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Hilbert–Schmidt Operators vs. Integrable Systems of Elliptic Calogero–Moser Type III. The Heun Case*

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Abstract. The Heun equation can be rewritten as an eigenvalue equation for an ordinary differential operator of the form $-d^2/dx^2 + V(g;x)$, where the potential is an elliptic function depending on a coupling vector $g \in \mathbb{R}^4$. Alternatively, this operator arises from the BC_1 specialization of the BC_N elliptic nonrelativistic Calogero–Moser system (a.k.a. the Inozemtsev system). Under suitable restrictions on the elliptic periods and on g, we associate to this operator a self-adjoint operator $H(g)$ on the Hilbert space $\mathcal{H} = L^2([0,\omega_1], dx)$, where $2\omega_1$ is the real period of $V(g;x)$. For this association and a further analysis of $H(g)$, a certain Hilbert–Schmidt operator $\mathcal{I}(g)$ on $\mathcal{H}$ plays a critical role. In particular, using the intimate relation of $H(g)$ and $\mathcal{I}(g)$, we obtain a remarkable spectral invariance: In terms of a coupling vector $c \in \mathbb{R}^4$ that depends linearly on g, the spectrum of $H(g(c))$ is invariant under arbitrary permutations $\sigma(c)$, $\sigma \in S_4$.

Key words: Heun equation; Hilbert–Schmidt operators; spectral invariance

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1 Introduction

In the first two papers of this series ([1, 2], henceforth referred to as I and II), we initiated a detailed study of spectral properties of operators arising in the context of elliptic Calogero–Moser type systems. The operators at issue are not only the Hamiltonians defining these systems, but also families of Hilbert–Schmidt (HS) operators that seem unrelated at first sight. The relation consists in the kernels of the HS operators being eigenfunctions of differences of the Hamiltonians. As we have first detailed in our lecture notes [3] and also sketched in I, this relation can be exploited to derive novel information about the Hamiltonians, viewed as Hilbert space operators (as opposed to differential or difference operators).

The present paper is concerned with the Heun Hamiltonian

$$H(g;x) \equiv -\frac{d^2}{dx^2} + V(g;x), \qquad g = (g_0, g_1, g_2, g_3) \in \mathbb{R}^4, \tag{1.1}$$

$$V(g;x) \equiv \sum_{t=0}^{3} g_t(g_t - 1)\wp(x + \omega_t; \pi/2r, i\alpha/2), \qquad r, \alpha > 0, \tag{1.2}$$

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$$\omega_0 \equiv 0, \qquad \omega_1 \equiv \pi/2r, \qquad \omega_2 \equiv i\alpha/2, \qquad \omega_3 \equiv -\omega_1 - \omega_2.$$

(For a lucid account of elliptic and allied functions we recommend [4], whose conventions and notation we mostly follow.) It can also be viewed as the BC_1 'nonrelativistic' elliptic Calogero–Moser Hamiltonian, the BC_N case being the Inozemtsev system [5]. There exists an extensive lore about the eigenfunctions of $H(g)$ viewed as a differential operator, cf. e.g. the reviews [6, 7, 8]. In recent years, this lore has been significantly enlarged by Takemura, whose work can be traced from his recent paper [9].

Here, however, we are concerned with spectral properties of $H(g)$, reinterpreted as a self-adjoint operator on the Hilbert space

$$\mathcal{H} \equiv L^2([0, \pi/2r], dx). \tag{1.3}$$

A few of the properties we derive (such as the occurrence of solely discrete spectrum for suitably restricted g) follow from well-known arguments in the theory of Schrödinger operators [10] (or, alternatively, from Sturm–Liouville theory and the theory of eigenfunction expansions). Our main tool for reobtaining known features and obtaining novel ones is the integral operator whose kernel is given by

$$\Psi(g; x, y) \equiv w(g; x)^{1/2} \mathcal{S}(g; x, y) w(g'; y)^{1/2}, \qquad x, y \in (0, \pi/2r). \tag{1.4}$$

This kernel and the definition of its ingredients can be found in I, cf. I Subsection 3.2. For ease of reference, however, we repeat the relevant definitions:

$$g' \equiv J_N g, \qquad J_N \equiv \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}, \tag{1.5}$$

$$\mathcal{S}(g; x, y) \equiv \exp(-s_g \ln[R(x+y)R(x-y)]), \tag{1.6}$$

$$s_g \equiv \frac{1}{2} \sum_{t=0}^{3} g_t = \frac{1}{2} \sum_{t=0}^{3} g'_t. \tag{1.7}$$

$$w(g; x) \equiv 1/c(g; x)c(g; -x), \tag{1.8}$$

$$c(g; x) \equiv R(x + i\alpha/2)^{-g_0} R(x + i\alpha/2 - \pi/2r)^{-g_1} R(x)^{-g_2} R(x - \pi/2r)^{-g_3},$$

$$R(z) \equiv \prod_{l=0}^{\infty} (1 - \exp[-(2l+1)r\alpha + 2irz])(1 - \exp[-(2l+1)r\alpha - 2irz]), \qquad z \in \mathbb{C}.$$

At this point we would like to mention that a close relative of the kernel $\mathcal{S}(g; x, y)$ has appeared in the literature before, a fact we only recently became aware of. Indeed, it can be tied in with the kernel that can be found on p. 124 of a monograph by Slavyanov and Lay [11]. (It seems this kernel was first introduced in a paper by Kazakov and Slavyanov [12].) More precisely, when a $\wp$-function substitution is made in the latter kernel, it can be factorized in terms of the Weierstrass σ-function; the nontrivial part is then an analytic continuation of $\mathcal{S}(g; x, y)$. Both in [11] and in Takemura's paper [13], this kernel is used to relate Heun functions defined on different intervals.

The entire function $R(z)$ plays a crucial role in this paper. It satisfies

$$R(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{\cos(2nrz)}{n \sinh(nr\alpha)} \right), \qquad |\Im z| < \alpha/2, \tag{1.9}$$

$$R(2z) = \prod_{\delta=+,-} R(z + i\delta\alpha/4)R(z + \pi/2r + i\delta\alpha/4), \tag{1.10}$$

$$R(z + i\alpha/2) = ipe^{-irz}s(z), \qquad p = 2r\prod_{k=1}^{\infty}(1 - e^{-2kr\alpha})^2. \tag{1.11}$$

Here, $s(z)$ is related to the σ-function by

$$s(z) = \exp\left(-\eta z^2 r/\pi\right)\sigma(z; \pi/2r, i\alpha/2), \tag{1.12}$$

so that

$$f(z + i\alpha) = -\exp(r\alpha - 2irz)f(z), \qquad f = s, R.$$

(For more details, see [14], where the functions s and R were introduced.)

From (1.11) it follows that the weight function (1.8) can also be written

$$w(g; x) = p^{2g_0+2g_1}s(x)^{2g_0}s(\pi/2r - x)^{2g_1}R(x)^{2g_2}R(\pi/2r - x)^{2g_3}. \tag{1.13}$$

It is positive on $(0, \pi/2r)$ for all $g \in \mathbb{R}^4$, but integrable on $[0, \pi/2r]$ only for $g_0, g_1 > -1/2$.

It now readily follows that for $\Psi(g; x, y)$ to belong to the kernel Hilbert space

$$\mathcal{H}_K \equiv L^2\left([0, \pi/2r]^2, dxdy\right), \tag{1.14}$$

the necessary and sufficient condition is that g belong to the parameter set

$$\Pi \equiv \left\{g \in \mathbb{R}^4 \mid g_0, g_1, g_0', g_1' > -1/2\right\}. \tag{1.15}$$

Hence it is clear that the integral operator

$$(\mathcal{I}(g)f)(x) \equiv \int_0^{\pi/2r} dy\Psi(g; x, y)f(y), \qquad f \in \mathcal{H}, \qquad g \in \Pi, \tag{1.16}$$

is well defined and HS.

