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A Lax Formalism for the Elliptic Difference Painlevé Equation*

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Abstract. A Lax formalism for the elliptic Painlevé equation is presented. The construction is based on the geometry of the curves on $\mathbb{P}^1 \times \mathbb{P}^1$ and described in terms of the point configurations.

Key words: elliptic Painlevé equation; Lax formalism; algebraic curves

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1 Introduction

The difference analogs of the Painlevé differential equations have been extensively studied in the last two decades (see [6, 12] for example). It is now widely recognized that some of the aspects of the Painlevé equations, in particular their algebraic or geometric properties, can be understood in universal way by considering differential and difference cases together.

In [12], Sakai studied the difference Painlevé equations from the point of view of rational surfaces and classified them into three categories: additive, multiplicative (q-difference) and elliptic[1]. The classification is summarized in the following diagram:

$$
\begin{array}{llllllllllllllll}
\text{ell.} & E_8^{(1)} & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & \mathbb{Z} & & \\
 & & & & & & & & & & & & \nearrow & & & \\
\text{mul.} & E_8^{(1)} & \to & E_7^{(1)} & \to & E_6^{(1)} & \to & D_5^{(1)} & \to & A_4^{(1)} & \to & (A_2+A_1)^{(1)} & \to & (A_1+A_1)^{(1)} & \to & A_1^{(1)} \to \mathcal{D}_6 \\
\text{add.} & E_8^{(1)} & \to & E_7^{(1)} & \to & E_6^{(1)} & \to & D_4^{(1)} & \to & A_3^{(1)} & \to & (A_1+A_1)^{(1)} & \to & A_1^{(1)} & \to & \mathbb{Z}_2 \\
 & & & & & & & & & & \searrow & & \searrow & & & \downarrow \\
 & & & & & & & & & & & A_2^{(1)} & \to & A_1^{(1)} & \to & 1
\end{array}
$$

Among them, Sakai's elliptic Painlevé equation [12] is the master equation of all the second order Painlevé equations. It has the affine Weyl group symmetry of type $E_8^{(1)}$ and all the other cases arise as its degenerations.

It is well known that the differential Painlevé equations describe iso-monodromy deformations of linear differential equations. Since the iso-monodromy interpretation of the Painlevé equations is a main source of variety of deep properties of the latter, it is an important problem to find Lax formalisms for difference cases.

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[1]The addition formulae of the trigonometric/elliptic functions are the typical examples of the multiplicative/elliptic difference equations.

In fact, for some of difference Painlevé equations, the Lax formulations have been known (see [1, 3, 4, 5, 7] for example). Let us give an example of q-difference case with symmetry of type $D_5^{(1)}$. The equation is Jimbo–Sakai's q-$P_{\rm VI}$ equation [7] which is a discrete dynamical system defined by the following birational transformation:

$$T: \begin{pmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \end{pmatrix}; f, g \mapsto \begin{pmatrix} qa_1, & qa_2, & a_3, & a_4 \\ qb_1, & qb_2, & b_3, & b_4 \end{pmatrix}; \dot{f}, \dot{g},$$

$$\dot{f}f = \frac{(\dot{g}-b_1)(\dot{g}-b_2)}{(\dot{g}-b_3)(\dot{g}-b_4)}a_3a_4, \qquad \dot{g}g = \frac{(f-a_1)(f-a_2)}{(f-a_3)(f-a_4)}b_3b_4, \tag{1}$$

where $(f,g) \in \mathbb{P}^1 \times \mathbb{P}^1$ are the dependent variables and $a_1, \dots, a_4, b_1, \dots, b_4$ are complex parameters with a constraint $q = a_3a_4b_1b_2/(a_1a_2b_3b_4)$.

