries

Let (X, d) be a metric space. A subset K of X is said to be proximinal if for each $x \in X$, there exists a point $y \in K$ such that $d(x, y) = D(x, K)$, where $D(x, A)$ denotes the ordinary distance between $x \in X$ and a non-empty subset A of X.

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We shall use the following notation and definitions:

$$\begin{aligned}
CL &= \{A : A \text{ is a non-empty closed subset of } X\},\\
CB(X) &= \{A : A \text{ is a non-empty closed and bounded subset of } X\},\\
C(X) &= \{A : A \text{ is a non-empty compact subset of } X\} \text{ and}\\
P_x(X) &= \{A : A \text{ is a non-empty proximinal subset of } X\}.
\end{aligned}$$

For $A, B \in CL(X)$ and $\varepsilon > 0$,
$N(\varepsilon, A) = \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A\}$,
$E_{A,B} = \{\varepsilon > 0 : A \subseteq N(\varepsilon, B) \text{ and } B \subseteq N(\varepsilon, A)\}$, and

$$H(A, B) = \begin{cases} inf\, E_{A,B}, & \text{if } E_{A,B} \neq \varnothing \\ +\infty & \text{if } E_{A,B} = \varnothing. \end{cases}$$

H is called the generalized Hausdorff distance function for $CL(X)$ induced by the metric d, and H defined on $CB(X)$ is said to be the Hausdorff-Pompeiu metric induced by d.

It is well known that $P_x(X) \subset CL(X)$ and $C(X) \subset P_x(X)$ [22].

Let $\{f_i\}$ be a family of single-valued mappings from X to itself and S, T be multi-valued mappings from X to the non-empty subsets of X.

Definition 2.1.
If, for $x_\circ \in X$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = f_{2n} x_{2n-1} \in S x_{2n} \text{ for every } n \in N,$$

and

$$y_{2n+1} = f_{2n+1} x_{2n} \in T x_{2n+1} \text{ for every } n \in N_\circ = N \cup \{0\},$$

then $0_{f_i}(x_\circ) = \{y_n : n = 0, 1, 2, \ldots\}$ is said to be the orbit for $(S, T : f_i)$ at $x_\circ$. Further, $0_{f_i}(x_\circ)$ is called a regular orbit for $(S, T; f)$ if

$$d(y_n, y_{n+1}) \leq \begin{cases} H(Sx_{n-1}, Tx_n), & \text{if } n \text{ is odd,} \\ H(Tx_{n-1}, Sx_n), & \text{if } n \text{ is even.} \end{cases}$$

Definition 2.2.
If, for $x_\circ \in X$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that every Cauchy sequence of the form $0_{f_i}(x_\circ)$ converges in X, then X is called $(S, T; f_i)$ -orbitally complete with respect to $x_\circ$ or simply $(S, T; f_i, x_\circ)$ -orbitally complete. If for each $i \in N$, f_i is an identity mapping on X, then $0_{f_i}(x_\circ)$ is denoted by $0(x_\circ)$ and $(S, T; f_i, x_\circ)$ -orbitally completeness by $(S, T; x_\circ)$ -orbital completeness.

Definition 2.3.
A pair (S, T) is said to be asymptotically regular at $x_\circ \in X$ if for any sequence $\{x_n\}$ in X and each sequence $\{y_n\}$ in X such that

$$y_n \in Sx_{n-1} \cup Tx_{n-1},\ \lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

We remark that Definitions 2.1–2.3 with $S = T$ and $f_i = f$ for each $i \in N$ reduce to Definitions 4, 6 and 7 of B.E. Rhoades et al. [15], respectively, and orbital completeness need not imply the completeness of the space. Evidently, a complete space is orbitally complete.

Definition 2.4. For each $i \in N$, the mappings f_i and S are said to be commuting at a point $x \in X$ [3] if $f_i Sx \subseteq Sf_i x$. The mappings f_i and S are said to be commuting on X if $f_i Sx \subseteq Sf_i x$ for every point $x \in X$.

In [22], S. Sessa introduced the concept of weak commutativity for single-valued mappings on a metric space, and Sessa et al. [25] extended this concept to the setting of a single-valued mapping and a multi-valued mapping on a metric space.

Definition 2.5.

For each $i \in N$, the mappings f_i and S are said to be weakly commuting at $x \in X$ if $H(f_i Sx,\, Sf_i x) \leq D(f_i x, Sx)$. The mappings f and S are said to be weakly commuting on X if they are weakly commuting at every $x \in X$.

It may be noted that commutativity implies weak commutativity, but the converse need not be true even in the case of single-valued mappings as shown in [20], [22]–[25] by suitable examples.

Let $\mathcal{F}$ be the family of all mappings $\varnothing$ from the set R^+ of non-negative real numbers to itself such that $\varnothing$ is upper semicontinuous, non-decreasing and $\varnothing(t) < t$ for any $t > 0$.

The existence of coincidence points of non-linear hybrid contractions have been shown by Popa [18] by generalizing the result of Hadžić [2]. In doing so, Popa used Hausdorff-Pompeiu metric as used by Hadžić. It may be remarked that the result of Popa [18] is independent of the main result of Kubiak [12].

Theorem 2.1 [18]. Let (X, d) be a complete metric space, S and T continuous mappings from X into X, A and B closed mappings from X into $CB(SX \cap TX)$ such that $ASx = SAx$, $BTx = TBx$, for every $x \in X$ and

$$(2.1)\qquad \begin{aligned} H^2(Ax, By) \leq{} & h \max\{d(Sx, Ax).\, d(Ty, By),\ d(Sx, By).\, d(Ty, Ax), \\ & \tfrac{1}{2}[d(Sx, Ax).d(Sx, By) + d(Ty, Ax).\, d(Ty, By)],\ d^2(Sx, Ty)\} \end{aligned}$$

for every $x, y \in X$ where $h \in (0, 1)$.

Then there exists a sequence $\{x_n\}_{n \in N}$ in X such that:

1. for every $n \in N, Sx_{2n+1} \in Bx_{2n}$, $Tx_{2n} \in Ax_{2n-1}$.
2. There exists
$$z = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sx_{n+1}.$$
3. $Sz \in\ Az$, $Tz \in Bz$.

On the other hand, Rhoades et al. [19] deal with the same existence problem by using generalized Hausdorff distance function:

Theorem 2.2 [19].

Let T be a multi-valued mapping from a metric space X into $CL(X)$. If there exists a mapping f from X into itself such that $T(X) \subseteq f(X)$, for each $x, y \in X$ and $\phi \in \mathcal{F}$,

(2.2) $H(Tx, Ty) \leq \phi(\max(D(fx, Tx),\ D(fy, Ty),\ D(fx, Ty),\ D(fy, Tx),\ d(fx, fy))$

(2.3) $\phi(t) \leq qt$ for each $t > 0$ and for some $0 < q < 1$,

(2.4) there exists a point $x_o \in X$ such that T is asymptotically regular of x_o and $f(X)$ is $(T;\ f, x_o)$–orbitally complete, then f and T have a coincidence point in X.

It is interesting to note that the commuting mappings f (not the identity mapping) and T satisfying the hypothesis of Theorem 2.2 need not have a common fixed point in X.

In [19], Rhoades et al. gave the solutions of the question "What additional conditions will guarantee the existence of a common fixed point of f and T?"

The main objective of this paper is to investigate different sets of conditions under which the fixed point equation

$$x = f_i x \in Sx \cap Tx \text{ for } x \in X \text{ and each } i \in N$$

possesses a solution.

Moreover, we observe that the following conditions are independent:

(a) There exists a constant q, $0 < q < 1$ with $\phi(t) < qt$ for each $t > 0$ such that
$$H(Sx, Ty) \leq \phi(\max(D(fx, Sx),\ D(gy, Ty),\ D(fx, Ty),\ D(gy, Sx),\ d(fx, gy))).$$

(b) There exists a constant q, $0 < q < 1$ with $\phi(t) < qt$ for each $t > 0$ such that
$$\begin{aligned} H^2(Sx, Ty) \leq{} & \phi(\max(D(fx, Sx).\, D(gy, Ty),\ D(fx, Ty).\, D(gy, Sx), \\ & \tfrac{1}{2}[D(fx, Sx).D(fx, Ty) + D(gy, Sx).\ D(gy, Ty)], d^2(fx, gy))). \end{aligned}$$

Example 1.

Let $X = \{1, 2, 3, 4\}$. Define a metric d for X and mappings f, g, S and T on X as follows:

$$\begin{aligned} d(1,4) &= \frac{5}{2},\ d(1,2) = d(1,3) = \frac{3}{2},\ d(2,3) = \frac{4}{5}, \\ d(3,4) &= \frac{7}{4},\ d(2,4) = 1, \\ S(1) &= \{3\},\ S(2) = S(3) = S(4) = \{4\}, \\ T(1) &= T(2) = T(4) = \{4\},\ T(3) = \{2\},\ \text{and } f = g = id. \end{aligned}$$

The f, g, S and T satisfy condition (a) for each q, $\frac{14}{15} \le q < 1$.

For $x = 1$ and $y = 2$

$$\begin{aligned} &H^2(S(1),\, T(2)) = D^2(S(1),\, T(2)) = \frac{49}{16} > \phi(\max(D(1,\, S(1).\, D(2,\, T(2)), \\ &D(1,\, T(2)).\, D(2,\, S(1)),\, \tfrac{1}{2}[D(1,\, S(1)).\, D(1,\, T(2)) + D(2,\, S(1)). \\ &D(2,\, T(2))],\, d^2(1,2)))) = \frac{91}{40}. \end{aligned}$$

Thus (b) is not satisfied.

Example 2.

Let $X = \{1, 2, 3\}$. Define a metric d for X by

$$d(1,2) = 2,\ d(2,3) = \frac{3}{2},\ d(1,3) = 2.$$

Let f, g, S, and T be the mappings on X such that

$$\begin{aligned} &S(1) = S(2) = S(3) = \{1\},\ T(1) = T(3) = \{1\}, \\ &T(2) = \{3\},\ f = g = id. \end{aligned}$$

Then f, g, S and T satisfy condition (b) for each q, $\frac{9}{16} \le q < 1$.

For $x = 1$ and $y = 2$ by (a)

$$\begin{aligned} &H(S(1),\, T(2)) = D(S(1),\, T(2)) = 2 \le \phi(\max(D(1,\, S(1)), D(2,\, T(2)), \\ &D(1,\, T(2)),\, D(2,\, S(1)),\, d(1,2))) = \phi(2) \le 2q \end{aligned}$$

which implies $q \le 1$. Thus (a) is not satisfied.

III. Main Results

Now we are ready to give our main theorems:

Theorem 3.1.

Let S and T be multi-valued mappings from a metric space X into $P_X(X)$ and let $\{f_i\}$ be a family of all continuous mappings from X into itself such that

(3.1) $S(X) \cup T(X) \subset f_i(X)$ for each $i \in N$,

(3.2)

$$\begin{aligned} H^2(Sx, Ty) \le &\phi(\max(D(f_i x, Sx).\ D(f_j y, Ty),\ \ D(f_i x, Ty).\ D(f_j y, Sx), \\ &\tfrac{1}{2}[D(f_i x,\, Sx).\ D(f_i x,\, Ty) + D(f_j y,\, Sx).\ D(f_j y,\, Ty)], \\ &d^2(f_i x,\, f_j y))), \end{aligned}$$

for each $x, y \in X, i, j \in N,\ i \neq j$ and $\phi \in \mathcal{F}$,

(3.3) $\phi(t) \leq qt$ for each $t > 0$ and for some $q \in (0,1)$,

(3.4) there exists a point $x_o \in X$ such that the pair (S, T) is asymptotically regular at x_o, and

(3.5) for each $i \in N$, $f_i(x)$ is (S, T, f_i, x_o)–orbitally complete.

Then (1) f_i, S and T have a coincidence point in X. If z is a coincidence point of f_i, S and T and $f_i z$ is a fixed point of f_i, then we have;

(2) $f_i z$ is also a fixed point of S provided f_i is weakly commuting with S at z for every even $i \in N$ for any odd $j \in N$.

(3) $f_i z$ is also a fixed point of T provided f_i is weakly commuting with T at z.

(4) $f_i z$ is a common fixed point of S and T provided f_i is weakly commuting with each of S and T at z.

Proof.

(1) Let x_o be a point in X satisfying (3.4). Since $S(X) \cup T(X) \subset f_i(X)$, for each $i \in N$, we can find sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$
\begin{aligned}
y_{2n} &= f_{2n} x_{2n-1} \in S x_{2n} \text{ for every } n \in N, \\
y_{2n+1} &= f_{2n+1} x_{2n} \in T x_{2n+1} \text{ for every } n \in N_\circ, \\
d(y_{2n},\, y_{2n+1}) &\leq q^{-1/4} H(S x_{2n},\, T x_{2n+1}) \qquad \text{and} \\
d(y_{2n+1},\, y_{2n+2}) &\leq q^{-1/4}\ H(T x_{2n+1},\, S x_{2n+2}).
\end{aligned}
$$

By (3.4), we have

$$\lim_{n \to \infty} d(y_n, y_{n+1}) = 0.$$

Now we wish to show that $\{y_{2n}\}$ is a Cauchy sequence in $f_i(X)$. Suppose that the sequence $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists a positive number $\in$ such that, for each positive integer $2k$, there exist integers $2n(k)$ and $2m(k)$ such that

(3.6) $2k \leq 2n(k) < 2m(k)$ and

(3.7) $d[y_{2n(k)},\, y_{2m(k)}) \geq \in$.

Let $d_{i,j} = d(y_i, y_j)$ and $d_i = d(y_i, y_{i+1})$ for each $i, j \in N$. Then, for each integer $2k$, we have

(3.8)
$$\in \leq d_{2n(k),\,2m(k)} \leq d_{2n(k),\,2m(k)-2} + d_{2m(k)-2} + d_{2m(k)-1}.$$

For each integer $2k$, let $2m(k)$ denote the smallest integer satisfying (2.1) and (2.2). Then we have $d_{2m(k),\,2m(k)-2} < \in$ and it follows from (2.3) that

(3.9)
$$\lim_{k \to \infty} d_{2n(k),2m(k)} = \in .$$

Using the triangle inequality, we get

$$
\begin{aligned}
&|d_{2n(k),\,2m(k)} - d_{2n(k),\,2m(k)}| \leq d_{2m(k)-1} \qquad \text{and} \\
&|d_{2n(k)+1,\,2m(k)-1} - d_{2n(k),\,2m(k)}| \leq d_{2n(k)} + d_{2m(k)-1},
\end{aligned}
$$

which yield

$$\lim_{k \to \infty} d_{2n(k),\,2m(k)-1} = \lim_{k \to \infty} d_{2n(k)+1,\,2m(k)-1} = \in$$

in view of (3.4) and (3.9) and so by (3.2), we have

$$
\begin{aligned}
d_{2n(k),\,2m(k)} \leq & d_{2n(k)} + d_{2n(k)+1,2m(k)} \\
\leq & d_{2n(k)} + q^{-1/4}\, H(Sx_{2m(k)},\, Tx_{2n(k)+1}) \\
\leq & d_{2n(k)} + q^{-1/4}\{\phi(\max(D(f_{2m(k)+1}\, x_{2m(k)},\, Sx_{2m(k)}). \\
& D(f_{2m(k)+2}\, x_{2m(k)+1},\, Tx_{2m(k)+1}),\, D(f_{2m(k)+1} \\
& x_{2m(k)},\, Tx_{2n(k)+1}).\, D(f_{2m(k)+2},\, x_{2m(k)+1},\, Sx_{2m(k)}), \\
& \tfrac{1}{2}[D(f_{2m(k)+1}\, x_{2m(k)},\, Sx_{2m(k)}).\, D(f_{2m(k)+1}\, x_{2m(k)}, \\
& Tx_{2m(k)+1} + D(f_{2m(k)+2}\, x_{2m(k)+1},\, Sx_{2m(k)}), \\
& D(f_{2m(k)+2}\, x_{2m(k)+1},\, Tx_{2m(k)+1})],\, d^2(f_{2m(k)+1}\, x_{2m(k)}, \\
& f_{2m(k)+2}\, x_{2m(k)+1})))\}^{\frac{1}{2}} \\
\leq & d_{2n(k)} + q^{-1/4}\{\phi(\max(d_{2m(k)},\, d_{2m(k)+1}, \\
& d_{2n(k)+1,\,2m(k)+1}\, d_{2m(k)+z,\,zm(k)}, \\
& \tfrac{1}{2}[d_{2m(k)}.\, d_{2m(k)+1} + d_{2m(k)+2,\,2m(k)}. \\
& d_{2m(k)+1}],\, d^2_{2m(k)+1})))\}^{\frac{1}{2}}.
\end{aligned}
$$

Using the semicontinuity of ϕ and letting $k \to \infty$, this inequality yields

$$
\in \leq q^{-1/4}\{\phi(\ ^2)\}^{\frac{1}{2}} \leq q^{-1/4}\,(q^2)^{\frac{1}{2}} < \in
$$

Since $\in > 0$ and $q^{1/4} < 1$, which contradicts the choice of $\in$ and so the sequence $\{y_{2n}\}$ is a Cauchy sequence. Similarly, we can prove that $\{y_{2n+1}\}$ is also a Cauchy sequence in $f_i(X)$.