It is not obvious, but true that its kernel is a zero-eigenvalue eigenfunction of a difference of Heun Hamiltonians:

$$(H(g; x) - H(g'; y))\Psi(g; x, y) = 0, \qquad x, y \in (0, \pi/2r). \tag{1.17}$$

(Indeed, this is a corollary of Proposition 3.4 in I, cf. I(3.115).) At face value, this eigenfunction identity seems to have no bearing on the Hilbert space eigenfunctions of the two (generically) distinct Hamiltonians. A principal result of this paper is, however, that for g in the restricted parameter set

$$\Pi_r \equiv \{g \in \Pi \mid s_g > 0\}, \tag{1.18}$$

the Hilbert space eigenvector ONBs (orthonormal bases) of $H(g)$ and $H(g')$ are given by functions $e_n(g; x)$ and $e_n(g'; x)$, resp., which yield the singular value decomposition of $\mathcal{I}(g)$ (cf. [15]):

$$\mathcal{I}(g) = \sum_{n=0}^{\infty}\nu_n(g)e_n(g) \otimes e_n(g'), \qquad \nu_0 \geq \nu_1 \geq \cdots > 0, \qquad \sum_{n=0}^{\infty}\nu_n^2 < \infty, \tag{1.19}$$
$$(e_m, e_n) = \delta_{mn}, \qquad m, n \in \mathbb{N}.$$

(All singular values are positive, since $\mathcal{I}(g)$ has trivial kernel and dense range for $g \in \Pi_r$, cf. (2.4). To appreciate the restriction on s_g, note that $\mathcal{I}(g)$ has rank 1 for $s_g = 0$, cf. (1.6).)

This result entails in particular that $H(g)$ and $H(g')$ have solely discrete spectrum, the two spectra being equal. To be sure, in the case at hand it follows from well-known Schrödinger operator theory that $H(g)$ can be reinterpreted as a self-adjoint operator that has an ONB of eigenvectors, provided g is suitably restricted. The main novelty is that these eigenvectors are just those occurring for $\mathcal{I}(g)$, and that this fixes the self-adjoint extensions of the Schrödinger operator $H(g)$ in case it is not essentially self-adjoint on $C_0^\infty((0,\pi/2r))$ (namely, for g_0 and/or g_1 in $(-1/2,3/2)$). As a bonus, this yields a hidden spectral invariance under permutations of four new parameters, which implies spectral consequences for special cases that are quite startling. Last but not least, the state of affairs for the present case gives rise to a simple paradigm for the 'relativistic' (analytic difference operator) Heun case, where the usual self-adjoint extension theory is of no help and no previous spectral results are known, cf. the fourth paper in this series [16].

We proceed to sketch the organization and results of this paper in more detail. We begin Section 2 by deriving features of functions in the range of $\mathcal{I}(g)$, assuming $g\in\Pi_r$. Then we show that $\mathcal{I}(g), g\in\Pi_r$, has dense range and trivial kernel. These crucial features readily follow from Lemma 2.1, which asserts that the integral operator with kernel $\mathcal{S}(g;x,y)$ has kernel $\{0\}$ on $L^1([0,\pi/2r])$ whenever s_g is positive. The characteristics of the range of $\mathcal{I}(g)$ are now used as a guide for a choice of domain for $H(g)$, namely the dense subspace $D(g_0,g_1)$ given by (2.7). Lemma 2.2 then asserts that $H(g)$ is symmetric on $D(g_0,g_1)$ for all g in the parameter space

$$\tilde{\Pi}\equiv\left\{g\in\mathbb{R}^4 \mid g_0,g_1>-1/2\right\}. \tag{1.20}$$

The proofs of these lemmas involve technicalities that are not enlightening, so we have relegated them to Appendix A.

Next, using the key identity (1.17), we show in Lemma 2.3 that for all $g\in\tilde{\Pi}$ the subspace $D(g_0,g_1)$ is a core (domain of essential self-adjointness [10]). This prepares us for studying the self-adjoint closure (again denoted $H(g)$) by using $\mathcal{I}(g)$. Only a few simple steps are then needed to reach the conclusion that the above ONB $\{e_n(g)\}$ may be chosen so that it is an eigenvector ONB for $H(g)$. Moreover, since the eigenvectors belong to $D(g_0,g_1)$, it easily follows from the Frobenius method that the spectrum of $H(g)$ is simple, cf. Theorem 2.4.

Whenever the potential $V(g;x)$ is bounded below, it is obvious that $H(g)$ is also bounded below. For g_0 and/or g_1 in $(0,1)$, however, $V(g;x)$ is not bounded below, so an extra argument is needed to show that for the pertinent self-adjoint extensions of (1.1) semi-boundedness is preserved. We prove this by a comparison argument involving the trigonometric BC_1 Calogero–Moser Hamiltonian

$$\begin{aligned}
&H_t(g_0,g_1;x)\equiv-\frac{d^2}{dx^2}+V_t(g_0,g_1;x),\\
&V_t(g_0,g_1;x)\equiv\frac{r^2g_0(g_0-1)}{\sin^2 rx}+\frac{r^2g_1(g_1-1)}{\cos^2 rx}.
\end{aligned}\tag{1.21}$$

The point is that this operator is well defined and essentially self-adjoint on $D(g_0,g_1)$, and that it has an ONB of eigenvectors in $D(g_0,g_1)$ involving the well-known Jacobi polynomials, cf. Theorem 2.5.

We would like to add that the trigonometric Hamiltonian (1.21) plays a key role in Takemura's paper [17], where he studies the Heun Hamiltonian $H(g;x)$ (1.1) with $g_0,g_1\geq 1$ from a Hilbert space point of view. His strategy is based on a perturbation expansion in the parameter $p=\exp(-\alpha r)$. (In this connection it should be pointed out that $H(g;x)$ with $g_2=g_3=0$ reduces to $H_t(g_0,g_1;x)$ for $\alpha\to\infty$.)

The main result of Theorem 2.6 is that for $g\in\Pi_r$ the spectra of $H(g)$ and $H(g')$ coincide. Unfortunately, we have not been able to show that the singular values in (1.19) are distinct for all $g \in \Pi_r$, a feature that seems plausible to us. If this is indeed the case, then it would follow that $e_n(g)$ is the $H(g)$-eigenvector with eigenvalue $E_n(g)$ in the obvious eigenvalue list

$$E_0(g) < E_1(g) < E_2(g) < \cdots, \qquad g \in \Pi_r.$$

In Section 3 we fix $g \in \Pi_r$ and address the question for which $\hat{g} \in \tilde{\Pi}$ the spectrum of $H(\hat{g})$ coincides with that of $H(g)$. From Theorem 2.6 we know this is true for $\hat{g} = g'$, but there are a few more involutory g-transformations for which this is valid. We determine the group G generated by these involutions in Theorem 3.1. It leaves the parameter set

$$\Pi_G \equiv \{g \in \Pi_r \mid g_0 + g_1 + 2 > g_2 + g_3\} \tag{1.22}$$

invariant, and is isomorphic to the permutation group S_4. (For ease of reference, we collect the four parameter sets (1.22), (1.18), (1.15) and (1.20) in a chain

$$\Pi_G \subset \Pi_r \subset \Pi \subset \tilde{\Pi},$$

and note that each inclusion is proper.) Indeed, defining new couplings

$$c_0 \equiv g_0 + g_3 - 1, \qquad c_1 \equiv g_1 + g_2 - 1, \qquad c_2 \equiv g_1 - g_2, \qquad c_3 \equiv g_0 - g_3,$$

the spectrum of

$$\hat{H}(c) \equiv H(g(c))$$

is invariant under arbitrary permutations of c_0, c_1, c_2, c_3.

In Section 4 we consider various special g-choices. For the sixteen g-values yielding $V(g;x)=0$, all of the relevant quantities can be determined in great detail. A striking consequence of our spectral invariance results is that there exist quite a few g-choices for which $V(g;x) \neq 0$, yet the spectrum of $H(g)$ equals that for one of the cases where $V(g;x) = 0$. We also consider the case $s_g = 0$, for which $\mathcal{I}(g)$ has rank 1. In particular, using results from I and [18], we verify that $\mathcal{I}(g)$ connects the ground state of $H(g')$ with the one of $H(g)$. Finally, we add some observations on the case where the π/r-periodic potential $V(g;x)$ has no poles for real x, a well-known setting that is for instance studied in considerable detail in [19].

The paper is concluded with Appendix A, in which we present the proofs of Lemmas 2.1 and 2.2.

2 The eigenvector ONBs of $H(g)$ and $H(g')$

Unless explicitly mentioned otherwise, we assume from now on that the coupling vector g belongs to the restricted parameter set Π_r (1.18). This entails in particular that the integral operator $\mathcal{I}(g)$ (1.16) is a well-defined HS operator on the Hilbert space $\mathcal{H}$ (1.3). We begin this section by obtaining features of functions in the range of $\mathcal{I}(g)$.