The q-$P_{\rm VI}$ equation (1) was originally derived as the compatibility of certain 2×2 matrix Lax pair:

$$Y(qz) = A(z)Y(z), \qquad T(Y(z)) = B(z)Y(z),$$

which is equivalent (up to a gauge transformation) with the following scalar Lax pair for the first component y of Y. One of the scalar Lax equations is

$$\frac{(a_1-z)(a_2-z)}{a_1a_2(z-f)}y(qz) - \left(c_0 + c_1z + \frac{c_2z}{z-f} + \frac{c_3z}{z-qf}\right)y(z) + \frac{a_1a_2(z-qa_3)(z-qa_4)}{b_3b_4q^2(z-qf)}y\left(\frac{z}{q}\right) = 0, \tag{2}$$

where $c_0 = -\frac{a_1a_2}{f}\left(\frac{1}{b_1}+\frac{1}{b_2}\right)$, $c_1 = \frac{1}{q}\left(\frac{1}{b_3}+\frac{1}{b_4}\right)$, $c_2 = \frac{(f-a_1)(f-a_2)}{qfg}$ and $c_3 = \frac{(f-a_3)(f-a_4)g}{b_3b_4f}$. The other one is

$$qgy(qz) - a_1a_2y(z) + z(z-f)T^{-1}(y(z)) = 0. \tag{3}$$

In elliptic case, it is natural to expect a scalar Lax pair which looks like

$$C_1y(z-\delta) + C_2y(z) + C_3y(z+\delta) = 0, \qquad C_4y(z-\delta) + C_5y(z) + C_6T(y(z)) = 0, \tag{4}$$

where the coefficients $C_1, \dots, C_6$ are elliptic functions on variables z and other parameters. However, the explicit construction of such Lax formalism has remained as a difficult problem because of the complicated elliptic dependence including many parameters.

In this paper, we give a Lax formulation for the elliptic Painlevé equation with $E_8^{(1)}$ symmetry using a geometric method. The main idea is to consider the Lax pair (4) as equations for algebraic curves with respect to the unknown variables of the Painlevé equation. We note that the Lax pair for the additive difference $E_8^{(1)}$ case was obtained by Boalch [4]. For the elliptic case, another approach has recently proposed by Arinkin, Borodin and Rains [2, 11].

This paper is organised as follows. In Section 2, a geometric description of the elliptic Painlevé equation is reviewed in $\mathbb{P}^1 \times \mathbb{P}^1$ formalism. In Section 3 the Lax pair for the elliptic Painlevé equation is formulated. Some properties of relevant polynomials are prepared in Section 4. Finally, in Section 5, the compatibility condition of the Lax pair is analyzed and its equivalence to the elliptic Painlevé equation is established (Theorem 1). In Appendix A, the differential case is discussed.

Before closing this introduction, let us look at an observation which may be helpful to motivate our construction. In this paper, we see the Lax equations like (2), (3) from two different viewpoints. One is a standard way, where we consider the equations as difference equations for unknown function $y(z)$, and variables (f,g) are regarded as parameters. The other is unusual viewpoint, where we consider these equations as equations of algebraic curves in variables $(f,g) \in \mathbb{P}^1 \times \mathbb{P}^1$, and $y(z), y(qz), y(z/q)$ or $T^{-1}(y(z))$ are regarded as parameters. In the second point of view, we have

Proposition 1. *The equation* (2) *is uniquely characterized as a curve of bi-degree* $(3,2)$ *in* $\mathbb{P}^1\times\mathbb{P}^1$ *passing through the* 12 *points:*

$$\begin{aligned}&(0,b_1/q),\quad (0,b_2/q),\quad (\infty,b_3),\quad (\infty,b_4),\quad (a_1,0),\quad (a_2,0),\quad (a_3,\infty),\quad (a_4,\infty),\\ &(z,\infty),\quad \left(\frac{z}{q},0\right),\quad \left(z,\frac{a_1a_2}{q}\frac{y(z)}{y(qz)}\right),\quad \left(\frac{z}{q},\frac{a_1a_2}{q}\frac{y(z/q)}{y(z)}\right).\end{aligned} \tag{5}$$

Similarly, the equation (3) *is also characterized as a curve of bi-degree* $(1,1)$ *passing through* 3 *points:*

$$(\infty,\infty),\quad \left(z,\frac{a_1a_2}{q}\frac{y(z)}{y(qz)}\right),\quad \left(z-\frac{a_1a_2}{z}\frac{y(z)}{T^{-1}y(z)},0\right).$$

In Sakai's theory, Painlevé equations are characterized by the 9 points configurations in $\mathbb{P}^2$ or equivalently by the 8 points configurations in $\mathbb{P}^1\times\mathbb{P}^1$. We note that the first 8 points in (5) are nothing but the configuration which characterize q-$P_{\rm VI}$. This kind of relations between the Lax equations and the point configurations have been observed also in other difference or differential cases [14] (See Appendix A for $P_{\rm VI}$ case). Hence, it is naturally expected that the Lax equations for the elliptic Painlevé equation will also be determined by suitable conditions as plane algebraic curves. This is what we will show in this paper.

