$\mathcal{PT}$ Symmetry and QCD: Finite Temperature and Density[⋆]

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Abstract. The relevance of $\mathcal{PT}$ symmetry to quantum chromodynamics (QCD), the gauge theory of the strong interactions, is explored in the context of finite temperature and density. Two significant problems in QCD are studied: the sign problem of finite-density QCD, and the problem of confinement. It is proven that the effective action for heavy quarks at finite density is $\mathcal{PT}$-symmetric. For the case of $1+1$ dimensions, the $\mathcal{PT}$-symmetric Hamiltonian, although not Hermitian, has real eigenvalues for a range of values of the chemical potential μ, solving the sign problem for this model. The effective action for heavy quarks is part of a potentially large class of generalized sine-Gordon models which are non-Hermitian but are $\mathcal{PT}$-symmetric. Generalized sine-Gordon models also occur naturally in gauge theories in which magnetic monopoles lead to confinement. We explore gauge theories where monopoles cause confinement at arbitrarily high temperatures. Several different classes of monopole gases exist, with each class leading to different string tension scaling laws. For one class of monopole gas models, the $\mathcal{PT}$-symmetric affine Toda field theory emerges naturally as the effective theory. This in turn leads to sine-law scaling for string tensions, a behavior consistent with lattice simulations.

Key words: $\mathcal{PT}$ symmetry; QCD

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1 $\mathcal{PT}$ symmetry and two difficult problems of QCD

Models with $\mathcal{PT}$ symmetry have emerged as an interesting extension of conventional quantum mechanics. There is a large class of models that are not Hermitian, but nevertheless have real spectra as a consequence of $\mathcal{PT}$ symmetry. Bender and Boettcher have shown that single-component quantum mechanical models with $\mathcal{PT}$-symmetric potentials of the form $-\lambda\,(-ix)^p$ have real spectra [1]. An extensive literature on $\mathcal{PT}$ symmetry and related matters now exists, and there are extensive review articles available [2, 3]. Here we explore the relevance of $\mathcal{PT}$ symmetry for two of the most difficult problems in quantum chromodynamics (QCD), the gauge theory of the strong interaction.

The sign problem of QCD arises in the Euclidean space approach to QCD at finite, i.e., non-zero, quark number density [4, 5]. There is broad interest, both theoretically and experimentally, in the properties of QCD at finite temperature and density. Finite density QCD is particularly important for exploring the possibility of color-superconducting quark matter in the interiors of neutron stars [6]. Lattice gauge theory has proven to be a powerful tool for exploring QCD and related models at finite temperature. Unfortunately, these results have been obtained largely for zero density. Non-zero quark density is implemented by introducing a chemical potential μ for quark number. Within the Euclidean space formalism, a non-zero temperature T is obtained

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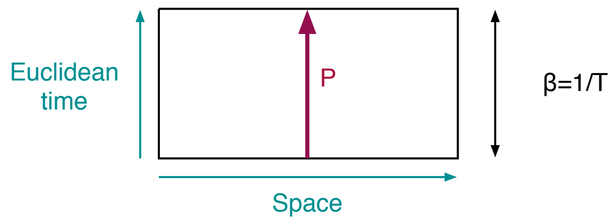

Figure 1. The Polyakov loop in Euclidean space-time.

by making the bosonic fields periodic in Euclidean time, with period $\beta = 1/T$. This is easy to implement in lattice simulations. Non-zero chemical potential, on the other hand, must be implemented in a way that makes the weight function used in the Feynman path integral complex. This is the so-called sign problem of finite density QCD. The complex weight assigned to Euclidean field configurations spoils the probabilistic interpretation of the Euclidean path integral, making the use of conventional importance-sampling algorithms impossible. While there have been impressive efforts to simulate finite-density QCD by extrapolating from $\mu = 0$, the sign problem remains a difficult, fundamental, and important problem. We will show below that QCD at finite density may be interpreted as a theory with $\mathcal{PT}$ symmetry. We will show explicitly how a (1+1)-dimensional gauge model can be reduced to a $\mathcal{PT}$-symmetric Hamiltonian over the gauge group, with real eigenvalues for a range of values of $\beta\mu$.

The other problem of modern strong-interaction physics we will consider is the origin of quark confinement. In many ways, it is the most important problem in QCD, because the confinement of quarks inside hadrons is the fundamental property of QCD not fully understood theoretically. A good overview of various approaches to this problem is provided by the review of Greensite [7]. Finite temperature gauge theories are advantageous in many aspects for the study of confinement. This is due largely to the utility and ubiquity of the Polyakov loop operator. Defined as a path-ordered exponential of the gauge field, in $3+1$ dimensions the Polyakov loop operator P is given by

$$P\left(\vec{x}\right) = \mathcal{P}\exp\left[i\int_0^\beta dt A_4\left(\vec{x},t\right)\right],$$

and represents the insertion of a static quark into a thermal system of gauge fields at a temperature $T = \beta^{-1}$. Fig. 1 shows the Polyakov loop in this geometry. Because of the periodic boundary conditions in the Euclidean time direction, the Polyakov loop is a closed loop, and its trace is gauge invariant. Also known as the Wilson line, the Polyakov loop represents the insertion of a static quark at a spatial point $\vec{x}$ in a gauge theory at finite temperature. In particular, the thermal average of the trace of P in an irreducible representation R of the gauge group is associated with the additional free energy F_R required to insert a static quark in the fundamental representation via

