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Ordering of Energy Levels for Extended $SU(N)$ Hubbard Chain*

Tigran HAKOBYAN

Yerevan State University, Alex Manoogian 1, Yerevan, Armenia
Yerevan Physics Institute, Alikhanian Br. 2, Yerevan, Armenia
E-mail: *hakob@yerphi.am*

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Abstract. The Lieb–Mattis theorem on the antiferromagnetic ordering of energy levels is generalized to $SU(N)$ extended Hubbard model with Heisenberg exchange and pair-hopping terms. It is proved that the minimum energy levels among the states from equivalent representations are nondegenerate and ordered according to the dominance order of corresponding Young diagrams. In particular, the ground states form a unique antisymmetric multiplet. The relation with the similar ordering among the spatial wavefunctions with different symmetry classes of ordinary quantum mechanics is discussed also.

Key words: Lieb–Mattis theorem; $SU(N)$ Hubbard model; ground state; dominance order; Schur–Weyl duality

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1 Introduction

The investigation of the ground state of quantum many-body systems is very important and relevant in many areas of condensed matter physics, in particular, in high-temperature superconductivity and magnetism. The quantum numbers associated with different symmetries and the degree of degeneracy are essential properties that characterize the ground state and the system as a whole.

In 1955, Marshall proved that the ground state of antiferromagnetic Heisenberg spin-1/2 ring with an even number of sites is a spin singlet [1]. This result was generalized to higher spins and dimensions [2]. The uniqueness of ground state was established also. Moreover, for Heisenberg antiferromagnets on bipartite lattices, Lieb and Mattis proved that the ground state is a unique multiplet of spin $S_{\rm gs} = |S_1 - S_2|$, where S_1, S_2 are the highest possible spin values of the two sublattices, which form the bipartite lattice. They proved also that the lowest energy $E(S)$ among all spin-S states is an increasing function for $S \geq S_{\rm gs}$ [3, 4]. This property of the spectrum is known as the antiferromagnetic ordering of the energy levels. In one dimension, the quantum mechanical system of interacting electrons without velocity- or spin-dependent forces and the Hubbard model possess this type of ordering too [5]. This fact is known as the absence of one-dimensional ferromagnetism.

The Lieb–Mattis theorem has been subsequently generalized to various spin and fermion lattice systems, such as the spin-1 chain with biquadratic interactions [6], the $t - J$ [7] and extended Hubbard [8] chains, Hubbard models on bipartite lattices at half filling [9], the Sutherland chain [10], frustrated spin-1/2 ladder systems [11]. The ferromagnetic ordering of the energy levels for spin-1/2 Heisenberg chain have been established also [12].

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In this article we formulate and prove the analogue of this theorem for $SU(N)$ symmetric extended Hubbard–Heisenberg chains with pair-hopping term. Recently, the spin and fermionic chains with such symmetry have been investigated intensively [10, 13, 14, 15]. This interest is motivated by their application in ultracold atoms [16, 17]. Recently, an interesting classification of the low-energy behavior of $SU(N)$ spin chains on three different types, which extends the well-known classes of integer and half-integer spin chains [18], has been conjectured and checked [13, 14].

In Section 2 we introduce the model and describe the symmetries. In Section 3 we construct a basis, in which all off-diagonal elements of the Hamiltonian are non-positive. Section 4 is devoted to the proof of the uniqueness of the lowest energy states in the weight subspaces of $SU(N)$ algebra. Here we also find the quantum numbers of such states. Based on the results of previous sections, in Section 5 we formulate and prove the analogue of the antiferromagnetic ordering of energy levels for the system under consideration. Its consequence for the ground state and its quantum numbers is analyzed in Section 6. In the last section we establish the similar ordering rule for the quantum mechanical system of fermions with $SU(N)$ degrees of freedom, and discuss the relation with $SU(N)$ Hubbard model.

2 $SU(N)$ symmetric fermionic chain

Consider the finite-size $SU(N)$ symmetric extended Hubbard chain described by the Hamiltonian

$$\begin{aligned} H = \sum_{x,\alpha} -t_x\big(c^+_{x+1,\alpha}c_{x,\alpha} + c^+_{x,\alpha}c_{x+1,\alpha}\big) + V(n_1,\dots,n_L) + \sum_{x,a} J_x\, T^a_x T^a_{x+1} \\ - \sum_{x,\alpha>\beta} K_x\big(c^+_{x+1,\alpha}c^+_{x+1,\beta}c_{x,\beta}c_{x,\alpha} + c^+_{x,\alpha}c^+_{x,\beta}c_{x+1,\beta}c_{x+1,\alpha}\big), \end{aligned} \tag{1}$$

where the open boundary conditions $\left(\sum_x = \sum_{x=1}^{L-1}\right)$ are supposed. The coefficients t_x, J_x and K_x are positive and dependent on the site x. There are N different flavors of fermions, which are numbered by α. The fermion creation $c^+_{x,\alpha}$ and annihilation $c_{x,\alpha}$ operators obey the standard anticommutation relations.

In the above Hamiltonian, $n_x = \sum_\alpha n_{x,\alpha} = \sum_\alpha c^+_{x,\alpha}c_{x,\alpha}$ is the fermion number at the x-th site. The form of the potential $V(n_1,\dots,n_L)$ does not matter, the only restriction is that it depends only on the local fermion numbers. The Hubbard potential is a particular case $V = U/2\sum_x n_x^2$. It is equivalent, up to the total particle number, to $U\sum_{x,\alpha>\beta} n_{x,\alpha}n_{x,\beta}$.

