Symmetry, Integrability and Geometry: Methods and Applications SIGMA 5 (2009), 045, 7 pages

The Explicit Construction of Einstein Finsler Metrics with Non-Constant Flag Curvature*

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Received December 08, 2008, in final form April 09, 2009; Published online April 14, 2009
doi:10.3842/SIGMA.2009.045

Abstract. By using the Hawking Taub-NUT metric, this note gives an explicit construction of a 3-parameter family of Einstein Finsler metrics of non-constant flag curvature in terms of navigation representation.

Key words: Finsler manifold; Einstein Randers metric; Ricci curvature

2000 Mathematics Subject Classification: 58E20

1 Introduction

The Ricci curvature of a Finsler metric F on a manifold is a scalar function $\mathrm{Ric} : TM \to \mathbb{R}$ with the homogeneity $\mathrm{Ric}(\lambda y) = \lambda^2 \mathrm{Ric}(y)$. See (3) below. A Finsler metric F on an n-dimensional manifold M is called an *Einstein metric* if there is a scalar function $K = K(x)$ on M such that

$$\mathrm{Ric} = (n-1)KF^2.$$

Recently, C. Robles studied a special class of Einstein Finsler metrics, that is, Einstein Randers metrics and obtained the following interesting result [1, Proposition 12.9]: Let F be a Randers metric on a 3-dimensional manifold. Then F is Einstein if and only if it has constant flag curvature. Note that the flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry and it furnishes the lowest order term for the Jacobi equation that governs the second variation of geodesics of the Finsler metric. Together with the classification theorem of Randers metrics of constant flag curvature due to Bao–Robles–Shen [2], three dimensional Einstein Randers metrics are completely determined.

The Randers metrics were introduced by physicist Randers in 1941 by modifying a Riemannian metric $\alpha := \sqrt{a_{ij}(x)y^i y^j}$ by a linear term $\beta := b_i(x)y^i$ [8]. By requiring $\|\beta\|_\alpha < 1$, we ensure that $\alpha + \beta$ is positive and strongly convex. The interested reader is referred to [1] for this result and a thorough treatment of Randers metrics.

The next problem is to describe four dimensional Einstein Randers metrics. This problem turns out to be very difficult. The very first step might be to construct as many examples as possible. In 2002 D. Bao and C. Robles constructed for the first time a family of non-constant

*This paper is a contribution to the Special Issue "Élie Cartan and Differential Geometry". The full collection is available at http://www.emis.de/journals/SIGMA/Cartan.html flag curvature Einstein Randers metrics on $\mathbb{C}P^2$ using Killing vector fields with respect to its Fubini–Study metric [1].

The main technique in [1] is described as follows. Given a Riemannian metric $g = g_{ij}(x)y^iy^j$ and a vector field $W = W^i\frac{\partial}{\partial x^i}$ on a manifold M with $g(x, W_x) < 1$, one can define a Finsler metric $F : TM \to [0, \infty)$ by

$$F(x, y) = \sqrt{g}\big(x, y - F(x, y)W_x\big). \tag{1}$$

Solving (1) for F, one obtains $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$ are given by

$$a_{ij} = \frac{g_{ij}}{\lambda} + \frac{W_iW_j}{\lambda^2}, \qquad b_j = -\frac{W_j}{\lambda}, \tag{2}$$

where $W_i = g_{ij}W^j$ and $\lambda = 1 - W_iW^i$.

In this paper, using (1) we are going to construct a 3-parameter family of Einstein Randers metrics with non-constant flag curvature.

Now let us describe our construction. Let (N^3, h) be an oriented constant curvature 3-manifold, set $M^4 = \mathbb{R} \times N^3$, and let $\varphi : M^4 \to N^3$ be the projection onto the second factor. Define a Riemannian metric g on M^4 by

$$g = u\varphi^*h + u^{-1}(dt + A)^2,$$

where u is a positive smooth function and A a 1-form on N^3. Then (M^4, g) is Einstein if and only if u and A are related by the monopole equation of mathematical physics

$$du = - * dA$$

and (N^3, h) is flat, in which case g is Ricci-flat, that is, g has zero Ricci curvature [4, 5]. For $a \geq 0$, define a harmonic function on $\mathbb{R}^3\backslash\{0\}$ by

$$u_a(y) = \frac{1}{4}\left(\frac{1}{|y|} + a\right).$$

Then the above construction gives the Hawking Taub-NUT Riemannian metric g_a $(a > 0)$ or the standard metric g_0 $(a = 0)$. Note that the metric g_a extends to the whole of $\mathbb{R}^4$; in fact it is given by the explicit formula [4, 5, 7, 11]

$$g_a = (a|x|^2 + 1)g_0 - \frac{a(a|x|^2 + 2)}{a|x|^2 + 1}\left(-x^2dx^1 + x^1dx^2 - x^4dx^3 + x^3dx^4\right)^2.$$

