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Higher Order Connections*

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Abstract. The purpose of this article is to present the theory of higher order connections on vector bundles from a viewpoint inspired by projective differential geometry.

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1 Introduction

We begin with a few well-known remarks on commonplace linear connections. Let E denote a smooth vector bundle on a smooth manifold M (throughout this article we work in the smooth category but all constructions go through *mutatis mutandis* in the holomorphic category). A *connection* on E may be defined as a splitting of the first jet exact sequence [13]

$$0 \to \Lambda^1 \otimes E \to J^1E \to E \to 0,$$

where Λ^1 is the bundle of 1-forms on M. Equivalently, a connection on E is a first order linear differential operator

$$\nabla : E \to \Lambda^1 \otimes E$$

whose symbol $\Lambda^1 \otimes E \to \Lambda^1 \otimes E$ is the identity. A connection on E induces a natural differential operator

$$\nabla : \Lambda^1 \otimes E \to \Lambda^2 \otimes E, \ \text{ characterised by } \nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla s$$

and the composition

$$E \xrightarrow{\nabla} \Lambda^1 \otimes E \xrightarrow{\nabla} \Lambda^2 \otimes E$$

is a homomorphism of vector bundles called the *curvature* of ∇.

In this article, a k^{th} *order connection* on E is a splitting of the $k^{\rm th}$ jet exact sequence

$$0 \to \bigodot^k \Lambda^1 \otimes E \to J^kE \to J^{k-1}E \to 0, \tag{1}$$

where $\bigodot^k \Lambda^1$ is the $k^{\rm th}$ symmetric power of Λ^1. Libermann [11, p. 155] considers these higher order connections (more their *semi-holonomic* counterparts) but does not pursue them so much along the lines done below. Other notions of higher order connections are due to various authors including Ehresmann [8], Virsik [14], and Yuen [15]. A $k^{\rm th}$ order connection, as above, is evidently equivalent to a $k^{\rm th}$ order linear differential operator

$$\nabla^{(k)} : E \to \bigodot^k \Lambda^1 \otimes E$$

whose symbol $\bigodot^k \Lambda^1 \otimes E \to \bigodot^k \Lambda^1 \otimes E$ is the identity. In the rest of this article, we motivate and extend the notion of curvature *et cetera* to these higher order connections. On the way, we shall encounter various useful constructions and assemble evidence for a final conjecture.

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2 Interlude on projective geometry

Let ∂_a denote the usual partial differential operator $\partial/\partial x^a$ on $\mathbb{R}^n$ with coördinates x^a. If a smooth 1-form ω_a is obtained as the exterior derivative $\partial_a f$ of a smooth function f, then $\partial_a\omega_b$ is necessarily symmetric in its indices and, conversely, this condition is locally sufficient to ensure that $\omega_a = \partial_a f$ for some f. Otherwise said, if we denote the skew part of a tensor by enclosing the relevant indices in square brackets, then being in the kernel of the operator $\omega_a \mapsto \partial_{[a}\omega_{b]}$ is the local integrability condition for the range of $f \mapsto \partial_a f$. Of course, these operators are the first two in the de Rham complex

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^{n-1} \xrightarrow{d} \Lambda^n.$$

Now consider the differential operator on $\mathbb{R}^n$

$$d^{(k)} : \Lambda^0 \to \bigodot^k \Lambda^1 \text{ given by } f \mapsto \partial_a\partial_b\cdots\partial_c f \ \ (k \text{ derivatives}).$$

Evidently, if a symmetric covariant tensor $\omega_{ab\cdots c}$ with k indices is of the form $\partial_a\partial_b\cdots\partial_c f$ for some f, then $\partial_{[a}\omega_{b]c\cdots d} = 0$. Just as in the case $k = 1$, this necessary condition is also locally sufficient to identify the range of $d^{(k)}$. This is a result from projective differential geometry, a full discussion of which may be found in [7]. Here, suffice it to give the following derivation. Let us define a connection ∇_a on the bundle $\mathbb{T} \equiv \Lambda^0 \oplus \Lambda^1$ by

$$\nabla_a \begin{bmatrix} f \\ \mu_b \end{bmatrix} \equiv \begin{bmatrix} \partial_a f - \mu_a \\ \partial_a \mu_b \end{bmatrix}.$$