Since $f_i(X)$ is $(S, T,\, f_i,\, x_\circ)$-orbitally complete, the Cauchy sequence $\{y_{2n}\}$ has a limit u in $f_1(X)$ for each $i \in N$. By condition (3.4), we have

$$
0 = \lim_{n\to\infty} d(y_{2n},\, y_{2n+1}) = \lim_{n\to\infty} d(u,\, y_{2n+1})
$$

and so $\{y_{2n+1}\}$ also converges to u.

Hence there exists a point $z \in X$ such that $f_i z = u$. By condition (3.2), for any even $i \in N$,

$$
\begin{aligned}
D(f_i z,\, Sz) \leq & d(f_i z,\, y_{2n+1}) = D(y_{2n+1},\, Sz) \\
\leq & d(f_i z,\, y_{2n+1} + H(Sz,\, Tx_{2n+1}) \\
\leq & d(f_i z, y_{2n+1}) + \{\phi(\max(D(f_i z,\, Sz).\, D(f_{2n+2}\, x_{2n+1}, \\
& Tx_{2n+1}), D(f_i z,\, Tx_{2n+1}).\, D(f_{2n+2}\, x_{2n+1},\, Sz), \\
& \tfrac{1}{2}[D(f_i z,\, Sz).\, D(f_i z,\, Tx_{2n+1}) + D(f_{2n+2}\, x_{2n+1},\, Sz). \\
& D\,(f_{2n+2}\, x_{2n+1},\, Tx_{2n+1})],\, d^2(f_i z,\, f_{2n+2}\, x_{2n+1})))\}^{\frac{1}{2}} \\
\leq & d(f_i z,\, y_{2n+1}) + \{\phi(\max(D(f_i z,\, Sz).\, D(y_{2n+2},\, y_{2n+1}), \\
& D(f_i z,\, y_{2n+1}).\, D(y_{2n+2},\, Sz),\, \tfrac{1}{2}[D(f_i z,\, Sz). \\
& D(f_i z,\, y_{2n+1}) + D(y_{2n+2},\, Sz).\, D(y_{2n+2},\, y_{2n+1})], \\
& d^2(f_i z,\, y_{2n+2})))\}.
\end{aligned}
$$

Letting $n \to \infty$, this inequality yields

$$
D(f_i z, Sz) \leq \{\phi(\max(0, 0, 0, 0))\}^{\frac{1}{2}} = 0.
$$

If $f_i z \in Sz$ for any even $i \in N$, then $D(f_i z,\, Sz) > 0$ and so the above inequality yields a contradiction.

Hence $f_i z \in Sz$ for every $i \in N$ since every proximinal set is closed. Similarly, we have $f_i z \in Tz$ for any odd $i \in N$. Therefore, z is a coincidence point of f_i, S and T for each $i \in N$.

(2) If for any even $i \in N$, $u = f_i z$ is a fixed point of f_i, then $u = f_i u = f_i f_i z \in f_i Sz$. If f_i is weakly commuting with S at z, then $f_i Sz = Sf_i z$ since $f_i z \in Sz$. Therefore, we have $u \in f_i Sz = Sf_i z = Su$; that is, u is a fixed point of S.

(3) It follows by the same techniques in (2) and by (2) and (3), we have (4).

Remark.

Theorem 3.1 is true even if we replace $P_X(X)$ by $CL(X)$.

By replacing the condition (3.1) of Theorem 3.1 by the orbital regularity, we have the following:

Theorem 3.2.

Let S and T be multi-valued mappings from a metric space X into $P_X(X)$ and let $\{f_i\}$ be the family of all continuous mappings from X into itself such that the conditions (3.2) holds,

(3.10) $\phi(t) < t$ for each $t > 0$ and some $\phi \in \mathcal{F}$.

(3.11) for some $x_\circ \in X$, and each $i \in N$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that the orbit $0_{f_i}(x_\circ)$ is regular, the pair (S, T) is asymptotically regular at $x_\circ$ and $f(X)$ is $(S, T, f_i, x_\circ)$–orbitally complete. Then we have:

(1) for each $i \in N$, f_i, S and T have a coincidence point in X,

(2) if the limit of $0_{f_i}(x_\circ)$ is a fixed point of f_i, then all the conclusions of (2) and (3) of Theorem 3.1 are also true.

It is well known that if S is a multi-valued mapping from X into $C(X)$, then for every $x, y \in X$ and $u \in Sx$, there exists a point $v \in Sy$ such that

$$d(u, v) \leq H(Sx, Sy).$$

It follows that if S and T are multi-valued mappings from X into $C(X)$, then the orbital regularity in Theorem 3.2 can be dropped. Thus we have the following:

Theorem 3.3. Let S and T be multi-valued mappings from a metric space X into $C(X)$ and let $\{f_i\}$ be a family of continuous mappings from X into itself such that the conditions (3.1), (3.2), (3.4), (3.5) and (3.6) hold. Then all the conclusions of Theorem 3.2 are also true.

In Theorem 3.1, if we replace (3.2) by the following condition

$$\begin{aligned} H^2(Sx, Ty) \leq &\phi(\max(D(f_i x, Sx).\, D(f_i y, Ty),\, D(f_i x, Ty).\, D(f_j y, Sx), \\ &\tfrac{1}{2}[D(f_i x, Sx).\, D(f_i x, Ty) + D(f_j y, Sx).\, D(f_i y, Ty)], \\ &d^2(f_i x, F_j y))) \end{aligned} \tag{3.8}$$

for each $x, y \in X$, $i, j \in N$, $i \neq j$, and $\phi \in \mathcal{F}$, then the asymptotic regularity of (S, T), (3.4) is not needed. In fact, we have the following:

Theorem 3.4. Let S and T be multi-valued mappings from a metric space X into $P_x(X)$ and let $\{f_i\}$ be a family of continuous mappings from X into itself such that the conditions (3.1), (3.5) and (3.8) hold. Then all the conclusions of Theorem 3.1 are also true.

In Theorem 3.1 and 3.3, taking $f = f_i$ on X for each $i \in N$ and defining $\phi(t) = qt$, $o < q < 1$, we have the following:

Corollary 3.5. Let S and T be multi-valued mappings from a metric space into $P_x(X)$ and let f be a continuous mapping from X into itself such that the condition (3.4) holds:

$$S(X) \cup T(X) \subset f(X) \tag{3.9}$$

$$\begin{aligned} H^2(Sx, Ty) \leq &q \max(D(fx, Sx).\, D(fy, Ty),\, D(fx, Ty).\, D(fy, Sx), \\ &\tfrac{1}{2}[D(fx, Sx).\, D(fx, Ty) + D(fy, Sx).\, D(fy, Ty)],\, d^2(fx, fy)) \end{aligned} \tag{3.10}$$

for each $x, y \in X$ and for some $0 < q < 1$,

(3.11) $f(X)$ is $(S, T;\, f, x_\circ)$–orbitally complete, and

(3.12) f is weakly commuting with each of S and T.

Then f, S, and T have a common fixed point in X.

Corollary 3.6. Let S and T be multi-valued mappings from a metric space X into $P_X(X)$ and let f be a continuous mapping from X into itself such that the condition (3.9) and (3.12) hold, and

$$\begin{aligned} H^2(Sx, Ty) \leq & q \max(D(fx, Sx).\, D(fy, Ty),\ \tfrac{1}{2}[D^2(fx, Ty) + D^2(fy, Sx)], \\ & \tfrac{1}{2}[D(fx, Sx).\, D(fx, Ty) + D(fy, Sx).\, D(fy, Ty)],\ d^2(fx, fy))) \end{aligned} \tag{3.13}$$

For each $x, y \in X$ and for some $0 < q < 1$. Then f, S and T have a common fixed point in X.

By using the technique of Kubiak [13], we also have the following theorem.

Theorem 3.7. Let S, T be multi-valued mappings from a metric space X into $P_X(f(X) \cap g(X))$ and f, g be continuous mappings from X into itself such that the condition (3.12) and (3.13) hold.

Then f, g, S and T have a common fixed point in X.

Remark. Our results differ from the corresponding theorems of Hadžić [3] and Sessa et al. [24] to the version hybrid contraction mappings.

IV. Some More Related Results

Let X be a reflexive Banach space and $WC(X)$ denote the family of non-empty weakly compact subset of X. Recall that every non-empty weakly compact subset of a reflexive Banach space is proximinal [26] and hence closed.

Our main theorem in this section is prefaced with the following lemma:

Lemma 4.1 [14].

Let $f \in \mathcal{F}$ and $t_\circ > 0$. If $t_{n+1} \leq f(t_n)$ for all $n \in N$, then the sequence $\{t_n\}$ converges to 0.

The following theorem is a variant of results due to Kaneko [9] and Khan et al. [11].

Theorem 4.2.

Let X be a reflexive Banach space and let $\{S_n\}$, $\{T_n\}$ be sequences of mappings from X into $WC(X)$ such that

$$\begin{aligned} H^2(S_m x,\, T_n y) \leq & \phi(\max(D(x,\, S_m x).\, D(y,\, T_n y),\ D(x, T_n y).\, D(y,\, S_m x), \\ & \tfrac{1}{2} D(x,\, S_m x).\, D(x,\, T_n y),\ \tfrac{1}{2} D(y,\, S_m x).\, D(y,\, T_n y),\ d^2(x, y)) \end{aligned} \tag{4.1}$$

for each $x, y \in X$ and for some $\phi \in \mathcal{F}$. Then there exists a point $z \in X$ such that $z \in S_m z \cap T_n z$ for each $m, n \in N$.

Proof.

Firstly, we assume that $h = 0$. Let $x_\circ \in X$ be an arbitrary point. Then there exists a point $x_1 \in S_1 x_\circ$ such that $d(x_\circ, x_1) = D(x_\circ, Sx_\circ)$. This is guaranteed due to the fact that $S_1 x_\circ$ is a non-empty weakly compact subset of a reflexive Banach space and so it is proximinal. Now, for all $n \in N$, we have

$$D(x_1,\, T_n x_1) \leq H(S_1 x_\circ,\, T_n x_1) = 0$$

which implies that $x_1 \in T_n x_1$. Similarly, for all $n \in N$,

$$D(x_1,\, S_1 x_1) \leq H(T_n x_1,\, S_1 x_1) = 0$$

yields $x_1 \in S_n x_1$. Thus x_1 is a common fixed point of the two sequences $\{S_n\}$ and $\{T_n\}$.

Next, we assume that $h \neq 0$. Let $x_\circ \in X$ be arbitrary but fixed. Then we construct a sequence $\{x_n\}$ in X such that

$$x_{2n-1} \in S_n x_{2n-2}, x_{2n} \in T_{2n} x_{2n-1}, d(x_{2n-2}, x_{2n-1}) = D(x_{2n-2},\\ S_n x_{2n-2}), \text{ and } d(x_{2n-1}, x_{2n}) = D(x_{2n-1}, T_n x_{2n}).$$

The existence of such a sequence in X is guaranteed by the proximinality of $S_m(x)$ and $T_n(x)$ for each $m, n \in N$.

Now, suppose that $x_n = x_{n+1}$ for some $n \in N$. If $n \in N$ is even, then we have $x_{2n} \in S_{n+1}\, x_{2n}$. Further, for each $m \in N$, we have

$$\begin{aligned} D^2(x_{2n},\, T_m x_{2n}) \leq & H^2(S_{n+1} x_{2n},\, T_m x_{2n}) \\ \leq & \phi(\max(D(x_{2n},\, S_{n+1} x_{2n}).\, D(x_{2n},\, T_m x_{2n}),\, D(x_{2n},\, T_m x_{2n}). \\ & D(x_{2n},\, S_{n+1} x_{2n}), \tfrac{1}{2} D(x_{2n},\, S_{2n+1}\, x_{2n}).\, D(x_{2n},\, T_m x_{2n}), \\ & \tfrac{1}{2} D(x_{2n},\, S_{2n+1} x_{2n}).\, D(x_{2n},\, T_m x_{2n})\, d^2(x_2 n, x_{2n}))) \\ = & \phi(\circ), \end{aligned}$$

which yields $x_{2n} \in T_m x_{2n}$ for each $m \in N$ similarly, $D^2(x_{2n},\, S_m x_{2n}) \leq H^2(S_m x_{2n},\, T_n x_{2n}) \leq \phi(\circ)$ gives $x_{2n} \in S_m x_{2n}$ for each $m \in N$.

In case, when an odd integer $n \in N$, we have the same results.

Therefore, in each case, we have a common fixed point for the sequences $\{S_m\}$ and $\{T_n\}$.

Now suppose that $x_n \neq x_{n+1}$ for all $n \in N$. We shall show that $\{x_n\}$ is a Cauchy sequence in X. We observe that

$$\begin{aligned} d^2(x_{2n},\, x_{2n+1}) = & D^2(x_{2n}, S_{n+1} x_{2n}) \\ \leq & H^2(T_n\, x_{2n-1},\, S_{n+1} x_{2n}) \\ \leq & \phi(\max(D(x_{2n},\, S_{n+1} x_{2n}).\, D(x_{2n-1},\, T_n x_{2n-1}), \\ & D(x_{2n-1},\, T_n x_{2n-1}).\, D(x_{2n},\, S_{2n+1} x_{2n}), \\ & \tfrac{1}{2} D(x_{2n-1},\, S_{n+1} x_{2n}).\, D(x_{2n},\, S_{n+1} x_{2n}), \\ & \tfrac{1}{2} D(x_{2n-1},\, S_{n+1} x_{2n}).\, D(x_{2n-1},\, T_n x_{2n-1}), \\ & d^2(x_{2n-1},\, x_{2n}))) \\ = & \phi(\max(d(x_{2n}\;\, x_{2n+1}).\, d(x_{2n-1},\, x_{2n}),\, d(x_{2n-1}, x_{2n}). \\ & d(x_{2n},\, x_{2n+1}), \tfrac{1}{2} d(x_{2n-1},\, x_{2n+1}).\, d(x_{2n},\, x_{2n+1}), \\ & \tfrac{1}{2} d(x_{2n-1},\, x_{2n+1}).\, d(x_{2n-1},\, x_{2n}),\, d^2(x_{2n-1},\, x_{2n}))) \\ \leq & \phi(\max(d(x_{2n-1},\, x_{2n}).\, d(x_{2n},\, x_{2n+1}),\, \tfrac{1}{2}\{d(x_{2n-1},\, x_{2n}) \\ & + d(x_{2n},\, x_{2n+1})\}\, d(x_{2n},\, x_{2n+1}),\, \tfrac{1}{2}\{d(x_{2n-1}, x_{2n}) + \\ & d(x_{2n},\, x_{2n+1})\}\, d(x_{2n-1},\, x_m),\, d^2(x_{2n-1},\, x_{2n}))). \end{aligned}$$

If $d(x_{2n-1}, x_{2n}) < d(x_{2n}, x_{2n+1})$, then the above inequality yields

$$d^2(x_{2n},\, x_{2n+1}) \leq \phi(d^2(x_{2n},\, x_{2n+1})) < d^2(x_{2n},\, x_{2n+1}),$$

a contradiction. Therefore, we get

$$d^2(x_{2n}, x_{2n+1}) \leq \phi(d^2(x_{2n-1}, x_{2n})).$$

.