$$\langle \mathrm{Tr}_R P\left(\vec{x}\right)\rangle = e^{-\beta F_R}.$$

Pure SU(N) gauge theories have a global $Z(N)$ symmetry $P \to zP$ where $z = e^{\frac{2\pi i}{N}}$ is the generator of $Z(N)$, the center of SU(N). This symmetry, if unbroken, guarantees that for the fundamental representation F, $\langle \mathrm{Tr}_F P\left(\vec{x}\right)\rangle = 0$. This is interpreted as F_F being infinite, and an infinite free energy is required to insert a heavy quark into the system. On the other hand, if the $Z(N)$ symmetry is spontaneously broken, the free energy required is finite. Thus confinement in pure gauge theories is associated with unbroken center symmetry, and broken symmetry with a deconfined phase. The Polyakov loop is the order parameter for the deconfinement transition in pure gauge theories $\langle \mathrm{Tr}_F P\rangle = 0$ in the confined phase and $\langle \mathrm{Tr}_F P\rangle \neq 0$ in the deconfined phase. The addition of dynamical quarks in the fundamental representation explicitly breaks this $Z(N)$ symmetry. Nevertheless, the Polyakov loop remains important in describing the behavior of the system, as we will see in our treatment of the sign problem.

In pure gauge theories, the Wilson loop operator is used to measure the string tension between quarks in the confined phase where F_R vanishes for representations transforming non-trivially under $Z(N)$. At non-zero temperature, a timelike string tension $\sigma_k^{(t)}$ between k quarks and k antiquarks can be measured from the behavior of the correlation function

$$\langle \mathrm{Tr}_F P^k(\vec{x})\, \mathrm{Tr}_F \left(P^+(\vec{y})\right)^k \rangle \simeq \exp\left[-\frac{\sigma_k^{(t)}}{T}|\vec{x}-\vec{y}|\right]$$

at sufficiently large distances. A confining phase is defined by two properties: the expectation value $\langle \mathrm{Tr}_R P\rangle$ is zero for all representations R transforming non-trivially under $Z(N)$, and the string tensions $\sigma_k^{(t)}$ must be non-zero for $k = 1$ to $N-1$. There are two kinds of model field theories, related to QCD, for which these two properties are known to hold. As we discuss below, $\mathcal{PT}$ symmetry plays an interesting role, which may extend to QCD.

2 The chemical potential and the sign problem

Perturbation theory can be used to calculate the one-loop free energy density f_q of quarks in $d+1$ dimensions in the fundamental representation with spin degeneracy s moving in a Polyakov loop background at non-zero temperature $T = \beta^{-1}$ and chemical potential μ

$$f_q = -sT\int \frac{d^dk}{(2\pi)^d}\,\mathrm{Tr}_R\left[\ln\left(1+Pe^{\beta\mu-\beta\omega_k}\right)+\ln\left(1+P^+e^{-\beta\mu-\beta\omega_k}\right)\right],$$

where $\omega_k = \sqrt{k^2+M^2}$ is the energy of the particle as a function of k and M is the mass of the particle [8, 9]. The expression for a bosonic field is similar. The logarithm can be expanded to give

$$f_q = sT\int \frac{d^dk}{(2\pi)^d}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[e^{n\beta\mu-n\beta\omega_k}\mathrm{Tr}_R P^n + e^{-n\beta\mu-n\beta\omega_k}\mathrm{Tr}_R P^{+n}\right].$$

This expression has a simple interpretation as a sum of paths winding around the timelike direction. With standard boundary conditions, which are periodic for bosons and antiperiodic for fermions, this one-loop free energy always favors the deconfined phase.

The effects of heavy quarks in the fundamental representation, with $\beta M \gg 1$, on the gauge theory can be obtained approximately from the $n = 1$ term in the free energy

$$f_q \approx -sT\int \frac{d^dk}{(2\pi)^d}\mathrm{Tr}_F\left[Pe^{\beta\mu-\beta\omega_k}+P^+e^{-\beta\mu-\beta\omega_k}\right],$$

because term with higher n are suppressed by a factor $e^{-n\beta M}$. In this approximation, bosons and fermions have the same effect at leading order. After integrating over k, the free energy f_q can be written as $f_q \approx -h_F\left[e^{\beta\mu}\mathrm{Tr}_F P + e^{-\beta\mu}\mathrm{Tr}_F P^+\right]$. The one-loop free energy density is the one-loop effective potential at finite temperature. Thus the free energy for the heavy quarks can be added to the usual gauge action to give an effective action which involves only the gauge fields. The effective action is given by

$$S_{\mathrm{eff}} = \int d^{d+1}x\left[\frac{1}{4g^2}\left(F_{\mu\nu}^a\right)^2 - h_F\left(e^{\beta\mu}\mathrm{Tr}_F(P)+e^{-\beta\mu}\mathrm{Tr}_F(P^+)\right)\right]$$

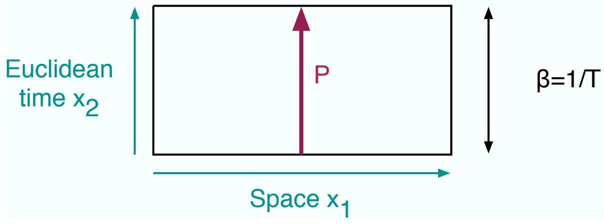

Figure 2. The Polyakov loop in $(1+1)$-dimensional space-time.