See [5] for discussion of g_1. For any $m, n \in \mathbb{R}^+$, we define $W_{m,n} \in \Gamma(T\mathbb{R}^4)$ by

$$W_{m,n} = -mx^2\frac{\partial}{\partial x^1} + mx^1\frac{\partial}{\partial x^2} - nx^4\frac{\partial}{\partial x^3} + nx^3\frac{\partial}{\partial x^4}.$$

Let $\Omega := \{x \in \mathbb{R}^4 \,|\, f(x) < 1\}$, where $f(x)$ is defined in (21). We obtain the following result:

Theorem 1. *Let $F = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i$ be any function in $T\Omega \to [0, \infty)$ on $T\Omega$ which is expressed by (2) in terms of the Hawking Taub-NUT metric g_a and vector field $W_{m,n}$. Then F has the following properties:*

(i) F is a Randers metric;

(ii) F is Einstein with Ricci constant zero;

(iii) F has non-constant flag curvature.

Our main approach is to show $W_{m,n}$ is the vector field induced by a one parameter isometric group with respect to the Hawking Taub-NUT metric g_a.

These examples show the existence of a large family of global Einstein Finsler metrics on $\mathbb{R}^4$ (taking $m = n < a$, then $\Omega = \mathbb{R}^4$). This is still far from a complete description, of course, but it gives an indication that this family is much larger than previously believed.

2 Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M. The second variation of geodesics gives rise to a family of endomorphisms $R_y = R^i_{\ k} dx^k \otimes \frac{\partial}{\partial x^i} : T_xM \to T_xM$, defined by

$$R^i_{\ k} = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k},$$

where G^i are the geodesic coefficients of F [6, 10]. F is said to be of *constant flag curvature* $K = \lambda$, if $R^i_{\ k} = \lambda(F^2\delta^i_{\ k} - FF_{y_k}y^i)$ where $F_{y^k} := \frac{\partial F}{\partial y^k}$ [6]. Finsler metrics of constant flag curvature are the natural extension of Riemannian metrics of constant sectional curvature.

The Ricci curvature Ric is defined to be the trace of R_y

$$\mathrm{Ric}(y) := R^k_{\ k}(x, y). \tag{3}$$

Ric is a well-defined scalar function on $TM\backslash\{0\}$. F is called an *Einstein metric* if there is a scalar function $K = K(x)$ on M such that $\mathrm{Ric} = (n-1)KF^2$.

Consider a Randers metric $F = \alpha + \beta$ on M. Let $g = g_{ij}y^iy^j$ be the Riemannian metric and $W = W^i\frac{\partial}{\partial x^i}$ be the vector field on M such that F is defined by [1]. Then D. Bao and C. Robles showed the following result [1, 9]:

Proposition 1. *F is Einstein with Ricci scalar* $\mathrm{Ric}(x) := (n-1)K(x)$ *if and only if*

(i) W satisfies $W_{i|j} + W_{j|i} = -4cg_{ij}$, where $W_i = g_{ij}W^j$, and

(ii) g is an Einstein metric, i.e. $\widetilde{\mathrm{Ric}} = (n-1)\{K(x) + c^2\}g$, where $c = \mathrm{const}$ and $\widetilde{\mathrm{Ric}}$ is the Ricci curvature tensor of g, in particular $K(x) = \mathrm{const}$ if $n \geq 3$.

3 One parameter transformation group

We rewrite the Hawking Taub-NUT metric g_a as

$$g_a = g_{ij}dx^idx^j,$$

where

$$(g_{ij}) = G(x) = \begin{pmatrix} B - A(x^2)^2 & Ax^1x^2 & -Ax^2x^4 & Ax^2x^3 \\ Ax^1x^2 & B - A(x^1)^2 & Ax^1x^4 & -Ax^1x^3 \\ -Ax^2x^4 & Ax^1x^4 & B - A(x^4)^2 & Ax^3x^4 \\ Ax^2x^3 & -Ax^1x^3 & Ax^3x^4 & B - A(x^3)^2 \end{pmatrix} \tag{4}$$

and

$$B = B(x) = a|x|^2 + 1, \qquad A = A(x) = a\left(1 + \frac{1}{B}\right). \tag{5}$$