Consider first for $f \in \mathcal{H}$ and $x \in [0, \pi/2r]$ the function

$$\begin{aligned} h(x) &\equiv \int_0^{\pi/2r} dy \mathcal{S}(g;x,y) w(g';y)^{1/2} f(y) \\ &= \int_0^{\pi/2r} dy \exp\left(2s_g \sum_{n=1}^{\infty} \frac{\cos(2nrx)\cos(2nry)}{n\sinh(nr\alpha)}\right) w(g';y)^{1/2} f(y), \end{aligned} \tag{2.1}$$

where we used (1.9). Since we have $w(g';\cdot)^{1/2} \in \mathcal{H}$, we deduce from the Schwarz inequality

$$w(g';y)^{1/2} f(y) \in L^1([0,\pi/2r], dy).$$

It readily follows that $h(x)$ extends to a function $h(z)$, $\Re z = x$, that is analytic in the strip $|\Im z| < \alpha/2$. Also, $h(z)$ is π/r-periodic and even. Now $R(z)$ is an entire, π/r-periodic and even function without zeros for $|\Im z| < \alpha/2$, so the function

$$s(x)^{-g_0} s(\pi/2r - x)^{-g_1} (\mathcal{I}(g)f)(x) = R(x)^{g_2} R(\pi/2r - x)^{g_3} h(x), \qquad x \in (0, \pi/2r), \tag{2.2}$$

extends to a function that is analytic for $|\Im z| < \alpha/2$, π/r-periodic and even.

Next, we point out the relations

$$g \in \Pi_r \ \Leftrightarrow \ g' \in \Pi_r, \qquad \mathcal{I}(g)^* = \mathcal{I}(g'), \tag{2.3}$$

which are easily verified. As a consequence, the orthocomplement of the range of $\mathcal{I}(g)$ equals the kernel of $\mathcal{I}(g')$. From (2.1) it is immediate that $\mathcal{I}(g), g \in \Pi$, has rank 1 when s_g vanishes. By contrast, for $s_g > 0$ the integral operator $\mathcal{I}(g)$ has dense range and trivial kernel:

$$\mathrm{Ker}(\mathcal{I}(g)) = \{0\}, \qquad \mathrm{Ran}(\mathcal{I}(g))^- = \mathcal{H}, \qquad \forall\, g \in \Pi_r. \tag{2.4}$$

This is clear from the above and the following lemma, which is proved in Appendix A.

Lemma 2.1. *Letting $s_g > 0$, suppose $\phi(y) \in L^1([0, \pi/2r], dy)$ satisfies*

$$\int_0^{\pi/2r} dy \mathcal{S}(g; x, y)\phi(y) = 0.$$

Then $\phi = 0$.

To proceed, we note that the kernel function $\Psi(g; x, y)$ is real-valued. From (2.4), (2.3) and the HS property we then deduce that for a fixed $g \in \Pi_r$, there exist two ONBs $\{e_n(g')\}_{n\in\mathbb{N}}$ and $\{e_n(g)\}_{n\in\mathbb{N}}$ of real-valued functions such that

$$\Psi(g; x, y) = \sum_{n=0}^{\infty} \nu_n(g) e_n(g; x) e_n(g'; y), \qquad \nu_0 \geq \nu_1 \geq \cdots > 0, \tag{2.5}$$

with $\{\nu_n\} \in l^2(\mathbb{N})$. (The choice of the first ONB uniquely determines the second one; later on, we will return to the specific choice we make.) Moreover, since $e_n(g)$ belongs to the range of $\mathcal{I}(g)$, the above analysis implies that it is of the form

$$e_n(g; x) = w(g; x)^{1/2} d_n(g; x), \tag{2.6}$$

where $d_n(g; x)$ extends to a function that is analytic in $|\Im z| < \alpha/2$, even and π/r-periodic.

We are now prepared to deal with the Heun Hamiltonian $H(g)$. We choose as initial domain for $H(g)$ the space

$$D(g_0, g_1) = s(x)^{g_0} s(\pi/2r - x)^{g_1} \mathcal{D}, \qquad g_0, g_1 > -1/2, \tag{2.7}$$

where

$$\mathcal{D} \equiv \left\{ f \in C^2([0, \pi/2r]) \mid f'(0) = f'(\pi/2r) = 0 \right\}. \tag{2.8}$$

This domain choice ensures that the ONB functions $e_n(g; x)$ belong to the domain. Indeed, from the paragraph containing (2.2) we see that the function on the r.h.s. of (2.2) belongs to $\mathcal{D}$, so that we have more generally

$$\mathrm{Ran}\,(\mathcal{I}(g)) \subset D(g_0, g_1). \tag{2.9}$$

From $e_n(g) \in D(g_0, g_1)$ it already follows that $D(g_0, g_1)$ is dense. But this feature can also be derived quite easily without invoking $\mathcal{I}(g)$. Indeed, $\mathcal{D}$ contains the functions $\cos(2nrx)$, $n \in \mathbb{N}$, which are well known to yield an orthogonal base of $\mathcal{H}$. Therefore, the assumption $f \perp D(g_0, g_1)$ entails

$$\int_0^{\pi/2r} dx s(x)^{g_0} s(\pi/2r - x)^{g_1} \cos(2nrx) f(x) = 0, \qquad \forall\, n \in \mathbb{N}.$$

From the orthogonal base property we now deduce $f(x) = 0$, so that $D(g_0, g_1)$ is dense.

We are now prepared for our second lemma, whose proof is relegated to Appendix A.

Lemma 2.2. *Assume g belongs to $\tilde{\Pi}$* (1.20)*. Then $H(g)$* (1.1) *is well defined and symmetric on the domain $D(g_0, g_1)$ given by* (2.7), (2.8).

The proof of this lemma does not involve the HS operator $\mathcal{I}(g)$. In the next lemma we do use $\mathcal{I}(g)$ to obtain a short proof of essential self-adjointness on $D(g_0, g_1)$.

Lemma 2.3. *Assuming $g \in \tilde{\Pi}$, the domain $D(g_0, g_1)$ is a core for $H(g)$.*

Proof. We begin by noting that the potentials $\wp(x + i\alpha/2)$ and $\wp(x - \pi/2r - i\alpha/2)$ yield bounded self-adjoint operators, so that they do not influence domain issues. Therefore, we need only prove the core property for one value of g_2 and g_3. We choose $g_2 = g_3 = 1$ (say), the point being that this entails $g \in \Pi_r$. (This is easily checked, cf. (1.5), (1.15) and (1.18).) Hence we can now use $\mathcal{I}(g)$.

Assume $\phi \in \mathcal{H}$ satisfies

$$(\phi, (H(g) + i)\psi) = 0, \qquad \forall\, \psi \in D(g_0, g_1).$$

By virtue of (2.9), this implies in particular

$$(\phi, (H(g) + i)\mathcal{I}(g)f) = 0, \qquad \forall\, f \in \mathcal{H}.$$

Thanks to (1.17) and (2.3), this inner product can be rewritten as

$$((H(g') - i)\mathcal{I}(g')\phi, f) = 0, \qquad \forall\, f \in \mathcal{H}.$$

This implies that the vector $\mathcal{I}(g')\phi$, which belongs to $D(g_0', g_1')$ by (2.9), is an eigenvector with eigenvalue i for the symmetric operator $H(g')$. Thus we must have $\mathcal{I}(g')\phi = 0$. By (2.4), this entails $\phi = 0$. Therefore $\mathrm{Ran}\,(H(g) + i)$ is dense in $\mathcal{H}$. Likewise, $\mathrm{Ran}\,(H(g) - i)$ is dense in $\mathcal{H}$. Hence the core property results. ■

We would like to point out that the dense subspace

$$\mathcal{C} \equiv C_0^\infty((0, \omega_1)) \subset D(g_0, g_1), \tag{2.10}$$

is already a core for $H(g)$ whenever $g_0, g_1 > 3/2$. This is basically known, but for completeness we sketch a proof. Choosing at first $g \in \mathbb{R}^4$, consider the differential operator $H(g)$ restricted to $\mathcal{C}$. This is obviously a well-defined symmetric operator on $\mathcal{H}$, and it is easy to verify that the domain of its closure contains the space

$$\begin{aligned} \mathcal{C}' \equiv \{&\psi \in C^1([0, \omega_1]) \mid \psi(0) = \psi(\omega_1) = \psi'(0) = \psi'(\omega_1) = 0, \\ &\psi'(x) \in C^1((0, \omega_1)), \psi''(x), x^{-2}\psi(x), (\omega_1 - x)^{-2}\psi(x) \in \mathcal{H}\}. \end{aligned}$$

Now assume ϕ is orthogonal to $(H(g)+i)\mathcal{C}$. Then ϕ is a weak solution to the ODE $(H(g)-i)\phi = 0$ on $(0, \omega_1)$, and by hypo-ellipticity it is a classical solution.