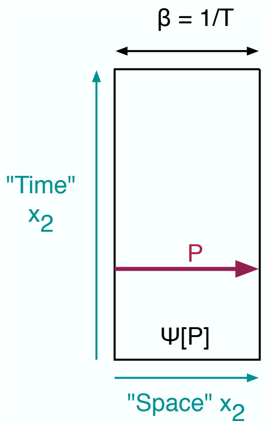

Figure 3. The Polyakov loop in a $(1+1)$-dimensional transfer matrix geometry.

and the structure and symmetries of the theory are obviously the same in any number of spatial dimensions. Because $\mathrm{Tr}_F P$ is complex for $N \geq 3$, the effective action for the gauge fields is complex. This is a form of the so-called sign problem for gauge theories at finite density: the Euclidean path integral involve complex weights. This problem is a fundamental barrier to lattice simulations of QCD at finite density.

3 Heavy quarks at $\mu \neq 0$ in $1+1$ dimensions

In $1+1$ dimensions, the field theory arising from the effective action can be reduced to a $\mathcal{PT}$-symmetric Hamiltonian acting on class functions of the gauge group. The effective action, including the effects of heavy quarks, is

$$S_{\mathrm{eff}} = \int d^2x \left[\frac{1}{4g^2} \left(F_{\mu\nu}^a\right)^2 - h_F \left(e^{\beta\mu} \mathrm{Tr}_F(P) + e^{-\beta\mu} \mathrm{Tr}_F(P^+) \right) \right],$$

where the gauge field A_μ now has two components. Fig. 2 shows the Polyakov loop in a $1+1$-dimensional geometry. It is convenient to work in a gauge where $A_1 = 0$; this is turn implies that A_2 depends only on x_1. After integration over x_2, we are left with a Lagrangian

$$L = \frac{\beta}{2g^2} \left(\frac{dA_2^a}{dx_1} \right)^2 - h_F \beta \left[e^{\beta\mu} \mathrm{Tr}_F(P) + e^{-\beta\mu} \mathrm{Tr}_F(P^+) \right],$$

which we regard as the Lagrangian for a system evolving as a function of a time coordinate x_1. This represents a change from a Euclidean time point of view to a transfer matrix geometry, as shown in Fig. 3. In this geometry, the Polyakov loop represents the insertion of an electric flux line in a box with periodic boundary conditions, and the free energy density is obtained from the lowest-lying eigenvalue of the transfer matrix.

The physical states of the system are gauge-invariant, meaning that they are class functions of P: $\Psi[P] = \Psi[gPg^+]$. The group characters form an orthonormal basis on the physical Hilbert space: $\Psi[P] = \sum_R a_R \mathrm{Tr}_R(P)$. The Hamiltonian H, obtained from L, acts on the physical states as

$$H = \frac{g^2\beta}{2}C_2 - h_F\beta\left[e^{\beta\mu}\mathrm{Tr}_F(P) + e^{-\beta\mu}\mathrm{Tr}_F(P^+)\right],$$

where C_2 is the quadratic Casimir operator for the gauge group, the Laplace–Beltrami operator on the group manifold. We have thus reduced the problem of heavy quarks at finite density in $1+1$ dimensions to one of quantum mechanics on the gauge group. Unfortunately, the Hamiltonian H is not Hermitian when $\mu \neq 0$, and thus cannot be relied upon to have real eigenvalues. This is a direct manifestation of the sign problem.

Although the Hamiltonian H is not Hermitian when $\mu \neq 0$, it is $\mathcal{PT}$-symmetric under the transformations

$$\mathcal{P}:\ x_2 \to -x_2, \quad A_2 \to -A_2, \qquad \mathcal{T}:\ i \to -i,$$

which should be regarded as parity and time-reflection in the transfer matrix geometry. Together these lead to

$$\mathcal{PT}:\ P \to P,$$

which leaves the Hamiltonian invariant. If this $\mathcal{PT}$ symmetry is unbroken, the eigenvalues of the Hamiltonian will be real, and there is no sign problem. The $\mathcal{PT}$ symmetry remains even in the high-density limit where the quark mass M and chemical potential μ are taken to infinity in such a way that antiparticles are suppressed and P^+ does not appear in H.

The simplest non-trivial gauge group is SU(3), because the cases of U(1) and SU(2) are atypical. For the gauge group U(1), the Hamiltonian H may be written as

$$H = -\frac{e^2\beta}{2}\frac{d^2}{d\theta^2} - h_F\beta\left(e^{\beta\mu+i\theta} + e^{-\beta\mu-i\theta}\right),$$

but a simple change of variable $\theta \to \theta + i\beta\mu$ eliminates μ:

$$H = -\frac{e^2\beta}{2}\frac{d^2}{d\theta^2} - h_F\beta\left(e^{+i\theta} + e^{-i\theta}\right).$$

This is very similar to the case of the two-dimensional $\mathcal{PT}$-symmetric sine-Gordon model considered in [10]. In the case of SU(2), all the irreducible representations are real, and the Hamiltonian is Hermitian:

$$H_{\mathrm{SU}(2)} = \frac{g^2\beta}{2}C_2 - 2h_F\cosh(\beta\mu)\,\chi_{j=1/2}(P).$$

This reality feature of SU(2) gauge theories at finite density holds in general, and has been exploited in lattice simulations with $\mu \neq 0$ [11, 12].