Similarly, we can show that

$$d^2(x_{2n+1},\, x_{2n+2}) \le \phi(d^2(x_{2n}, x_{2n+1})).$$

It follows from the above relation that

$$d^2(x_n, x_{n+1}) \le \phi(d^2(x_{2n}, x_{2n+1}))$$

for all $n \in N$. By Lemma 4.1, we have

$$\lim_{n\to\infty} d^2(x_n,\, x_{n+1}) = 0,$$

which in turn implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is a Banach space, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z.$$

Then we have

$$\begin{aligned} D^2(x_{2n-1}, T_m z) \le & H^2(S_n x_{2n-2},\, T_m z) \\ \le & \phi(\max(D(x_{2n-2},\, S_n x_{2n-2}).\, D(z,\, T_m z),\, D(x_{2n-2},\, T_m z). \\ & D(z,\, S_n x_{2n-2}),\, \tfrac{1}{2} D(x_{2n-2},\, S_n x_{2n-2}).\, D(x_{2n-2},\, T_m z), \\ & \tfrac{1}{2} D(z,\, S_n x_{2n-2}).\, D(z,\, T_m z),\, d^2(x_{2n-2},\, z))) \\ \le & \phi(\max(d(x_{2n-2},\, x_{2n-1}).\, D(z,\, T_m z),\, D(x_{2n-2},\, T_m z), \\ & d(z, x_{2n-1}),\, \tfrac{1}{2} d(x_{2n-2},\, x_{2n-1}).\, D(x_{2n-2},\, T_m z), \\ & \tfrac{1}{2} d(z,\, x_{2n-1}).\, D(z,\, T_m z),\, d^2(x_{2n-2} z))). \end{aligned}$$

Letting $n \to \infty$, the above inequality yields

$$D^2(z,\, T_m z) = 0$$

Since $T_m z$ is closed, $z \in T_m z$ for all $m \in N$. Similarly, we have also $z \in S_m z$ for all $m \in M$.
Therefore, z is a common fixed point of the sequences $\{S_n\}$ and $\{T_n\}$. This completes the proof.

In the last two decades, a large number of fixed point theorems appeared for commuting and weakly commuting mappings (see, for instance, [5]–[7] and [19]–[24]). By using the technique of Khan et al. [11], we have the following results.

Corollary 4.3.
Let f, g, S and T be single-valued mappings from a complete metric space X into itself such that

$$\begin{aligned} d^2(fx, gy) \le & \phi(\max(d(fx, Sx),\, d(gy, Ty), d(fx, Ty).\, d(gy, Sx), \\ & \tfrac{1}{2}[d(fx, Sx).\, d(fx, Ty) + d(gy, Sx).\, d(gy, Ty)]\, d^2(Sx, Ty))), \end{aligned} \tag{4.2}$$

for each $x, y \in X$ and for some $\phi \in \mathcal{F}$, and

$$\{f, S\} \text{ and } \{g, T\} \text{ are weakly commuting pairs.} \tag{4.3}$$

Then f, g, S and T have a unique common fixed point in X.

Corollary 4.4.

Let S and T be two continuous mappings from a complete metric space X into itself. S and T have a common fixed point in X if and only if there are two self-mappings A and B on X such that

$$A(X) \cup B(X) \subset S(X) \cap T(X) \tag{4.4}$$

$$\begin{aligned} d^2(Ax, Bx) \leq & \phi(\max(d(Sx, Ax), d(Ty, By), d(Sx, By).\, d(Ty, Ax), \\ & \tfrac{1}{2}[d(Sx, Ax).\, d(Ty, Ax) + d(Sx, By).\, d(Ty, By)], d^2(Sx, Ty))), \end{aligned} \tag{4.5}$$

for each $x, y \in X$ and for some $\phi \in \mathcal{F}$, and

$$\{A, S\} \text{ and } \{B, T\} \text{ are weakly commuting pairs.} \tag{4.6}$$

Further, z is the unique common fixed point of A, B, S and T.

Theorem 4.5.

Let X be a compact metric space and let S, T be multi-valued mappings from X into $P_X(X)$ such that either S or T is continuous,

$$\begin{aligned} H^2(Sx, Ty) \leq & \phi(\max(d(x, Sx)\, d(y, Ty), d(y, Sx)\, d(x, Ty), \tfrac{1}{2} d(x, Sx) \\ & d(x, Ty), \tfrac{1}{2} d(y, Sx)\, d(y, Ty), d^2(x, y))) \end{aligned} \tag{4.7}$$

for each $x, y \in X$, $x \neq y$, and for some $\phi \in \mathcal{F}$. Then either S or T has a fixed point in X.

References

1. Čiri č Lj. B., *Fixed points for generalized multivalued contractions*, Math. Vesnik **9 (24)** (1972), 265–272.
2. Hadžić O., *A coincidence theorem for multivalued mappings in metric space*, Studia Univ. Babes-Bolyai, Mathematica XXVI **4** (1981), 65–67.
3. Hadžić O., *Common fixed point theorems for family of mappings in complete metric spaces*, Math. Japon. **29** (1984), 127–134.
4. Itoh S. and Takahashi W., *Single-valued mappings, Multi-valued mappings and fixed point theorems*, J. Math. Anal. Appl. **59** (1977), 261–263.
5. Jungck G., *Commuting mapping and fixed points*, Amer. Math. Monthly **83** (1976), 261–263.
6. Jungck G., *Periodic and fixed points, and commuting mappings*, Proc. Amer. Math. Soc. **76** (1979), 333–338.
7. Jungck G., *Common fixed points for commuting and comatible maps on compacts*, Proc. Amer. Math. Soc. **103** (1988), 977–983.
8. Kaneko H., *Single-valued and multi-valued f-contractions*, Boll. Un. Math. Ital. **6, 4-A** (1985), 29-33.
9. Kaneko H., *A comparison of contractive conditions for multi-valued mappings*, Kobe J. Math **3** (1986), 37–45.
10. Khan M. S., *Common fixed point theorems for multi-valued mappings*, Pacific J. Math **95** (1981), 337–347.
11. Khan M. S., Y. J. Cho, W. T. Part and T. Mumtaz, *Coincidence and common fixed points of hybrid contraction*, J. Austral. Math. Soc.
12. Kubiak T., *Two coincidence theorems for contractive type multi-valued mappings*, Studia Univ. Babes-Bolyai, Mathematica XXX (1985), 65–68.
13. Kubiak T., *Fixed point theorems for contractive type multi-valued mappings*, Math. Japon. **30** (1985), 89–101.
14. Meade B. A. and Singh S. P., *On common fixed point theorems*, Bull. Austral. Math. Soc. **16** (1977), 49–53.
15. Mukherjee R. N., *On fixed points of single-valued and set-valued mappings*, J. Indian Acad. Math. **4** (1982), 101–103.
16. Nadler S. B., *Multivalued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.
17. Naimpally S. A., Singh S. L. and Whitfield J. H. M., *Coincidence theorems for hybrid contractions*, Math. Nachr. **127** (1986), 177–180.
18. Popa V., *A coincidence theorem for multi-functions*, Review of Research, Univ. u Novom Sadu, Zb. Rad. Prirod–Mat. Fak. Ser. Mat., 18 **1** (1988), 149–156.
19. Rhoades B. E., Singh S. L. and Kulshrestha C., *Coincidence theorems for some multi-valued mappings*, Internat. J. Math. Sci. **7** (1984), 429–434.

20. Rhoades B. E. and Sessa S., *Common fixed point theorems for three mappings under a weak commutativity condition*, Indian J. Pure Appl. Math., 17 **1** (1986), 47–57.
21. Rhoades B. E., Sessa S., Khan M. S. and Swaleh M., *On fixed points of asymptotically regular mappings*, J. Austral. Math. Soc., Series A **43** (1987), 328–346.
22. Sessa S., *On a weak commutativity condition in fixed point considerations*, Publ. Inst. Math., 32 **46** (1982), 149–153.
23. Sessa S. and Fisher B., *On common fixed points of weakly commuting mappings and set-valued mappings*, Internat. J. Math. Soc., 9 **2** (1986), 323–329.
24. Sessa S., Mukherjee R. N. and Som T., *A common fixed point theorem for weakly commuting mappings*, Math. Japon., 31 **2** (1986), 235–245.
25. Sessa S. , Khan M. S. and Imdad M., *A common fixed point theorem with a weak commutativity condition*, Glasnik Mat., 21 **41** (1986), 225–235.
26. Singer I., *The theory of best approximation and functional analysis*, Regional Conference Series in Appl. Math, 13 SIAM (1974).
27. Simthon R. E., *Fixed points for contractive multifunctions*, Proc. Amer. Math. Soc. **27** (1971), 192–194.
28. Wegrzyk R., *Fixed point theorems for multi-valued functions and their applications to functional equations*, Dissertations Math., Warazawa **201** (1982), 1–28.

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When is a Poset Isomorphic to the Poset of Connected Induced Subgraphs of a Graph?

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Abstract

Suppose G is a finite connected graph. Let $C(G)$ denote the inclusion ordering on the connected vertex-induced subgraphs of G. Resolving a problem posed by Steve Steinsaltz, we characterize posets that arise in this way. In particular, we show that, for a finite poset $\mathbf{P}$, there exists a finite graph G such that $C(G) \cong \mathbf{P}$ if and only if $\mathbf{P}$ is atom-height and properly semimodular. We present a similar characterization for infinite posets.

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1 Introduction

Suppose G is a finite connected graph. Let $C(G)$ denote the inclusion ordering on the connected vertex-induced subgraphs of G.

Partially ordered sets arising in this manner have come under study within the last ten years [7, 9, 8, 10]. In recent work with Michael Jacobson [5] the authors proved that these posets need not be Sperner. In this paper we solve a problem posed by S. Steinsaltz [9] by characterizing those posets that arise as the inclusion ordering on the connected vertex-induced subgraphs of some graph. Our characterization links commonly studied properties of posets with commonly studied properties of graphs.

In the next section we give preliminary definitions and notation. In section three, we prove that for a finite poset $\mathbf{P}$, there exists a finite graph G such that $C(G) \cong \mathbf{P}$ if and only if $\mathbf{P}$ is atom-height and properly semimodular. In section four we integrate the familiar lattice concepts of meet semidistributivity and local distributivity into this characterization. In section five we extend the characterization to infinite posets.

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2 Preliminaries

A *partially ordered set* (or *poset*) $\mathbf{P} = (P, \leq)$ is a set P of elements together with a binary relation $\leq$ on P that is reflexive, antisymmetric, and transitive. An element $x \in P$ is an *upper bound* for a subset $S \subseteq P$ if $s \leq x$ for all $s \in S$. An element $x \in P$ is a *least upper bound* for $S \subseteq P$ if x is an upper bound for S and, for any upper bound $z \in P$ for S, $x \leq z$. *Lower bounds* and *greatest lower bounds* for S are defined dually. If a subset $S \subseteq P$ has a least upper bound (greatest lower bound) then we denote it by $\bigvee S$ ($\bigwedge S$). A *lattice* is a poset in which every pair of elements has a least upper bound and greatest lower bound. If arbitrary subsets of elements have least upper bounds and greatest lower bounds, then the lattice is *complete*. A lattice is *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, for all elements a, b, and c in the poset.

If the poset $\mathbf{P}$ has a unique least element, we will refer to it as 0. Given two elements a and b of a poset, if $a \leq b$ and $a \leq c \leq b$ implies $c \in \{a, b\}$, then we say that b *covers* a and write $a \prec b$ or $b \succ a$. If a poset $\mathbf{P}$ has a 0 and $0 \prec a \in P$, then a is said to be an *atom* of $\mathbf{P}$.

Two elements $a, b \in P$ are *comparable* if $a \leq b$ or $b \leq a$; otherwise they are *incomparable*. A subset of pairwise comparable elements is called a *chain*. A chain C is *saturated* if there does not exist a $z \in P - C$ such that $x \leq z \leq y$ for some $x, y \in C$. The *length* of a chain is one less than the number of elements in the chain. The *interval* between two comparable elements $a, b \in P$ is defined as $[a, b] = \{c \in P : a \leq c \leq b\}$. A poset is *graded* (or *ranked*) if, for all comparable pairs $a \leq b$, all saturated chains with least element a and greatest element b have the same length. In a finite, graded poset with 0, the *height* of the element p, denoted $\mathrm{ht}(p)$ is the length of a saturated chain from 0 to p.

Two posets $\mathbf{P}$ and $\mathbf{Q}$ are *isomorphic* if there exists an *order-preserving* bijection $\phi : P \to Q$; that is,

$$x \leq y \qquad \Leftrightarrow \qquad \phi(x) \leq \phi(y).$$

If $\mathbf{P}$ and $\mathbf{Q}$ are isomorphic, then we write $\mathbf{P} \cong \mathbf{Q}$.

A lattice $\mathbf{L} = (L, \leq)$ is *semimodular* if for any $a, b, c \in L$, $a \prec b$, $a < c$ and $b \not\leq c$ implies $c \prec b \vee c$. It is well-known that semimodular finite lattices are graded. The converse is not true.

Given an element p from a poset $\mathbf{P}$, the symbol $p\!\uparrow$ denotes the *principal filter* of $\mathbf{P}$ determined by p; that is, $p\!\uparrow = \{q \in P : p \leq q\}$. The set of atoms below an element $x \in P$ is denoted A_x. Hence $A_x = \{a \in P : a \leq x \text{ and } a \text{ is an atom of } \mathbf{P}\}$. A poset $\mathbf{P}$ is *atomic* if $x = \bigvee A_x$, for all $x \in P$.

We now give two definitions specific to this paper. A graded, atomic poset is *atom-height* if for all $x \in P$, $\mathrm{ht}(x) = |A_x|$. If $\mathcal{P}$ is a lattice property, then a poset is said to be *properly* $\mathcal{P}$ if every proper, principal filter is a lattice and satisfies $\mathcal{P}$.

3 A characterization of $C(G)$ posets

Suppose that $G = (V, E)$ is a (not necessarily finite) connected graph with vertex set V and edge set E. Let $C(G)$ denote the inclusion ordering on the connected vertex-induced subgraphs of G. In this section we characterize posets that arise as the inclusion ordering on the connected vertex-induced subgraphs of some finite graph.

Theorem 3.1 *Let* $\mathbf{P}$ *be a finite poset. The following are equivalent:*

(i) *There exists a finite, connected graph* G *such that* $\mathbf{P} \cong C(G)$*,*
(ii) $\mathbf{P}$ *is atom-height and properly semimodular.*

Our proof of Theorem 3.1 requires several lemmas.

We shall consistenly represent elements of $C(G)$ with lower case letters. Vertices of G are denoted by **bold** lower case letters. If $x \in C(G)$, the corresponding subset of V is denoted by V_x. In the special case when x is an atom of $C(G)$, $\mathbf{x}$ will denote the vertex of G corresponding to x. If $S \subset V(G)$, then the subgraph of G induced by S is denoted $G[S]$.

A few simple observations: for any $x \in C(G)$, $|V_x| = |A_x|$; and, $x \leq y$ in $C(G)$ if and only $V_x \subseteq V_y$. Suppose $0 \neq x \in C(G)$. For any y, z contained in $x\uparrow$, the induced subgraph $G[V_y \cup V_z]$ is connected because both $G[V_y]$ and $G[V_z]$ are connected, and $V_x \subseteq V_y \cap V_z$. Indeed, for any y, z contained in $x\uparrow$ (for $x > 0$), we have

$$\begin{aligned} V_{y \vee z} &= V_y \cup V_z & (1) \\ V_{y \wedge z} &= \max\{S : V_x \subseteq S \subseteq V_y \cap V_z \text{ and } G[S] \text{ is connected.}\} & (2) \end{aligned}$$

Observe that the maximum taken in (2) exists and does not depend upon the finiteness of G.

Lemma 3.2 *If* G *is a connected graph, then* $x \prec y$ *in* $C(G)$ *if and only if* $|V_y - V_x| = 1$ *and* $x \leq y$*.*

Proof. Suppose that $x \prec y$ in $C(G)$. Clearly $V_y \neq V_x$. If $|V_y - V_x| > 1$, then because V_y is connected and contains V_x, there must be some vertex $\mathbf{u}$ in G such that $H = V_x \cup \{\mathbf{u}\}$ induces a connected graph. No element of $C(G)$ could correspond to H because any such element $z \in C(G)$ must satisfy $x < z < y$, contradicting $x \prec y$. We conclude that $|V_y - V_x| = 1$.

Conversely, suppose that $|V_y - V_x| = 1$ and $x \leq y$. Any subset of V_y properly containing V_x must contain V_y. The result follows. □

Lemma 3.3 *If* G *is a finite, connected graph, then* $C(G)$ *is an atom-height poset.*

Proof. Suppose $u \in C(G)$. Because u is an upper bound for each element in A_u and $|V_u| = |A_u|$, it follows that $u = \bigvee A_u$. So $C(G)$ is atomic.

Suppose $x, y \in C(G)$ with $x < y$. By lemma 3.2, $x \prec y$ if and only if $|V_y - V_x| = 1$. By straight-forward induction on $|V_x|$, it also follows that any saturated chain from 0 to x has length $|V_x| = |A_x|$. In other words, $C(G)$ is graded, and for $x \in C(G)$, $\mathrm{ht}(x) = |A_x|$. This proves that $C(G)$ is atom-height. □

Lemma 3.4 *Suppose that G is a connected graph. For any $0 \neq x \in C(G)$, the principal filter generated by x is a complete semimodular lattice.*

Proof. Suppose $0 \neq x \in C(G)$. For any $\{x_i\}_{i\in I} \subseteq x\uparrow$, the join $\bigvee_{i\in I} x_i$ exists because $G\left[\bigcup_{i\in I} V_{x_i}\right]$ is connected. The presence of a unique least element in $x\uparrow$ (namely x) and the existence of arbitrary joins guarantees the existence of arbitrary meets: $\bigwedge_{i\in I} x_i = \bigvee\{y \in x\uparrow : y \leq x_i \text{ for all } i \in I\}$. Hence $x\uparrow$ is a complete lattice.