Choosing next $g_0, g_1 > 3/2$, it follows from the Frobenius theory that near 0 we have

$$\phi(x) = x^{g_0} h_0(x),$$

with $h_0(x)$ analytic at $x = 0$. (Indeed, x^{1-g_0} is not in L^2 near 0.) Likewise, near $x = \omega_1$,

$$\phi(x) = (\omega_1 - x)^{g_1} h_1(x),$$

with $h_1(x)$ analytic at $x = \omega_1$. But then we deduce that ϕ has all of the properties defining $\mathcal{C}'$, and since the closure of $H(g)$ is symmetric on $\mathcal{C}'$, it cannot have an eigenvector with nonreal eigenvalue. Thus $\phi = 0$ and the core property follows.

After this digression we proceed to obtain our first main result, for which we again need to require $g \in \Pi_r$. We denote the domain of the self-adjoint closure of $H(g)$ by $D(g)$, but keep the notation $H(g)$ for the closure. We now use the key identity (1.17) as in the proof of Lemma 2.3 to deduce

$$(\phi, H(g)\mathcal{I}(g)\psi) = (H(g')\mathcal{I}(g')\phi, \psi), \qquad \forall\, \phi, \psi \in \mathcal{H}. \tag{2.11}$$

Choosing $\lambda \in \mathbb{C}$ with $\Im\lambda > 0$, this entails in particular

$$(\phi, (H(g) + \lambda)\mathcal{I}(g)\psi) = ((H(g') + \overline{\lambda})\mathcal{I}(g')\phi, \psi), \qquad \forall\, \phi \in D(g), \qquad \forall\, \psi \in D(g'). \tag{2.12}$$

Now for $\phi \in D(g)$ and $\psi \in D(g')$ there exist $e, f \in \mathcal{H}$ such that

$$\phi = (H(g) + \overline{\lambda})^{-1} e, \qquad \psi = (H(g') + \lambda)^{-1} f,$$

so (2.12) implies

$$(e, \mathcal{I}(g)(H(g') + \lambda)^{-1} f) = (\mathcal{I}(g')(H(g) + \overline{\lambda})^{-1} e, f), \qquad \forall\, e, f \in \mathcal{H}.$$

Hence we obtain the relation

$$\mathcal{I}(g)(H(g') + \lambda)^{-1} = (H(g) + \lambda)^{-1}\mathcal{I}(g).$$

From this we readily deduce

$$[(H(g) + \lambda)^{-1}, T(g)] = 0,$$

where

$$T(g) \equiv \mathcal{I}(g)\mathcal{I}(g)^* = \sum_{n=0}^{\infty} \nu_n^2(g) e_n(g) \otimes e_n(g). \tag{2.13}$$

The upshot is that the self-adjoint trace class operator $T(g)$ commutes with the $H(g)$-resolvent. Since $T(g)$ has trivial kernel by (2.4), all of its eigenspaces are finite-dimensional, and the $H(g)$-resolvent leaves these finite-dimensional spaces invariant. Thus we can choose the eigenvectors of $T(g)$ to be eigenvectors of the $H(g)$-resolvent, hence of $H(g)$ as well. We now summarize and extend these findings.

Theorem 2.4. *Let $g \in \Pi_r$. Then the positive trace class operator $T(g)$ (2.13) has an ONB $\{e_n(g)\}_{n\in\mathbb{N}}$, of eigenvectors that are also eigenvectors of $H(g)$. The functions $e_n(g; x)$ are real-valued and belong to $D(g_0, g_1)$. They are of the form (2.6), where $d_n(g; x)$ extends to a function that is analytic in $|\Im z| < \alpha/2$, even and π/r-periodic. Moreover, we have*

$$d_n(g; 0) \neq 0, \qquad d_n(g; \pi/2r) \neq 0, \tag{2.14}$$

and the spectrum of $H(g)$ is nondegenerate.

Proof. We have just proved the first assertion. The second and third one have been shown earlier too, cf. the paragraphs containing (2.6) and (2.8). To prove the last one, we first note that it follows from $e_n(g) \in D(g_0, g_1)$ that $e_n(g; x)$ is a classical solution to the ODE $H(g)e_n = \lambda_n e_n$ for a certain $\lambda_n \in \mathbb{R}$. From the Frobenius method it then follows that $d_n(g; x)$ satisfies (2.14). Moreover, any solution to the same ODE that is linearly independent of $e_n(g; x)$ does not have the features encoded in (2.6), so the eigenvalue λ_n is simple. ■

Quite likely, for $g \in \tilde{\Pi} \setminus \Pi_r$ the $H(g)$-eigenvectors still belong to $D(g_0, g_1)$ and have the same features as for $g \in \Pi_r$. (If so, it follows as before that $\sigma(H(g))$ is simple.) However, we cannot prove this via the above arguments, since they hinge on exploiting the HS operator $\mathcal{I}(g)$ with g restricted to Π_r.

It is a corollary of Theorem 2.4 that there exist distinct real numbers $\lambda_n(g)$ such that

$$H(g)e_n(g) = \lambda_n(g)e_n(g), \qquad n \in \mathbb{N}, \qquad g \in \Pi_r. \tag{2.15}$$

At this stage, however, we have no other information on the $H(g)$-eigenvalues. In the following theorem, we obtain further results on the $H(g)$-spectrum that do not involve $\mathcal{I}(g)$, and which are valid for all $g \in \tilde{\Pi}$. As a preparation, note that $H(g)$ is obviously bounded below for $g_0, g_1 \geq 1$. (Indeed, this entails that $V(g; x)$ is bounded below, cf. (1.2).) For g_0 and/or g_1 in $(0, 1)$, however, $V(g; x)$ diverges to $-\infty$ as $x \downarrow 0$ and/or $x \uparrow \pi/2r$. Even so, our next theorem implies that $H(g)$ is still bounded below. Its proof is based on a comparison argument involving the trigonometric BC_1 Calogero–Moser Hamiltonian (1.21).

Theorem 2.5. *For all $g \in \tilde{\Pi}$, the spectrum of the operator $H(g)$ is a discrete set that is bounded below. Moreover, each eigenvalue has finite multiplicity.*

Proof. The trigonometric and elliptic differential operators (1.21) and (1.1) differ by a real-valued potential

$$V_d(g; x) \equiv V_t(g_0, g_1; x) - V(g; x),$$

that is bounded on $[0, \pi/2r]$. Hence we can associate to (1.21) the operator on $D(g_0, g_1)$ given by

$$H_t(g_0, g_1) \equiv H(g) + V_d(g; \cdot). \tag{2.16}$$

This operator is essentially self-adjoint on $D(g_0, g_1)$, since $H(g)$ is e.s.a. on $D(g_0, g_1)$ by virtue of Lemma 2.3 and the multiplication operator $V_d(g; \cdot)$ is bounded and self-adjoint. We now introduce the trigonometric weight function

$$w_t(g_0, g_1; x) \equiv (\sin rx)^{2g_0}(\cos rx)^{2g_1},$$

and observe that we may rewrite $D(g_0, g_1)$ as

$$D(g_0, g_1) = w_t(g_0, g_1; x)^{1/2}\mathcal{D}.$$

(Indeed, $\mathcal{D}$ (2.8) is closed under products and we have

$$\left(\frac{s(x)}{\sin rx}\right)^{\lambda}, \quad \left(\frac{s(\pi/2r - x)}{\cos rx}\right)^{\mu} \in \mathcal{D}, \qquad \lambda, \mu \in \mathbb{R},$$

as is easily checked.)

Next, we calculate the similarity-transformed differential operator

$$A_t(g_0, g_1; x) \equiv w_t(g_0, g_1; x)^{-1/2} H_t(g_0, g_1; x) w_t(g_0, g_1; x)^{1/2}$$

$$= -\frac{d^2}{dx^2} - 2r(g_0 \cot rx - g_1 \tan rx)\frac{d}{dx} + r^2(g_0 + g_1)^2.$$

This yields an essentially self-adjoint operator $A_t(g_0, g_1)$ on the dense subspace $\mathcal{D}$ (given by (2.8)) of the Hilbert space

$$\mathcal{H}_t \equiv L^2([0, \pi/2r], w_t(g_0, g_1; x)dx).$$

The crux is now that $A_t(g_0, g_1)$ has an orthogonal base of eigenvectors belonging to $\mathcal{D}$ with nonnegative eigenvalues, explicitly given by

$$d_{m,t}(g_0, g_1; x) = {}_2F_1(-m, g_0 + g_1 + m; g_0 + 1/2; \sin^2 rx), \qquad m \in \mathbb{N}, \tag{2.17}$$

$$A_t(g_0, g_1)d_{m,t}(g_0, g_1) = r^2(g_0 + g_1 + 2m)^2 d_{m,t}(g_0, g_1), \qquad m \in \mathbb{N}. \tag{2.18}$$

Indeed, the r.h.s. of (2.17) yields polynomials of degree m in $\sin^2 rx$, and these polynomials amount to the Jacobi polynomials

$$P_m^{(\alpha,\beta)}(y), \qquad \alpha \equiv g_0 - 1/2, \qquad \beta \equiv g_1 - 1/2, \qquad y \equiv \cos 2rx, \qquad m \in \mathbb{N},$$

which form an orthogonal base in

$$\mathcal{H}_P \equiv L^2\big([-1, 1], (1 - y)^\alpha(1 + y)^\beta dy\big),$$

see for instance [20].