Thus $N = 3$ is the first non-trivial case for SU(N) gauge groups. We have calculated the lowest eigenvalues of H using finite dimensional approximants. It is convenient to work in the group character basis. The Casimir operator C_2 is diagonal in this basis, and characters act as raising and lowering operators. For example, in the 4×4 subspace spanned by the 1, 3, $\bar{3}$, and 8 representations of SU(3), the Hamiltonian takes the form

$$\begin{pmatrix} 0 & e^{-\beta\mu}h_F\beta & e^{\beta\mu}h_F\beta & 0 \\ e^{\beta\mu}h_F\beta & \frac{4}{3}\cdot\frac{g^2\beta}{2} & e^{-\beta\mu}h_F\beta & e^{\beta\mu}h_F\beta \\ e^{-\beta\mu}h_F\beta & e^{\beta\mu}h_F\beta & \frac{4}{3}\cdot\frac{g^2\beta}{2} & e^{-\beta\mu}h_F\beta \\ 0 & e^{-\beta\mu}h_F\beta & e^{\beta\mu}h_F\beta & 3\cdot\frac{g^2\beta}{2} \end{pmatrix}.$$

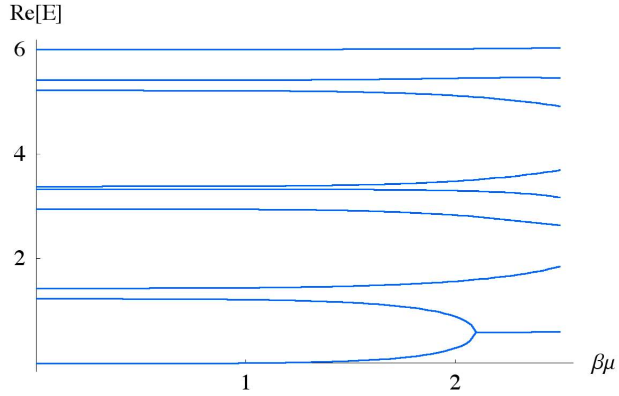

Figure 4. Spectrum for $2h_F/g^2 = 0.1$.

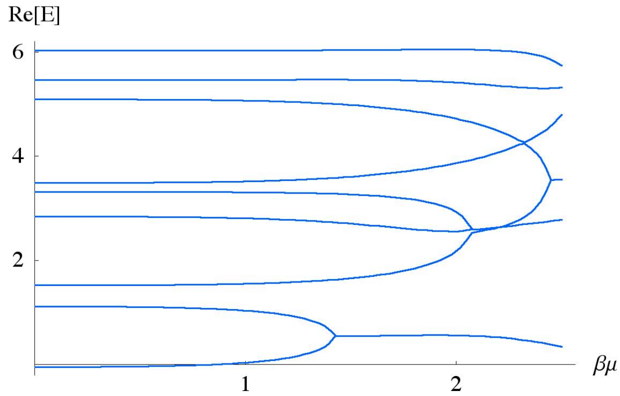

Figure 5. Spectrum for $2h_F/g^2 = 0.2$.

If h_F is set to zero, we see that the eigenvalues are proportional to Casimir invariants 0, 4/3, 4/3, and 3 for the 1, 3, $\bar{3}$, and 8 representations of SU(3). We have therefore removed an overall factor of $g^2\beta/2$, so the overall strength of the potential term is controlled by the dimensionless parameter $2h_F/g^2$. The resulting dimensionless energy eigenvalues are thus normalized to give the quadratic Casimir operator when $2h_F/g^2 = 0$. The lowest eigenvalues have been calculated numerically using a basis of dimension nine or larger, with the stability of the lowest eigenvalues checked by changing the basis size.

When $\mu = 0$, the Hamiltonian is Hermitian and all eigenvalues are guaranteed to be real. As we see in Fig. 4, for $2h_F/g^2 \ll 1$ and $\mu = 0$, the eigenvalues are close to the quadratic Casimir values. Even when $\mu = 0$, the effect of the quarks is to split the degeneracies found in the pure gauge theory. Thus, linear combinations of the 3 and $\bar{3}$ representations show a splitting. As μ increases from zero, the eigenvalues remain real. Eventually, the two lowest energy eigenvalues approach one another, forming a complex conjugate pair indicating the breaking of $\mathcal{PT}$ symmetry. As shown in Figs. 4–10, increasing the parameter $2h_F/g^2$ decreases the value of $\beta\mu$ at which the coalescence of the two lowest eigenvalues occurs. Unlike the $(-ix)^p$ models where the lowest energy eigenvalues are the last to become complex as the parameter p decreases, here $\mathcal{PT}$ symmetry breaking appears to occur in the lowest energies first. Figs. 4–10 show that the eigenvalues above the ground state have a similar complicated behavior as a function of $\beta\mu$. However, the free energy density in the limit of infinite spatial dimension only depends on the ground state energy in the transfer matrix geometry. It is not yet clear what, if any, is the physical meaning of the breaking of $\mathcal{PT}$ symmetry in this context. Nevertheless, it is clear that there is a range of values for $\beta\mu$ for which the sign problem is avoided, due to $\mathcal{PT}$ symmetry.

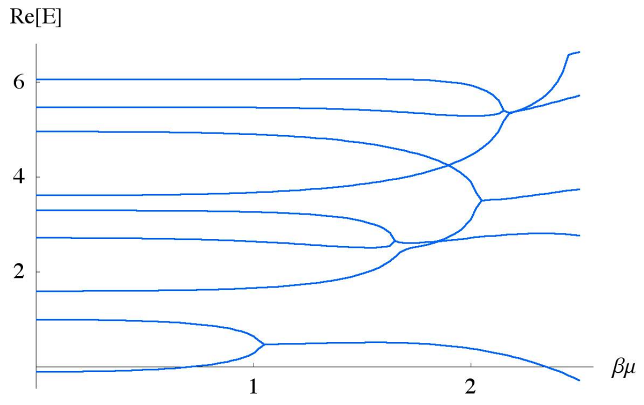

Figure 6. Spectrum for $2h_F/g^2 = 0.3$.