2 The elliptic Painlevé equation

Let $P_1,\ldots,P_8$ be points on $\mathbb{P}^1\times\mathbb{P}^1$. We assume that the configuration of the points $P_1,\ldots,P_8$ is generic, namely the curve C_0 of bi-degree (2,2) passing through the eight points is unique and it is a smooth elliptic curve. We denote the equation of the curve $C_0:\varphi_{22}(f,g)=0$, where (f,g) is an inhomogeneous coordinate of $\mathbb{P}^1\times\mathbb{P}^1$. Let X be the rational surface obtained from $\mathbb{P}^1\times\mathbb{P}^1$ by blowing up the eight points $P_1,\ldots,P_8$. Its Picard lattice $\mathrm{Pic}(X)$ is given by

$$\mathrm{Pic}(X)=\mathbb{Z}H_1\oplus\mathbb{Z}H_2\oplus\mathbb{Z}E_1\oplus\cdots\oplus\mathbb{Z}E_8,$$

where H_i $(i=1,2)$ is the class of lines corresponding to i-th component of $\mathbb{P}^1\times\mathbb{P}^1$ and E_j $(j=1,\ldots,8)$ is the exceptional divisors. The nontrivial intersection pairings for these basis are given by

$$(H_1,H_2)=(H_2,H_1)=1,\qquad (E_j,E_j)=-1.$$

Note that the surface X is birational equivalent with the 9 points blown-up of $\mathbb{P}^2$.

In the most generic situation, the group of Cremona transformations on the surface X is the affine Weyl group of type $E_8^{(1)}$ and its translation part $\mathbb{Z}^8$ gives the elliptic Painlevé equations [12, 10]. A choice of $E_8^{(1)}$ simple roots $\alpha_0,\ldots,\alpha_8$ in $\mathrm{Pic}(X)$ is $\alpha_0=E_1-E_2$, $\alpha_1=H_1-H_2$, $\alpha_2=H_2-E_1-E_2$, $\alpha_i=E_{i-1}-E_i$ $(i=3,\ldots,8)$. The null root δ $(=-K_X$: the anti-canonical divisor of $X)$ is

$$\delta=2H_1+2H_2-E_1-E_2-\cdots-E_8. \tag{6}$$

The action of the translation T_α on $\mathrm{Pic}(X)$ is given by the Kac's formula

$$T_\alpha(\beta)=\beta+(\delta,\beta)\alpha-\left((\delta,\beta)\frac{(\alpha,\alpha)}{2}+(\alpha,\beta)\right)\delta.$$

For instance, for the translation $T=T_{E_i-E_j}$ along the direction E_i-E_j $(1\le i\ne j\le 8)$, we have

$$T(H_i)=H_i+2(E_i-E_j)+2\delta,\qquad i=1,2,$$

$$\begin{aligned}
&T(E_j) = E_i, \\
&T(E_i) = E_i + (E_i - E_j) + 2\delta, \\
&T(E_k) = E_k + (E_i - E_j) + \delta, \qquad k \neq i, j.
\end{aligned} \tag{7}$$

Similar to the case of the 9 points blown-up of $\mathbb{P}^2$ [8], the above type of translations $T_{E_i - E_j}$ admit simple geometric description as follows.

(i) Points $P_1, \ldots, P_8$ are transformed as

$$\begin{aligned}
&T(P_k) = P_k, \qquad k \neq i, j, \\
&P_1 + \cdots + P_{i-1} + T(P_i) + P_{i+1} + \cdots + P_8 = 0, \\
&T(P_i) + T(P_j) = P_i + P_j,
\end{aligned} \tag{8}$$

with respect to the addition on the elliptic curve C_0 passing through $P_1, \ldots, P_8$.