From the above it follows that the spectrum of $H_t(g_0, g_1)$, $g_0, g_1 > -1/2$, is bounded below, so by (2.16) the same is true for $H(g)$, $g \in \tilde{\Pi}$. Furthermore, it is clear from the eigenvalues (2.18) that the resolvent of $H_t(g_0, g_1)$ is compact. By the second resolvent identity this also holds true for the resolvent of $H(g)$, so the theorem follows. ■

This theorem also reveals the special character of the (at most four) self-adjoint extensions of $H(g)$ restricted to $\mathcal{C}$ (2.10) that arise via $\mathcal{I}(g)$: They are the extensions whose trigonometric limits yield the (at most four) self-adjoint extensions of $H_t(g_0, g_1)$ restricted to $\mathcal{C}$ that are associated with the Jacobi polynomials. Of course, this is particularly clear when $V(g; x)$ and $V_t(g_0, g_1; x)$ vanish identically, cf. Section 4.

For the remainder of this section we assume $g \in \Pi_r$. It follows from Theorems 2.4 and 2.5 that $H(g)$ has nondegenerate eigenvalues

$$E_0(g) < E_1(g) < E_2(g) < \cdots .$$

Also, since $H(g)$ and $H_t(g_0, g_1)$ differ by a bounded self-adjoint operator (cf. (2.16)), it follows from (2.18) that the eigenvalues have asymptotics

$$E_m(g) \sim 4r^2m^2, \qquad m \to \infty.$$

By Theorem 2.4 the associated eigenvectors $u_m(g)$ are also eigenvectors of $T(g)$, and they are just the scalar multiples of $e_n(g)$ for some $n \in \mathbb{N}$. Hence we may and will fix the ordering choice of the $T(g)$-eigenvectors for a given degenerate eigenvalue (if any occur), attained for example for $\nu_j = \nu_{j+1} = \cdots = \nu_{j+k}$, by requiring that the associated $H(g)$-eigenvalues satisfy $\lambda_j < \lambda_{j+1} < \cdots < \lambda_{j+k}$. With this ordering of the $e_n(g)$ understood from now on, there is a uniquely determined permutation τ_g of the nonnegative integers such that

$$\lambda_{\tau_g(m)}(g) = E_m(g), \qquad m \in \mathbb{N}, \tag{2.19}$$

cf. (2.15).

Next, we observe that we still have not completely fixed the choice of the basis vectors $e_n(g)$. Indeed, we have required real-valuedness of $e_n(g;x)$, but this leaves a sign undetermined. We may and will fix this ambiguity by requiring

$$d_n(g;0) > 0, \qquad \forall\, n \in \mathbb{N},$$

cf. (2.14).

At this point the alert reader might (or rather should) object that the sign choice just detailed may be in conflict with the decomposition (2.5). Indeed, once the ONB-functions $e_n(g;x)$ have been fixed, the ONB-functions $e_n(g';x)$ are uniquely determined, and the signs of the latter might differ from the above sign convention.

It follows from the last theorem of this section that this contingency does not arise. Its main result is, however, that $H(g)$ and $H(g')$ have the same spectrum.

Theorem 2.6. *The integral operator $\mathcal{I}(g)$ is real-analytic in g on Π_r in the Hilbert–Schmidt norm. With the above sign convention for $e_n(g;x)$ understood, it has singular value decomposition* (1.19)*. Furthermore, we have an equality of spectra*

$$\sigma(H(g)) = \sigma(H(g')), \qquad g \in \Pi_r.$$

Proof. Analyticity of $\mathcal{I}(g)$ in g with respect to the HS norm amounts to analyticity of its kernel $\Psi(g;x,y)$ in g in the strong topology of the Hilbert space $\mathcal{H}_K$ (1.14). From the explicit formula (1.4) it is readily verified that the kernel is strongly analytic in g in a complex neighborhood of Π_r, so the first assertion follows.

Next, we fix $g \in \Pi_r$ and note that the vectors

$$b_m(g;x) \equiv e_{\tau_g(m)}(g;x), \qquad m \in \mathbb{N},$$

yield an ONB with $H(g)$-eigenvalues $E_m(g)$, cf. (2.19). Recalling (2.11) and (2.3), we now deduce

$$H(g')\mathcal{I}(g')b_m(g) = \mathcal{I}(g')H(g)b_m(g) = E_m(g)\mathcal{I}(g')b_m(g).$$

In view of (2.4), this entails that the vectors $\mathcal{I}(g')b_m(g)$ are eigenvectors of $H(g')$ with eigenvalues $E_m(g)$, and that they are total in $\mathcal{H}$. Hence,

$$E_m(g) = E_m(g'), \qquad \forall\, m \in \mathbb{N},$$

and so the last assertion follows. Moreover, it follows that we have

$$\tau_g = \tau_{g'},$$

and

$$\mathcal{I}(g')b_m(g) = \mu_m(g')b_m(g'), \qquad \mu_m(g') \in \mathbb{R}^*.$$

To prove the theorem, it remains to show that all of the numbers $\mu_m(g)$ are positive. Now they do not vanish and are continuous on Π_r, since their squares yield eigenvalues of the self-adjoint trace class operator $T(g)$, which (by the first assertion) is continuous in g on Π_r in the trace norm topology. As Π_r is a connected set, it suffices to check positivity for one g in Π_r. For $g = (0,0,1,1)$ we have $g' = g$ and $\mathcal{I}(g)$ is self-adjoint. Thus the numbers $\mu_m(g)$ are eigenvalues of $\mathcal{I}(g)$. These are easily determined explicitly, and they are all positive, cf. (4.1). ∎

For all of the special $g \in \Pi_r$ where we have calculated the numbers $\mu_m(g)$ explicitly (cf. Section 4), they are distinct and decreasing in m:

$$\mu_0(g) > \mu_1(g) > \mu_2(g) > \cdots > 0.$$

This entails in particular that the permutation τ_g is the identity permutation. It is an open problem whether these features hold true on all of Π_r.

3 A hidden S_4 symmetry

In this section our purpose is to study, for a given coupling vector $g \in \Pi_r$, the couplings $\hat{g}$ for which the spectra of $H(g)$ and $H(\hat{g})$ are equal. To this end we note that we can establish equality of spectra by exploiting the core property of $D(g_0, g_1)$, cf. Lemma 2.3. First, since the action of the differential operator $H(g)$ on the domain $D(g_0, g_1)$ is invariant under taking $g_t \to 1 - g_t$, and the domain does not depend on g_2 and g_3, we get equality of spectra (and even equality of operators) under the transformations

$$g_2 \to 1 - g_2, \qquad g_3 \to 1 - g_3. \tag{3.1}$$

(A caveat is in order: these maps can take g out of Π_r.) Less trivially, consider the permutation

$$(g_0, g_1, g_2, g_3) \to (g_1, g_0, g_3, g_2). \tag{3.2}$$

Defining the 'mirror' unitary

$$(Mf)(x) \equiv f(\pi/2r - x), \qquad x \in [0, \pi/2r], \qquad f \in \mathcal{H}, \tag{3.3}$$

we clearly get

$$MD(g_0, g_1) = D(g_1, g_0).$$

This implies

$$MH(g_0, g_1, g_2, g_3)M = H(g_1, g_0, g_3, g_2),$$

so that (3.2) yields operators with equal spectra; in this case the two operators are different for generic g.

Next, we note that the matrix J_N (1.5) can be viewed as the reflection with respect to the unit vector $(1, -1, 1, -1)/2$:

$$J_N = \mathbf{1}_4 - 2u \otimes u, \qquad u \equiv (1, -1, 1, -1)/2.$$

Thus the self-dual couplings are given by

$$g' = g \;\Leftrightarrow\; g_0 + g_2 = g_1 + g_3,$$

and on this 3-dimensional submanifold of Π_r the HS operator $\mathcal{I}(g)$ is self-adjoint. The maps (3.1) and (3.2) just considered do not involve J_N. But it is clear that even when we start from a self-dual g, the maps (3.1) give rise to new couplings $\hat{g}$ that are generically not self-dual. When also $\hat{g} \in \Pi_r$, we can act with J_N to obtain $\hat{g}' \in \Pi_r$ for which $H(\hat{g}')$ has the same spectrum as $H(g)$ (by virtue of Theorem 2.6).