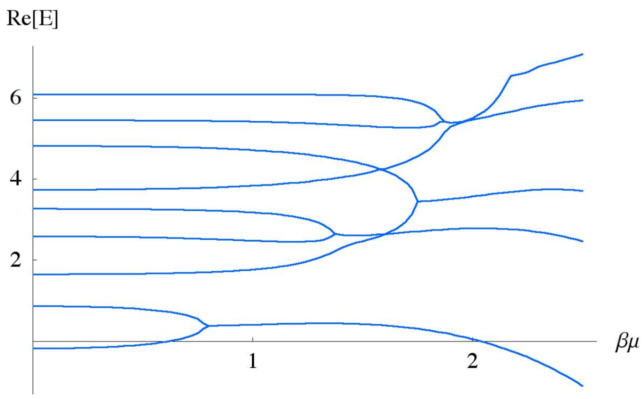

Figure 7. Spectrum for $2h_F/g^2 = 0.4$.

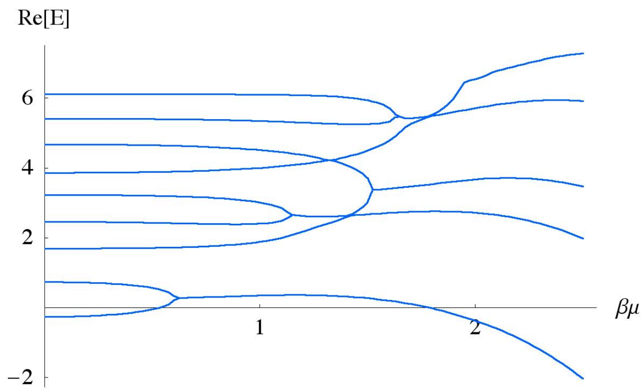

Figure 8. Spectrum for $2h_F/g^2 = 0.5$.

The partition function associated with the effective Lagrangian L can also be interpretated as a classical statistical mechanical system. For simplicity, consider the case of the U(1) gauge group. The partition function can be expanded as a power series in βh_F, and the path integral over A_2 performed order by order. The result in the U(1) case is

$$Z = \sum_{n=0}^{\infty} \frac{(\beta h_F)^{2n}}{(n!)^2} \int dy_1 \cdots dy_n dz_1 \cdots dz_n$$

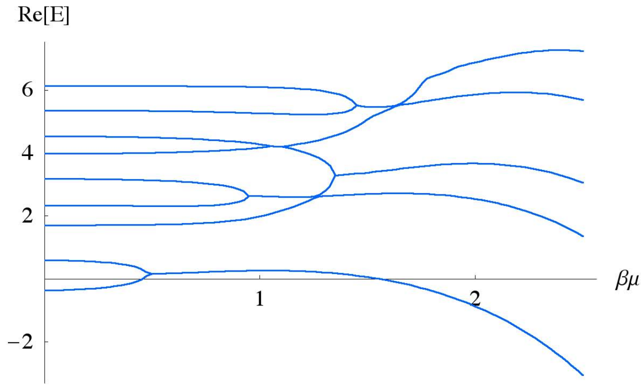

Figure 9. Spectrum for $2h_F/g^2 = 0.6$.

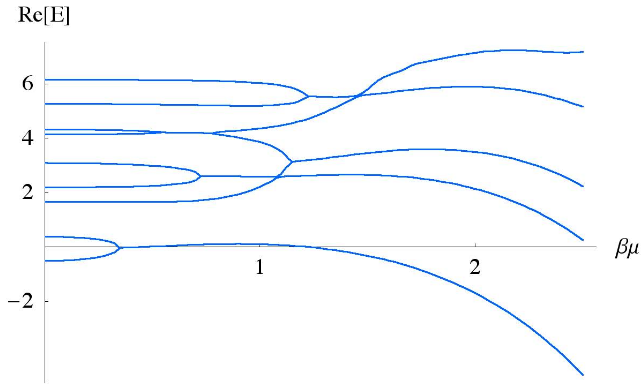

Figure 10. Spectrum for $2h_F/g^2 = 0.75$.

$$\times \exp\left\{\frac{e^2\beta}{2}\sum_{j\neq k}\left[G\left(y_j - z_k\right) - G\left(y_j - y_k\right) - G\left(z_j - z_k\right)\right]\right\},$$

where $G(x)$ is the one-dimensional Green's function $G(x) = -(1/2)\,|x|$. This is an example of the familiar equivalence between field theories of sine-Gordon type and the classical Coulomb gas. Although first derived in one dimension [13], the equivalence holds in all dimensions. More complicated gauge groups result in a similar, but more complicated expansion with a non-trivial dependence on μ. As will be discussed below, generalized non-Hermitian sine-Gordon models are also relevant for the study of quark confinement.

4 Confinement

It is remarkable that there are two classes of $Z(N)$-invariant systems that are confining at arbitrarily high temperatures, evading the transition to the deconfined phase found in the pure gauge theory. Both classes obtain a high-temperature confined phase from a pure gauge theory by the addition of fermions in the adjoint representation of SU(N), with the non-standard choice of periodic boundary conditions for the fermions in the timelike direction. One class consists of $\mathcal{N} = 1$ supersymmetric gauge theories [14, 15], where the periodic boundary conditions on the gauginos is necessary to preserve supersymmetry. The perturbative contribution to the effective potential for the Polyakov loop is identically zero, because the gauge field contribution is cancelled exactly by the gaugino contribution. However, the non-perturbative contribution to the effective potential can be calculated exactly, and leads to a single, $Z(N)$-invariant, confining phase. These ideas have recently been extended to a second class of models where the effect of adjoint fermions dominates the contribution of the gauge fields [16, 17, 18]. If the number of adjoint fermion flavors N_f is not too large, these systems are asymptotically free at high temperature, and therefore the effective potential for P is calculable using perturbation theory. The system will lie in the confining phase if the fermion mass m is sufficiently light and $N_f > 1/2$. In this case, electric string tensions can be calculated perturbatively from the effective potential. In both this case and the supersymmetric case, magnetic string tensions arise semiclassically from non-Abelian magnetic monopoles. This provides a realization of one of the oldest ideas about the origin of confinement. Moreover, lattice simulations [16] indicate that this high-temperature confining region is smoothly connected to the low-temperature confining phase of the pure gauge theory as the temperature is lowered and the fermion mass is increased.