Notice that

$$\nabla_a\nabla_b \begin{bmatrix} f \\ \mu_c \end{bmatrix} = \nabla_a \begin{bmatrix} \partial_b f - \mu_b \\ \partial_b \mu_c \end{bmatrix} = \begin{bmatrix} \partial_a(\partial_b f - \mu_b) - \partial_b\mu_a \\ \partial_a\partial_b \mu_c \end{bmatrix} = \begin{bmatrix} \partial_a\partial_b f - \partial_a\mu_b - \partial_b\mu_a \\ \partial_a\partial_b \mu_c \end{bmatrix}$$

is symmetric in ab. In other words, the connection ∇_a on $\mathbb{T}$ is flat. It follows immediately, that the coupled de Rham complex

$$\Lambda^0 \otimes \mathbb{T} \xrightarrow{\nabla} \Lambda^1 \otimes \mathbb{T} \xrightarrow{\nabla} \Lambda^2 \otimes \mathbb{T} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Lambda^{n-1} \otimes \mathbb{T} \xrightarrow{\nabla} \Lambda^n \otimes \mathbb{T}$$

is locally exact. In particular, we conclude that locally

$$\begin{bmatrix} \phi_a \\ \omega_{ab} \end{bmatrix} = \begin{bmatrix} \partial_a f - \mu_a \\ \partial_a \mu_b \end{bmatrix} \text{ for some } \begin{bmatrix} f \\ \mu_b \end{bmatrix} \text{ if and ony if } \begin{bmatrix} \partial_{[a}\phi_{b]} + \omega_{[ab]} \\ \partial_{[a}\omega_{b]c} \end{bmatrix} = 0.$$

In particular, if we take $\phi_a = 0$ and ω_{ab} to be symmetric, then this statement reads

$$\omega_{ab} = \partial_a\partial_b f \text{ for some } f \text{ if and only if } \partial_{[a}\omega_{b]c} = 0,$$

as required in case $k = 2$. The general case may be similarly derived from the induced flat connection on $\bigodot^{k-1}\mathbb{T}$. In order further to untangle the consequences of the local exactness of these coupled de Rham sequences, it is useful to define various additional tensor bundles. Let $\Theta^{p,q}$ be the bundle (used here on $\mathbb{R}^n$ but let us maintain the same notation on a general manifold) whose sections are covariant tensors satisfying the following symmetries

$$\phi_{\underbrace{a\cdots b}_{p}\underbrace{cd\cdots e}_{q}} = \phi_{[a\cdots b](cd\cdots e)} \text{ such that } \phi_{[a\cdots bc]d\cdots e} = 0,$$

where enclosing indices in round brackets means to take the symmetric part. These include the bundles we have encountered so far

$$\Lambda^p = \Theta^{p,0} \quad \text{and} \quad \bigodot^k\Lambda^1 = \Theta^{1,k-1}$$

and also accommodate the local integrability conditions for the range of $f \mapsto \partial_a\partial_b\cdots\partial_c f$. Sorting out the meaning of local exactness for the coupled de Rham complex $\Lambda^\bullet \otimes \bigodot^{k-1}\mathbb{T}$, we find that there are locally exact complexes on $\mathbb{R}^n$

$$\Lambda^0 \xrightarrow{\partial^{(k)}} \Theta^{1,k-1} \xrightarrow{\partial} \Theta^{2,k-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Theta^{n-1,k-1} \xrightarrow{\partial} \Theta^{n,k-1} \tag{2}$$

for all $k \geq 1$ with the case $k = 1$ being the de Rham complex itself. Details are left to the reader. These complexes are special cases of the Bernstein–Gelfand–Gelfand (BGG) complex on real projective space $\mathbb{RP}_n$ viewed in a standard affine coördinate patch $\mathbb{R}^n \hookrightarrow \mathbb{RP}_n$. Details may be found in [7]. An independent construction was given by Olver [12].

3 Curvature

The integrability conditions found in § 2 provide the motivation for the following construction.

Theorem 1. *A k^{th} order connection $\nabla^{(k)} : E \to \bigodot^k\Lambda^1 \otimes E = \Theta^{1,k-1} \otimes E$ canonically induces a first order operator $\nabla : \Theta^{1,k-1} \otimes E \to \Theta^{2,k-1} \otimes E$ characterised by the following two properties*

- *its symbol $\Lambda^1 \otimes \bigodot^k\Lambda^1 \otimes E \to \Theta^{2,k-1} \otimes E$ is $\delta \otimes \mathrm{Id}$ where $\delta : \Lambda^1 \otimes \bigodot^k\Lambda^1 \to \Theta^{2,k-1}$ is the tensorial homomorphism $\phi_{abc\cdots d} \xmapsto{\delta} \phi_{[ab]c\cdots d}$;*
- *the composition $E \xrightarrow{\nabla^{(k)}} \bigodot^k\Lambda^1 \otimes E \xrightarrow{\nabla} \Theta^{2,k-1} \otimes E$ has order $k-1$.*