The third term is the Heisenberg interaction of $SU(N)$ spins (flavors) given in the Schwinger representation

$$T^a_x = \sum_{\alpha,\beta} c^+_{x,\alpha}\mathcal{T}^a_{\alpha\beta}c_{x,\beta}, \qquad a = 1,\dots,N^2-1 = \dim SU(N), \tag{2}$$

where $\mathcal{T}^a_{\alpha\beta}$ are the generators of $SU(N)$ Lie algebra in the defining representation, which are orthogonal with respect to the trace. Using the completeness relation for $SU(N)$ matrices

$$\sum_a \mathcal{T}^a_{\alpha\beta}\mathcal{T}^a_{\alpha'\beta'} = 2\delta_{\alpha\beta'}\delta_{\alpha'\beta} - \frac{2}{n}\delta_{\alpha\beta}\delta_{\alpha'\beta'},$$

one can rewrite this term in the following form [19]:

$$\sum_a T^a_x T^a_{x+1} = 2\sum_{\alpha,\beta} c^+_{x,\alpha}c_{x,\beta}c^+_{x+1,\beta}c_{x+1,\alpha} - \frac{2}{n}n_x n_{x+1}$$

$$= -2\sum_{\alpha,\beta} c^+_{x,\alpha}c_{x+1,\alpha}c^+_{x+1,\beta}c_{x,\beta} + 2n_x - \frac{2}{n}n_x n_{x+1}, \tag{3}$$

The first term above just exchanges the flavors between adjacent sites [21]. The last two terms depend only on n_x and may be included in the potential.

The last term in (1) describes the hopping of fermion pairs.

The Hamiltonian preserves the $U(1)$ total charge, which corresponds to the total number of particles

$$n = \sum_x n_x = \sum_{x,\alpha} c^+_{x,\alpha}c_{x,\alpha}.$$

It is also invariant with respect to $SU(N)$ generators

$$T^a = \sum_x T^a_x = \sum_{x,\alpha,\beta} c^+_{x,\alpha}\mathcal{T}^a_{\alpha\beta}c_{x,\beta}.$$

The total symmetry group is $U(N) = SU(N)\times U(1)$. The spin-raising, spin-lowering, and Cartan generators are given by the upper triangular, lower triangular, and diagonal matrixes. The corresponding basis is presented below both in the defining and multi-particle representations.

$$\left(\mathcal{F}^{\alpha\beta}\right)_{\alpha'\beta'} = \delta^\alpha_{\alpha'}\delta^\beta_{\beta'}, \qquad F^{\alpha\beta} = \sum_x c^+_{x,\alpha}c_{x,\beta}, \qquad F^{\alpha\alpha} = \sum_x n_{x,\alpha}. \tag{4}$$

The current system is a $SU(N)$ generalization of the $SU(2)$ Hamiltonian, for which the Lieb–Mattis theorem has been established and proven already [8]. For the Hubbard potential, the first three terms of the Hamiltonian (1) correspond to the $SU(N)$ Hubbard–Heisenberg chain introduced in [19]. The $SU(4)$ system without pair-hopping and Heisenberg terms has been proven to have nondegenerate ground state and gapless excitations [20].

Each site has 2^N different states with fermion number varying from zero to N. According to (2) or (4), the one-particle states $c^+_\alpha|0\rangle$ form the defining representation of $U(N)$. Similarly, due to the anticommutation of the fermionic operators, the multi-particle states $c^+_{\alpha_1}\cdots c^+_{\alpha_k}|0\rangle$ form the $\binom{N}{k}$-dimensional antisymmetric representation. There are two singlets ($k = 0, N$), which correspond to the vacuum $|0\rangle$ and completely filled states.

In this article we follow the standard way in order to establish and prove the generalized Lieb–Mattis theorem for the described system [3, 11]. First, we construct a basis, in which all nonzero off-diagonal matrix elements of the Hamiltonian are negative. Next, we confine ourselves to the subspaces, where the Hamiltonian is connected, and due to the Perron–Frobenius theorem, the lowest-energy state is unique and positive. Employing the positivity, we compare this state with a simple trial state and detect in this way the containing multiplet. Finally, using the representation theory of $SU(N)$ group, we express the ordering of the lowest energy levels of different multiplets in terms of the dominance order of the corresponding Young diagrams.

3 Nonpositive basis

The natural basis for the Hamiltonian (1) is formed by the fixed particles. The coordinates are given by the set $\{x^\alpha\}_{\alpha=1}^N$ consisting of N subsets. Each subset

$$\{x^\alpha\} = \{x^\alpha_1, \ldots, x^\alpha_{M_\alpha} | x^\alpha_1 < \cdots < x^\alpha_{M_\alpha}\}$$

describes the positions of the fermions carrying the flavor α, and $M_\alpha = \#\{x^\alpha\}$ is their number. Let $M = \sum_\alpha M_\alpha$ be the total number of particles. It appears that for each state a sign factor can be chosen in order to make the nonzero off-diagonal elements of the Hamiltonian negative.

First, we observe from (1) and (3) that the off-diagonal part is built of the nearest-neighbor hoppings $c^+_{x\pm1,\alpha}c_{x,\alpha}$ and their products. Due to the appropriate signs of the constants $-t_x$, $-K_x$ and J_x, the positive values for all nonzero matrix elements of these hoppings imply the non-positive values for the off-diagonal matrix elements of the Hamiltonian. The correct sign factor is encoded in the following arrangement of the fermion creation operators:

$$|\{x^1\},\ldots,\{x^N\}\rangle = \prod_{\alpha=1}^{N}(c^+_{x^\alpha_1,\alpha}c^+_{x^\alpha_2,\alpha}\cdots c^+_{x^\alpha_{M_\alpha},\alpha})|0\rangle, \tag{5}$$

where the product is taken from the left to the right. In other words, we group together the particles of the same flavor ordering them according to their position. Due to the Fermi–Dirac statistics, the hopping term $c^+_{x\pm1,\alpha}c_{x,\alpha}$ acts nontrivially on the states having one and only one α-fermion per two adjacent states. In that case, it produces a similar state (5) with $c^+_{x,\alpha}$ just replaced by $c^+_{x\pm1,\alpha}$ without any additional sign factor. Therefore, the off-diagonal matrix elements of the Hamiltonian (1) in the above basis are non-positive.