All of the maps studied thus far are involutions. The question now arises which group is generated by these involutions. We proceed to answer this question.

For this purpose it is expedient to switch to different couplings. We do this in two steps. First, we set

$$\lambda = g - \zeta/2, \qquad \zeta \equiv (1, 1, 1, 1), \tag{3.4}$$

so that

$$H(g) \to -\frac{d^2}{dx^2} + \sum_t \left(\lambda_t^2 - \frac{1}{4}\right) \wp(x + \omega_t). \tag{3.5}$$

Also, (3.1) and (3.2) become

$$\lambda_2 \to -\lambda_2, \qquad \lambda_3 \to -\lambda_3, \tag{3.6}$$
$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \to (\lambda_1, \lambda_0, \lambda_3, \lambda_2), \tag{3.7}$$

and we have

$$\lambda = \lambda' \ \Leftrightarrow \ \lambda_0 + \lambda_2 = \lambda_1 + \lambda_3,$$

since ζ is self-dual.

The second step may seem unmotivated at first sight. It consists in defining couplings

$$c_0 = \lambda_0 + \lambda_3, \qquad c_3 = \lambda_0 - \lambda_3, \qquad c_1 = \lambda_1 + \lambda_2, \qquad c_2 = \lambda_1 - \lambda_2. \tag{3.8}$$

The point of this definition is that the maps (3.6) become permutations

$$c_0 \leftrightarrow c_3, \qquad c_1 \leftrightarrow c_2, \tag{3.9}$$

while (3.7) turns into the permutation

$$(c_0, c_1, c_2, c_3) \to (c_1, c_0, c_3, c_2). \tag{3.10}$$

Moreover, the J_N-action transforms into the permutation

$$c_2 \leftrightarrow c_3, \tag{3.11}$$

as is easily checked.

Since the three transpositions (3.9) and (3.11) already generate S_4, it follows that (3.10) may be viewed as a suitable product of these maps. The symmetries (3.9), (3.10) generate an 8-element S_4-subgroup isomorphic to the dihedral group $I_2(4)$. This subgroup gives rise to a decomposition of S_4 into three cosets. Specifically, as coset-representants we may choose the three permutations

$$(c_0, c_1, c_2, c_3) \ \to \ (c_0, c_1, c_2, c_3), \ (c_0, c_1, c_3, c_2), \ (c_0, c_2, c_3, c_1). \tag{3.12}$$

Now that we have clarified the character of the symmetry group by exploiting the new couplings c, we may and will transform back to g. In particular, for a given g the three maps (3.12) are

$$g \to g, g', \tilde{g},$$

where

$$\tilde{g}_0 \equiv \frac{1}{2}(g_0 + g_1 + g_2 + g_3 - 1), \qquad \tilde{g}_1 \equiv \frac{1}{2}(g_0 + g_1 - g_2 - g_3 + 1),$$
$$\tilde{g}_2 \equiv \frac{1}{2}(-g_0 + g_1 - g_2 + g_3 + 1), \qquad \tilde{g}_3 \equiv \frac{1}{2}(g_0 - g_1 - g_2 + g_3 + 1).$$

We now summarize and extend the above analysis in the following theorem.

Theorem 3.1. *The group G generated by the maps*

$$(g_0, g_1, g_2, g_3) \to (g_0, g_1, 1 - g_2, g_3), \tag{3.13}$$
$$(g_0, g_1, g_2, g_3) \to (g_0, g_1, g_2, 1 - g_3), \tag{3.14}$$

and

$$g \to J_N g, \tag{3.15}$$

with J_N *given by* (1.5), *is isomorphic to* S_4 *and contains the mirror map*

$$(g_0, g_1, g_2, g_3) \to (g_1, g_0, g_3, g_2).$$

Now assume g *belongs to the set* Π_G, *cf.* (1.22). *Then the* G*-orbit of* g *belongs to* Π_G *and one has*

$$\sigma(H(g)) = \sigma(H(w(g))), \qquad \forall\, w \in G. \tag{3.16}$$

Proof. We have already shown the first assertion. Next, we recall that Π_r is defined by the five linear constraints

$$2g_0 > -1, \qquad 2g_1 > -1, \qquad g_0 + g_1 + g_2 + g_3 > 0, \qquad g_0 + g_1 - g_2 + g_3 > -1,$$
$$g_0 + g_1 + g_2 - g_3 > -1.$$

Together with the extra constraint defining Π_G, i.e.,

$$g_0 + g_1 - g_2 - g_3 > -2,$$

this yields six constraints that transform into

$$c_\mu + c_\nu > -2, \qquad \mu, \nu \in \{0, 1, 2, 3\}, \qquad \mu \neq \nu, \tag{3.17}$$

as is readily verified from (3.8) and (3.4). Since the latter are manifestly permutation invariant, the G-orbit assertion follows. Finally, since (3.16) holds for the generators (3.13)–(3.15) and Π_G is invariant under G, it holds for all $w \in G$. ■

Rewriting (3.5) in terms of c, we obtain

$$\hat{H}(c) = -\frac{d^2}{dx^2} - \frac{1}{4}\sum_{t=0}^{3} \wp(x + \omega_t) + \frac{1}{4}\left(c_0^2 + c_3^2\right) \sum_{t=0,3} \wp(x + \omega_t) + \frac{1}{2} c_0 c_3 (\wp(x) - \wp(x + \omega_3))$$
$$+ \frac{1}{4}\left(c_1^2 + c_2^2\right) \sum_{t=1,2} \wp(x + \omega_t) + \frac{1}{2} c_1 c_2 (\wp(x + \omega_1) - \wp(x + \omega_2)).$$

Requiring (3.17), the spectral invariance under the $I_2(4)$ subgroup is readily understood in this guise as well. But the invariance under any other c-permutation looks bizarre.

4 Special cases

It is plain from (1.1), (1.2) that the Heun differential operator reduces to $-d^2/dx^2$ for the sixteen g's obtained by choosing $g_t \in \{0, 1\}$, $t = 0, 1, 2, 3$. Viewing $-d^2/dx^2$ as an operator on $\mathcal{H}$ with domain $D(g_0, g_1)$, it is also clear which ONB $\{b_n\}_{n \in \mathbb{N}}$ of eigenvectors in $D(g_0, g_1)$ arises. Specifically, with the normalization constant

$$\mathcal{N} \equiv (4r/\pi)^{1/2},$$

the following ONBs are involved:

(i) $g_0 = g_1 = 0 \;\Rightarrow\; b_0 = \mathcal{N}/\sqrt{2}, \quad b_n = \mathcal{N} \cos 2nrx, \quad n > 0,$

(ii) $g_0 = g_1 = 1 \;\Rightarrow\; b_n = \mathcal{N} \sin(2n+2)rx$,

(iii) $g_0 = 1,\; g_1 = 0 \;\Rightarrow\; b_n = \mathcal{N} \sin(2n+1)rx$,

(iv) $g_0 = 0,\; g_1 = 1 \;\Rightarrow\; b_n = \mathcal{N} \cos(2n+1)rx$.

We proceed to study the above g-choices in relation to $\mathcal{I}(g)$. For case (i) we must choose $g_2 = g_3 = 1$ to obtain $g \in \Pi_r$. (For $g_2 = g_3 = 0$ we get $s_g = 0$, and for $g_2 = 0$, $g_3 = 1$ and $g_2 = 1$, $g_3 = 0$ we get $g_1' = -1/2$ and $g_0' = -1/2$, resp.) Then we obtain from (1.4)–(1.6) and (1.13)

$$\Psi((0,0,1,1);x,y) = R(x)R(\omega_1 - x)R(y)R(\omega_1 - y)/R(x+y)R(x-y).$$

It now follows from Theorem 2.4 that the ONB (i) consists of eigenvectors for $\mathcal{I}((0,0,1,1))$. In fact, this can be readily verified by a contour integration, which also yields the eigenvalues explicitly. (To simplify the residues, one needs the duplication formula for the R-function, cf. (1.10).) Specifically, we obtain

$$\nu_n = \pi/p\cosh(nr\alpha), \qquad g = (0,0,1,1), \qquad c = (0,0,-1,-1), \tag{4.1}$$

with p the infinite product in (1.11).