Up to a point, both classes of models can be treated similarly, but we will largely focus on the second, non-supersymmetric case. The one-loop effective potential for a boson in a representation R with spin degeneracy s moving in a Polyakov loop background P at non-zero temperature and density is given by [8, 9]

$$V_b = sT \int \frac{d^d k}{(2\pi)^d} \mathrm{Tr}_R \left[\ln\left(1 - P e^{\beta\mu - \beta\omega_k}\right) + \ln\left(1 - P^+ e^{-\beta\mu - \beta\omega_k}\right) \right].$$

Periodic boundary conditions are assumed. With standard boundary conditions (periodic for bosons, antiperiodic for fermions), 1-loop effects always favor the deconfined phase. For the case of pure gauge theories, the one-loop effective potential can be written in the form

$$V_{\text{gauge}}\left(P, \beta, m, N_f\right) = \frac{-2}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{\mathrm{Tr}_A P^n}{n^4}.$$

This series is minimized, term by term if $P \in Z(N)$, so $Z(N)$ symmetry is spontaneously broken at high temperature. The same result is obtained for any bosonic field with periodic boundary conditions or for fermions with antiperiodic boundary conditions.

The addition of fermions with periodic boundary conditions can restore the broken $Z(N)$ symmetry. Consider the case of N_f flavors of Dirac fermions in the adjoint representation of $\mathrm{SU}(N)$. Periodic boundary conditions in the timelike direction imply that the generating function of the ensemble, i.e., the partition function, is given by

$$Z = \mathrm{Tr}\left[(-1)^F e^{-\beta H} \right],$$

where F is the fermion number. This ensemble, familiar from supersymmetry, can be obtained from an ensemble at chemical potential μ by the replacement $\beta\mu \to i\pi$. In perturbation theory, this shifts the Matsubara frequencies from $\beta\omega_n = (2n+1)\pi$ to $\beta\omega_n = 2n\pi$. The one loop effective potential is like that of a bosonic field, but with an overall negative sign due to fermi statistics [19]. The sum of the effective potential for the fermions plus that of the gauge bosons gives

$$V_{\text{1-loop}}\left(P, \beta, m, N_f\right) = \frac{1}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{\mathrm{Tr}_A P^n}{n^2} \left[2N_f \beta^2 m^2 K_2(n\beta m) - \frac{2}{n^2} \right].$$

Note that the first term in brackets, due to the fermions, is positive for every value of n, while the second term, due to the gauge bosons, is negative.

The largest contribution to the effective potential at high temperatures is typically from the $n = 1$ term, which can be written simply as

$$\frac{1}{\pi^2 \beta^4} \left[2N_f \beta^2 m^2 K_2(\beta m) - 2 \right] \left[\left| \mathrm{Tr}_F P \right|^2 - 1 \right],$$

where the overall sign depends only on N_f and βm. If $N_f > 1/2$ and βm is sufficiently small, this term will favor $\mathrm{Tr}_F P = 0$. On the other hand, if βm is sufficiently large, a value of P from the center, $Z(N)$, is preferred. Note that an $\mathcal{N} = 1$ super Yang–Mills theory would correspond to $N_f = 1/2$ and $m = 0$, giving a vanishing perturbative contribution for all n [14, 15]. In this case, it is necessary to calculate the non-perturbative contribution to the effective potential. This suggests that it should be possible to obtain a $Z(N)$ symmetric, confining phase at high temperatures using adjoint fermions with periodic boundary conditions or some equivalent deformation of the theory.

This possibility has been confirmed in SU(3), where both lattice simulations and perturbative calculations have been used to show that a gauge theory action with an extra term of the form $\int d^4x\, a_1 \mathrm{Tr}_A P$ is confining for sufficiently large a_1 at arbitrarily high temperatures [16]. This simple, one-term deformation is sufficient for SU(2) and SU(3). However, in the general case, a deformation with at least $\left[\frac{N}{2}\right]$ terms is needed to assure confinement for representations of all possible non-zero k-alities. Thus the minimal deformation necessary is of the form

$$\sum_{k=1}^{\left[\frac{N}{2}\right]} a_k \mathrm{Tr}_A P^k,$$

which is analyzed in detail in [20]. If all the coefficients a_k are sufficiently large and positive, the free energy density

$$V_{\text{1-loop}}\left(P, \beta, m, N_f\right) = \frac{-2}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{\mathrm{Tr}_A P^n}{n^4} + \sum_{k=1}^{\left[\frac{N}{2}\right]} a_k \mathrm{Tr}_A P^k$$

will be minimized by a unique set of Polyakov loop eigenvalues corresponding to exact $Z(N)$ symmetry.