(ii) The transformation of the Painlevé dependent variable $P = (f, g)$ can be found as follows. Let C be the elliptic curve passing through $P_1, \ldots, P_{i-1}$, $P_{i+1}, \ldots, P_8$ and P. It is easy to see that $T(P_i)$ lies on C. Define $T(P)$ by

$$T(P_i) + T(P) = P_j + P, \tag{9}$$

with respect to the addition on C.

In this paper, we employ the rules (i) and (ii) as the definition of the elliptic Painlevé equation. Then the relations (7) are consequence of them.

It is convenient to introduce a Jacobian parametrization of the point $P_u = (f_u, g_u)$ on C_0 in such a way that (1) $P_u + P_v = P_{u+v}$, and (2) Let C_{mn} be a curve of bi-degree (m, n) and let P_{x_i} ($i = 1, \ldots, 2mn$) be the intersections $C_{mn} \cap C_0$, then

$$mh_1 + nh_2 - x_1 - \cdots - x_{2mn} = 0, \qquad \text{(mod. period)}, \tag{10}$$

where h_1, h_2 are constant parameters.

We put[2] $\delta = 2h_1 + 2h_2 - u_1 - \cdots - u_8$ where u_i is the parameter corresponding to the point $P_i = P_{u_i}$. Note that $f_u = f_{h_1 - u}$ and $g_u = g_{h_2 - u}$. An example of such parametrization is

$$f_u = \frac{[u+a][u-h_1-a]}{[u+b][u-h_1-b]}, \qquad g_u = \frac{[u+c][u-h_2-c]}{[u+d][u-h_2-d]},$$

where $[u]$ is an odd theta function and a, b, c, d are constants. An expression of the elliptic Painlevé equation on $\mathbb{P}^1 \times \mathbb{P}^1$ using a parametrization in terms of the Weierstrass $\wp$ function was given by Murata [10].

In this paper, we will consider the case $T = T_{E_2 - E_1}$ as an example, and we use the notation:

$$\dot{x} = T_{E_2 - E_1}(x),$$

for any variables x. From equation (8), we have

$$\dot{u_k} = u_k, \qquad k \neq 1, 2, \qquad \dot{u_1} = u_1 - \delta, \qquad \dot{u_2} = u_2 + \delta.$$

In our construction, various polynomials and curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are defined through their degree and vanishing conditions. Let us introduce a notation to describe them.

Definition 1. Let $\Phi_{mn}(p_1^{m_1} p_2^{m_2} \cdots)$ be a linear space of polynomials in (f, g) of bi-degree (m, n) which vanish at point $p_i \in \mathbb{P}^1 \times \mathbb{P}^1$ with multiplicity m_i.

[2]This $\delta \in \mathbb{C}$ is different from $\delta \in \mathrm{Pic}(X)$ in equation (6).

Common zeros of $F \in \Phi_{mn}(p_1^{m_1} p_2^{m_2} \cdots)$ are called the base points of the family. Note that there may be some un-assigned base points besides to the assigned ones $p_1, p_2, \ldots$.

For convenience, we also use an extended notation such as

$$\Phi^d_{mn}\left(p_1^{m_1} p_2^{m_2} \cdots \,|\, p'^{n_1}_1 p'^{n_2}_2 \cdots\right).$$

Where d and $p'^{n_1}_1 p'^{n_2}_2 \cdots$ indicate the additional information: the dimension

$$d = \dim \Phi_{mn}\left(p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} \mid \cdots\right) = (m+1)(n+1) - \sum_{i=1}^{k} \frac{m_i(m_i+1)}{2},$$

and the un-assigned base points p'_i with multiplicity n_i.

3 The Lax equations

In this section we define a pair of 2nd order linear difference equations (the Lax pair for the elliptic Painlevé equation).