Turning to case (ii), we get two self-dual subcases. For the first one,

$$\Psi((1,1,0,0);x,y) = p^4 s(x)s(\omega_1 - x)s(y)s(\omega_1 - y)/R(x+y)R(x-y),$$

the eigenvalues of the corresponding integral operator $\mathcal{I}(g)$ on the ONB (ii) can be calculated in the same way as before. The result is

$$\nu_n = \pi e^{r\alpha}/p\cosh((n+1)r\alpha), \qquad g = (1,1,0,0), \qquad c = (0,0,1,1).$$

Using (1.10) and (1.11), the second subcase can be rewritten as

$$\Psi((1,1,1,1);x,y) = p^2 s(2x)s(2y)/R^2(x+y)R^2(x-y).$$

With due effort (involving in particular a suitable use of (1.11)), the eigenvalues for this g-choice can again be determined explicitly, yielding

$$\nu_n = \frac{2\pi(n+1)re^{r\alpha}}{p^2\sinh((n+1)r\alpha)}, \qquad g = (1,1,1,1), \qquad c = (1,1,0,0).$$

It should be noted that the $\mathcal{I}(g)$-spectra of these two subcases do not coincide, even though they are related by a c-permutation.

The remaining two subcases are not self-dual, but they are related by mirror symmetry, cf. (3.3). Hence we only study

$$g = (1,1,0,1),\; c = (1,0,1,0) \;\Leftrightarrow\; g' = \frac{1}{2}(3,1,1,1),\; c' = (1,0,0,1). \tag{4.2}$$

For this choice it is clear that $H(g')$ has a nontrivial potential. Its eigenvector ONB is related to the ONB (ii) via $\mathcal{I}(g')$. Specifically, we must have

$$b_n(g';x) = \mathcal{N}_n \int_0^{\pi/2r} dy \frac{[w(g';x)w(g;y)]^{1/2}}{[R(x+y)R(x-y)]^{3/2}} \sin(2n+2)ry, \qquad g = (1,1,0,1),$$

where the normalization constant $\mathcal{N}_n$ ensures that $b_n(g';x)$ has norm 1. Also, the $H(g')$-eigenvalues are given by

$$E_n((3,1,1,1)/2) = (2n+2)^2 r^2. \tag{4.3}$$

The cases (iii) and (iv) are related by mirror symmetry, so we only consider case (iii). First we choose

$$g = (1, 0, 1, 1),\ c = (1, 0, -1, 0) \ \Leftrightarrow \ g' = \frac{1}{2}(1, 1, 1, 3),\ c' = (1, 0, 0, -1).$$

Thus $H(g')$ has a nontrivial potential. Just as for (4.2), its eigenvector ONB is given by

$$b_n(g'; x) = \mathcal{N}_n \int_0^{\pi/2r} dy \frac{[w(g'; x)w(g; y)]^{1/2}}{[R(x + y)R(x - y)]^{3/2}} \sin(2n + 1)ry, \qquad g = (1, 0, 1, 1).$$

The analog of (4.3) is

$$E_n((1, 1, 1, 3)/2) = (2n + 1)^2 r^2.$$

We continue with

$$g = (1, 0, 0, 0),\ c = (0, -1, 0, 1) \ \Leftrightarrow \ g' = \frac{1}{2}(1, 1, -1, 1),\ c' = (0, -1, 1, 0).$$

Here we obtain the $H(g')$-ONB

$$b_n(g'; x) = \mathcal{N}_n \int_0^{\pi/2r} dy \frac{[w(g'; x)w(g; y)]^{1/2}}{[R(x + y)R(x - y)]^{1/2}} \sin(2n + 1)ry, \qquad g = (1, 0, 0, 0),$$

with eigenvalues

$$E_n((1, 1, -1, 1)/2) = (2n + 1)^2 r^2.$$

Next we consider the self-dual choice $g_2 = 0$, $g_3 = 1$. Then we get

$$\Psi((1, 0, 0, 1); x, y) = p^2 s(x)R(\omega_1 - x)s(y)R(\omega_1 - y)/R(x + y)R(x - y).$$

As before, the eigenvalues of $\mathcal{I}(g)$ on the ONB (iii) can be explicitly determined, yielding

$$\nu_n = \pi e^{r\alpha/2}/p \cosh((n + 1/2)r\alpha), \qquad g = (1, 0, 0, 1), \qquad c = (1, -1, 0, 0).$$

Finally, we study the choice

$$g = (1, 0, 1, 0),\ c = (0, 0, -1, 1) \ \Leftrightarrow \ g' = (0, 1, 0, 1),\ c' = (0, 0, 1, -1),$$

for which

$$\Psi((1, 0, 1, 0); x, y) = p^2 s(x)R(x)s(\omega_1 - y)R(\omega_1 - y)/R(x + y)R(x - y).$$

Once more, a contour integration yields the expected result

$$\sin(2n + 1)rx = \nu_n \int_0^{\pi/2r} dy \Psi((1, 0, 1, 0); x, y) \cos(2n + 1)ry,$$

with the singular values ν_n given by

$$\nu_n = \pi e^{r\alpha/2}/p \sinh((n + 1/2)r\alpha), \qquad g = (1, 0, 1, 0), \qquad c = (0, 0, -1, 1).$$

We conclude this section with some remarks on special cases of a different character. We recall first that $\mathcal{I}(g)$ reduces to a rank-one operator for $g \in \Pi$ with $s_g = 0$, cf. (1.4) and (1.6). By continuity in g, it therefore follows that the functions $w(g; x)^{1/2}$ and $w(g'; x)^{1/2}$ are eigenfunctions of $H(g)$ and $H(g')$, resp., with the same eigenvalue. We point out that this is not at all obvious. With due effort, however, this can be directly verified.

Specifically, it follows from I (3.79), (3.113) and (3.114) that the eigenfunction property is equivalent to constancy of the function

$$\sum_{t=0}^{3} g_t^2 \wp(x+\omega_t) - \left(\sum_{t=0}^{1} \left(g_t \frac{s'(x+\omega_t)}{s(x+\omega_t)} + g_{t+2} \frac{R'(x+\omega_t)}{R(x+\omega_t)} \right) \right)^2, \qquad \sum_{t=0}^{3} g_t = 0.$$

We encountered this functional identity before on p. 253 of [18], where we supplied a (short) proof. Using next I (3.105)–(3.106), the constant (which amounts to the ground state eigenvalue) can be calculated explicitly:

$$E_0(g) = \sum_{j=1}^{3} g_j e_j (g_j + 2g_0), \qquad e_j \equiv \wp(\omega_j), \qquad j = 1, 2, 3, \qquad \sum_{t=0}^{3} g_t = 0.$$

From this formula one readily verifies equality to $E_0(g')$.

Finally, we comment on the case where $V(g; x)$ is real-analytic on $\mathbb{R}$. For this we need to choose $g_0, g_1 \in \{0, 1\}$, cf. (1.2). Then one need only combine the $\mathcal{H}$-ONB's $b_n((0, 0, g_2, g_3); x)$ and $b_n((1, 1, g_2, g_3); x)$ (consisting of real-analytic π/r-periodic functions that are even and odd, resp.) to obtain an ONB of π/r-periodic functions for

$$\mathcal{H}_b \equiv L^2([-\pi/2r, \pi/2r], dx).$$

Likewise, an ONB of π/r-antiperiodic functions for $\mathcal{H}_b$ results from combining the $\mathcal{H}$-ONB $b_n((0, 1, g_2, g_3); x)$ of even functions and the $\mathcal{H}$-ONB of odd functions $b_n((1, 0, g_2, g_3); x)$. In terms of Floquet theory, therefore, one can only arrive at the $\mathcal{H}_b$-ONB's associated with the multipliers 1 and -1. Whenever the four g-values are all in Π_r, one can invoke the spectral invariance under taking $g \to g'$. Note that $V(g'; x)$ may be singular at $x = 0$ and/or $x = \pi/2r$, as we have already seen in the above special cases where $V(g; x)$ vanishes identically.

A Proofs of Lemmas 2.1 and 2.2

Proof of Lemma 2.1. We recall that the function

$$h(x) \equiv \int_0^{\pi/2r} dy \mathcal{S}(g; x, y)\phi(y), \qquad \phi \in L^1([0, \pi/2r]), \qquad x \in [0, \pi/2r],$$

extends to an even analytic function in $|\mathrm{Im}\, z| < \alpha/2$, cf. the paragraph containing (2.1). Thus its vanishing is equivalent to vanishing of the coefficients in its Taylor expansion

$$h(x) = \sum_{k=0}^{\infty} c_k x^{2k}, \tag{A.1}$$

at $x = 0$. We proceed to study these coefficients.