Proof. If we choose an arbitrary local trivialisation of E and local coördinates on M, then

$$s \xmapsto{\nabla^{(k)}} \overbrace{\partial_{(b}\partial_c\partial_d\cdots\partial_{e)}}^{k} s + \Gamma_{bcd\cdots e}{}^{fg\cdots h}\overbrace{\partial_f\partial_g\cdots\partial_h}^{k-1} s + \text{lower order terms}$$

for a uniquely defined tensor $\Gamma_{bcd\cdots e}{}^{fg\cdots h}$ symmetric in both its lower and upper indices and having values in $\mathrm{End}(E)$. But then

$$\omega_{bcd\cdots e} \xmapsto{\nabla} \partial_{[a}\omega_{b]cd\cdots e} + \Gamma_{cd\cdots e[a}{}^{fg\cdots h}\omega_{b]fg\cdots h}$$

is forced by the two characterising properties of ∇. ■

Definition 1. We shall refer to the composition

$$E \xrightarrow{\nabla\circ\nabla^{(k)}} \Theta^{2,k-1} \otimes E$$

as the *curvature* of $\nabla^{(k)}$. Of course, when $k = 1$ this is the usual notion of curvature $E \to \Lambda^2 \otimes E$ for a (first order) connection on E. The operator $\partial^{(k)}$ on $\mathbb{R}^n$ has zero curvature.

An alternative construction, both of the operator $\nabla : \bigodot^k\Lambda^1 \otimes E \to \Theta^{2,k-1} \otimes E$ and the curvature $E \to \Theta^{2,k-1} \otimes E$, may be given by expressing higher order connections in terms of commonplace connections on the jet bundle $J^{k-1}E$ as follows. Recall that the *Spencer operator* is a canonically defined first order differential operator $\mathcal{S} : J^\ell E \to \Lambda^1 \otimes J^{\ell-1}E$ characterised by the following properties [9, Propositions 4 and 5]:

- its symbol is $\Lambda^1 \otimes J^\ell E \xrightarrow{\mathrm{Id}\otimes\pi} \Lambda^1 \otimes J^{\ell-1}E$ where π is the canonical jet projection;
- the sequence $E \xrightarrow{j^\ell} J^\ell E \xrightarrow{\mathcal{S}} \Lambda^1 \otimes J^{\ell-1}E$, where j^ℓ is the universal ℓ^{th} order differential operator, is locally exact.

As a splitting of (1), we may regard a k^{th} order connection as a homomorphism $h : J^{k-1}E \to J^kE$ such that $\pi \circ h = \mathrm{Id}$. Composing with the Spencer operator

$$J^{k-1}E \xrightarrow{h} J^kE \xrightarrow{\mathcal{S}} \Lambda^1 \otimes J^{k-1}E \tag{3}$$

gives a first order differential operator $\nabla \equiv \mathcal{S} \circ h$ whose symbol $\Lambda^1 \otimes J^{k-1}E \to \Lambda^1 \otimes J^{k-1}E$ is the identity, in order words a connection on $J^{k-1}E$. (In the holomorphic category, Jahnke and Radloff already observed [10] that a splitting of the jet exact sequence (1) implied the vanishing of the Atiyah obstruction [2] to $J^{k-1}E$ admitting a connection but (3) is stronger in actually creating the desired connection.)

Theorem 2. *A k^{th} order connection on a vector bundle E induces a commonplace connection on the jet bundle $J^{k-1}E$ with the following properties*

- *the composition $J^{k-1}E \xrightarrow{\nabla} \Lambda^1 \otimes J^{k-1}E \xrightarrow{\mathrm{Id}\otimes\pi} \Lambda^1 \otimes J^{k-2}E$ is the Spencer operator;*
- *its curvature $\kappa : J^{k-1}E \to \Lambda^2 \otimes J^{k-1}E$ has values in $\Lambda^2 \otimes \bigodot^{k-1}\Lambda^1 \otimes E \hookrightarrow \Lambda^2 \otimes J^{k-1}E$.*

Conversely, a connection on $J^{k-1}E$ with these two properties uniquely characterises a k^{th} order connection on E.

Proof. As observed in [9], the Spencer operator induces first order differential operators

$$\mathcal{S} : \Lambda^1 \otimes J^\ell E \to \Lambda^2 \otimes J^{\ell-1}E \text{ defined by } \mathcal{S}(\omega \otimes s) = d\omega \otimes \pi s - \omega \wedge \mathcal{S}s \tag{4}$$

and there is a commutative diagram [9, (31)]