We chose a generic point P_z on a curve C_0. The variable z plays the role of dependent variable of the Lax equations. Unknown function of the Lax equations is denoted by $y = y(z)$. For simplicity, we use the following notation:

$$\overline{F}(z) = F(z+\delta), \qquad \underline{F}(z) = F(z-\delta).$$

Then our Lax pair takes the following form:

$$\begin{aligned} &(\mathrm{L1}) \quad L_1 = C_1 \underline{y} + C_2 y + C_3 \overline{y} = 0, \\ &(\mathrm{L2}) \quad L_2 = C_4 \underline{y} + C_5 y + C_6 \dot{y} = 0. \end{aligned} \tag{11}$$

Here $\dot{y} = T(y)$, and the coefficients $C_1, \ldots, C_6$ depend on $P_1, \ldots, P_8, P_z$ and $P = (f, g)$.

The main idea of our construction is to consider the equations like (11) as equations of curves in variables $(f, g) \in \mathbb{P}^1 \times \mathbb{P}^1$.

The first Lax equation (L1) is defined as follows.

Definition 2. Let Q_z and $Q_{\underline{z}}$ be points in $\mathbb{P}^1 \times \mathbb{P}^1$ defined in the inhomogeneous coordinates (f, g) as

$$\begin{aligned} Q_z &= \{f = f_z\} \cap \{(g - g_z) y = (g - g_{h_1 - z}) \overline{y}\}, \\ Q_{\underline{z}} &= \{f = f_{\underline{z}}\} \cap \{(g - g_{\underline{z}}) \underline{y} = (g - g_{h_1 - \underline{z}}) y\}. \end{aligned} \tag{12}$$

Note that these points depend on $\overline{y}$, y, $\underline{y}$ besides the dependence on $P_1, \ldots, P_8$ and z. Then the curve $L_1 = 0$ is defined by the following conditions:

(L1a) $L_1 \in \Phi^3_{32}(P_1 \cdots P_8 P_z \,|\, P_{\delta + h_1 - z})$,

(L1b) the curve $L_1 = 0$ passes through Q_z and $Q_{\underline{z}}$.

Lemma 1. *The conditions* (L1a), (L1b) *determine the curve* $L_1 = 0$ *uniquely and it is of the form* (L1) *in equation* (11).

Proof. Polynomial of bi-degree $(3, 2)$ has 12 free parameters. The condition (L1a) determines 9 of them and we have 3 parameter (2 dimensional) family of curves

$$c_1 G_1(f, g) + c_2 G_2(f, g) + c_3 G_3(f, g) = 0$$

satisfying the condition (L1a). The condition (L1b) adds two more linear equations on the coefficients c_1, c_2, c_3, hence the curve $L_1 = 0$ is unique up to an irrelevant overall factor. To see the resulting equation is linear in $\underline{y}$, y, $\overline{y}$, we take the following basis of the above family:

$$G_1 = (f - f_z)\varphi_{22}(f, g), \qquad G_2 = \varphi_{32}(f, g), \qquad G_3 = (f - f_{\underline{z}})\varphi_{22}(f, g).$$

Where, $\varphi_{22} = 0$ is the equation of the curve C_0 and φ_{32} is a polynomial of bi-degree (3,2) which is tangent to the lines $f = f_z$ and $f = f_{\underline{z}}$ at P_z and $P_{h_1-\underline{z}}$ respectively. Then we have

$$\begin{aligned} &G_1 = 0, \qquad G_2 \propto (g - g_z)^2, \qquad G_3 \propto (g - g_z)(g - g_{h_1-z}), \qquad \text{for} \quad f = f_z, \\ &G_1 \propto (g - g_{\underline{z}})(g - g_{h_1-\underline{z}}), \qquad G_2 \propto (g - g_{h_1-\underline{z}})^2, \qquad G_3 = 0, \qquad \text{for} \quad f = f_{\underline{z}}, \end{aligned}$$

and hence, $c_1 \propto \underline{y}$, $c_2 \propto y$, $c_3 \propto \overline{y}$. ■

The 2nd Lax equation (L2) in (11) is defined in a similar way.