To show that $x\uparrow$ is semimodular, suppose that $i, j, k \in x\uparrow$ and $i \prec j$, $i \leq k$, $j \not\leq k$. Now $V_i \subseteq V_j$ and $V_i \subseteq V_k$. Because $i \prec j$, lemma 3.2 implies $|V_j - V_i| = 1$. From (1) we know that the vertex set of $j \vee k$ is $V_j \cup V_k$. Because $|V_j \cup V_k - V_k| = 1$, lemma 3.2 guarantees that $k \prec k \vee j$ as desired. $\square$

Observe that this last result actually does not depend upon the finiteness G. Indeed we shall use this result later in the case that G is infinite.

Consider a finite, atom-height, properly semimodular poset $\mathbf{P}$. From $\mathbf{P}$ construct a graph G_P whose vertices are the atoms of $\mathbf{P}$, and whose edges are pairs of distinct atoms $a, b \in P$ for which the join $a \vee b$ exists and has height two. In the following two lemmas, we show that there is a natural isomorphism $\mathbf{P} \cong C(G_P)$.

Lemma 3.5 *Suppose that $\mathbf{P}$ is an atom-height, properly semimodular poset. If $x \in P$, then $G_P[A_x]$ is connected.*

Proof. The proof is an induction on the $\mathrm{ht}(x)$. Recall that A_x is the set of atoms of $\mathbf{P}$ less than or equal to x. If $\mathrm{ht}(x) = 1$, then $A_x = \{x\}$ and so the subgraph of G_P induced by this set is clearly connected. Similarly, if $\mathrm{ht}(x) = 2$, then $|A_x| = 2$ because $\mathbf{P}$ is atom-height. Let $\{a, b\} = A_x$. Since $\mathbf{P}$ is atomic, $x = \bigvee A_x = a \vee b$. By definition of the graph G_P, $\mathbf{ab}$ is an edge of G_P; hence $G[\{\mathbf{a}, \mathbf{b}\}]$ is connected.

Suppose that the lemma is true for all elements of P with height at most n. Let $x \in P$ with $\mathrm{ht}(x) = n + 1 > 2$.

We claim that if $w \in P$ and $\mathrm{ht}(w) \geq 2$, then either w is the unique largest element of P or $|\{y \in P : y \prec w\}| \geq 2$. If not, a non-maximum element $w \in P$ exists with $\mathrm{ht}(w) \geq 2$ and $|\{y \in P : y \prec w\}| \leq 1$. Let y be the unique element covered by w. Now $A_w = A_y$, but because P is atomic $w = \bigvee A_w = \bigvee A_y = y$, contradicting $y \prec w$.

By the previous paragraph, we may assume that there are two distinct elements $w, y \in P$ such that $w \prec x$ and $y \prec x$. Because $\mathbf{P}$ is atom-height, $|A_w| = |A_y| = n$. By the induction hypothesis, the subgraphs of G_P induced by the vertex sets A_w and A_y are connected. Because $|A_x| = n + 1$, $A_w \cup A_y = A_x$ and $|A_w| = |A_y| = n > 2$, we conclude that $A_w \cap A_y \neq \emptyset$. Hence the subgraph of G_P induced by $A_w \cup A_y = A_x$ is connected. $\square$

Lemma 3.6 *Suppose that $\mathbf{P}$ is an atom-height, properly semimodular poset. If $\mathcal{S}$ is a set of atoms of $\mathbf{P}$ and $G_P[\mathcal{S}]$ is connected, then there exists some $x \in P$ such that $A_x = \mathcal{S}$.*

Proof. The proof is by induction on $|\mathcal{S}|$. If $|\mathcal{S}| = 1$, then $\mathcal{S} = \{a\}$, for some atom $a \in P$; hence $\mathcal{S} = A_a$. If $|\mathcal{S}| = 2$, then $\mathcal{S} = \{a, b\}$, for some pair of atoms $a, b \in P$ such that **ab** is an edge of G_P. Because **ab** is an edge of G_P, the element $a \vee b$ exists and has height 2. By the atom-height property, $A_{a \vee b} = \{a, b\} = \mathcal{S}$.

Suppose the lemma is true for all sets of atoms with at most $n \geq 2$ elements. Consider a set $\mathcal{S}$ of $n + 1$ atoms from P such that $G_P[\mathcal{S}]$ is connected. By considering a spanning tree of $G_P[\mathcal{S}]$, we can find two leaves of $G_P[\mathcal{S}]$, and hence we can find two atoms a_1, a_2 of P such that $G_P[\mathcal{S} - \{\mathbf{a}_1\}]$, $G_P[\mathcal{S} - \{\mathbf{a}_2\}]$, and $G_P[\mathcal{S} - \{\mathbf{a}_1, \mathbf{a}_2\}]$ are all connected graphs. Note that $n \geq 2$ implies that $G_P[\mathcal{S} - \{\mathbf{a}_1, \mathbf{a}_2\}]$ contains at least one vertex. By the induction hypothesis, there exist $x, y, z \in P$ such that $\mathcal{S} - \{a_1\} = A_x$, $\mathcal{S} - \{a_2\} = A_y$, and $\mathcal{S} - \{a_1, a_2\} = A_z$. Because **P** has the atom-height property, $\mathrm{ht}(x) = n = \mathrm{ht}(y)$, $z \prec x$, and $z \prec y$. In particular, $z \neq 0$ so the proper semimodularity of P implies $z\!\uparrow$ is a semimodular lattice. Hence, $x \prec x \vee y$, $y \prec x \vee y$, and $\mathrm{ht}(x \vee y) = n + 1 = |A_{x \vee y}|$. Observing that $A_x \subset A_{x \vee y}$ and $A_y \subset A_{x \vee y}$, we conclude that $A_{x \vee y} = \mathcal{S}$. $\square$

Theorem 3.1: *Proof.* By Lemma 3.3 and Lemma 3.4, if G is a finite connected graph, then $C(G)$ is an atom-height, properly semimodular poset.

Conversely, suppose that **P** is an atom-height, properly semimodular poset. By Lemma 3.5, the mapping $\phi : P \to C(G_P)$ defined by $\phi(x) = A_x$ is well-defined. By Lemma 3.6, the mapping ϕ is onto $C(G_P)$. Because **P** is atomic, $A_x = A_y$ implies that $x = y$ so ϕ is one-to-one. If $x \leq y$, then $A_x \subseteq A_y$. Conversely, if $A_x \subseteq A_y$, then $x \leq y$ because P is atomic. Therefore, $C(G_P) \cong \mathbf{P}$. $\square$

4 $SD_\wedge$ and Local Distributivity

Meet semidistributivite ($SD_\wedge$) and locally distributive lattices often arise when graph theory and combinatorics is the background setting (see [4, 6, 8]). We now integrate such lattices into our description of $C(G)$ posets.

A lattice $\mathbf{L} = (L, \leq)$ is *meet semidistributive* ($SD_\wedge$) if $a \wedge b = a \wedge c$ implies $a \wedge b = a \wedge (b \vee c)$, for all elements a, b, c, d from the lattice.

Lemma 4.1 *Let G be a connected simple graph. The poset $C(G)$ is properly $SD_\wedge$.*

Proof. Suppose that $0 \neq u \in C(G)$. We may assume that u is an atom of $C(G)$. Consider $u\!\uparrow$, and let $x, y, z \in u\!\uparrow$ such that $x \wedge y = x \wedge z$. We must prove that $x \wedge (y \vee z) = x \wedge y$, or equivalently $V_{x \wedge (y \vee z)} = V_{x \wedge y}$. The vertex set $V_{x \wedge y}$ is the maximal set in

$$\{S \subset V(G) : U \subseteq S \subseteq V_x \cap V_y \text{ and } G[S] \text{ is connected.}\}.$$

as noted in (2). Because $V_{x \wedge y} = V_{x \wedge z}$, it follows that the vertex set $V_{x \wedge y}$ is contained in $V_x \cap V_y \cap V_z$. Because of (1), the vertex set $V_{y \vee z}$ is $V_y \cup V_z$. This means that $V_{x \wedge (y \vee z)}$ is a subset of $V_x \cap (V_y \cup V_z) = (V_x \cap V_y) \cup (V_x \cap V_z)$.

Let $\{\mathbf{u}\} = V_u$, and let $\mathbf{v} \in V_{x\wedge(y\vee z)}$. By symmetry, we may assume that $\mathbf{v} \in V_x \cap V_y$. Because $\mathbf{u}$ and $\mathbf{v}$ are in $V_{x\wedge(y\vee z)}$, there must be a path from $\mathbf{u}$ to $\mathbf{v}$ using only vertices in this set. Let

$$\mathbf{u} = \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_n = \mathbf{v}$$

be the sequence of vertices in this path, where $\mathbf{v}_i \in (V_x \cup V_y) \cup (V_x \cap V_z)$ and $\mathbf{v}_i\mathbf{v}_{i+1} \in E(G)$ for each i. Note that both $\mathbf{u} = \mathbf{v}_1$ and $\mathbf{v} = \mathbf{v}_n$ are vertices of $V_x \cap V_y$.

The subgraph of $G[V_u \cup \{\mathbf{v}_2\}]$ is connected and contains $\mathbf{u}$. Suppose that w is the element of $u\!\uparrow$ in $C(G)$ corresponding to this induced subgraph. If $\mathbf{v}_2 \in V_x \cap V_y$, then it follows that $w \leq x \wedge y = x \wedge z$. In particular, $\{\mathbf{u}, \mathbf{v}_2\} \subseteq V_x \cap V_z$. We have shown that $\mathbf{v}_2 \in V_x \cap V_y$ implies that $\mathbf{v}_2 \in V_x \cap V_z$. By induction, it follows that $\mathbf{v} \in V_x \cap V_y \cap V_z$. That is, $V_{x\wedge(y\vee z)}$ is contained in $V_x \cap V_y \cap V_z$ and contains $\mathbf{u}$. Maximal among connected vertex subsets containing $\mathbf{u}$ and contained in $V_x \cap V_y \cap V_z$ is $V_{x\wedge y\wedge z} = V_{x\wedge y} = V_{x\wedge z}$. Hence, $x \wedge (y \vee z) \leq x \wedge y = x \wedge z$. Consequently, $x \wedge (y \vee z) = x \wedge y$, as desired. □

Let $\mathbf{L} = (L, \leq)$ be a lattice. For $a \in L$, define $C_a = \{b \in L : a \prec b\}$. Let $a^+ = \bigvee C_a$. If for all $a \in L$, the interval $[a, a^+]$ is distributive, then L is said to be *locally distributive.* (In the literature, locally distributive lattices are sometims called *upper locally distributive.* See the paper by Monjardet [6] for more information.) The following result is well-known.

Proposition 4.2 *If* $\mathbf{L}$ *is a finite lattice, then* $\mathbf{L}$ *is locally distributive if and only if* $\mathbf{L}$ *is both semimodular and* $SD_\wedge$*.*

Here is an extension of Theorem 3.3.

Theorem 4.3 *For a finite poset* $\mathbf{P}$ *the following are equivalent:*

- (i) *There exists a connected graph* G *such that* $\mathbf{P} \cong C(G)$*,*
- (ii) $\mathbf{P}$ *is an atom-height, properly semimodular poset,*
- (iii) $\mathbf{P}$ *is an atom-height, properly semimodular, and properly* $SD_\wedge$ *poset,*
- (iv) $\mathbf{P}$ *is an atom-height, and properly locally distributive poset.*

Proof. By Theorem 3.3, (i) and (ii) are equivalent. By Lemma 4.1 and Theorem 3.3, (i) implies (iii). Proposition 4.2 and Theorem 3.3 shows that (iv) implies (i). □

Using (1) and Lemma 3.2, one can show directly that, for any graph $C(G)$ is locally distributive. Infinte locally distributive lattices are not in general semimodular or $SD_\wedge$; hence, for infinite graphs, Lemmas 3.4 and 4.1 are not direct consequences of the local distributivity of $C(G)$ as they are in the finite case via Proposition 4.2.

5 The Infinite Case

In this section we characterize those infinite partial orders that arise as the inclusion ordering on the connected vertex-induced subgraphs of an infinite graph. It is natural to try to reduce the infinite case to the finite case via compactness. Here, compactness arises in the context of lattices, and we are able to describe $C(G)$ posets (G infinite), using Theorem 4.3 (the finite case) along with a compactness condition In this way we avoid having to deal with infinite cardinals.

By a connected infinite graph G we mean an infinite graph in which, for any pair of vertices u and v of G, there exists a finite connected subgraph of G containing u and v. A poset is *graphic* if it is isomorphic the inclusion ordering on the connected vertex-induced subgraphs of a graph.

It makes sense here to define the *height* of an element in a poset with a 0 to be the length of the *shortest* saturated chain from 0 to that element (Note that the usual definition of height of an element is the length of the *longest* saturated chain from 0 to that element). With this definition of height, we say that an infinite poset $\mathbf{P}$ with 0 is *finitely graphic* if for any element $a \in P$ with finite height, the interval $[0, a]$ is graphic.

Recall that a lattice $\mathbf{L} = (L, \leq)$ is *complete* if $\bigvee S$ and $\bigwedge S$ exists for all $S \subseteq L$. Suppose that $\mathbf{L}$ is a complete lattice. An element $k \in L$ is *compact* if, for every $S \subseteq L$, $k \leq \bigvee S$ implies that $k \leq \bigvee T$, for some finite subset $T \subset S$. A lattice $\mathbf{L}$ is *algebraic* if, for each $a \in L$,

$$a = \bigvee \{k \in L | k \text{ is compact and } k \leq a\}.$$

Now we are ready to state the characterization in the infinite case.

Theorem 5.1 *Suppose that* $\mathbf{P}$ *is a poset with* 0*. The following are equivalent:*

(i) $\mathbf{P} \cong C(G)$*, for some connected graph* G

(ii) $\mathbf{P}$ *satisfies the following three conditions:*

- (a) $\mathbf{P}$ *is finitely graphic*
- (b) $\mathbf{P}$ *is atomic*
- (c) *for* $0 < x \in P$*,* $x\!\uparrow$ *is an algebraic semimodular lattice whose compact elements are the elements of finite height.*

Proof. "$(i) \Rightarrow (ii)$" Suppose that $\mathbf{P}$ is a poset with 0, and G is a connected graph such that $\mathbf{P} \cong C(G)$. For an element $x \in P$ with finite height, the interval $[0, x]$ in P is isomorphic to $C(X)$, where X is the subgraph of G corresponding to x. Hence P is finitely graphic.

For an arbitrary element $0 < x \in P$, the connected subgraph of G corresponding to x has a vertex set that is simply the union of its vertices. Therefore P is atomic. The completeness and semimodularity of $x\!\uparrow$ follows from Lemma 3.4. Because every element of $x\!\uparrow$ corresponds to the vertex set of a connected subgraph of G containing x, and such vertex sets are unions of finite subgraphs containing x, if follows that the compact elements of $x\!\uparrow$ are precisely the elements of finite height. The filter $x\!\uparrow$ is algebraic for the same reasons.

"$(ii) \Rightarrow (i)$" Suppose that P is a poset with 0 satisfying conditions $(a), (b)$, and (c). Recall that, given a poset $\mathbf{P}$, we define the graph G_P whose vertices are the atoms of $\mathbf{P}$, and whose edges are pairs of distinct atoms $a, b \in P$ for which the join $a \vee b$ exists and has height two. We prove (ii) implies (i) by showing $\mathbf{P} \cong G_P$ in two steps. The first step is an analog of Lemma 3.5.

Step 1: For all $w \in P$, $G_P[A_w]$ is connected.

If w has finite height, then condition (a) implies that $G_P[A_w]$ is connected. Assume w does not have finite height. By condition (b), w is the join of atoms. In particular, w is in the filter of some atom a, so condition (c) guarantees that $w = \bigvee_{i \in I} p_i$, where the p_i's have finite height in $a\uparrow$. Consequently, the p_i's have finite height in $\mathbf{P}$. Let A_i be the set of atoms below p_i, for $i \in I$. Condition (a) guarantees that each $G_P[A_i]$ is connected. Hence,

$$G_P[A_w] = G_P[\bigcup_{i \in I} A_i]$$

The last subgraph is connected because it is the union of connected graphs containing the vertex $\mathbf{a}$.