For any $m, n \in \mathbb{R}^+$, we define $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\phi_\theta = \left(\phi^1_\theta, \phi^2_\theta, \phi^3_\theta, \phi^4_\theta\right), \qquad \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix} = \phi^T_\theta = A_\theta \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}, \tag{6}$$

where A_θ is given by

$$A_\theta = \begin{pmatrix} A_{\theta,m} & ,0 \\ 0 & A_{\theta,n} \end{pmatrix} \tag{7}$$

and

$$A_{\theta,m} = \begin{pmatrix} \cos(m\theta) & -\sin(m\theta) \\ \sin(m\theta) & \cos(m\theta) \end{pmatrix}, \qquad A_{\theta,n} = \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix}. \tag{8}$$

It is easy to see that

$$|y|^2 := \sum_{i=1}^{4} \left(y^i\right)^2 = \sum_{i=1}^{4} \left(x^i\right)^2 = |x|^2. \tag{9}$$

Furthermore, ϕ_θ is a one-parameter transformation group.

4 Killing fields

In this section, we explicitly construct a two-parameter family of Killing fields of the Hawking Taub-NUT Riemannian metrics. First, we prove that ϕ_θ in (6) is an isometry. It is equivalent to prove that

$$A_\theta^T G(y) A_\theta = G(x). \tag{10}$$

Let

$$H(x) := \begin{pmatrix} H_1(x) & H_2(x) \\ H_2^T(x) & H_3(x) \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned} H_1(x) &:= \begin{pmatrix} x^2 \\ -x^1 \end{pmatrix} \left(x^2, -x^1\right), \qquad H_2(x) := \begin{pmatrix} x^2 \\ -x^1 \end{pmatrix} \left(x^4, -x^3\right), \\ H_3(x) &:= \begin{pmatrix} x^4 \\ -x^3 \end{pmatrix} \left(x^4, -x^3\right). \end{aligned} \tag{12}$$

It follows from (4), (11) and (12) that

$$G(x) = B(x)I - A(x)H(x). \tag{13}$$

Plugging (9) into (5) yields

$$A(x) = A(y), \qquad B(x) = B(y). \tag{14}$$

Note that A_θ satisfies that

$$A_\theta^T = A_\theta^{-1}. \tag{15}$$

By using (13), (14) and (15), we obtain that (10) is equivalent to

$$H(x) = A_\theta^T H(y) A_\theta. \tag{16}$$

In order to check (16), from (7), (8) and (11), it is enough to check 3 matrix equations of order 2×2 as follows:

$$H_1(x) = A_{\theta,m}^T H_1(y) A_{\theta,m}, \qquad H_2(x) = A_{\theta,m}^T H_2(y) A_{\theta,n}, \qquad H_3(x) = A_{\theta,n}^T H_3(y) A_{\theta,n}, \tag{17}$$

where $A_{\theta,m}$ and $A_{\theta,n}$ are defined in (8). Hence

$$A_{\theta,m}^T H_1(y) A_{\theta,m} = A_{\theta,m}^T \begin{pmatrix} y^2 \\ -y^1 \end{pmatrix} \left(y^2, -y^1\right) A_{\theta,m} = \begin{pmatrix} x^2 \\ -x^1 \end{pmatrix} \left(x^2, -x^1\right) = H_1(x),$$

thus we obtain the first equation of (17), the others are completely analogous.

For any fixed $p = (p^1, p^2, p^3, p^4) \in \mathbb{R}^4$, we have

$$\frac{d\phi_\theta^1(p)}{d\theta} = -m(p^1 \sin(m\theta) + p^2 \cos(m\theta)) = -m\phi_\theta^2(p),$$
$$\frac{d\phi_\theta^2(p)}{d\theta} = m(p^1 \cos(m\theta) - p^2 \sin(m\theta)) = m\phi_\theta^2(p).$$

Similarly, we have

$$\frac{d\phi_\theta^3(p)}{d\theta} = -n\phi_\theta^3(p), \qquad \frac{d\phi_\theta^4(p)}{d\theta} = n\phi_\theta^4(p).$$

It follows that the vector field induced by ϕ_θ is given by

$$\begin{aligned} W_p &= \frac{d}{d\theta}[\phi_\theta(p)]\Big|_{\theta=0} = \frac{d\phi_\theta^j(p)}{d\theta}\Big|_{\theta=0} \frac{\partial}{\partial x^j}\Big|_p \\ &= -mp^2 \frac{\partial}{\partial x^1}\Big|_p + mp^1 \frac{\partial}{\partial x^2}\Big|_p - np^4 \frac{\partial}{\partial x^3}\Big|_p + np^3 \frac{\partial}{\partial x^4}\Big|_p. \end{aligned} \tag{18}$$

Note that ϕ_θ is an isometry. It follows that W is of Killing type with respect to the Hawking Taub-NUT metric g_a.