Definition 3. Let Q_{u_1} be a point on $\mathbb{P}^1 \times \mathbb{P}^1$ given in inhomogeneous coordinate (f, g) as

$$Q_{u_1} = \{f = f_{u_1}\} \cap \{(g - g_{u_1})y = (g - g_{h_1-u_1})\dot{y}\}, \tag{13}$$

which depends on the variables y, $\dot{y}$. Then the curve $L_2 = 0$ is defined as

(L2a) $L_2 \in \Phi^3_{32}(P_1 P_3 \cdots P_8 P_{z+u_2-u_1} P_{h_1+\delta-z} \,|\, P_1)$,[3]

(L2b) the curve $L_2 = 0$ passes through $Q_{\underline{z}}$ in equation (12) and Q_{u_1}.

The fact that the curve specified above is unique and is of the form (L2) in equation (11) can be proved in a similar way as Lemma 1. In this case, (L2) takes the form

$$c_1(f - f_1)\varphi_{22}\underline{y} + c_2 F_{32}(h_1 - \underline{z})y + c_3(f - f_{\underline{z}})\varphi_{22}\dot{y} = 0,$$

where the curve $F_{32}(h_1 - \underline{z}) = 0$ is tangent to the lines $f = f_1$ and $f = f_{\underline{z}}$ at P_1 and $P_{h_1-\underline{z}}$ respectively. Then the curve $F_{32}(h_1 - \underline{z}) = 0$ is tangent both $f = f_1$ and C_0 at P_1, i.e. it has a node at P_1. Hence $F_{32}(h_1 - \underline{z}) \in \Phi^2_{32}(P_1^2 P_3 \cdots P_8 P_{h_1-\underline{z}})$.

In what follows, this polynomial $F_{32}(z)$ (a polynomial in (f, g) with parameter z) plays important role. Its defining properties are

$$\begin{aligned} &F_{32}(z) \in \Phi^2_{32}(P_1^2 P_3 \cdots P_8 P_z), \\ &F_{32}(z) = 0 \text{ is tangent to the line } f = f_z \text{ at } P_z. \end{aligned} \tag{14}$$

Under these conditions, $F_{32}(z)$ is unique up to normalization.

4 Some useful relations

In this section, we prepare several formulas satisfied by f, g, $\dot{f}$ and $\dot{g}$. Some results (Lemmas 3, 5 and 10) will be used to analyze the compatibility of the Lax equations in the next section.

Lemma 2. *For generic $Q = (x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$, let $F = F(f, g)$ be a polynomial such that $F \in \Phi^1_{54}(P_1^4 P_3^2 \cdots P_8^2 Q)$. Then $F = 0$ for $\dot{f} = \dot{x}$.*

[3] Due to (10), the intersection $(L_2 = 0) \cap C_0$ at P_1 is of multiplicity 2. This means the curve $L_2 = 0$ is tangent to C_0 at P_1, but is not a node in general.

Proof. From equation (7), the evolution $\dot{P} = (\dot{f}, \dot{g})$ of $P = (f, g)$ takes the following form

$$\dot{f} = \frac{F_1(f,g)}{F_2(f,g)}, \qquad \dot{g} = \frac{G_1(f,g)}{G_2(f,g)},$$

where $F_1, F_2 \in \Phi^2_{54}(P_1^4 P_3^2 \cdots P_8^2)$. Then the polynomial $F \in \Phi^1_{54}(P_1^4 P_3^2 \cdots P_8^2 Q)$ is given by $F \propto F_1(P)F_2(Q) - F_2(P)F_1(Q)$. Then we have $F = 0 \Leftrightarrow \dot{f} = F_1(P)/F_2(P) = F_1(Q)/F_2(Q) = \dot{x}$ for $F_2(P) \neq 0$ and $F_2(Q) \neq 0$. ■

From equation (9) we have

$$\dot{P}_z = P_{z+u_1-u_2-\delta}. \tag{15}$$

Then, putting $Q = P_{z+\delta-u_1+u_2}$ (i.e. $\dot{Q} = P_z$) in the above Lemma, we have

Lemma 3. *Let $\varphi_{54}(z) \in \Phi^1_{54}(P_1^4 P_3^2 \cdots P_8^2 P_{z+\delta-u_1+u_2} \,|\, P_{h_1+\delta-z-u_1+u_2})$. Then $\varphi_{54}(z) = 0$ for $\dot{f} = f_z$.*