The next step is an analog of Lemma 3.6.

Step 2: Suppose that C is a set of atoms in $\mathbf{P}$. If $G_P[C]$ is connected, then some $x \in P$ exists such that $A_x = C$.

Suppose first that $|C|$ is finite. We prove the result by induction on $|C|$. The basis cases are $|C| = 1, 2$. If $|C| = 1$, then $C = A_a$ where $C = \{a\}$. If $|C| = 2$, then $C = \{a, b\}$ for atoms $a \neq b$ of $\mathbf{P}$ such that $a \vee b$ exists and $a \vee b$ has height two. Condition (a) implies that $[0, a \vee b]$ is graphic and, consequently $C = A_{a \vee b}$. If $|C| \geq 3$, then $G_P[C]$ contains a vertex a such that C is the disjoint union of two sets C_1 and C_2 with $C_1 \cap C_2 = \{a\}$ and $G_P[C_i]$ is connected for $i = 1, 2$ (consider a spanning tree T of $G_P[C]$, and let b be a leaf T with neighbor a. Let $C_1 = \{a, b\}$ and let $C_2 = C - \{b\}$.) By the inductive hypothesis, $C_1 = A_w$ and $C_2 = A_z$, where $w, z \geq a$ and both w and z have finite height. Because $a\uparrow$ is semimodular, it follows that $w \vee z$ has finite height. Therefore $A_{w \vee z}$ contains C. Moreover, $G_P[C]$ is a connected subgraph of $G_P[A_{w \vee z}]$ so $C = A_{w \vee z}$.

Suppose now that $|C|$ is infinite. Let $a \in C$ and consider $a\uparrow$. Define

$$\mathcal{C} = \{S \subset C : |S| \text{ is finite, } G_P[S] \text{ is connected, } a \in S\}.$$

By the previous paragraph, for all $S \in \mathcal{C}$, an element $s \in P$ exists such that $S = A_s$ and $s \geq a$. Because $a\uparrow$ is complete, the join $\bigvee\{s \in P | A_s \in \mathcal{C} \text{ and } s \geq a\}$ exists; let $w \in P$ equal this join. We want to show $A_w = C$. Clearly $C \subseteq A_w$. It suffices to show $A_w \subseteq C$. Step 1 guarantees that $G_P[A_w]$ is connected. Furthermore, $G_P[A_w]$ contains C. Suppose, for a contradiction, that there exists some element $b \in A_w - C$. We may assume that $\mathbf{b}$ has a neighbor $\mathbf{x}$ in $G_P[A_w]$ such that $x \in C$. Consider the set $\{s \in P | A_s \in \mathcal{C} \text{ and } s \geq a \text{ and } s \geq x\} = \{w_i\}_{i \in I}$. Note that $\bigvee_{i \in I} w_i = w$ because every $s \in P$ such that $A_s \in \mathcal{C}$ and $s \geq a$ is below some w_i. By assumption, $b \in A_w$. Hence $b \leq w = \bigvee_{i \in I} w_i$. Consequently,

$$\begin{aligned} x \vee b &\leq x \vee \left(\bigvee_{i \in I} w_i \right) \\ &\leq \bigvee_{i \in I} (x \vee w_i) \end{aligned}$$

$$\le \bigvee_{i\in I} w_i \qquad \text{(because } x \le w_i, \text{ for all } i \in I)$$
$$\le \bigvee_{i\in J\subset I, |J|<\infty} w_i \qquad \text{(because } x{\uparrow} \text{ is compact)}$$
$$\le v$$

where $v = \bigvee\{w_i : i \in J\}$. The semimodularity of $x{\uparrow}$ ensures that v has finite height. Because v has finite height, it is graphic. Hence $A_v \in \mathcal{C}$, and $b \in C$. This shows that $A_w = C$.

The theorem now follows from steps 1 and 2 by the same reasoning as the proof of Theorem 2.1 (define a mapping $\phi : P \to C(G_P)$ via $\phi(x) = A_x$ and show it is an order-preserving bijection). $\square$

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References

[1] Anderson I., *Combinatorics of Finite Sets*, Oxford University Press, New York, (1987).

[2] Chartrand G. and Lesniak L. , *Graphs and Digraphs,* 2nd Edition, Wadsworth Inc., Belmont, California, 1985.

[3] Davey B. A., and Priestley H. A., *Introduction to Lattices and Order*, Cambridge University Press, (1990).

[4] Edelman P. H. and Jamison R., The theory of convex geometries, *Geom. Dedicata*, 19 (1985) 247-274.

[5] Jacobson M. S., Kézdy A. E., and Seif S., The poset on connected induced subgraphs of a graph need not be Sperner, Order, 12 (1995) 315-318.

[6] Monjardet B., The Consequences of Dilworth's Work on Lattices with Unique Irreducible Decompositions, in: *The Dilworth Theorems* (eds. K. P. Bogart, R. Freese, and J. P. S. Kung), Birhauser Verlag, (1990) 192-199.

[7] Steelman J. H., The Lattice of Connected Subgraphs of a Graph, solution to Problem E3281, *Amer. Math. Monthly,* 97 (1990) 924-925, proposed by S. Klavžar and M. Petkovšek, *Amer. Math. Monthly,* 95 (1988) 655.

[8] Trotter, W. T. , and Moore J.I., Some theorems on graphs and posets, *Discrete Math.*, 15 (1976) 79-84.

[9] West D. B., Open Problems, *Newsletter of the SIAM Activity Group on Disc. Math.,* Vol. 2 No. 3: Spring (1992).

[10] West D. B., Open Problems, *Newsletter of the SIAM Activity Group on Disc. Math.,* Vol. 4 No. 1: Fall (1993).

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COMMON FIXED POINTS FOR δ-COMPATIBLE MAPPINGS

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ABSTRACT. In this paper, using the condition of δ-compatibility between a set-valued mapping and a single-valued mapping due to Jungck and Rhoades [6], we generalize a common fixed point theorem of Khan and Kubiaczyk [7] in a complete metric space. Their Theorem extends well known results due to Chang [1], Khan et al. [8], and Singh-Whitfield [13]. We give an example to support our extension.

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1. Basic Preliminaries.

Let (X,d) be a metric space, and B(X) be the set of all nonempty, bounded subsets of X.

As in [3,4], let $\delta(A,B)$ be the function defined by

$$\delta(A,B) = sup\{d(a,b) : a \in A, b \in B\},$$

for all A, B in B(X).

If A consists of a single point a, we write $\delta(A,B) = \delta(a,B)$. If B also consists of a single point b, we write $\delta(A,B) = d(a,b)$.

It follows immediately from the definition that

$$\begin{aligned}
&\delta(A,B) = \delta(B,A) \geq 0,\\
&\delta(A,B) \leq \delta(A,C) + \delta(C,B),\\
&\delta(A,B) = 0 \quad iff \quad A = B = \{a\},\\
&\delta(A,A) = diamA,
\end{aligned}$$

for all A, B, C in B(X).

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Definition 1.1 [3]. A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if

(i) given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for n=1,2, ..., and a_n converges to a.

(ii) given $\epsilon > 0$ there exists a positive integer N such that $A_n \subseteq A_\epsilon$ for $n > N$ where A_ϵ is the the union of all open spheres with centers in A and radius ϵ.

Lemma 1.1 [3,4]. If $\{A_n\}$ and $\{B_n\}$ are sequences in B(X) converging to A and B in B(X) respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.

Lemma 1.2 [4]. Let $\{A_n\}$ be a sequence in B(X) and y be a point of X such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in B(X).

Definition 1.2 [4,12]. A set-valued mapping F of X into B(X) is said to be continuous at $x \in X$ if the sequence $\{Fx_n\}$ in B(X) converges to Fx whenever $\{x_n\}$ is a sequence in X converging to x in X. F is said to be continuous on X if it is continuous at every point of X.

The following Lemma was proved in [4].

Lemma 1.3. Let $\{A_n\}$ be a sequence of nonempty subsets of X and let $z \in X$ such that

$$\lim_{n\to\infty} a_n = z,$$

z being independent of the particular choice of each $a_n \in A_n$. If a selfmap I of X is continuous, then Iz is the limit of the sequence $\{IA_n\}$.

Definition 1.3 [12]. The mappings $F : X \to B(X)$ and $I : X \to X$ are said to be weakly commuting on X if $IFx \in B(X)$ and

$$\delta(FIx, IFx) \leq max\{\delta(Ix, Fx), \quad diamIFx\} \tag{1.1}$$

for all x in X.

Note that if F is a single-valued mapping, then the set $\{IFx\}$ consists of a single point and therefore diam IFx=0 for all $x \in X$ and condition (1.1) therefore reduces to the condition given by Sessa [11], that is

$$d(FIx, IFx) \leq d(Ix, Fx) \tag{1.2}$$

for all x in X.

Two commuting mappings F and I clearly commute but two weakly commuting F and I do not necessarily commute as shown in [12].

In a recent paper Jungck [5] made an extension of weak commutativity in the following way

Definition 1.4.

Two single-valued mappings f and g of a metric space (X, d) into itself. are *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some z in X.

It can be seen that two *weakly commuting mappings* are *compatible* but the converse is false. Examples supporting this fact can be found in [5].

In [6], Jungck and Rhoades extended the definition 1.4 of compatibility to a set-valued setting as follows.

Definition 1.5.

The mappings $I : X \to X$ and $F : X \to B(X)$ are *weakly δ-compatible* if they commute at coincidence points, i.e., for each point u such that Fu=Iu, we have FIu=IFu. (Note that the equation Fu=Iu implies that Fu is a singleton.)

Definition 1.6. The mappings $I : X \to X$ and $F : X \to B(X)$ are *δ-compatible* if $\lim_{n\to\infty} \delta(FIx_n, IFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $IFx_n \in B(X)$,

$$Fx_n \to \{t\} \quad Ix_n \to t$$

for some t in X.

Two *weakly commuting mappings* are *δ-compatible* but the converse is not true as is shown in the following example.

Example 1.1. Let $X = [0, \infty)$ with the Euclidean metric d and define

$$Ix = x^3 + 2x^2, \quad Fx = [0, x^2]$$

for all x in X. We have

$$Ix_n \to 0, \quad Fx_n \to \{0\} \quad if \quad x_n \to 0$$

and

$$\delta(FIx_n, IFx_n) \to 0 \quad as \quad x_n \to 0.$$

Also $IFX_n \in B(X)$, we see that I and F are δ-compatible. But I and F are not weakly commuting, hence are not commuting because for x=1, we have

$$\delta(FI1, IF1) = 9 > 3 = max\{\delta(I1, F1), diamIF1\},$$

and

$$FI1 = [0, 9] \neq [0, 3] = IF1.$$

The following proposition due to Jungck and Rhoades [6] is useful in the sequel.

Proposition 1.1. Let (X,d) be a complete metric space. Suppose $I : X \to X$ and $F : X \to B(X)$, and (I,F) is δ-compatible,

(1) Suppose that the sequence $\{Fx_n\}$ converges to $\{t\}$ and $\{Ix_n\}$ converges to t. If I is continuous, then $FIx_n \to \{It\}$.

(2) If $\{It\} = Ft$ for some $t \in X$, then FIt=IFt.

Throughout this paper, (X,d) stands for a complete metric space. Further Φ is the set of functions $\phi : [0, \infty)^5 \longrightarrow [0, \infty)$ which satisfies the following conditions

(i) ϕ is upper semi-continuous from the right and non-decreasing in each coordinate variable,

(ii) For each $t > 0$,

$$\Psi(t) = max\{\phi(t, t, t, t, t), \phi(t, t, t, 2t, 0), \phi(t, t, t, 0, 2t)\} < t.$$

The following Lemma is due to Matkowski [9].

Lemma 1.4. For every $t > 0$,

$$\Psi(t) < t \quad iff \quad \lim_{n\to\infty} \Psi^n(t) = 0.$$

M.S.Khan and I.Kubiaczyk [7] proved the following Theorem.

Theorem 1.1. Let $I : X \to X$ and $F, G : X \to B(X)$ such that
(I) I, F and G are continuous,

$$FX \subset IX \quad and \quad GX \subset IX, \tag{1.3}$$

$$IF = FI \quad and \quad IG = GI, \tag{1.4}$$

For all x, y in X

$$\delta(Fx, Gy) \leq \phi(d(Ix, Iy), \delta(Ix, Fx), \delta(Iy, Gy), D(Ix, Gy), D(Iy, Fx)), \tag{1.5}$$

where ϕ satisfies (i) and

$$\phi(t, t, t, at, bt) < t,$$

for each $t > 0$, and $a \geq 0$, $b \geq 0$ with $a + b \leq 2$. Then I, F and G have a unique common fixed point u such that $u = Iu \in Fu \cap Gu$.

Now there arises a natural question. Is it possible to replace the commutativity of mappings in Theorem 1.1 by the δ-compatibility of a pair of mappings (Definition 1.6) ? The answer is in the affirmative.

2.Main results.

As an improvement of Theorem 1.1 we have the following.

Theorem 2.1. Let $I, J : X \to X$ and $F, G : X \to B(X)$ be mappings such that

$$FX \subset JX \quad and \quad GX \subset IX, \tag{2.1}$$

$$\delta(Fx, Gy) \leq \phi(d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), D(Ix, Gy), D(Jy, Fx)), \tag{2.2}$$

for all $x, y \in X$ and $\phi \in \Phi$, either

(II) $\{F, I\}$ are δ -compatible, F or I is continuous and (G,J) are weakly δ -compatible, or

(III)$\{G, J\}$ are δ -compatible, G or J is continuous and $\{F, I\}$ are weakly δ -compatible.

Then I, J, F and G have a unique common fixed point u. Moreover $Fu = Gu = \{u\}$ is the unique common fixed point of F and I and of G and J.

Proof. Let $x_0 \in X$ be an arbitrary point in X. By (2.1) we choose a point x_1 in X such that $Jx_1 \in Fx_0 = Z_0$ and for this point x_1 there exists a point x_2 in X such that $Ix_2 \in Gx_1 = Z_1$, and so on. Continuin
g in this manner we can define a sequence $\{x_n\}$ as follows:

$$Jx_{2n+1} \in Fx_{2n} = Z_{2n+1}, Ix_{2n+2} \in Gx_{2n+1} = Z_{2n+2}. \tag{2.3}$$

For simplicity, we Put $V_n = \delta(Z_n, Z_{n+1})$, for n=0, 1, 2 ... By (2.2) and (2.3), we have

$$\begin{aligned}
&V_{2n+1} = \delta(Z_{2n+1}, Z_{2n+2}) = \delta(Fx_{2n}, Gx_{2n+1}) \\
&\le \phi(d(Ix_{2n}, Jx_{2n+1}), \delta(Ix_{2n}, Fx_{2n}), \delta(Jx_{2n+1}, Gx_{2n+1}), \\
&D(Ix_{2n}, Gx_{2n+1}), D(Jx_{2n+1}, Fx_{2n})) \\
&\le \phi(V_{2n}, V_{2n+1}, V_{2n}, V_{2n} + V_{2n+1}, 0).
\end{aligned}$$

If $V_{2n+1} > V_{2n}$, then we have

$$V_{2n+1} \le \phi(V_{2n+1}, V_{2n+1}, V_{2n+1}, 2V_{2n+1}, 0) \le \Psi(V_{2n+1}) < V_{2n+1},$$

which ia a contradiction. Thus,

$$V_{2n+1} \le \phi(V_{2n}, V_{2n}, V_{2n}, 2V_{2n}, 0) \le \Psi(V_{2n}). \tag{2.4}$$

Similarly, we have

$$V_{2n+2} \le \Psi(V_{2n+1}). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$V_n \le \psi(V_{n-1}) \le ... \le \Psi^n(V_0), \tag{2.6}$$

where

$$V_0 = \delta(Z_0, Z_1) = \delta(Fx_0, Gx_1) \le \phi(V_0, V_0, V_0, 2V_0, 0).$$

By (2.6) and Lemma 1.4 we obtain

$$\lim_{n\to\infty} V_n = 0. \tag{2.7}$$

Let z_n be an arbitrary point in Z_n for n=0, 1, 2, ... Then

$$\lim_{n\to\infty} d(z_n, z_{n+1}) \le \lim_{n\to\infty} \delta(Z_n, Z_{n+1}) = 0.$$

In order to show that $\{z_n\}$ is a Cauchy sequence, it is sufficient to show that $\{z_{2n}\}$ is a Cauchy sequence. Suppose that $\{z_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that, for each even integer 2k, there exist even integers 2m(k) and 2n(k) such that

$$\delta(Z_{2m(k)}, Z_{2n(k)}) > \epsilon \tag{2.8}$$

for $m(k) > n(k) \ge k$.