The basis (5) has been used before to study the degeneracy of the ground states. It was described by Affleck and Lieb and used for the construction of non-positive basis for the antiferromagnetic $SU(N)$ Heisenberg chains [21]. An explicit expression similar to (5) was written for the extended $t-J$ and $SU(2)$ Hubbard models in [7, 8], where it was applied for the proof of the Lieb–Mattis theorem. For multicomponent $t-J$ model with $SU(N_b)\times SU(N_f)$ symmetry, the similar basis ensures the nondegeneracy of the relative ground states [22]. For $SU(4)$ Hubbard model, it has been used for the study of ground state and the proof of the Lieb–Schultz–Mattis theorem [20].

It is interesting to extract the sign factor, which was encoded in the particular arrangement of fermion operators in (5), in case of the pure Heisenberg system. Note that the Heisenberg interaction preserves the number of particles on each site. Therefore, it may be restricted to smaller subspace, where each site contains only one of $N-1$ fundamental (antisymmetric) representations of $SU(N)$. It is easy to express the basis (5) in terms of the usual Ising basis of the Heisenberg model. Set, for the simplicity, one fermion per site, which corresponds to the defining N-dimensional representation. Then the usual Ising (Potts) basis is

$$|\alpha_1,\ldots,\alpha_L\rangle = c^+_{1,\alpha_1}c^+_{2,\alpha_2}\cdots c^+_{L,\alpha_L}|0\rangle. \tag{6}$$

The Heisenberg exchange (3) acts on these states as

$$\sum_a T^a_x T^a_{x+1} = 2P_{x,x+1} - \frac{2}{n},$$

where $P_{x,y}$ is the permutation of two sites. The states (5) can be obtained by an appropriate rearrangement of the states (6). Since the fermions of the same flavor are already ordered by coordinates in (5), we get:

$$|\{x^1\},\ldots,\{x^N\}\rangle = (-1)^{p_{\alpha_1\cdots\alpha_L}}|\alpha_1,\ldots,\alpha_L\rangle, \qquad p_{\alpha_1\ldots\alpha_L} = \{\,\#(x<y)\mid \alpha_x>\alpha_y\,\}.$$

Here $p_{\alpha_1\ldots\alpha_L}$ is the number of inversions in the sequence. The nonpositive basis in this form have been used in the study of Heisenberg chains with higher symmetries [10, 23].

Note that for the systems with reflection symmetry, there is another approach referred as a reflection positivity. It can be applied for more general class of systems, like frustrated spin models [4, 25, 9]. The spin flip on the half of the lattice is performed in order to construct a new basis (distinct from (5)) from the usual one. The ground state wavefunction becomes positive in the new basis. The method is problematic for higher rank $SU(N)$ spins, since most of the multiplets, including the defining one, are not invariant under the reflection. As a result, one must either use the reflected (conjugate) representations for the half of the lattice, or confine itself to self-conjugate ones, which are not the cases considered in this article.

4 Relative ground states

The Hamiltonian is invariant on any M_α-subspace, which is made up of the basic states (5) with the same number of fermions of each type. It can be considered also as a weight subspace under the $U(N)$ action. The weight is given by the set $\{M_\alpha\}_{\alpha=1}^{N}$ composed of the eigenvalues of the diagonal generators from (4). Note that according to the Fermi–Dirac statistics, the volume of the chain must be large enough in order to contain all particles:

$$L \geq \max_{1\leq\alpha\leq N} M_\alpha. \tag{7}$$

This is the condition of the existence of M_α-subspace for the L-size chain.

It is easy to see that any two basic states from the same subspace are connected by the kinetic terms of the Hamiltonian. Then, according to the Perron–Frobenius theorem [24],

- The relative ground state of the Hamiltonian in any M_α-subspace is unique, and its all coefficients in the basis (5) are straightly positive:

$$\Omega_{M_1\ldots M_N} = \sum_{\substack{\{x^1\},\ldots,\{x^N\}\\ \#\{x^\alpha\}=M_\alpha}} \omega_{\{x^1\}\ldots\{x^N\}} |\{x^1\},\ldots,\{x^N\}\rangle, \qquad \omega_{\{x^1\}\ldots\{x^N\}} > 0. \tag{8}$$

The positivity can be used in order to trace the type of the $U(N)$ multiplet, which contains the above state. From the basis (5), we choose a trial state $\Psi_{M_1\ldots M_N}$, where the fermions of each flavor occupy successively the sites starting from the first one:

$$\Psi_{M_1\ldots M_N} = \prod_{\alpha=1}^{N} (c^+_{1,\alpha} c^+_{2,\alpha} \cdots c^+_{M_\alpha,\alpha})|0\rangle. \tag{9}$$

For the sake of completeness, below we present in terms of fermions some aspects of the representation theory of the unitary group, which are essential in the following discussions. The irreducible representations of $U(N)$ are labeled by the Young diagrams $\mathbb{Y}$ with at most N rows [27]. Every box of $\mathbb{Y}$ is associated with a single particle, and the number of boxes is equal to the number of particles. Symmetrize the flavors over the rows, then antisymmetrize over the columns. The spatial coordinates are kept fixed during this process. The states constructed in this way from all possible distributions of flavors along the boxes of $\mathbb{Y}$ form an irreducible representation of the unitary group. The states, where flavors do not decrease along the rows from left to right and increase along the columns from top to bottom, form the standard basis of the multiplet. Among them there is the highest weight state, in which the i-th rows is filled by the particles with the flavor $\alpha = i$. This fact can be easily verified acting on it by the raising generators from (4).