$$\begin{array}{ccccccccc}
 & & 0 & & E & = & E & & \\
 & & \downarrow & & {\scriptstyle j^k}\downarrow & & {\scriptstyle j^{k-1}}\downarrow & & \\
0 & \to & \bigodot^k\Lambda^1 \otimes E & \to & J^kE & \xrightarrow{\pi} & J^{k-1}E & \to & 0 \\
 & & \downarrow & & {\scriptstyle \mathcal{S}}\downarrow & & {\scriptstyle \mathcal{S}}\downarrow & & \\
0 & \to & \Lambda^1 \otimes \bigodot^{k-1}\Lambda^1 \otimes E & \to & \Lambda^1 \otimes J^{k-1}E & \xrightarrow{\mathrm{Id}\otimes\pi} & \Lambda^1 \otimes J^{k-2}E & \to & 0 \\
 & & \downarrow & & {\scriptstyle \mathcal{S}}\downarrow & & {\scriptstyle \mathcal{S}}\downarrow & & \\
0 & \to & \Lambda^2 \otimes \bigodot^{k-2}\Lambda^1 \otimes E & \to & \Lambda^2 \otimes J^{k-2}E & \xrightarrow{\mathrm{Id}\otimes\pi} & \Lambda^2 \otimes J^{k-3}E & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{5}$$

with exact rows of vector bundle homomorphisms and locally exact columns of linear differential operators (apart from the first column, which consists of homomorphisms starting with $-\iota \otimes \mathrm{Id}$ where $\iota : \bigodot^k\Lambda^1 \hookrightarrow \Lambda^1 \otimes \bigodot^{k-1}\Lambda^1$ is the natural inclusion). For the first characterising property of ∇, we compute from (3) and (5):

$$(\mathrm{Id} \otimes \pi) \circ \nabla = (\mathrm{Id} \otimes \pi) \circ \mathcal{S} \circ h = \mathcal{S} \circ \pi \circ h = \mathcal{S} \circ \mathrm{Id} = \mathcal{S},$$

as required. To compute the curvature κ of ∇ we must consider the induced operator

$$\nabla : \Lambda^1 \otimes J^{k-1}E \to \Lambda^2 \otimes J^{k-1}E \text{ defined by } \nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla s.$$

Composing this formula with $\mathrm{Id} \otimes \pi : \Lambda^2 \otimes J^{k-1}E \to \Lambda^2 \otimes J^{k-2}E$ gives

$$(\mathrm{Id} \otimes \pi) \circ \nabla(\omega \otimes s) = d\omega \otimes \pi s - \omega \wedge (\mathrm{Id} \otimes \pi) \circ \nabla s = d\omega \otimes \pi s - \omega \wedge \mathcal{S}s,$$

by the first property of ∇ established above. From (4) we conclude that

$$(\mathrm{Id} \otimes \pi) \circ \nabla = \mathcal{S} : \Lambda^1 \otimes J^{k-1}E \to \Lambda^2 \otimes J^{k-2}E. \tag{6}$$

Therefore,

$$(\mathrm{Id}\otimes\pi)\circ\kappa=(\mathrm{Id}\otimes\pi)\circ\nabla\circ\nabla=\mathcal{S}\circ\nabla=\mathcal{S}\circ\mathcal{S}\circ h=0,\quad\text{because }\mathcal{S}\circ\mathcal{S}=0.$$

From the exactness of $0\to\Lambda^2\otimes\bigodot^{k-1}\Lambda^1\otimes E\to\Lambda^2\otimes J^{k-1}E\to\Lambda^2\otimes J^{k-2}E\to 0$, it follows that κ takes values in $\Lambda^2\otimes\bigodot^{k-1}\Lambda^1\otimes E$, as required.

Conversely, given a connection ∇ on $J^{k-1}E$ satisfying the two properties in the statement of the theorem, let us define a k^{th} order differential operator

$$\nabla^{(k)}:E\to\Lambda^1\otimes J^{k-1}E\ \text{ as the composition }\nabla\circ j^{k-1}.$$

From the first property of ∇ we observe that

$$(\mathrm{Id}\otimes\pi)\circ\nabla^{(k)}=(\mathrm{Id}\otimes\pi)\circ\nabla\circ j^{k-1}=\mathcal{S}\circ j^{k-1}=0$$

and, with reference to (5), deduce that, in fact,

$$\nabla^{(k)}:E\to\Lambda^1\otimes\bigodot^{k-1}\Lambda^1\otimes E.$$

It is easy to check that its symbol $\bigodot^k\Lambda^1\otimes E\to\Lambda^1\otimes\bigodot^{k-1}\Lambda^1\otimes E$ is the natural inclusion $\iota\otimes\mathrm{Id}$. Therefore, to show that $\nabla^{(k)}$ is, in fact, a k^{th} order connection, it suffices to show that $\nabla^{(k)}$ takes values in $\bigodot^k\Lambda^1\otimes E$ and, with reference to (5), for this it suffices to show that $\mathcal{S}\circ\nabla^{(k)}=0$. For this we may compute using (6) and our definition of $\nabla^{(k)}$:

$$\mathcal{S}\circ\nabla^{(k)}=(\mathrm{Id}\otimes\pi)\circ\nabla\circ\nabla\circ j^{k-1}=(\mathrm{Id}\otimes\pi)\circ\kappa\circ j^{k-1}=0\circ j^{k-1}=0,$$

as required. Finally, we must check that this construction does indeed provide an inverse to setting $\nabla\equiv\mathcal{S}\circ h$. From (5), the usual splitting rigmarole produces

$$j^k-h\circ j^{k-1}:E\to\bigodot^k\Lambda^1\otimes E\hookrightarrow J^kE.$$

But, viewing via $\bigodot^k\Lambda^1\otimes E\xrightarrow{\iota\otimes\mathrm{Id}}\Lambda^1\otimes\bigodot^{k-1}\Lambda^1\otimes E\hookrightarrow\Lambda^1\otimes J^{k-1}E$ (as we were doing) gives

$$-\mathcal{S}\circ(j^k-h\circ j^{k-1})=\mathcal{S}\circ h\circ j^{k-1}=\nabla\circ j^{k-1}.$$

Therefore, the combined effect of $\nabla^{(k)}\rightsquigarrow h\rightsquigarrow\nabla\equiv\mathcal{S}\circ h\rightsquigarrow\nabla^{(k)}\equiv\nabla\circ J^{(k-1)}$ is to end up back where we started. To check that $\nabla\rightsquigarrow\nabla^{(k)}\equiv\nabla\circ J^{(k-1)}\rightsquigarrow h\rightsquigarrow\nabla\equiv\mathcal{S}\circ h$ is also the identity is a similar unravelling of definitions and is left to the reader. ■

Any construction starting with a k^{th} order connection $\nabla^{(k)}:E\to\bigodot^k\Lambda^1\otimes E$ may, of course, be carried out using a commonplace connection on the jet bundle $J^{k-1}E$ in accordance with Theorem 2. Consider, for example, the composition

$$\bigodot^k\Lambda^1\otimes E\xrightarrow{\iota\otimes\mathrm{Id}}\Lambda^1\otimes\bigodot^{k-1}\Lambda^1\otimes E\hookrightarrow\Lambda^1\otimes J^{k-1}E\xrightarrow{\nabla}\Lambda^2\otimes J^{k-1}E \tag{7}$$

where $\Lambda^1\otimes J^{k-1}E\xrightarrow{\nabla}\Lambda^2\otimes J^{k-1}E$ is the usual induced first order operator. By (6), if we further compose with $\mathrm{Id}\otimes\pi:\Lambda^2\otimes J^{k-1}E\to\Lambda^2\otimes J^{k-2}E$ then we obtain

$$\bigodot^k\Lambda^1\otimes E\xrightarrow{\iota\otimes\mathrm{Id}}\Lambda^1\otimes\bigodot^{k-1}\Lambda^1\otimes E\hookrightarrow\Lambda^1\otimes J^{k-1}E\xrightarrow{\mathcal{S}}\Lambda^2\otimes J^{k-2}E,$$

which vanishes by dint of (5). Therefore (7) actually has range in $\Lambda^2\otimes\bigodot^{k-1}\otimes E$ and, in fact, has range in $\Theta^{2,k-1}\otimes E$ as follows. According to the commutative square

$$\begin{array}{ccc}\Lambda^2\otimes\bigodot^{k-1}\Lambda^1\otimes E&\hookrightarrow&\Lambda^2\otimes J^{k-1}E\\ \downarrow&&\mathcal{S}\downarrow\\ \Lambda^3\otimes\bigodot^{k-2}\Lambda^1\otimes E&\hookrightarrow&\Lambda^3\otimes J^{k-2}E,\end{array}$$

we must show that composing (7) with $\mathcal{S} : \Lambda^2 \otimes J^{k-1}E \to \Lambda^3 \otimes J^{k-2}E$ gives zero. But

$$\begin{array}{ccc} \Lambda^1 \otimes J^{k-1}E & \xrightarrow{\nabla} & \Lambda^2 \otimes J^{k-1}E \\ \kappa\downarrow & & \mathcal{S}\downarrow \\ \Lambda^3 \otimes J^{k-1}E & \xrightarrow{\mathrm{Id}\otimes\pi} & \Lambda^3 \otimes J^{k-2}E \end{array}$$

also commutes and we see that $\mathcal{S} \circ \nabla = 0$ from the curvature restriction imposed by the second condition in Theorem 2. In summary, the composition (7) takes values in $\Theta^{2,k-1} \otimes E$ and, of course, it is the first order differential operator characterised in Theorem 1. Similarly, the two conditions imposed by Theorem 2 on a connection on $J^{k-1}E$ imply that its curvature

$$\kappa : J^{k-1}E \to \Lambda^2 \otimes J^{k-1}E$$

actually takes values in $\Theta^{2,k-1} \otimes E$ and as a $(k-1)^{\mathrm{st}}$ order differential operator on E it coincides with $\nabla \circ \nabla^{(k)}$.