Obviously, we need information on the coefficients in the expansion of $\mathcal{S}(g; x, y)$ at $x = 0$. To this end we first consider the expansion

$$F(x) \equiv -\frac{1}{2}(\ln R(x+y) + \ln R(x-y)) = \sum_{j=0}^{\infty} \frac{F^{(2j)}(0)}{(2j)!} x^{2j}.$$

Using

$$-\frac{d^2}{dz^2}\ln R(z) = \tilde{\wp}(z) + 2\eta r/\pi, \qquad \tilde{\wp}(z) \equiv \wp(z + i\alpha/2),$$

(cf. (1.11), (1.12)), we obtain

$$F(0) = -\ln R(y), \qquad F^{(2)}(0) = \tilde{\wp}(y) + 2\eta r/\pi,$$
$$F^{(2k)}(0) = \tilde{\wp}^{(2k-2)}(y), \qquad k > 1.$$

Now from standard elliptic function lore [4] we deduce

$$\wp^{(2k-2)}(z) = (2k-1)!\wp(z)^k + \text{l.d.}, \qquad k > 1,$$

where l.d. stands for a polynomial in $\wp(z)$ of degree $< k$. Thus we have

$$F(x) = -\ln R(y) + \sum_{j=1}^{\infty} a_j x^{2j},$$

with

$$a_k = \frac{1}{2k}\tilde{\wp}(y)^k + \text{l.d.} \tag{A.2}$$

We are now prepared to consider the expansion of $\mathcal{S}$. It is given by

$$\mathcal{S}(g; x, y) = R(y)^{-2s_g} \exp\left(2s_g \sum_{j=1}^{\infty} a_j x^{2j}\right) = R(y)^{-2s_g} \sum_{k=0}^{\infty} b_k x^{2k}, \tag{A.3}$$

so that the first few b_k read

$$b_0 = 1, \qquad b_1 = 2s_g a_1, \qquad b_2 = 2s_g a_2 + 2s_g^2 a_1^2, \qquad b_3 = 2s_g a_3 + 4s_g^2 a_2 a_1 + 4s_g^3 a_1^3/3.$$

More generally, it is easily seen that b_k is the solution to the recurrence

$$kb_k = 2s_g \sum_{j=1}^{k} j a_j b_{k-j}, \qquad k > 0, \qquad b_0 = 1.$$

In view of (A.2), this implies that b_k is given by

$$b_k = Q^{(k)}(s_g)\tilde{\wp}(y)^k + \text{l.d.}, \tag{A.4}$$

where $Q^{(k)}(u)$ is a polynomial of the form

$$Q^{(k)}(u) = \sum_{l=1}^{k} p_l^{(k)} u^l, \qquad p_l^{(k)} \in (0, \infty).$$

We are now in the position to use our assumption $s_g > 0$. It entails that the coefficient $Q^{(k)}(s_g)$ in (A.4) is positive:

$$b_k = q_k \tilde{\wp}(y)^k + \text{l.d.}, \qquad q_k \in (0, \infty), \qquad \forall\, k \in \mathbb{N}. \tag{A.5}$$

Returning to (A.1), we first note that (A.3) entails

$$c_k = \int_0^{\pi/2r} dy b_k(y)\psi(y),$$

where

$$\psi(y) \equiv R(y)^{-2s_g}\phi(y). \tag{A.6}$$

Since the coefficients c_k vanish by assumption, it follows recursively from (A.5) that we have

$$\int_0^{\pi/2r} dy \tilde{\wp}(y)^k \psi(y) = 0, \qquad \forall\, k \in \mathbb{N}.$$

Now $\tilde{\wp}(y)$ is monotone increasing on $[0,\pi/2r]$. Thus we can invoke the Stone-Weierstrass theorem [15] to conclude that the span of the functions $1, \tilde{\wp}(y), \tilde{\wp}(y)^2, \tilde{\wp}(y)^3, \dots$ is dense (in the supremum norm) in the space of real-valued continuous functions on $[0,\pi/2r]$. Since $\psi(y)$ is an L^1-function, it follows by dominated convergence that we have

$$\int_0^{\pi/2r} dy C(y)\psi(y) = 0, \qquad \forall\, C \in C_{\mathbb{R}}([0,\pi/2r]).$$

Hence $\psi = 0$, so by (A.6) $\phi = 0$. ■

Proof of Lemma 2.2. Letting $e(x) \in D(g_0, g_1)$, we first show that the action of the differential operator $H(g)$ (1.1) on $e(x)$ yields a function in $\mathcal{H}$. To this end we use the relations

$$e(x) = s(x)^{g_0} s(x-\omega_1)^{g_1} d(x), \qquad d \in \mathcal{D},$$

and

$$\frac{s'(z)^2}{s(z)^2} = \frac{s''(z)}{s(z)} + \wp(z) + \eta\omega_1,$$

to calculate

$$\begin{aligned} e''(x) &= e(x) \sum_{t=0,1} \left([g_t^2 - g_t][\wp(x+\omega_t) + \eta\omega_1] + g_t^2 \frac{s''(x+\omega_t)}{s(x+\omega_t)} \right) \\ &\quad + e(x) \left(Q(x) + 2\frac{d'(x)}{d(x)} \sum_{t=0,1} g_t \frac{s'(x+\omega_t)}{s(x+\omega_t)} + \frac{d''(x)}{d(x)} \right), \end{aligned}$$

where

$$Q(x) \equiv 2g_0 g_1 \frac{s'(x)}{s(x)} \frac{s'(x+\omega_1)}{s(x+\omega_1)}. \tag{A.7}$$

Subtracting the potential term

$$e(x) \sum_{t=0}^{3} g_t(g_t - 1)\wp(x+\omega_t),$$

we obtain a sum of terms that are manifestly in $\mathcal{H}$, save for the terms

$$e(x) \left(Q(x) + \sum_{t=0,1} g_t^2 \frac{s''(x+\omega_t)}{s(x+\omega_t)} \right) + 2s(x)^{g_0} s(x-\omega_1)^{g_1} \sum_{t=0,1} g_t d'(x) \frac{s'(x+\omega_t)}{s(x+\omega_t)}. \tag{A.8}$$

Now $s(z)$ is odd and has a first order zero for $z = 0$, so $s''(z)$ vanishes at $z = 0$ at least to first order. Thus the two terms in the first sum are smooth for $x \in \mathbb{R}$. Next, $s'(z)$ vanishes at

$z = \omega_1$, since $s(z)$ is odd and $2\omega_1$-antiperiodic. Therefore $Q(x)$ (A.7) is also smooth. Finally, since $d'(0) = d'(\omega_1) = 0$, the two terms in the second sum are continuous on $[0, \omega_1]$. Hence the function given by (A.8) is in $\mathcal{H}$, so that $H(g)$ is well defined on $D(g_0, g_1)$.

It remains to show that $H(g)$ is symmetric on $D(g_0, g_1)$. To this end, let $d_1, d_2 \in \mathcal{D}$ and consider

$$\begin{aligned}&(s(\cdot)^{g_0} s(\cdot + \omega_1)^{g_1} d_1, H(g) s(\cdot)^{g_0} s(\cdot + \omega_1)^{g_1} d_2)\\&\qquad - (H(g) s(\cdot)^{g_0} s(\cdot + \omega_1)^{g_1} d_1, s(\cdot)^{g_0} s(\cdot + \omega_1)^{g_1} d_2).\end{aligned} \tag{A.9}$$

From the above calculation we see that this equals

$$\begin{aligned}&\int_0^{\omega_1} dx s(x)^{2g_0} s(x+\omega_1)^{2g_1} \Bigg(\overline{d_1''(x)} d_2(x) - \overline{d_1(x)} d_2''(x)\\&\qquad + 2[\overline{d_1'(x)} d_2(x) - \overline{d_1(x)} d_2'(x)] \sum_{t=0,1} g_t \frac{s'(x+\omega_t)}{s(x+\omega_t)} \Bigg).\end{aligned}$$

Clearly, this can be rewritten as

$$\int_0^{\omega_1} dx \left(-\overline{d_1(x)} \frac{d}{dx} [s(x)^{2g_0} s(x+\omega_1)^{2g_1} d_2'(x)] + d_2(x) \frac{d}{dx} [s(x)^{2g_0} s(x+\omega_1)^{2g_1} \overline{d_1'(x)}] \right). \tag{A.10}$$

Now for all $d \in \mathcal{D}$ we have

$$d'(x) = \int_0^x d''(t) dt = O(x), \quad x \downarrow 0, \qquad d'(x) = -\int_x^{\omega_1} d''(t) dt = O(x - \omega_1), \quad x \uparrow \omega_1.$$

Integrating by parts in (A.10), it follows that the boundary terms vanish, whereas the new integrand vanishes identically. As a result, the difference (A.9) vanishes, so that $H(g)$ is indeed symmetric on $D(g_0, g_1)$. ■

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