In case of two particles at sites $x \neq y$ the aforementioned procedure extracts the symmetric and antisymmetric multiplets:

$$(c^+_{x,\alpha} c^+_{y,\beta} + c^+_{x,\beta} c^+_{y,\alpha})|0\rangle = \begin{array}{|c|c|}\hline {}^{x}\alpha & {}^{y}\beta \\ \hline \end{array}\,, \qquad (c^+_{x,\alpha} c^+_{y,\beta} - c^+_{x,\beta} c^+_{y,\alpha})|0\rangle = \begin{array}{|c|}\hline {}^{x}\alpha \\ \hline {}^{y}\beta \\ \hline \end{array}\,.$$

The site index is mentioned at the upper left corner of the box. Since we deal with the fermions, all sites on the same row must differ in order to get a nonzero state. For the state below, the $SU(N)$ spins in the brackets are symmetrized, then the corresponding spins of each group are antisymmetrised giving rise to the presented Young tableau:

$$(c^+_{x,\alpha} c^+_{y,\beta} c^+_{z,\gamma})(c^+_{x,\mu} c^+_{z,\nu})c^+_{x,\delta}|0\rangle \quad \longrightarrow \quad \begin{array}{|c|c|c|}\hline {}^{x}\alpha & {}^{y}\beta & {}^{z}\gamma \\ \hline {}^{x}\mu & {}^{z}\nu & \multicolumn{1}{c}{} \\ \cline{1-2} {}^{x}\delta & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}\,. \tag{10}$$

Note that the symmetrization along a row is trivial if it contains particles of one species like the rows of the highest weight state. This would happen with the first row in the above tableau if $\alpha = \beta = \gamma$. Similarly, the antisymmetrization along a column is trivial if its particles are on the same site like the first column in (10).

Consider now the trial state $\Psi_{M_1 \dots M_N}$. Construct the Young tableau with the row lengths given by the set $\{M_\alpha\}$, the j-th column containing particles from the j-th site. Fill the first row by α_1-type fermions, where M_{α_1} is the largest number from the set, then the second row by α_2-type fermions, where M_{α_2} is the next largest number, and so on. As was argued above, the entire symmetrization-antisymmetrization procedure is trivial (not needed). Therefore, the trial state (9) is really given by the constructed Young tableau, and it is a part of a multiplet described by the similar Young diagram. Note that like the Fermi sea, the trial state is the most compact one: it occupies M_{α_1} sites, and due to (7) exists for any M_α-subspace. Therefore, the state (9) belongs to the multiplet related to the constructed Young diagram. Note that the last property is peculiar: the common basic state (5), in general, is not a part of a single multiplet. Below are the examples of the trial states for $SU(3)$ chain:

$$\Psi_{3,2,1} = \begin{array}{|c|c|c|}\hline {}^1 1 & {}^2 1 & {}^3 1 \\ \hline {}^1 2 & {}^2 2 \\ \cline{1-2} {}^1 3 \\ \cline{1-1}\end{array}\,, \qquad \Psi_{2,3,1} = \begin{array}{|c|c|c|}\hline {}^1 2 & {}^2 2 & {}^3 2 \\ \hline {}^1 1 & {}^2 1 \\ \cline{1-2} {}^1 3 \\ \cline{1-1}\end{array}\,, \qquad \Psi_{2,0,4} = \begin{array}{|c|c|c|c|}\hline {}^1 3 & {}^2 3 & {}^3 3 & {}^4 3 \\ \hline {}^1 1 & {}^2 1 \\ \cline{1-2}\end{array}\,.$$

In the first state, the first site of the chain is filled completely making up a singlet, the second site has two fermions with $\alpha = 1, 2$, and the third one is occupied by one $\alpha = 1$ fermion. The remaining sites are empty.

Due to $U(N)$ symmetry and orthogonality of nonequivalent representations, the projections of $\Omega_{M_1 \dots M_N}$ on different sectors corresponding to the nonequivalent multiplets produce orthogonal states with the same energy. Due to the uniqueness of the relative ground state, it must belong to only one sector. According to (8), the state $\Psi_{M_1 \dots M_N}$ being a basic state participates in the decomposition of the relative ground state $\Omega_{M_1 \dots M_N}$ overlapping it. So, both states are the members of equivalent multiplets. We conclude that

- The relative ground state state in M_α-subspace belongs to a single irreducible $U(N)$ representation characterized by the Young diagram $\mathbb{Y}_{M_\alpha}$ with row lengths given by the nonzero numbers from the set $\{M_\alpha\}$.

5 Ordering of energy levels

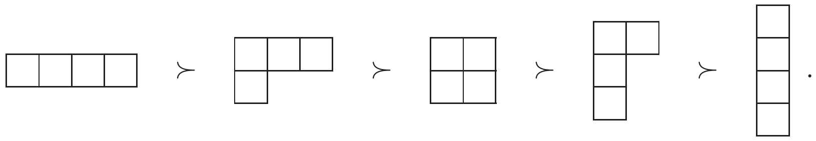
Due to $U(N)$ symmetry, the Hamiltonian of the extended Hubbard chain (1) remains invariant on the individual sectors combining the equivalent representations. These sectors are labeled by the Young diagrams. The number of boxes is the quantum number of the $U(1)$ subgroup and corresponds to the total number of particles.