4 Application to prolongation

In joint work in progress with Rod Gover, higher order connections are used as follows. Suppose $D : E \to F$ is a linear differential operator of order k with surjective symbol. The diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & K & \to & \mathbb{T} & \to & J^{k-1}E & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \bigodot^k \Lambda^1 \otimes E & \to & J^k E & \to & J^{k-1}E & \to & 0 \\ & & \sigma(D)\downarrow & & D\downarrow & & & & \\ & & F & = & F & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{8}$$

with exact rows and columns defines the bundles K and $\mathbb{T}$. Furthermore, a splitting of

$$0 \to K \to \mathbb{T} \to E \to 0 \tag{9}$$

evidently splits the middle row of (8). Thus, a splitting of (9) induces a k^{th} order connection $\nabla^{(k)}$ on E such that $D = \sigma(D) \circ \nabla^{(k)}$. In combination with Theorem 2, we see that

$$D\phi = 0 \iff \nabla^{(k)}\phi = \omega \iff \nabla\tilde{\phi} = \omega,$$

where ω is some section of K and $\tilde{\phi} = j^{k-1}\phi \in \Gamma(M, J^{k-1}E)$. Now, according to Theorem 1 and the discussion at the end of the previous section, the operator $\nabla : \bigodot^k \Lambda^1 \otimes E \to \Theta^{2,k-1} \otimes E$ applied to ω may be written as $\kappa\tilde{\phi}$ where κ is the curvature of the connection ∇ on $J^{k-1}E$. The upshot of this reasoning is that the equation $D\phi = 0$ may be rewritten as the following system

$$\begin{aligned} \nabla\tilde{\phi} &= \omega, \\ \nabla\omega &= \kappa\tilde{\phi}, \end{aligned}$$

for $(\tilde{\phi}, \omega)$ a section of $J^{k-1}E \oplus K$. This is the first step in prolonging the equation $D\phi = 0$. With more care, this first step may be taken more invariantly, ending up with a well-defined first order differential operator on $\mathbb{T}$ independent of our choice of splitting of (9). We shall see an example of this phenomenon in the following section.

5 Other BGG-like sequences

In Theorem 1, we saw the start of a sequence of differential operators modelled on the complex (2) from projective differential geometry. In fact, it is not too hard to extend this sequence all the way

$$E \xrightarrow{\nabla^{(k)}} \Theta^{1,k-1} \otimes E \xrightarrow{\nabla} \Theta^{2,k-1} \otimes E \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Theta^{n-1,k-1} \otimes E \xrightarrow{\nabla} \Theta^{n,k-1} \otimes E$$

as a coupled version of (2). The ingredients for this construction are the connection ∇ on $J^{k-1}E$ from Theorem 1 and its relation with Spencer operators $\mathcal{S}$ coming from (5). Details are left to the reader.

Another BGG complex on $\mathbb{R}^n \hookrightarrow \mathbb{RP}_n$ starts with the operator $\Lambda^1 \to \bigodot^2 \Lambda^1$ given by

$$\phi_a \longmapsto \partial_{(a}\phi_{b)} \tag{10}$$

and continues with the second order operator (sometimes called the Saint Venant operator)

$$h_{ab} \longmapsto \partial_a\partial_c h_{bd} - \partial_b\partial_c h_{ad} - \partial_a\partial_d h_{bc} + \partial_b\partial_d h_{ac}. \tag{11}$$

This suggests that if we are given an arbitrary first order differential operator

$$D : \Lambda^1 \otimes E \to \bigodot^2 \Lambda^1 \otimes E$$

whose symbol is

$$\Lambda^1 \otimes \Lambda^1 \otimes E \xrightarrow{\_\odot\_\otimes \mathrm{Id}} \bigodot^2 \Lambda^1 \otimes E,$$

then there should be a canonically defined second order operator

$$\bigodot^2 \Lambda^1 \otimes E \to \Xi^{2,2} \otimes E$$

with the same symbol as (11), where $\Xi^{p,q}$ is the bundle whose sections are covariant tensors satisfying the following symmetries

$$\phi_{\underbrace{a\cdots b}_{p}\underbrace{cd\cdots e}_{q}} = \phi_{[a\cdots b][cd\cdots e]} \text{ such that } \phi_{[a\cdots bc]d\cdots e} = 0.$$