The unique set of eigenvalues of P invariant under $Z(N)$ is $\left\{w, wz, wz^2, \ldots, wz^{N-1}\right\}$, where $z = e^{2\pi i/N}$ is the generator of $Z(N)$, and w is a phase necessary to ensure unitarity [21]. A matrix with these eigenvalues, such as $P_0 = w \cdot \mathrm{diag}\left[1, z, z^2, \ldots, z^{N-1}\right]$, is gauge-equivalent to itself after a $Z(N)$ symmetry operation: $zP_0 = gP_0 g^+$. This guarantees that $\mathrm{Tr}_F\left[P_0^k\right] = 0$ for any value of k not divisible by N, indicating confinement for all representations transforming non-trivially under $Z(N)$.

To find the conditions under which P_0 is a global minimum of the effective potential, we use the high-temperature expansion for the one-loop free energy of a particle in an arbitrary background Polyakov loop gauge equivalent to the matrix $P_{jk} = \delta_{jk} e^{i\phi_j}$. The first two terms have the form [19]

$$\begin{aligned} V_{\text{1-loop}} \approx & \sum_{j,k=1}^{N} \left(1 - \frac{1}{N}\delta_{jk}\right) \frac{2\left(2N_f - 1\right)T^4}{\pi^2} \left[\frac{\pi^4}{90} - \frac{1}{48\pi^2}\left(\phi_j - \phi_k\right)^2 \left(\phi_j - \phi_k - 2\pi\right)^2\right] \\ & - \sum_{j,k=1}^{N} (1 - \frac{1}{N}\delta_{jk}) \frac{N_f m^2 T^2}{\pi^2} \left[\frac{\pi^2}{6} + \frac{1}{4}\left(\phi_j - \phi_k\right)\left(\phi_j - \phi_k - 2\pi\right)\right]. \end{aligned}$$

The T^4 term dominates for $m/T \ll 1$, and has P_0 as a minimum provided $N_f > 1/2$. Even if the adjoint fermion mass is enhanced by chiral symmetry breaking, as would be expected in a confining phase, it should be of order gT or less, and the second term in the expansion of $V_{\text{1-loop}}$ can be neglected at sufficiently high temperature. It is interesting to note that for $N_f = 1/2$, any $m > 0$ will give a perturbative term that leads to a deconfined phase.

5 String tension scaling laws

The timelike string tension $\sigma_k^{(t)}$ between k quarks and k antiquarks can be measured from the behavior of the correlation function

$$\langle \mathrm{Tr}_F P^k(\vec{x})\, \mathrm{Tr}_F \left(P^+(\vec{y})\right)^k \rangle \simeq \exp\left[-\frac{\sigma_k^{(t)}}{T}|\vec{x}-\vec{y}|\right]$$

at sufficiently large distances. Two widely-considered scaling behaviors for string tensions are Casimir scaling, characterized by

$$\sigma_k = \sigma_1 \frac{k(N-k)}{N-1},$$

and sine-law scaling, given by

$$\sigma_k = \sigma_1 \frac{\sin[\pi k/N]}{\sin[\pi/N]}.$$

For a review, see reference [7].

At non-zero temperatures, time-like and space-like string tensions may be different. Time-like string tensions may be measured by Polyakov loop correlation functions. while spatial string tensions are measured by space-like Wilson loops. For the supersymmetric case, both string tensions obey sine-law scaling. This is a consequence of the close connection of this class of models with the affine Toda field theory, which is a $\mathcal{PT}$-symmetric model [22]. We will explore the string tension scaling laws for the second class of models, with $N_f > 1/2$, and then return to the affine Toda models and their possible connection to QCD.

Timelike string tensions are calculable perturbatively in the high-temperature confining region for $N_f > 1/2$ from small fluctuations about the confining minimum of the effective potential [23]. The scale is naturally of order gT:

$$\left(\frac{\sigma_k^{(t)}}{T^2}\right)^2 = g^2 N \frac{N_f m^2}{\pi^2} \sum_{j=0}^{\infty} \big[K_2((k+jN)\beta m) + K_2((N-k+jN)\beta m) - 2K_2((j+1)N\beta m)\big] - g^2 N \frac{T^2}{3N^2}\left[3\csc^2\left(\frac{\pi k}{N}\right) - 1\right].$$

These string tensions are continuous functions of βm. The $m=0$ limit is simple:

$$\left(\frac{\sigma_k^{(t)}}{T}\right)^2 = \frac{(2N_f-1)\,g^2T^2}{3N}\left[3\csc^2\left(\frac{\pi k}{N}\right) - 1\right]$$

and is a good approximation for $\beta m \ll 1$. This scaling law is not at all like either Casimir or sine-law scaling, because the usual hierarchy $\sigma_{k+1}^{(t)} \geq \sigma_k^{(t)}$ is here reversed. Because we expect on the basis of SU(3) simulations that the high-temperature confining region is continuously connected to the conventional low-temperature region, there must be an inversion of the string tension hierarchy between the two regions for all $N \geq 4$.

The confining minimum P_0 of the effective potential breaks SU(N) to U$(1)^{N-1}$. This remaining unbroken Abelian gauge group naively seems to preclude spatial confinement, in the sense of area law behavior for spatial Wilson loops. However, as first discussed by Polyakov in the case of an SU(2) Higgs model in $2+1$ dimensional gauge systems, instantons can lead to nonperturbative confinement [24]. In the high-temperature confining region, the dynamics of the magnetic sector are effectively three-dimensional due to dimensional reduction. The Polyakov loop plays a role similar to an adjoint Higgs field, with the important difference that P lies in the gauge group, while a Higgs field would lie in the gauge algebra. The standard topological analysis [25] is therefore slightly altered, and there are N fundamental monopoles in the finite temperature gauge theory [26, 27, 28, 29, 30] with charges proportional to the affine roots of $\mathrm{SU}(N)$, given by $2\pi\alpha_j/g$ where $\alpha_j = \hat{e}_j - \hat{e}_{j+1}$ for $j = 1$ to $N-1$ and $\alpha_N = \hat{e}_N - \hat{e}_1$. Monopole effects will be suppressed by powers of the Boltzmann factor $\exp\left[-E_j/T\right]$ where E_j is the energy of a monopole associated with α_j.