For each even integer 2k, let 2m(k) be the least even integer exceeding 2n(k) satisfying (2.8), that is

$$\delta(Z_{2n(k)}, Z_{2m(k)-2}) \le \epsilon \quad and \quad \delta(Z_{2n(k)}, Z_{2m(k)}) > \epsilon \tag{2.9}$$

Then for each even integer 2k,

$$\epsilon < \delta(Z_{2n(k)}, Z_{2m(k)}) \le \delta(Z_{2n(k)}, Z_{2m(k)-2}) + V_{2m(k)-2} + V_{2m(k)-1}.$$

The foregoing inequality implies

$$\lim_{n\to\infty}\delta(Z_{2n(k)}, Z_{2m(k)}) = \epsilon. \tag{2.10}$$

Also by the triangle inequality, we have

$$|\delta(Z_{2n(k)}, Z_{2m(k)-1}) - \delta(Z_{2n(k)}, Z_{2m(k)})| \leq V_{2m(k)-1},$$

and

$$|\delta(Z_{2n(k)+1}, Z_{2m(k)-1}) - \delta(Z_{2n(k)}, Z_{2m(k)})| \leq V_{2m(k)-1} + V_{2n(k)}.$$

Using (2.10) we get

$$\delta(Z_{2n(k)}, Z_{2m(k)-1}) \to \epsilon \tag{2.11}$$

and

$$\delta(Z_{2n(k)+1}, Z_{2m(k)-1}) \to \epsilon. \tag{2.12}$$

From (2.2) and (2.3), we have

$$\begin{aligned}
&\delta(Z_{2n(k)}, Z_{2m(k)}) \leq V_{2n(k)} + \delta(Fx_{2n(k)}, Gx_{2m(k)-1})\\
&\leq V_{2n(k)} + \phi(d(Ix_{2n(k)}, Jx_{2m(k)-1}), \delta(Ix_{2n(k)}, Fx_{2n(k)}),\\
&\delta(Jx_{2m(k)-1}, Gx_{2m(k)-1}), D(Ix_{2n(k)}, Gx_{2m(k)-1}), D(Jx_{2m(k)-1}, Fx_{2n(k)}))\\
&\leq V_{2n(k)} + \phi(\delta(Z_{2n(k)}, Z_{2m(k)-1}), V_{2n(k)}, V_{2m(k)-1},\\
&\delta(Z_{2n(k)}, Z_{2m(k)}), \delta(Z_{2m(k)-1}, Z_{2n(k)+1})).
\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\epsilon \leq \phi(\epsilon, 0, 0, \epsilon, \epsilon) \leq \Psi(\epsilon) < \epsilon,$$

which is a contradiction. Therefore the sequence $\{z_n\}$ is a Cauchy sequence in X and it has a limit u in X. So the sequences $\{Ix_{2n}\}$ and $\{Jx_{2n+1}\}$ converge to u and further, the sequences of sets $\{Fx_{2n}\}$ and $\{Gx_{2n+1}\}$ converge to the set $\{u\}$.

Now supose that I is continuous. Then

$$I^2x_{2n} \to Iu, IFx_{2n} \to \{Iu\}$$

by Lemma 1.3. But I and F are δ-compatible. Therefore $FIx_{2n} \longrightarrow \{Iu\}$ by proposition 1.1 (1).

Using inequality (2.2), we have

$$\begin{aligned}
&\delta(FIx_{2n}, Gx_{2n+1}) \leq \phi(d(I^2x_{2n}, Jx_{2n+1}), \delta(I^2x_{2n}, FIx_{2n}),\\
&\delta(Jx_{2n+1}, Gx_{2n+1}), D(I^2x_{2n}, Gx_{2n+1}), D(Jx_{2n+1}, FIx_{2n})).
\end{aligned}$$

for $n \in N$. As $n \to \infty$ we obtain by Lemma 1.1

$$d(Iu, u) \leq \phi(d(Iu, u), 0, 0, d(Iu, u), d(Iu, u)) \leq \Psi(d(Iu, u)) < d(Iu, u)$$

which is a contradiction. Thus Iu=u.

Further

$$\delta(Fu, Gx_{2n+1}) \leq \phi(d(Iu, Jx_{2n+1}), \delta(Iu, Fu),$$
$$\delta(Jx_{2n+1}, Gx_{2n+1}), D(Iu, Gx_{2n+1}), D(Jx_{2n+1}, Fu)).$$

Letting $n \longrightarrow \infty$, it follows by Lemma 1.1 that

$$\begin{aligned}\delta(Fu, u) &\leq \phi(0, \delta(Fu, u), 0, 0, \delta(u, Fu)) \\ &\leq \Psi(\delta(u, Fu)) < \delta(u, Fu),\end{aligned}$$

which is a contradiction. So $Fu = \{u\}$.

Since $F(X) \subset J(X)$, there exists a point $w \in X$ such that $\{u\} = Fu = \{Jw\}$.

We shall now show that $Gw = \{Jw\}$. From (2.2),

$$\delta(Fu, Gw) \leq \phi(d(Iu, Jw), \delta(Iu, Fu), \delta(Jw, Gw), D(Iu, Gw), D(Jw, Fu)),$$

or

$$\delta(Jw, Gw) \leq \phi(\delta(Jw, Gw)),$$

which implies that $Gw = \{Jw\}$. Thus Gw is a singleton and w is a coincidence point for G and J.

Since they are weakly δ -compatible, GJw=JGw.

It then follows that $\{Ju\} = JGw = GJw = Gu$. Using (2.2),

$$\delta(Fu, Gu) \leq \phi(d(Iu, Ju), \delta(Iu, Fu), \delta(Ju, Gu), D(Iu, Gu), D(Ju, Fu))$$

or

$$\delta(u, Ju) \leq \phi(\delta(u, Ju)),$$

which implies that u=Ju. Since $\{Ju\} = Gu$, u is a common fixed point of F,G, I and J.

The proof assuming the continuity of J is similar, as is the proof of (III).

Now suppose that F and I have a second common fixed point v such that $Fv = \{v\} = \{Iv\}$. Then, using inequality (2.2), we obtain

$$d(v, u) = \delta(Fv, Gu) \leq \phi(d(v, u), 0, 0, d(v, u), d(v, u)) \leq \Psi(d(v, u)) < d(v, u),$$

which is a contradiction. So v=u, and u is the unique common fixed point of F and I such that $Fu = \{u\}$.

Similarly, it can be shown that u is the unique common fixed point of G and J such that $Gu = \{u\}$.

Remark. In Theorem 2.1, if F and G are single-valued mappings, we obtain Theorem 2.2 of [10].

Now , we give an example to support our extension.

Example. Let $X = [0, \infty)$ with the Euclidean metric d. Define $F, G : X \to B(X)$ and $I, J : X \to X$ by

$$Fx = [0, \frac{x^4}{2}], Gx = [0, \frac{x^2}{2}]$$
$$Ix = x^4 + 2x^2, Jx = x^8 + x^4 + x^2$$

for all x, y in X. Now:

$$F(X) \subset I(X), G(X) \subset J(X),$$

and F, G, I and J are continuous. Moreover, for any sequence $\{x_n\}$ in X, we have

$$Ix_n \to 0, Fx_n \to \{0\} \quad as \quad x_n \longrightarrow 0,$$

$IFx_n \in B(X)$,

$$\delta(FIx_n, IFx_n) = max\{\frac{(x_n^4 + 2x_n^2)^4}{2}, (\frac{x_n^4}{2})^2 + 2(\frac{x_n^4}{2})^2\} \to 0 \quad as \quad x_n \to 0,$$

thus F and I are δ -compatible . Similarly, one can show that G and J are δ -compatible.

Assuming $\phi \in \Phi$ as

$$\phi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{2} max\{t_2, t_3\}$$

for every $t_i \in R^+$, i=1,2,3,4,5.

We have for any $x, y \in X$

$$\begin{aligned}
\delta(Fx, Gy) &= max\{\frac{x^4}{2}, \frac{y^2}{2}\} \\
&= \frac{1}{2} max\{x^4, y^2\} \\
&\leq \frac{1}{2} max\{x^4 + 2x^2, y^8 + y^4 + y^2\} \\
&= \frac{1}{2} max\{\delta(Ix, Fx), \delta(Jy, Gy)\} \\
&= \phi(d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), \delta(Ix, Gy), \delta(Jy, Fx)).
\end{aligned}$$

Since ϕ obviously satisfies the conditions (i) and (ii), all the assumptions of Theorem 2.1 are verified and 0 is the unique common fixed point of F, G, I and J.

Note that Theorem 1.1 is not applicable even if F=G and I=J because F and I are not weakly commuting mappings at x=2, for

$$\begin{aligned}
&\delta(FI1, IF1) = \frac{81}{2} > 3 = max\{\delta(F1, I1), diamIF1\}, \\
&\delta(GJ1, JG1) = \frac{9}{2} > 3 = max\{\delta(G1, J1), diamJG1\}, \\
&FI1 \neq IF1, GJ1 \neq JG1.
\end{aligned}$$

3. A common fixed point theorem in compact metric space.

Mativated by the paper [2], we have the following.

Theorem 3.1. Let (X,d) be a compact metric space and let I, J be functions from X into X, $F, G : X \to B(X)$ be two set-valued functions with $FX \subset JX$ and $GX \subset IX$ such that for all x, y in X and $\phi \in \Phi$, the inequality

$$\delta(Fx, Gy) < \phi(d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), D(Ix, Gy), D(Jy, Fx)),$$

holds whenever the right hand side of (3.1) is positive. If the pairs (F,I) and (G,J) are δ - compatible, and if the functions F and I are continuous, then there is a unique fixed point u in X such that

$$Fu = Gu = \{u\} = \{Iu\} = \{Ju\}.$$

Proof. Let $\eta = \inf_{x \in X} \delta(Ix, Fx)$. Since X is compact metric space, there is a convergent sequence $\{x_n\}$ with limit x_0 in X such that

$$\delta(Ix_n, Fx_n) \to \eta \quad as \quad n \to \infty.$$

Since

$$\delta(Ix_0, Fx_0) \le d(Ix_0, Ix_n) + \delta(Ix_n, Fx_n) + \delta(Fx_n, Fx_0),$$

then by the continuity of F and I and $\lim_{n\to\infty} x_n = x_0$ we get $\delta(Ix_0, Fx_0) \le \eta$ and thus $\delta(Ix_0, Fx_0) = \eta$.

Since $FX \subset JX$, then there exists a point y_0 in X with $Jy_0 \in Fx_0$ and $d(Ix_0, Jy_0) \le \eta$.

If $\eta > 0$ then

$$\begin{aligned}\delta(Jy_0, Gy_0) &\le \delta(Fx_0, Gy_0)\\ &< \phi(d(Ix_0, Jy_0), \delta(Ix_0, Fx_0),\\ &\delta(Jy_0, Gy_0), D(Ix_0, Gy_0), D(Jy_0, Fx_0))\\ &\le \phi(\eta, \eta, \delta(Jy_0, Gy_0), \eta + \delta(Jy_0, Gy_0), 0).\end{aligned}$$

If $\delta(Jy_0, Gy_0) > \eta$ in the above inequality, then we have

$$\begin{aligned}&\delta(Jy_0, Gy_0)\\ &< \phi(\delta(Jy_0, Gy_0), \delta(Jy_0, Gy_0), \delta(Jy_0, Gy_0), 2\delta(Jy_0, Gy_0), 0)\\ &\le \Psi(\delta(Jy_0, Gy_0)) < \delta(Jy_0, Gy_0),\end{aligned}$$

which is a contradiction. Thus $\delta(Jy_0, Gy_0) \le \eta$.

Since $GX \subset IX$, then there is a point z_0 in X such that $Iz_0 \in Gy_0$ and $d(Iz_0, Jy_0) \le \eta$. Hence we have

$$\begin{aligned}&\eta \le \delta(Iz_0, Fz_0) \le \delta(Fz_0, Gy_0)\\ &< \phi(d(Iz_0, Jy_0), \delta(Iz_0, Fz_0), \delta(Jy_0, Gy_0), D(Iz_0, Gy_0), D(Jy_0, Fz_0))\\ &\le \phi(\eta, \delta(Iz_0, Fz_0), \eta, 0, \eta + \delta(Iz_0, Fz_0)). \qquad (3.1)\end{aligned}$$

If $\delta(Iz_0, Fz_0) > \eta$ in the last inequality, then we get

$$\begin{aligned}\eta &\le \delta(Iz_0, Fz_0)\\ &< \phi(\delta(Iz_0, Fz_0), \delta(Iz_0, Fz_0), \delta(Iz_0, Fz_0), 0, 2\delta(Iz_0, Fz_0))\\ &\le \Psi(\delta(Iz_0, Fz_0)) < \delta(Iz_0, Fz_0))\end{aligned}$$

a contradiction. Thus $\delta(Iz_0, Fz_0) \le \eta$.

Substituting into (3.1) yields

$$\eta \le \phi(\eta, \eta, \eta, 0, 2\eta) \le \Psi(\eta) < \eta,$$

a contradiction. Therefore $\eta = 0$ and hence

$$Ix_0 = Jy_0 = Iz_0 = \{Gy_0\} = \{Fz_0\}.$$

If $\delta(Jy_0, Gy_0) > 0$, then

$$\begin{aligned}&\delta(Jy_0, Gy_0) \le \delta(Fx_0, Gy_0)\\ &< \phi(d(Ix_0, Jy_0), \delta(Ix_0, Fx_0), \delta(Jy_0, Gy_0), D(Ix_0, Gy_0), D(Jy_0, Fx_0))\\ &= \phi(0, 0, \delta(Jy_0, Gy_0), 0, 0) \le \Psi(\delta(Jy_0, Gy_0)) < \delta(Jy_0, Gy_0),\end{aligned}$$

a contradiction. Therefore

$$\{Gy_0\} = \{Jy_0\} = \{Fx_0\} = \{Ix_0\}.$$

Since F and I are δ -compatible and $\delta(Ix_0, Fx_0) = 0$, we get $\delta(FIx_0, IFx_0) = 0$, and we get

$$\delta(F^2x_0, I^2x_0) = \delta(FIx_0, I^2x_0) = \delta(IFx_0, I^2x_0) = \delta(I^2x_0, I^2x_0) = 0.$$

We have

$$F^2x_0 = FIx_0 = IFx_0 = \{I^2x_0\}.$$

If $I^2x_0 \ne Ix_0$ then we have

$$\begin{aligned}&d(I^2x_0, Ix_0) = \delta(F^2x_0, I^2x_0)\\ &< \phi(\delta(IFx_0, Jy_0), \delta(IFx_0, F^2x_0), \delta(Jy_0, Gy_0), D(IFx_0, Gy_0), D(Jy_0, F^2x_0)\\ &= \phi(d(I^2x_0, Ix_0), 0, 0, d(I^2x_0, Ix_0), d(I^2x_0, Ix_0)\\ &\le \Psi(d(I^2x_0, Ix_0) < d(I^2x_0, Ix_0),\end{aligned}$$

which is a contradiction, and so $I^2x_0 = Ix_0$. Hence

$$FIx_0 = \{Ix_0\} = \{I^2x_0\}.$$

Similarly, we have

$$GJy_0 = \{Jy_0\} = \{J^2y_0\}.$$

Let $u = Ix_0 = Jy_0$. Then

$$Fu = \{u\} = \{Iu\} = \{Ju\} = Gu.$$

Suppose that the point y in X is a common fixed point of F,G, I and J with $y \ne u$

$$\begin{aligned}d(u, y) &= \delta(u, y) = \delta(Fu, Gy)\\ &< \phi(d(Iu, Jy), \delta(Iu, Fu), \delta(Jy, Gy), D(Iu, Gy), D(Jy, Fu))\\ &= \phi(d(u, y), 0, 0, d(u, y), d(u, y)) \le \Psi(d(u, y)) < d(u, y),\end{aligned}$$

whence u is the unique common fixed point of F, G, I and J.