In fact, inspired by the full BGG-complex on $\mathbb{RP}_n$, we might expect a coupled sequence

$$\Lambda^1 \otimes E \xrightarrow{D} \bigodot^2 \Lambda^1 \otimes E \to \Xi^{2,2} \otimes E \to \Xi^{3,2} \otimes E \to \cdots \to \Xi^{n-1,2} \otimes E \to \Xi^{n,2} \otimes E. \tag{12}$$

This is, indeed, the case. Although it is not clear what should be the counterpart to Theorem 1, we may canonically construct the desired operators as follows. Consider what becomes of (8):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \Lambda^2 \otimes E & \to & \mathbb{T} & \to & \Lambda^1 \otimes E & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \Lambda^1 \otimes \Lambda^1 \otimes E & \to & J^1(\Lambda^1 \otimes E) & \to & \Lambda^1 \otimes E & \to & 0 \\
 & & \downarrow & & {\scriptstyle D}\downarrow & & & & \\
 & & \bigodot^2 \Lambda^1 \otimes E & = & \bigodot^2 \Lambda^1 \otimes E & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{13}$$

In particular, this diagram defines $\mathbb{T}$ and also shows that a splitting of

$$0 \to \Lambda^2 \otimes E \to \mathbb{T} \to \Lambda^1 \otimes E \to 0$$

not only enables us to write sections of $\mathbb{T}$ as

$$\left[\begin{array}{c} \phi_a \\ \mu_{ab} \end{array}\right] \quad \text{for} \quad \left\{\begin{array}{l} \phi_a \in \Gamma(\Lambda^1 \otimes E), \\ \mu_{ab} \in \Gamma(\Lambda^2 \otimes E), \end{array}\right.$$

but also splits the middle row of (13), i.e. defines a connection ∇_a on $\Lambda^1 \otimes E$. In terms of this connection, the operator D is simply $\phi_a \stackrel{D}{\longmapsto} \nabla_{(a}\phi_{b)}$. Now consider the operator

$$\mathbb{T} \ni \left[\begin{array}{c} \phi_a \\ \mu_{ab} \end{array}\right] \mapsto \left[\begin{array}{c} \nabla_a\phi_b - \mu_{ab} \\ \nabla_{[a}\mu_{b]c} - \kappa_{abc}{}^d\phi_d - \nabla_{[a}\mu_{c]b} + \kappa_{acb}{}^d\phi_d - \nabla_{[b}\mu_{c]a} + \kappa_{bca}{}^d\phi_d \end{array}\right] \in \Lambda^1 \otimes \mathbb{T},$$

where $\kappa : \Lambda^1 \otimes E \to \Lambda^2 \otimes \Lambda^1 \otimes E$ is the curvature of ∇. It is a connection on $\mathbb{T}$ and a tedious computation verifies that it is independent of choice of splitting of $\mathbb{T}$. Now consider the coupled de Rham sequence with values in $\mathbb{T}$ derived from this connection

$$\begin{array}{ccccccccc} \mathbb{T} & \xrightarrow{\nabla} & \Lambda^1 \otimes \mathbb{T} & \xrightarrow{\nabla} & \Lambda^2 \otimes \mathbb{T} & \xrightarrow{\nabla} & \Lambda^3 \otimes \mathbb{T} & \xrightarrow{\nabla} & \cdots \\ \| & & \| & & \| & & \| & & \\ \Lambda^1 \otimes E & & \Lambda^1 \otimes \Lambda^1 \otimes E & & \Lambda^2 \otimes \Lambda^1 \otimes E & & \Lambda^3 \otimes \Lambda^1 \otimes E & & \\ \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus & & \\ \Lambda^2 \otimes E & & \Lambda^1 \otimes \Lambda^2 \otimes E & & \Lambda^2 \otimes \Lambda^2 \otimes E & & \Lambda^3 \otimes \Lambda^2 \otimes E, & & \end{array} \tag{14}$$

noticing that the restricted operators $\nearrow$ are simply homomorphisms given by

$$\mu_{\underbrace{a\cdots b}_{p}cd} \mapsto -\mu_{[a\cdots bc]d}.$$

When $p = 0$ this homomorphism is injective. When $p = 1$ it is an isomorphism. For $p \geq 2$ it is surjective with $\Xi^{p,2} \otimes E$ as kernel. It is now just diagram chasing to extract (12) from (14).