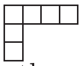
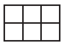
Denote by $E(\mathbb{Y})$ the lowest energy level among all multiplets of the same equivalence class $\mathbb{Y}$. In fact, the relative ground state $\Omega_{M_1 \dots M_N}$ has the lowest energy level $E(\mathbb{Y}_{M_\alpha})$ because any $\mathbb{Y}_{M_\alpha}$ multiplet has a representative in the corresponding M_α-subspace. The nondegeneracy of the level $E(\mathbb{Y}_{M_\alpha})$ follows directly form the uniqueness of the relative ground state.

Note that the relative ground states in all M'_α-subspaces obtained by rearrangements of the set $\{M_\alpha\}$ are related to the same Young diagram. In fact, all they are members of the same $\mathbb{Y}_{M_\alpha}$ multiplet [10]. This fact reflects the discrete symmetry of the Hamiltonian with respect to the permutation group of N flavors, which is a discrete subgroup of the unitary group. Therefore, one can consider without loosing the generality, the M_α-subspaces with nonascending sequences $M_1 \geq \dots \geq M_N$. The corresponding relative ground states are the highest weight states of the lowest energy $\mathbb{Y}_{M_\alpha}$ multiplet.

Consider now two different Young diagrams $\mathbb{Y}_{M_\alpha}$ and $\mathbb{Y}_{M'_\alpha}$ and try to compare the related minimal energies. Suppose that the highest weight vector of the first multiplet is also a weight (evidently, not highest) of the second one. This means that the irreducible representation generated by the relative ground state $\Omega_{M'_1 \dots M'_N}$ has also a representative in the M_α-subspace. Of course, both states differ. Then, due to the uniqueness of the relative ground state, this representative, together with the whole multiplet, has a higher energy than the $\Omega_{M_1 \dots M_N}$ has, i.e. $E(\mathbb{Y}_{M'_\alpha}) > E(\mathbb{Y}_{M_\alpha})$.

This relation introduces some ordering among the representations, which can be formulated in an elegant way in terms of Young diagrams. Namely, in this case $\mathbb{Y}_{M_\alpha}$ may be obtained from $\mathbb{Y}_{M'_\alpha}$ by the displacement of some of its boxes from the upper rows to the lower ones, which we note shortly as $\mathbb{Y}_{M'_\alpha} \succ \mathbb{Y}_{M_\alpha}$ [10]. In the representation theory of the symmetric group, this is known as a dominance order [28]. For example, for unitary group with $N \geq 4$ we have:



The dominance order is a partial one. There are Young diagrams, which are not related by this order for higher ($N > 2$) algebras and higher ($M > 5$) box numbers, like the following ones:  and . The Young diagrams with different number of boxes are not related to each other also.

In summary, we have proved that for the extended Hubbard model (1),

- The minimum energy levels in the sectors characterized by different Young diagrams satisfy the ordering rule

$$E(\mathbb{Y}_2) > E(\mathbb{Y}_1) \qquad \text{if} \quad \mathbb{Y}_2 \succ \mathbb{Y}_1; \tag{11}$$

- The levels $E(\mathbb{Y})$ are nondegenerate, up to the trivial $SU(N)$ degeneracy.

These results are in agreement with those obtained for the $SU(2)$ system by Xiang and d'Ambrumenil [8]. In that case, the Young diagram is labeled by the spin quantum number S, and the usual Lieb–Mattis ordering rule $E(S_2) > E(S_1)$ if $S_2 > S_1$ is fulfilled [3]. For the pure Heisenberg system in the defining representation, the system is reduced to the Sutherland chain [26]. A similar ordering rule for that system has been formulated and proven in [10].

The described ordering was used already in the one-dimensional many-particle quantum mechanics by Lieb and Mattis in order to compare the minimum energies of the wavefunctions with different permutation symmetries [5]. This is not surprising, because these symmetry classes are also described by the Young diagrams. The pouring principle obtained in [5] is just the "reverse" version of the energy level ordering obtained above (see (16) below). We show in the last section that for the particles with $SU(N)$ spin degrees of freedom, the spatial and spin parts of the fermionic wavefunction are described by conjugate Young diagrams. This leads to the "direct" ordering for the spins in agreement with our results above.

6 Ground state

Although the dominance order is partial, there is a lowest diagram $\mathbb{Y}_{\rm gs}$ among all diagrams containing the same amount of boxes: $\mathbb{Y} \succ \mathbb{Y}_{\rm gs}$. All columns in $\mathbb{Y}_{\rm gs}$ have the maximal length N besides the last one having

$$m = M \mod N \tag{12}$$

boxes, where M is the total box number. $\mathbb{Y}_{\rm gs}$ corresponds to the m-th order antisymmetric representation. According to the ordering rule (11), the sector defined by $\mathbb{Y}_{\rm gs}$ has the lowest energy value amon other sectors: $\mathbb{Y}_{\rm gs}$. So,

- The ground states of the extended Hubbard chain (1) with M particles form a unique $\binom{N}{m}$-dimensional antisymmetric $SU(N)$ multiplet.
- In particular, if the number of particles is the multiplicity of N, the ground state is a unique singlet.