5 Construction of Einstein–Finsler metrics

We rewrite our Killing field W as

$$W = W_{m,n} = \sum_{j=1}^{4} W^j \frac{\partial}{\partial x^j}.$$

Then, from (18), we have

$$W^1 = -mx^2, \qquad W^2 = mx^1, \qquad W^3 = -nx^4, \qquad W^4 = nx^3.$$

Together with (4) we get

$$W_j = g_{ji} W^i = \begin{cases} W^j \left(B - \dfrac{\sigma A}{m}\right), & j = 1, 2, \\[2ex] W^j \left(B - \dfrac{\sigma A}{n}\right), & j = 3, 4, \end{cases}$$

where

$$\sigma := m\left(x^1\right)^2 + m\left(x^2\right)^2 + n\left(x^3\right)^2 + n\left(x^4\right)^2 \tag{19}$$

and A and B are defined in (5). It follows that

$$\lambda = 1 - |W|^2,$$

where

$$|W|^2 = \sum_{j=1}^{4} W_j W^j = \left[\left(W^1\right)^2 + \left(W^2\right)^2\right]\left(B - \frac{\sigma A}{m}\right) + \left[\left(W^3\right)^2 + \left(W^4\right)^2\right]\left(B - \frac{\sigma A}{n}\right). \quad (20)$$

We are going to find the sufficient condition producing Einstein–Finsler metrics in terms of navigation representation, i.e. $|W| < 1$.

Put

$$p := \max\{m, n\}, \qquad q := \min\{m, n\}, \qquad \text{then} \qquad p - q = |m - n|.$$

By (19), we obtain

$$\frac{\sigma}{m} \geq \frac{q}{p}|x|^2, \qquad \frac{\sigma}{n} \geq \frac{q}{p}|x|^2.$$

Thus we have

$$\begin{aligned}
&\left[\left(W^1\right)^2 + \left(W^2\right)^2\right]\left(B - \frac{\sigma A}{m}\right) + \left[\left(W^3\right)^2 + \left(W^4\right)^2\right]\left(B - \frac{\sigma A}{n}\right) \\
&\quad \leq p|x|^2\left(1 + a|x|^2 - \frac{aq}{p}|x|^2\frac{2 + a|x|^2}{1 + a|x|^2}\right) \\
&\quad = \frac{|x|^2}{1 + a|x|^2}\left(p + 2a|m - n||x|^2 + a^2|m - n||x|^4\right) := f(x). \qquad (21)
\end{aligned}$$

Proof of Theorem 1. From (20) and (21), we obtain that $f(x) < 1$ implies $|W_{m,n}| < 1$. It ensures that $(g_a, W_{m,n})$ produces Randers metrics in terms of navigation representation. It is easy to check that $\sum_{i=1}^{4} \tilde{F}_{x^i y^1} y^i \neq \tilde{F}_{x^1}$, where $\tilde{F} := \sqrt{g_{ij}(x) y^i y^j}$ and $g_{ij} = g_a\big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\big)$. According to [3], g_a is not locally projectively flat. Recall that a Finsler metric $\bar{F}$ on a k-dimensional manifold M is *projectively flat* if every point of M has a neighborhood U that can be embedded into $\mathbb{R}^k$ in such a way that it carries the $\bar{F}$-geodesics in U to straight line segments. In the Riemannian case, by a theorem of Bonnet–Beltrami, projective flatness is equivalent to having constant sectional curvature. Hence g_a does not have constant sectional curvature. Note that $F := \alpha + \beta$ has constant flag curvature if and only if g_a has constant sectional curvature and $W_{m,n}$ is a homothetic vector field [2]. It follows that F does not have constant flag curvature and we obtain *(iii)* of Theorem 1. Note that the Hawking Taub-NUT metric g_a on $\mathbb{R}^4$ is an Einstein metric for all $a > 0$ [7, 11]. From Section 4 we see that $W_{m,n}$ is of Killing type with respect to g_a for all $(m, n) \in (\mathbb{R}^+, \mathbb{R}^+)$. Now *(ii)* of Theorem 1 is an immediate consequence of Proposition 1. ■

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