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REFERENCES

1. Cheng Chun Chang, *On a fixed point theorem of contractive type*, Comment. Math. Univ. St. Paul. **32, No.1** (1983), 15-19.
2. Tong-Huei Chang, *Fixed point theorems for contractive type set-valued mapppings*, Math. Japonica **38, No.4** (1993), 675-690.
3. Fisher B., *Common fixed points of mapppings and set-valued mappings*, Rostick. Math. Kollaq. **18,** (1981), 69-77.
4. Fisher B. and Sessa S., *Two common fixed point theorems for weakly commuting mapppings*, Periodica Math. Hungarica **20, No.3** (1989), 207-218.
5. Jungck G., *Compatible mapppings and common fixed points (2)*, Internat. J. Math. & Math. Sci. **11,** (1986), 285-288.
6. Jungck G. and B. E. Rhoades, *Some fixed point theorems for compatible maps*, Internat. J. Math. & Math. Sci. **16, No.3** (1993), 417-428.
7. Khan M. S. and Kubiaczyk I., *Fixed point theorems for point to set maps*, Math. Japonica **33, No.3** (1988), 409-415.
8. Khan M. S. , Khan M. D. and Kubiaczyk I., *Some common fixed point theorems for multi-valued mappings*, Demonstratio Mathematica **17, Mo.4** (1984), 997-1002.
9. Matkowski J., *Fixed point theorems for mappings with a contractive iterate at a point*, Proc.Amer. Math. Soc. **62, No.2** (1977), 344-348.
10. Rashwan R.A., *A common fixed point theorem for compatible mappings*, to appear in Demonstratio Mathematica.
11. Sessa S., *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. (Beograd) **32, No. 46** (1982), 149-153.
12. Sessa S., Khan M. S. and M. Imdad, *Common fixed point theorem with a weak commutativity condition*, Glasnik Mat. **21, No.41** (1986), 225-235.
13. Singh K. L. and Whitfield J.H.M., *Fixed points for contractive type multi-valued mappings*, Math. Japonica **27** (1982), 117-124.

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Hyperbolic Functions that Linearize Inverted Manifold Surfaces

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Abstract

In this paper a control law that linearizes inverted dynamics is developed. The equation for inverted circular dynamics is derived from differential method and an integral method. By making alternate approximative assumptions of the inverted circular manifold with hyperbolic functions, an inverting control function is derived. Computer simulation results show that the alternate manifold linearizes the inverted dynamics. Analog simulation results show that the developed control law in this paper linearizes inverted dynamics. The method can be applied other nonlinear systems that have a gain term for the position of the angle in the sinusoid nonlinearity such as the space station. Because of torque requirements of the actual inverted systems such as the space station and inverted pendulum, a nonlinear adaptive gain is added. The adaptive gain is derived from a hyperbolic function which also results in a linearizing inverting function.

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1 Introduction

The results of this work linearize nonlinear differential equations that are inverted with sinusoid nonlinearity by inputing hyperbolic forcing functions. The applications of the results in this paper can be applied to robotics, aerospace engineering, and machinery that can be described by circular manifold. It will be shown that there are

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hyperbolic functions that approximate the inverse system solution for the inverted circular dynamics. Stability theory is applied to show that this solution is valid. Computer and analog simulation verify these results. The control law is derived from the sliding chain model. An application of the derived control law using hyperbolic functions is attitude control of space vehicles employing control moment gyro (CMG) is an interesting problem. The equations of motion of the space station are described by nonlinear differential equations. Often, attitude control system design using linear control theory [3, 6, 4] is obtained. However linear control systems are designed based on the assumption that the perturbation in attitude angles are small. The controller of [2] is based on the inversion of a nonlinear input-output map. The adaptive nonlinear control law that is derived by making alternate assumption for the potential of the manifold. The adaptive control law inversely proportional to cosh$\alpha\theta$. The resulting control law is ideal for the space station employing control moment gyros because momentum is bounded. In this paper, for simplicity, CMG's are considered as ideal torquers; however, the CMG gimbal dynamics should be included in the further development of control systems [5].

Approximate solution for inverted dynamics is derived in sections 2 and 3. The results are applied to the nonlinear inverted dynamics. Stability proof is given in section 4. Simulation for the claims are done in software and by mechanical analog. Described in this paper the dynamics of circular inverted dynamics and the space station dynamics.

2 Fixed Point Inverted Circular Dynamics

The pendulum dynamics are developed here by integral equation method. The derivation for a pendulum consisting of a mass m suspended on a for a rigid rod of length l, and angle θ. The forces acting are $F_n = ml\omega^2 = ml\dot{\theta}$ and $F_t = ml\ddot{\theta}$. The forces are defined as $F_n = s - mgcos\theta$ and $F_t = -mgsin\theta$. The equations of motion are[7]:

$$s - mgcos\theta = ml\dot{\theta}^2$$

$$-mgsin\theta = ml\ddot{\theta}$$

The first equation gives $s = mgcos\theta + ml\dot{\theta}^2$, which determines s if θ can be found as a function of time. The second equation gives the equation of motion:

$$\ddot{\theta} + \frac{g}{l}sin\theta = 0$$

With forcing function input:

$$\ddot{\theta} + \frac{g}{l}sin\theta = -u$$

An alternate assumption on the tangential force F_t.

$$F_t = ml\ddot{\theta} = -mgsin\theta \approx -mgK_{scaling}(e^{sin\theta} - e^{-sin\theta}) \tag{1}$$

Where for proper choice of scaling factor the approximation is nearly exact. Because sinθ is a solution for the circular manifold about a stable equilibrium point, then the feedback control law is able to regulate:

$$u = Ksinh\theta \tag{2}$$

Where u is a feedback that can subtract out the nonlinear sinθ term. This assumption was applied in computer simulation to the input of the inverted pendulum dynamics a bounded oscillatory response occured.

3 Derivation of Dynamics from Volterra Equation

From analog simulation of inverted pendulum cart dynamics with sinhθ terms, it became apparent that some form of gain scheduling would be needed [8]. This is because the cart had a radius of gyration that was ± 45 deg, and the driving torque of the motor was limited. The controller was a digital computer with sampling rate was high enough that discretization error was approximately equal to zero. To solve the control problem for a large angle, some form of adaptive model reference gain would need to be proportional to inverse of the position. From section 2, the potential of the inverted system is:

$$V = 1 - mgcos\theta \tag{3}$$

An approximation of this curve for ± 90 deg is $cosh\theta$. The adaptive time varying gain is then:

$$Model\ Reference\ Gain = \frac{1}{cosh\theta} \tag{4}$$

The resulting control law:

$$u = Ktanh(\alpha\theta) \tag{5}$$

Where K and α are constants.

In this section, an integral equation will be derived from sliding chain [1], and prove the above claim. This will be applied to inverted dynamics. By an alternate approximation for the shape of the surface, the control law is developed. The equation of the curve will be referred to as:

$$z = \phi(s) \tag{6}$$

Where $\phi(s)$ is assumed to be a smooth function. Let σ be the length of the arc between the points O and A. Let λ be the distance MA. To describe the curve only for the portion $s = \sigma - \lambda$. This yields

$$z = \phi(\sigma - \lambda) \tag{7}$$

Assume that $f(\lambda)$ is the linear density function of the pendulum at the point M, then $f(\lambda)d\lambda$ is the mass of the pendulum at point M, of length dλ. The work corresponding to the movement from z to z+dz is

$$gf(\lambda)d\lambda = g\phi^{'}(\sigma - \lambda)f(\lambda)d\lambda \tag{8}$$

Where the total work done over the interval 0 to σ

$$W_{tot} = gd\sigma \int_0^{\sigma} \phi^{'}(\sigma - \lambda)f(\lambda)d\lambda dz \tag{9}$$

The kinetic energy energy over the time t to t+dt (over the period necessary to move z to z+dz).

$$T = \frac{1}{2}Mv^2 \tag{10}$$

Where $M = \int_0^a f(\lambda)d\lambda$ the total mass of the pendulum, and since $v = ds/dt = d\sigma/dt$.

$$dT = M\frac{d\sigma}{dt}\frac{d^2\sigma}{dt^2}dt = M\frac{d^2\sigma}{dt^2}d\sigma \tag{11}$$

The resulting equation of motion is

$$M\frac{d^2\sigma}{dt^2} = g\int_0^{\sigma} \phi^{'}(\sigma - \lambda)f(\lambda)d\lambda \tag{12}$$

The law of motion $d^2\sigma = F(\sigma)dt^2$, then

$$\int_0^{\sigma} \phi^{'}(\sigma - \lambda)f(\lambda)d\lambda dz = g^{-1}MF(\sigma) \tag{13}$$

which constitutes an integral equation of Volterra type of the first kind.

To apply these equations to our control problem, assume that λ is small with respect to σ. The width of MA is small compared to the arc OC.

$$d\theta = d\lambda \tag{14}$$

The equation of the curve is

$$z = Rsin^2\theta + Rcos^2\theta \tag{15}$$

$$\phi^{'} = -2sin\theta + 2cos\theta \tag{16}$$

The linear density function $f(\lambda)$ is a constant K. Assume that R=1. The distance OC are regulating to is from 0 to θ. So $\theta = \sigma$.

$$\frac{d^2\theta}{dt^2} = gM^{-1}K\int_0^{\theta}(-2sin\theta + 2cos\theta)d\theta \tag{17}$$

For small angles the $cos\theta$ term is dominant.

$$\frac{d^2\theta}{dt^2} = gM^{-1}K_{proportional}\int_0^\theta cos\theta d\theta \tag{18}$$

Where $K_{proportional}$ equals $K \times 2$.

$$\frac{d^2\theta}{dt^2} = gM^{-1}K_{proportional} \times sin\theta \tag{19}$$

An approximation of the cosine term with hyperbolic function for the interval $\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

$$cos\theta = sech^2\theta \tag{20}$$

Substituting:

$$\frac{d^2\theta}{dt^2} = gM^{-1}K_{proportional}\int_0^\theta sech^2\theta d\theta \tag{21}$$

The resulting differential equation is

$$\frac{d^2\theta}{dt^2} = gM^{-1}K_{proportional} \times tanh\theta \tag{22}$$

For dynamics that differ by a gain term in the sinusoid angle. The equation of the curve is

$$z = Rsin^2(\alpha\theta) + Rcos^2(\alpha\theta) \tag{23}$$

The linear density function $f(\lambda)$ is a constant K. Assume that R=1. The distance OC are regulating to is from 0 to θ. So $\theta = \sigma$.

$$\frac{d^2\theta}{dt^2} = gM^{-1}K\int_0^\theta (-2\alpha sin(\alpha\theta) + 2\alpha cos(\alpha\theta))d\theta \tag{24}$$

For small angles:

$$\frac{d^2\theta}{dt^2} = gM^{-1}K_{proportional}\int_0^\theta \alpha cos(\alpha\theta)d\theta \tag{25}$$

$$\frac{d^2\theta}{dt^2} = gM^{-1}\alpha K_{proportional} \times sin(\alpha\theta) \tag{26}$$

The approximation of the cosine and sine term with a sech function in the preceding integral equation:

$$\frac{d^2\theta}{dt^2} = gM^{-1}K_{proportional}\int_0^\theta \alpha sech^2(\alpha\theta)d\theta \tag{27}$$

The resulting differential equation is

$$\frac{d^2\theta}{dt^2} = gM^{-1}\alpha K_{proportional} \times tanh(\alpha\theta) \tag{28}$$

Definition 1 *A Manifold is topological space which is locally Euclidean; there are four types; topological, piecewise linear, differentiable, and complex, depending on whether the local coordinate systems are obtained from continuous, piecewise linear, differentiable, or complex analytic functions of those in Euclidean space; intuitively a surface[9].*

Applying this definition, $\sinh \alpha\theta$ is an approximation of the inverse dynamics of second order inverted manifold systems described in this paper.

4 Stability Proof for the Pitch Axis Dynamics

In this section stability proof for the control law developed in sections 3 and 4 is given [10].

Theorem 1 *The equilibrium point* $\mathbf{0}$ *at time* t_0 *of a nonlinear differential equation is stable if there exists a continuously differentiable l.p.d.f.* V *such that*

$$\dot{V}(t, \mathbf{x}) \leq 0 \ \ \forall t \geq t_0, \forall \mathbf{x} \in \mathbf{B_r} \tag{29}$$

for some ball $\mathbf{B_r}$

Conjecture 1 *The control law*

$$u = K_{gain} sinh(\alpha\theta)$$

for proper choice of K_{gain} *and* α *results in stable dynamics for inverted systems of the type:*

$$\ddot{\theta} - K_{gain} sin(\alpha\theta) = -u$$

Proof 1 *For the control given the potential surface is*

$$V(t, \theta) = \alpha K_{gain} cosh(\alpha\theta) \tag{30}$$

$$\dot{V}(t, \theta) = \alpha K_{gain} sinh(\alpha\theta) \tag{31}$$

For the inverted equilibrium point, the Liapunov function bounds the surface and

$$\dot{V}(t, \mathbf{x}) \leq 0 \ \ \forall t \geq t_0, \forall \mathbf{x} \in \mathbf{B_r}$$

Conjecture 2 *The nonlinear feedback* $u = K_{gain} sinh(\alpha\theta)$ *is a feedback linearization of inverted dynamics of the form:*

$$\ddot{\theta} - \frac{g}{l} sin(\alpha\theta) = -u \tag{32}$$

Where g*,* l*, and* α *are constants.*

Proof 2 *From preceding sections it is assumed that* $sinh(\alpha\theta)$ *is an approximation for the surface manifold. The control is for a region* $\frac{\pi}{2}$ *to* $\frac{-\pi}{2}$ *with proper choice of gain value* K_{gain}*, and* α*. For the condition of stability shown in preceding conjecture, a solution can be bounded sinusoid. Therefore for this case:*

$$K_{gain} sinh(\alpha sin(\alpha\theta)) \approx \frac{g}{l} sin(\alpha\theta)$$

Substituting into the nonlinear differential equation for proper choice of gain K_{gain} *and* α*.*

$$\ddot{\theta} + \epsilon\theta = 0 \tag{33}$$

Where ϵ *is assumed to be approximately a constant or a small value with respect to* $\frac{g}{l} sin(\alpha\theta)$*.*

Conjecture 3 *The nonlinear feedback* $u = K_{gain} tanh(\alpha\theta)$ *is a feedback linearization of inverted dynamics of the form:*

$$\ddot{\theta} - \frac{g}{l} sin(\alpha\theta) = -u \tag{34}$$

Where g*,* l*, and* α *are constants.*

Proof 3 *This is a result of for the interval* $-\frac{\pi}{2\alpha}$ *to* $\frac{\pi}{2\alpha}$*:*

$$K_{gain} tanh(\alpha\theta) \approx \frac{g}{l} sin(\alpha\theta) \tag{35}$$

Substituting into the nonlinear differential equation for proper choice of gain K_{gain} *and* α*.*

$$\ddot{\theta} + \epsilon\theta = 0 \tag{36}$$

Where ϵ *is assumed to be approximately a constant or a small value with respect to* $\frac{g}{l} sin(\alpha\theta)$*.*

The resulting output is oscillatory the control laws developed using hyperbolic functions described. Oscillatory dynamics are easily described by 2 poles on the complex axis using describing function method.

4.1 Stability and Control of Resulting Dynamics

Theoretical conditions for stability are discussed and applied to the control problem for oscillatory dynamics. The conditions for stability is a result of La Salle's Theorem for periodic systems. Which is defined as [10]:

Definition 2 *A limit cycle of dynamics such as*

$$\dot{x_1} = f_1[x_1(t), x_2(t)]$$

$$\dot{x_2} = f_2[x_1(t), x_2(t)]$$

is a periodic solution of these same described dynamics.

La Salle's Theorem for periodic systems.

Theorem 2 *Suppose that system $\dot{x}(t) = f[t, x(t)]$ for $t \geq 0$ is periodic, i.e., suppose*

$$f(t,x) = f(t+T,x), \ \ \forall\, t, \ \forall\, x \in \ R^n,$$

for some $T > 0$. Suppose $V : R_+ \times R^n \to R$ is a continuously differentiable p.d.f., with

$$V(t,x) = V(t+T,x), \ \ \forall\, x \in R^n, \forall\, t \geq 0$$

Define

$$S = \{x \in R^n : \dot{V}(t,x) = 0, \ \forall\, t \geq 0\} \tag{37}$$

Suppose $\dot{V}(t,x) \leq 0$, $\forall\, t \geq 0$, $\forall\, x \in R^n$, and that S contains no nontrivial trajectories of $\dot{x}(t) = f[t,x(t)]$. Under these conditions, the equilibrium point **0** *is globally asymptotically stable.*

In order to regulate the system to **0** the control

$$u = K_1\theta + K_2\dot{\theta} + K_3 sinh(\alpha\theta) \tag{38}$$

or

$$u = K_1\theta + K_2\dot{\theta} + K_3 tanh(\alpha\theta) \tag{39}$$

is used such that K_1, K_2, and K_3 are chosen such that $\dot{V}(t,x) \leq 0$.