This lemma gives a characterization of $\dot{f}$ which will be used in the next section (Lemma 13). For the later use, we should also prepare a characterization of $\dot{g}$ using some properties of the polynomial $F_{32}(z)$. To do this, let us introduce an involution r on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$r: \ (f, g) \mapsto \big(f, \tilde{g}(f, g)\big), \tag{16}$$

defined as follows. For generic $Q = (x, y)$, let $F(f, g) \in \Phi^1_{2,2}(P_1 \dot{P}_2 P_3 \cdots P_8 Q)$. The equation $F(x, g) = 0$ have two solutions, one is trivial $g = y$, and the other solution $g = \tilde{g}(x, y)$ gives the desired birational transformation. The action of the involution r on the $\mathrm{Pic}(X)$ is given by

$$r(H_1) = H_1, \qquad r(H_2) = 4H_1 + H_2 - E_1 - \cdots - E_8, \qquad r(E_i) = H_1 - E_i. \tag{17}$$

Hence, $\tilde{g}(f, g)$ is a fractional linear transformation of g with coefficients depending on f. Specialized to generic point on the curve C_0, we have

$$r(P_z) = P_{h_1 - z}.$$

The basic property of the transformations r and T is

$$rT(\lambda) = \lambda, \qquad \lambda = 3H_1 + 2H_2 - 2E_1 - E_3 - \cdots - E_8, \tag{18}$$

which follows from equations (7) and (17). More precisely, we have the following

Lemma 4. *Let $\{F_1, F_2, F_3\}$ be a basis of polynomials $\Phi^3_{3,2}(P_1^2 P_3 \cdots P_8)$, then the equation ($P$ is given and P' is unknown)*

$$(F_1(P) : F_2(P) : F_3(P)) = (F_1(P') : F_2(P') : F_3(P')) \tag{19}$$

has unique unassigned solution $P' = rT(P)$.

Proof. The equation (19) is equivalent to $F_i(P)F_3(P') = F_3(P)F_i(P')$ $(i = 1, 2)$, which are of bi-degree (3,2) and have 12 solutions. 11 of them are assigned ones P_1 (multiplicity $2^2 = 4$), $P_3, \ldots, P_8$ and trivial one $P' = P$, and hence there exist one unassigned solution, which is given by $rT(P)$ by the above formula (18). ■

The following is a special case of the above lemma.

Lemma 5. *Let $\{F_1, F_2\}$ be a basis of polynomials $\Phi^2_{3,2}(P_1^2 P_3 \cdots P_8 P_z)$, then we have*

$$T\left(\frac{F_1}{F_2}\right) = r\left(\frac{F_1}{F_2}\right), \qquad \forall\, z.$$

In the remaining part of this section, we will study a special polynomial $\mathcal{F}$. Its property (Lemma 10) will play crucial role in the next section. As a polynomial in (f, g), $\mathcal{F}$ is defined by the conditions:

$$\mathcal{F} \in \Phi^2_{32}\left(P_1^2 P_3 \cdots P_8 Q\right), \qquad \frac{\partial \mathcal{F}}{\partial g}\Big|_{P=Q} = 0. \tag{20}$$

Then $\mathcal{F}$ is unique up to normalization factor. Note that the specialization $\mathcal{F}|_{Q=P_z}$ satisfy the defining property of $F_{32}(z)$ in equation (14). The normalization of $\mathcal{F}$ may depend on Q. We fix it so that $\mathcal{F}$ is a polynomial in $Q = (x, y)$ of minimal degree. Then we have

Lemma 6. *As a polynomial in $Q = (x, y)$, $\mathcal{F}$ has bi-degree $(5, 2)$ and has zeros at P_1 (double point), $P_3, \ldots, P_8$, P. Moreover it satisfy the following properties:*

$$\frac{\partial \mathcal{F}}{\partial y}\Big|_{Q=P_i} = 0, \qquad i = 3, \ldots, 8.$$