In the high-temperature confining region, monopoles interact with each other through both their long-ranged magnetic fields, and also via a three-dimensional scalar interaction, mediated by A_4. The scalar interaction is short-ranged, falling off with a mass of order gT. The long-range properties of the magnetic sector may be represented in a simple form by a generalized sine-Gordon model which generates the grand canonical ensemble for the monopole/anti-monopole gas [18]. The action for this model represents the Abelian dual form of the magnetic sector of the $\mathrm{U}(1)^{N-1}$ gauge theory. It is given by

$$S_{\mathrm{mag}} = \int d^3x \left[\frac{T}{2} (\partial\rho)^2 - 2\xi \sum_{j=1}^{N} \cos\left(\frac{2\pi}{g} \alpha_j \cdot \rho \right) \right],$$

where ρ is the scalar field dual to the $\mathrm{U}(1)^{N-1}$ magnetic field. The monopole fugacity ξ is given by $\exp\left[-E_j/T\right]$ times functional determinantal factors [31].

This Lagrangian is a generalization of the one considered by Polyakov for SU(2), and the analysis of magnetic confinement follows along the same lines [24]. The Lagrangian has N degenerate inequivalent minima $\rho_{0k} = g\mu_k$ where the μ_k's are the simple fundamental weights, satisfying $\alpha_j \cdot \mu_k = \delta_{jk}$. Note that $e^{2\pi i \mu_k} = z^k$. A spatial Wilson loop

$$W\left[\mathcal{C}\right] = \mathcal{P} \exp\left[i \oint_{\mathcal{C}} dx_j \cdot A_j \right]$$

in the x-y plane introduces a discontinuity in the z direction in the field dual to B. Moving this discontinuity out to spatial infinity, the string tension of the spatial Wilson loop is the interfacial energy of a one-dimensional kink interpolating between the vacua ρ_{0k}. The calculation is similar to that of the 't Hooft loop in the deconfined phase, where the kinks interpolate between the N different solutions associated with the spontaneous breaking of $\mathrm{SU}(N)$. The main technical difficulty lies in finding the correct kink solutions. A straight line ansatz through the Lie algebra [32] using $\rho(z) = g\mu_k q(z)$ gives

$$\sigma_k^{(s)} = \frac{8}{\pi} \left[\frac{g^2 T \xi}{N} k\left(N-k\right) \right]^{1/2}.$$

This result is exact for $N = 2$ or 3, but may be only an upper bound for $N > 3$. The square-root-Casimir scaling behavior obtained differs somewhat from both Casimir and sine-law scaling, and is inconsistent with lattice simulation results for $N \geq 4$ pure gauge theories [33]. It is not completely surprising that the addition of additional particles, in this case adjoint fermions, might change string tension scalings laws.

Nevertheless, the behavior of spatial string tensions in the high-temperature confining region is similar to what we think might occur in the low-temperature confining phase of a pure gauge theory. Therefore it is interesting to ask whether there are classes of models, corresponding to different monopole gases and having different string tension scaling laws. The model just discussed is an affine sine-Gordon model with a sum over the affine simple roots, both positive and negative. If we sum instead over the affine positive roots, we have the affine Toda model, a non-Hermitian but $\mathcal{PT}$-symmetric, model. The action is

$$S_{\text{Toda}} = \int d^3x \left[\frac{T}{2} (\partial\rho)^2 - \xi \sum_{j=1}^{N} \exp\left(i\frac{2\pi}{g}\alpha_j \cdot \rho \right) \right].$$

This is an effective field theory for a gas of monopoles, but no anti-monopoles. As shown by Hollowood [22], the kink solutions of this model have a sine-law mass spectrum:

$$\sigma_k^{(s)} = \frac{2N}{\pi} \left[g^2 T\xi \right]^{1/2} \sin\left(\frac{\pi k}{N} \right).$$

Diakonov and Petrov have shown that this effective theory may be plausibly obtained from SU(N) gauge theories at finite temperature if anti-monopoles are excluded from the ensemble of field configurations considered in the path integral [34]. We can also consider a sine-Gordon model with a sum over all roots. In this case, the string tension exhibits Casimir scaling:

$$\sigma_k^{(s)} = \frac{8}{\pi} \left[\frac{g^2 T\xi}{N} \right]^{1/2} k\,(N-k).$$

The similarity of these three sets of results, and the closeness of lattice simulation results to both Casimir and sine-law scaling, suggest the possibility of a crossover from one string tension scaling law to another as the character of the monopole gas changes. The details of how this might happen, however, are not clear.

6 Conclusions

The common thread connecting $\mathcal{PT}$ symmetry to applications to QCD at non-zero temperature and density is the use of generalized sine-Gordon models to represent the statistical mechanics of objects carrying non-Abelian electric and magnetic charge. Heavy quarks at non-zero density give rise to a $\mathcal{PT}$-symmetric effective action. This is turn gives us a new way of looking at the sign problem. In the problem of quark confinement, there are models where monopole gases are responsible for confinement. Depending on the specific model, the effective action for the monopole gas may be Hermitian, or non-Hermitian but $\mathcal{PT}$-symmetric. In addition to the connection of $\mathcal{PT}$ symmetry to QCD, these models also suggest interesting possibilities for $\mathcal{PT}$ symmetry more generally in statistical physics.

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