The Killing operator in Riemannian geometry provides a good example of a first order linear differential operator to which the reasoning above may be applied. In this example, the bundle E is trivial and

$$D : \Lambda^1 \to \bigodot^2\Lambda^1 \quad \text{is given by } \phi_a \mapsto \nabla_{(a}\phi_{b)},$$

where ∇_a is the Levi-Civita connection. It is a straightforward generalisation of (10). Similarly, the flat operator (11) is modified by replacing ∂_a by ∇_a but also by adding suitable zeroth order curvature terms. Details may be found in [7]. Both of these differential operators have geometric interpretations. The Killing operator itself gives the infinitesimal change in the Riemannian metric g_{ab} due to the flow of a vector field ϕ^a. The next operator

$$\bigodot^2\Lambda \xrightarrow{\nabla^{(2)}} \Xi^{2,2}$$

gives the infinitesimal change in the Riemann curvature tensor due to a perturbation of g_{ab} by an arbitrary symmetric tensor (i.e. replace g_{ab} by $g_{ab} + \epsilon h_{ab}$ for sufficiently small ϵ, compute the Riemannian curvature for this new metric, differentiate in ϵ, and then set $\epsilon = 0$). The next operator is an infinitesimal manifestation of the Bianchi identity. This particular BGG complex on $\mathbb{RP}_n$

$$\Lambda^1 \xrightarrow{\nabla} \bigodot^2\Lambda^1 \xrightarrow{\nabla^{(2)}} \Xi^{2,2} \xrightarrow{\nabla} \Xi^{3,2} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Xi^{n-1,2} \xrightarrow{\nabla} \Xi^{n,2}$$

was also constructed by Calabi [4] as the Riemannian deformation complex for the constant curvature metric (only constant curvature metrics are projectively flat).

In three dimensions, the deformation of a Riemannian metric coincides with the mathematical formulation of elasticity in continuum mechanics (see, e.g. [5]). In three dimensions, we may also choose a volume form ϵ_{abc} to effect an isomorphism $\Xi^{2,2} \cong \bigodot^2\Lambda^1$ (a reflection of the fact that in three dimensions there is only Ricci curvature) and rewrite (11) as

$$\textstyle\bigodot^2\Lambda^1 \ni h_{ab} \longmapsto \epsilon_a{}^{cd}\epsilon_b{}^{ef}\partial_c\partial_e h_{df} \in \bigodot^2\Lambda^1 \quad \text{(sometimes written as } h \mapsto \operatorname{curl}\operatorname{curl} h\text{)}.$$

Also $\Xi^{3,2} \cong \Lambda^1$ and the *linearised elasticity complex* becomes

$$\Lambda^1 \xrightarrow{\nabla} \textstyle\bigodot^2\Lambda^1 \xrightarrow{\nabla^{(2)}} \bigodot^2\Lambda^1 \xrightarrow{\nabla} \Lambda^1,$$

usually interpreted as *displacement* $\mapsto$ *strain* $\mapsto$ *stress* $\mapsto$ *load*. A derivation of the complex in this form on $\mathbb{RP}_3$ (by means of a coupled de Rham complex as above) is given in [6]. The close link between BGG complexes and coupled de Rham complexes (in the flat case) has recently been modified and then used by Arnold, Falk, and Winther [1] to give new stable finite element schemes applicable to numerical elasticity.

Having seen two examples thereof, it is natural to conjecture that there are canonically defined sequences of differential operators modelled on the general projective BGG complex. However, this remains a conjecture. Notice that there is no direct link between projective differential geometry and the constructions in this article (and we are not using that the Killing operator considered above happens to be projectively invariant when suitably interpreted [7]). More challenging cases of this conjecture are to ask, for $s \geq 2$, if

$$D : \textstyle\bigodot^s\Lambda^1 \otimes E \to \bigodot^{s+1}\Lambda^1 \otimes E$$

is an arbitrary first order differential operator whose symbol is

$$\Lambda^1 \otimes \textstyle\bigodot^s\Lambda^1 \otimes E \xrightarrow{\_\odot\_\otimes\mathrm{Id}} \bigodot^{s+1}\Lambda^1 \otimes E,$$

whether there is a canonically defined $(s+1)^{\mathrm{st}}$ order operator

$$\textstyle\bigodot^{s+1}\Lambda^1 \otimes E \to \Xi^{s+1,s+1} \otimes E$$

whose symbol is $\mathrm{proj} \otimes \mathrm{Id}$, where

$$\textstyle\bigodot^{s+1}\Lambda^1 \otimes \bigodot^{s+1}\Lambda^1 \to \Xi^{s+1,s+1}$$

is induced by the canonical projection of $\mathrm{GL}(n,\mathbb{R})$-modules

$$\begin{array}{|c|c|c|c|}\hline \; & \; & \cdots & \; \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|}\hline \; & \; & \cdots & \; \\ \hline \end{array} \to \begin{array}{|c|c|c|c|}\hline \; & \; & \cdots & \; \\ \hline \; & \; & \cdots & \; \\ \hline \end{array}\,.$$

Even restricting attention to first order operators D, there are many more examples of this conjecture that might be considered. In general, the conjecture applies to operators whose symbol is induced by a *Cartan product* as in [3].

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