For $SU(2)$ case, the ground state is a spin singlet for even number of particles, while for odd number it is a spin doublet in agreements with the results [8]. For $SU(3)$ symmetric chain, depending on the value of the remainder (12), the ground state is a singlet, a three-dimensional defining representation $\mathbf{3}$, or its complex conjugate one $\bar{\mathbf{3}}$. They are presented below in case of six, seven, and eight particles.

$$\mathbb{Y}^{\mathbf{0}}_{\rm gs} = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \ & \ \\ \hline \end{array}\,, \qquad \mathbb{Y}^{\mathbf{3}}_{\rm gs} = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ & \multicolumn{1}{c}{} \\ \cline{1-2} \ & \ & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\,, \qquad \mathbb{Y}^{\bar{\mathbf{3}}}_{\rm gs} = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ & \ \\ \hline \ & \ & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\,. \tag{13}$$

Note that using the described method we can not compare the ground states having different amount of particles.

Consider now the solvable free fermion case when only the hopping term survives in the Hamiltonian (1), and the ground state has a very simple form. The digitalization of the bilinear Hamiltonian is reduced to the digitalization of $L \times L$ matrix composed from the coefficients t_x. There are L energy eigenvalues ε_k, which we arrange is ascending order: $\varepsilon_1 < \cdots < \varepsilon_L$. Under the periodic boundary conditions and translation invariance, they are reduced to $\varepsilon_k = 4t\sin^2(\pi(k-1)/L)$. Note that here there is the twofold degeneracy $\varepsilon_k = \varepsilon_{L-k}$ due to the reflection invariance. This degeneracy is removed in the general case, but the degeneracy on the flavor quantum number still remains. In the ground state with M-fermion, all lowest levels ε_k are completely filled by N fermions of different flavors up to the Fermi level k_F, which is filled partially by $m = M - k_F N$ particles. The fermions with the completely filled levels form, of course, singlets, while the remaining fermions form m-th order antisymmetric multiplet on the Fermi level. This is exactly the same picture as we obtained for the interacting system. In the examples considered in (13), the ground states for noninteracting system are described by the following Young tableaux:

$$\Omega^{\mathbf{0}}_{\rm gs} = \begin{array}{|c|c|}\hline {}^1 1 & {}^2 1 \\ \hline {}^1 2 & {}^2 2 \\ \hline {}^1 3 & {}^2 3 \\ \hline \end{array}\,, \qquad \left(\Omega^{\mathbf{3}}_{\rm gs}\right)_\alpha = \begin{array}{|c|c|c|}\hline {}^1 1 & {}^2 1 & {}^3 \alpha \\ \hline {}^1 2 & {}^2 2 & \multicolumn{1}{c}{} \\ \cline{1-2} {}^1 3 & {}^2 3 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\,, \qquad \left(\Omega^{\bar{\mathbf{3}}}_{\rm gs}\right)_{\alpha\beta} = \begin{array}{|c|c|c|}\hline {}^1 1 & {}^2 1 & {}^3 \alpha \\ \hline {}^1 2 & {}^2 2 & {}^3 \beta \\ \hline {}^1 3 & {}^2 3 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\,.$$

Now the number at the upper left corners is the energy quantum number k, but not the space position as before. The α, β are $SU(3)$ flavors which label the three states of the multiplet.

The relative ground states $\Omega_{M_1 \ldots M_N}$ can be constructed in the same way. First choose the largest number M_α from the set and fill the first level by M_α fermions of flavor α, then choose the next largest number M_β and fill the second level by M_β fermions of flavor β, and so on. This state is similar to the trial state (9) used to detect the type of the representation of $\Omega_{M_1 \ldots M_N}$ for the interacting system. The only difference is that again, instead of coordinates one must use the energy quantum number. The $SU(N)$ structure of these tree states is the same. Moreover, the relative ground state of noninteracting system can be used as a trial state instead of (9) as it was done for $SU(2)$ Hubbard model [5, 8]. Indeed, it overlaps with the interacting state since both states are positive in the basis (5) (see (8)).

7 Energy level ordering for the quantum mechanical system with interacting $SU(N)$ fermions

Consider the quantum mechanical system of one-dimensional identical fermions with $SU(N)$ internal degrees of freedom and the interaction depending on the spatial coordinates only:

$$H_{\mathrm{QM}} = -\frac{1}{2m}\sum_{i=1}^{M}\frac{\partial^2}{\partial x_i^2} + V(x_1, \ldots, x_M). \tag{14}$$

The potential is invariant under the particle exchange and must be integrable. For the $SU(2)$ system, Lieb and Mattis proved the antiferromagnetic ordering of the energy levels [5] (see also [29]). In this section we apply their result to higher unitary symmetries and discuss the relation with the results obtained in the previous sections.

For the distinguishable particles, the spatial and spin degrees of freedom are decoupled. The stationary states are the products of the spatial and spin parts, and the spectrum is determined by the former. Due to the exchange invariance of the Hamiltonian H_{QM}, the spatial eigenfunctions are classified by their symmetries with respect to the permutations $\mathcal{S}_M^{\mathrm{space}}$ of the spatial coordinates. According to the representation theory of the symmetric group, the symmetry classes are described by Young diagrams $\mathbb{Y}'$ [5, 30, 27]. For a given distribution of the coordinates along the boxes, it defines a similar symmetrization-antisymmetrization procedure as for $SU(N)$ case. The defined map is a projector, and the projectors constructed from the different Young diagrams are mutually orthogonal. The function $\phi_{\mathbb{Y}'}(x_1, \ldots, x_M)$ has the symmetry class $\mathbb{Y}'$, if the associated projector does not change it. It generates an irreducible representation of the symmetric group. The standard basic states correspond to the distributions, in which the indexes increase along the rows from left to right and along the columns from top to bottom. The totally symmetric and antisymmetric representations are one-dimensional and are described by one-row and one-column Young diagram respectively, while the others have higher dimensions.