The equation is linearized by use of describing function technique. From simulation when these functions are used as control inputs to the dynamics for the pitch axis, bounded oscillation results. By use of describing function method, the oscillation is an approximation of $Asin\beta\theta$ (where A the amplitude and β are constants). No other dominant oscillations exist from this control input. The oscillation is easily described by the 2 dominant poles on the $j\omega$ axis in the complex S plane. So the describing function for this system in the inverted state space:

$$\ddot{\theta} + \beta\theta = 0 \tag{40}$$

This is then a linear system with β term describing the period of the oscillation. In the inverted systems state space:

$$x_1 = \theta \tag{41}$$

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -\beta x_1$$

If terms $K_1\theta + K_2\dot{\theta}$ are negatively fed back into this system:

$$x_1 = \theta \tag{42}$$

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -\beta x_1 - K_1 x_1 - K_2 x_2$$

This is of the form [10]:

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -f(x_2) - g(x_1)$$

Where $f(x_2)$ represents the friction and $g(x_1)$ the restoring properties of the linearized system. Where the Liapunov function is:

$$V(x_1, x_2) = \frac{{x_2}^2}{2} + \int_0^{x_0} g(\sigma) d\sigma \tag{43}$$

For this case $g(\sigma) = (\beta - K_1)x_1$. And

$$\dot{V}(x_1, x_2) = x_2 \dot{x_2} + g(x_1)\dot{x_1} \tag{44}$$

$$= x_2[-f(x_2) - g(x_1)] + g(x_1)x_2 = -x_2 f(x_2)$$

$$\dot{V}(x_1, x_2) = -K_2 {x_2}^2 \tag{45}$$

Where $\dot{V}(x_1, x_2) \leq 0 \ \forall$ stable feedback pole pairs when control feedbacks from $\sinh\alpha\theta$ and $\tanh\alpha\theta$ produce bounded oscillations.

5 Simulation Results

The equations for control were derived mathematically. Computer simulation and analog models were used to test the control laws. The simulation described:

1. Nonlinear pitch axis dynamics of space station with control input $u_2 = sinh(2\theta_2)$.

2. Nonlinear pitch axis dynamics of space station with control input $u_2 = tanh(2\theta_2)$.

3. Nonlinear pitch axis dynamics of space station with control input $u_2 = tanh(2\theta_2) + .2\dot{\theta}$.

4. A nonlinear inverted cart with control input $u = -60tanh(2\theta)$. The $\theta(0) = 4°$. Note that although inverted circular dynamics had a α term of 1, other values can be used so long as the potential function bounds.

5. Computer simulation of the inverted cart with control input $u = 44tanh\theta_{angle} + 37\dot{\theta}_{angle} + 225\theta_{cart} + 67.3\dot{\theta}_{cart}$

The Space Station Parameters are: (Inertia slug-ft^2)

$$I_{11} = 50.28E6,$$
$$I_{22} = 10.80E6,$$
$$I_{33} = 58.57E6,$$

Where $I_i \triangleq I_{ii}$; i=1,2,3 The computer simulation for the space station nonlinear pitch axis dynamics in fortran used the pitch axis dynamics:

$$I_2\ddot{\theta}_2 + 3n^2(I_1 - I_3)sin\theta_2 cos\theta_2 = -u_2 \tag{46}$$

which simplifies to:

$$I_2\ddot{\theta}_2 + \frac{3}{2}n^2(I_1 - I_3)sin(2\theta_2) = -u_2$$

6 Conclusion

By use of hyperbolic functions to describe differential surfaces, it is possible to linearize inverted second order systems with sinusoid nonlinearity in position and small omega. The derivation is for inverted circular dynamics, but the solution is made general by making the hyperbolic function approximate other types of sinusoid nonlinearity. The results were applied to various nonlinear systems, and the dynamics were linearized.

The control laws are the result of alternate assumptions for the potential surface of the dynamics of inverted circular fixed point dynamics. Computer simulation verified these assumptions for the space station dynamics. Simulation on mechanical analog that is similar to the inverted dynamics of the space station, the inverted pendulum cart are presented. An adaptive gain which was proportional to the inverse of the potential surface is suggested, and is a result of analog testing the control law. The technique for feeding back nonlinear terms into a hyperbolic function that approximates the nonlinearity has applications in all areas of circular manifold and related dynamics.

References

[1] Corduneanu C. *Integral Equations and Applications.* Cambridge University Press, Cambridge, New York.

[2] Dwyer A. W. and Batten A. L. Exact spacecraft detumbling and reorientation maneuvers with gimbaled thrusters and reaction wheels. *Journal of the Astronautical Sciences*, 33:217–232, 1983.

[3] A Greensite. L. *Analysis and Design of Space Vehicle Flight Control Systems Properties of Functions.* Sparton, New York, N.Y., 1970.

[4] Iyer A. and Singh S. N. Mfds of spinning satellite and attitude control using gyrotorquers. *IEEE Transactions on Aerospace and Electronic Systems*, 25:611–620, 1989.

[5] Kennel H. F. Double-gimbaled control moment gyro steering law update. Technical report, May 1986.

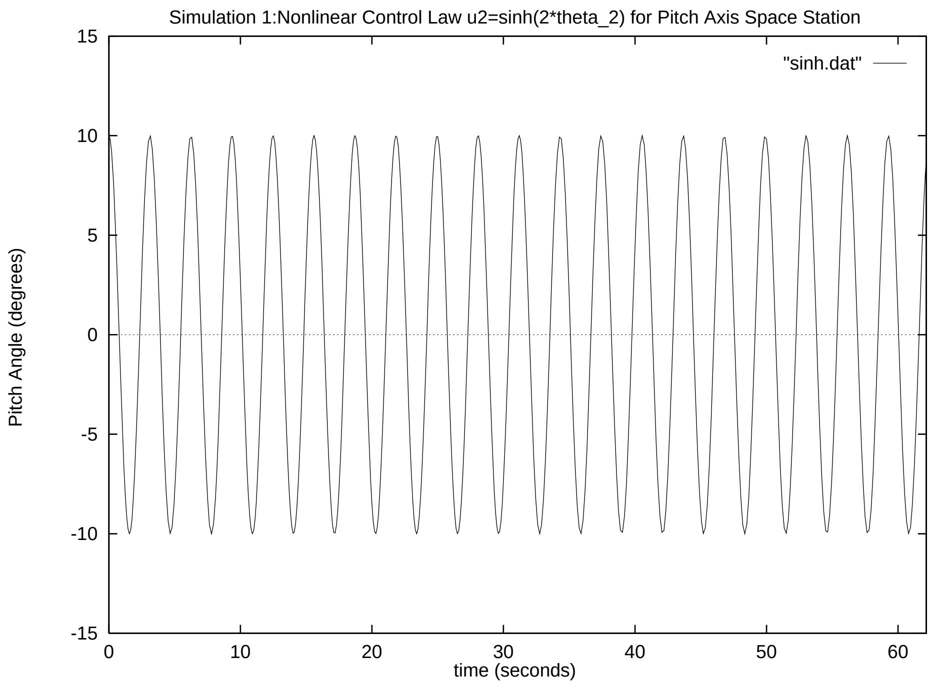
[6] Wie B., Bynn K.W., Warrem V.W., Geller D., Long D. and Sunkel J. New approach to attitude / momentum control for the space station. *Journal of Guidance, Control and Dynamics*, 12:714–722, 1989.

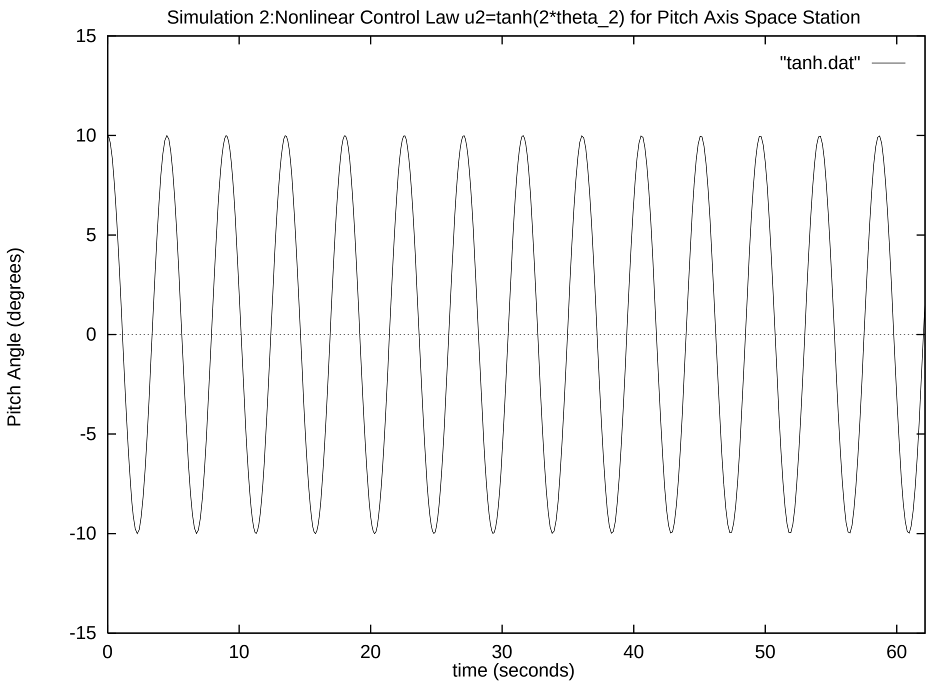
[7] Maltbaek J. C.. *Dynamics in Engineering*. Halsted Press, Chichester, West Sussex, England.

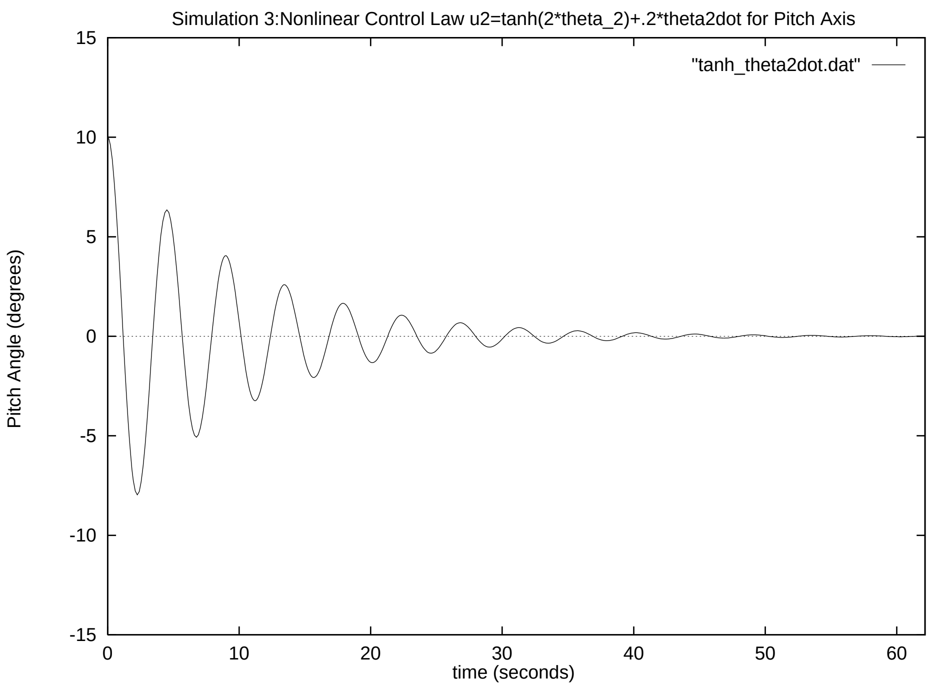
[8] Mauer G.F. Technical conversation, 1994.

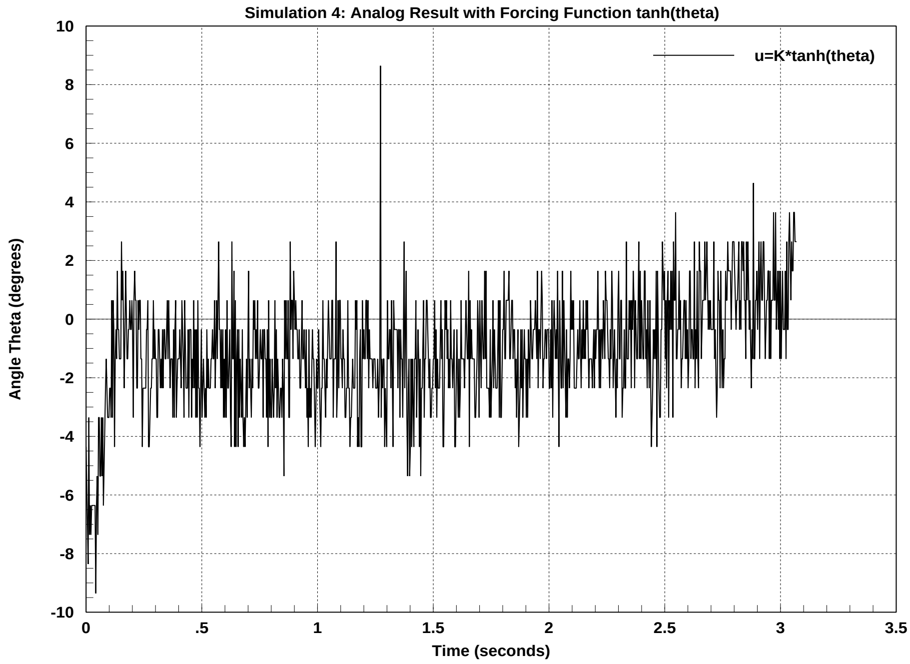
[9] Nijmeijer H. and Der Schaft A. *Nonlinear Dynamical Control Systems*. Springer-Verlag, New York, N.Y., 1990.

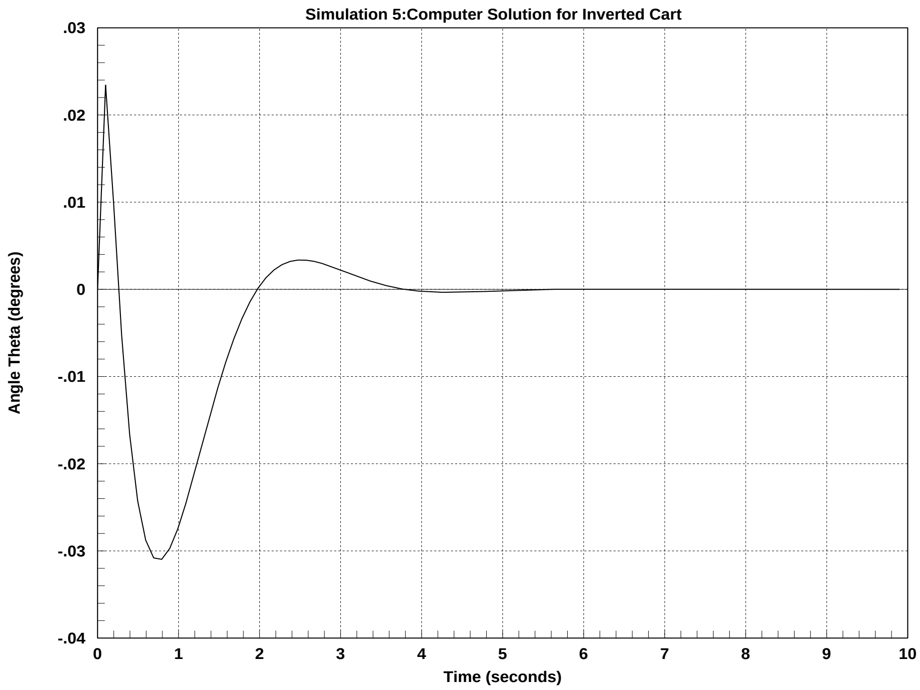
[10] Vidyasagar M.. *Nonlinear Systems Analysis*. Prentice-Hall, Englewood Cliffs, N.J., 1978.


Simulation 1:Nonlinear Control Law u2=sinh(2*theta_2) for Pitch Axis Space Station
"sinh.dat"
Pitch Angle (degrees)
time (seconds)
15
10
5
0
-5
-10
-15
0
10
20
30
40
50
60


Simulation 2:Nonlinear Control Law u2=tanh(2*theta_2) for Pitch Axis Space Station
"tanh.dat"
Pitch Angle (degrees)
time (seconds)
15
10
5
0
-5
-10
-15
0
10
20
30
40
50
60


Simulation 3:Nonlinear Control Law u2=tanh(2*theta_2)+.2*theta2dot for Pitch Axis
"tanh_theta2dot.dat"
Pitch Angle (degrees)
time (seconds)
15
10
5
0
-5
-10
-15
0
10
20
30
40
50
60


Simulation 4: Analog Result with Forcing Function tanh(theta)
u=K*tanh(theta)
Angle Theta (degrees)
Time (seconds)
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
.5
1
1.5
2
2.5
3
3.5


Simulation 5:Computer Solution for Inverted Cart
Angle Theta (degrees)
Time (seconds)
.03
.02
.01
0
-.01
-.02
-.03
-.04
0
1
2
3
4
5
6
7
8
9
10

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