The function $\phi_{\mathbb{Y}'}(x_1, \ldots, x_M)$ is separately antisymmetric in the variables related to the same column and satisfies the following equations:

$$\left[1 - \sum_{j:\,\mathrm{col}(j)=c} P_{ij}\right]\phi_{\mathbb{Y}'}(x_1, \ldots, x_M) = 0 \qquad \text{for any } i \text{ with } \mathrm{col}(i) > c. \tag{15}$$

Here $\mathrm{col}(i)$ is the index of the column in $\mathbb{Y}'$ containing x_i, these are counted from left to right. P_{ij} permutes x_i and x_j. These relations mean that the antisymmetrization of the column variables with any variable located on the right hand side must vanish, since it has been already symmetrized with a variable from that column in the Young projector.

Denote now by $E(\mathbb{Y}')$ the lowest energy level among the states, which belong to the symmetry class defined by $\mathbb{Y}'$. These levels are nondegenerate (up to the coordinate permutations) and obey the following ordering rule [5]:

$$E(\mathbb{Y}'_1) > E(\mathbb{Y}'_2) \qquad \text{if} \quad \mathbb{Y}'_1 \prec \mathbb{Y}'_2. \tag{16}$$

In particular, the highest energy level is totally antisymmetric in spatial coordinates, while the lowest level is totally symmetric. Here and in the following, the notation $\phi_{\mathbb{Y}'}$ will be used for the lowest-energy state with $\mathbb{Y}'$ symmetry.

For indistinguishable fermions, according to the Pauli exclusion principle, the wavefunction of the entire system must be antisymmetric under the interchange of individual particles. This can be achieved by the selection of appropriate spin wavefunctions. For the symmetric spatial part, the spin part must me antisymmetric, and vice versa. For more general symmetry classes, the entire wavefunction is “entangled”, i.e. a superposition of the products of spin and spatial parts. The general construction can be figured out using the representation theory.

The Hamiltonian (14) has the trivial unitary $SU(N)$ and permutation $\mathcal{S}_M^{\text{spin}}$ symmetries, which act on the spin variables and are mutually independent. The representations of both groups are classified by the Young diagrams $\mathbb{Y}$ with with M boxes and at most N rows. According to the Schur–Weyl duality, their joint action decomposes into a direct sum of tensor products of irreducible modules:

$$\sum_{\mathbb{Y}} \pi_{\mathbb{Y}} \otimes \rho_{\mathbb{Y}}.$$

Here $\pi_{\mathbb{Y}}$ and $\rho_{\mathbb{Y}}$ are the irreducible representations of $\mathcal{S}_M^{\text{spin}}$ and $SU(N)$ correspondingly. This property implies, in particular, that the highest weight vectors of all $SU(N)$ multiplets of type $\rho_{\mathbb{Y}}$ form an irreducible $\mathcal{S}_M^{\text{spin}}$-representation of type $\pi_{\mathbb{Y}}$. Similarly, the action of the common symmetry group $\mathcal{S}_M^{\text{space}} \times \mathcal{S}_M^{\text{spin}} \times SU(N)$ on the total space of the states decomposes as

$$\sum_{\mathbb{Y}',\mathbb{Y}} \pi_{\mathbb{Y}'} \otimes \pi_{\mathbb{Y}} \otimes \rho_{\mathbb{Y}}. \tag{17}$$

Here $\pi_{\mathbb{Y}'}$ the (unique) irreducible representation of the symmetric group formed by the lowest-energy spatial wavefunctions with the symmetry class $\mathbb{Y}'$. The physical space of states in our case is the subspace of (17) formed by the antisymmetric wavefunctions. It forms the antisymmetric representation of the symmetric group corresponding to the image of the injective diagonal homomorphism $\mathcal{S}_M \rightarrowtail \mathcal{S}_M^{\text{space}} \times \mathcal{S}_M^{\text{spin}}$. The representation $\pi_{\mathbb{Y}'} \otimes \pi_{\mathbb{Y}}$ can be treated as a tensor product (or Kronecker product) of two $\mathcal{S}_M$ modules. According to the representation theory of the symmetric group, the antisymmetric representation appears only in the tensor product of two conjugate representations, while the symmetric representation appears in the products of two equivalent ones [5, 27]. Their multiplicities are equal to one. The conjugate Young diagrams $\mathbb{Y}$ and $\tilde{\mathbb{Y}}$ are reflections of each other with respect to the main diagonal, i.e. the rows (columns) of $\mathbb{Y}$ are replaced by columns (rows) of $\tilde{\mathbb{Y}}$. Both $\pi_{\mathbb{Y}}$ and $\pi_{\tilde{\mathbb{Y}}}$ have the same dimension, and one can obtained from another by multiplication on the antisymmetric representation.

So, only the terms with $\mathbb{Y}' = \tilde{\mathbb{Y}}$ are relevant for the fermionic states in the sum (17). Extracting them and applying the formal Clebsch–Gordan series, we arrive at

$$\sum_{\mathbb{Y}} \pi_{\tilde{\mathbb{Y}}} \otimes \pi_{\mathbb{Y}} \otimes \rho_{\mathbb{Y}} = \sum_{\mathbb{Y}} \left[\pi_{\text{asym}}^{\tilde{\mathbb{Y}}\times\mathbb{Y}} \otimes \rho_{\mathbb{Y}} + \sum_{\mathbb{Y}'\neq\text{asym}} \pi_{\mathbb{Y}'}^{\tilde{\mathbb{Y}}\times\mathbb{Y}} \otimes \rho_{\mathbb{Y}} \right].$$

Thus, the fermionic wavefunctions constructed from $\phi_{\tilde{\mathbb{Y}}}$ form a unique $\rho_{\mathbb{Y}}$-type $SU(N)$ multiplet. Its highest weight state $\Phi_{\text{asym}}^{\tilde{\mathbb{Y}}\times\mathbb{Y}}$ has a simple “canonical” form, which was constructed and applied in case of $SU(2)$ fermions in [5].

Let the M_i be the length of the i-th row of $\mathbb{Y}$. Suppose, for certainty, that the first M_1 variables of $\phi_{\tilde{\mathbb{Y}}}(x_1, \ldots, x_M)$ are positioned on the first column in the downward direction, the second M_2 ones are on the second column, and so on. Then we have (up to a normalization factor):

$$\Phi_{\text{asym}}^{\tilde{\mathbb{Y}}\times\mathbb{Y}} = \sum_{P\in\mathcal{S}_M} (-1)^P \pi_{\tilde{\mathbb{Y}}}(P) \phi_{\tilde{\mathbb{Y}}}(x_1, \ldots, x_M) \pi_{\mathbb{Y}}(P) |\underbrace{1\ldots1}_{M_1}\,\underbrace{2\ldots2}_{M_2}\ldots\ldots\underbrace{N\ldots N}_{M_N}\rangle. \tag{18}$$

It belongs to the Kronecker product $\pi_{\tilde{\mathbb{Y}}} \otimes \pi_{\mathbb{Y}}$ and does not vanish, since the coefficient in front of the ordered spin state is proportional to $\phi_{\tilde{\mathbb{Y}}}(x_1, \ldots, x_M)$. Due to the factor $(-1)^P$ (the parity of P), the above state is totally antisymmetric under an interchange of two particles. Finally, its $SU(N)$ weight coincides with the highest weight of $\rho_{\mathbb{Y}}$. From the uniqueness condition, we conclude that the expression (18) is correct.

The fact that the wavefunction (18) is the highest weight state, i.e. the spin-raising generators (given in (4), $\alpha < \beta$) annihilate it, can be verified independently by direct calculations. Below we demonstrate this for F^{12}, the other generators can be handled in the same way.

$$\begin{aligned}
\rho_{\mathbb{Y}}\big(F^{12}\big)\Phi^{\tilde{\mathbb{Y}}\times\mathbb{Y}}_{\rm asym} &= \sum_{P\in\mathcal{S}_M}(-1)^P P\phi_{\tilde{\mathbb{Y}}}(x_1,\dots,x_M)\sum_{i=M_1+1}^{M_2} P\mathcal{F}^{12}_i|\underbrace{1\dots1}_{M_1}\underbrace{2\dots2}_{M_2}\dots\rangle \\
&= \sum_{P\in\mathcal{S}_M}(-1)^P P\phi_{\tilde{\mathbb{Y}}}(x_1,\dots,x_M)P\left[1+\sum_{i=M_1+2}^{M_2}P_{M_1+1\,i}\right]|\underbrace{1\dots1}_{M_1+1}\underbrace{2\dots2}_{M_2-1}\dots\rangle \qquad (19)\\
&= \sum_{P\in\mathcal{S}_M}(-1)^P P\left[1-\sum_{i=M_1+2}^{M_2}P_{M_1+1\,i}\right]\phi_{\tilde{\mathbb{Y}}}(x_1,\dots,x_M)P|\underbrace{1\dots1}_{M_1+1}\underbrace{2\dots2}_{M_2-1}\dots\rangle = 0.
\end{aligned}$$

Here the matrix $\mathcal{F}^{12}_i$ acts on i-th spin. In the first equation we have used the commutativity of unitary and symmetric groups. The second equation employs the definition of $\mathcal{F}^{12}$ in (4). In the third equation we have changed the summation index $\left[\sum_P \to \sum_{PP_{M_1+1\,i}}\right]$, which alters also the sing in the square brackets since $(-1)^{PP_{M_1+1\,i}} = -(-1)^P$. The last equation in (19) is the consequence of the relations (15).

So, we come to the conclusion that the spin and spatial parts of the fermionic wavefunction are described by the conjugate Young diagrams. It is clear that the conjugation inverts the dominance order: if $\mathbb{Y}_1 \succ \mathbb{Y}_2$ then $\tilde{\mathbb{Y}}_1 \prec \tilde{\mathbb{Y}}_2$ and vice versa [5]. Therefore, in terms of the $SU(N)$ representations, the "reverse" ordering rule (16) changes to the "direct" one, which corresponds to the antiferromagnetic ordering (11) established for $SU(N)$ Hubbard chain in the previous sections. For the Sutherland chain the similar ordering has been established in [10]. Recall that the spin Young diagrams have no more than N rows, hence the spatial ones must have no more than N columns. For usual $SU(2)$ spin they correspond to two-row and two-column diagrams respectively [5].

Note that for the indistinguishable bosons, the reverse ordering rules (16) takes place. The total wavefunctionnow must be symmetric, which must be composed from the equivalent representations of the spatial and spin symmetric groups (must apply $\mathbb{Y}' = \mathbb{Y}$ in (17), replace $\tilde{\mathbb{Y}} \to \mathbb{Y}$ and omit the parity factor in (18)).

Finally, we mention that although the ordering of energy levels for both lattice (1) and quantum mechanical (14) systems are similar, there is an essential difference between these two models. The second system has permutation symmetry with respect to the coordinate exchange, which leads to the separation of the spin and spatial degrees of freedom. The second system does not possess the spacial symmetry at all (despite of possible reflection invariance in case of the appropriate choice of constants), it has spin interactions, and the lowest-energy state (8) can not presented in a factorized form like (18).

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