Proof. Consider the following 12×12 determinant:

$$D = m_{P_1} \wedge \frac{\partial m_{P_1}}{\partial f} \wedge \frac{\partial m_{P_1}}{\partial g} \wedge m_{P_3} \wedge \cdots \wedge m_{P_8} \wedge m_P \wedge m_Q \wedge \frac{\partial m_Q}{\partial y}, \tag{21}$$

where

$$m_{(f,g)} = \left\{\left(1, f, f^2, f^3\right), \left(1, f, f^2, f^3\right) g, \left(1, f, f^2, f^3\right) g^2\right\} \in \mathbb{C}^{12}$$

is a vector of monomials of bi-degree $(3, 2)$. As a polynomial in (f, g), it is easy to see that this determinant D has the desired property (20) as $\mathcal{F}$. As a polynomial in $Q = (x, y)$, the bi-degree of D is apparently $(6, 4)$. The degree in variable y is actually 2, since the y dependent part $m_Q \wedge \frac{\partial m_Q}{\partial y}$ in the determinant can be reduced to

$$\left\{\left(1, x, x^2, x^3\right), \left(1, x, x^2, x^3\right)\frac{y}{2}, (0, 0, 0, 0)\right\} \wedge \left\{(0, 0, 0, 0), \left(1, x, x^2, x^3\right), \left(1, x, x^2, x^3\right) 2y\right\}.$$

Moreover, the determinant D is factorized by $(x - f_1)$ where $P_1 = (f_1, g_1)$. This follows form the relation

$$2m_{P_1} + (y - g_1)\frac{\partial m_{P_1}}{\partial g_1} - 2m_Q + (y - g_1)\frac{\partial m_Q}{\partial y} = 0 \qquad \text{at} \quad x = f_1.$$

Hence, one can take $\mathcal{F} = D/(x - f_1)$ which is of degree $(5, 2)$ in variables (x, y). Its desired vanishing conditions are easily checked from the structure of the determinant (21). Since we have 17 vanishing conditions, the degree $(5, 2)$ is minimal. ■

Lemma 7. *For the determinant D in (21), we have*

$$D = (g - g_1)^2 (x - f_1)^2 G \qquad \text{at} \quad f = f_1.$$

Where G is independent of $P = (f, g)$ and is a polynomial in $Q = (x, y)$ of degree $(4, 2)$. It satisfy

$$G = \frac{\partial G}{\partial y} = 0 \qquad \text{at} \quad Q = P_1, P_3, \ldots, P_8.$$

Proof. It is enough to show that

$$D = \frac{\partial D}{\partial x} = 0 \qquad \text{at} \quad f = x = f_1.$$

To see this, let M_i be the i-th vector in determinant D in equation (21). Then for $f = x = f_1$ we have the following linear relations:

$$(g-y)(g+y-2g_1)M_1 + (g-y)(g-g_1)(y-g_1)M_3 + (y-g_1)^2 M_{10} = (g-g_1)^2 M_{11},$$
$$2(g_1-y)M_1 + (g-g_1)(g+g_1-2y)M_3 + 2(y-g_1)M_{10} = (g-g_1)^2 M_{12}.$$

Hence, $M_{11} \wedge M_{12}$ and $\frac{\partial}{\partial x}(M_{11} \wedge M_{12})$ vanishes when multiplied with $M_1 \wedge M_3 \wedge M_{10}$. ■

Lemma 8. *Let G be the polynomial in the above Lemma 7 and let $A = A(x)$, $B = B(x)$, $C = C(x)$ be the coefficient of the fractional linear transformation:*

$$\tilde{y} = \tilde{y}(x, y) = -\frac{A + By}{B + Cy}, \tag{22}$$

where $(x, \tilde{y})$ is the image of (x, y) under the involution r (16). Then we have $G = A + 2By + Cy^2$ up to a normalization factor.

Proof. Let $\phi_{22}(f, g) = \phi_{22}(f, g; x, y)$ be a polynomial of $P = (f, g)$ belonging to $\Phi^1_{22}(P_1P_3 \cdots P_8Q)$ with $Q = (x, y)$. By the definition of the involution r, we have

$$\phi_{22}(x, \tilde{y}; x, y) = (y - \tilde{y})(A + B(y + \tilde{y}) + Cy\tilde{y}).$$

On the other hand, the polynomial $\phi_{22}(f, g; x, y)$ can be represented as the following 9×9 determinant:

$$\phi_{22}(f, g; x, y) = m'_{P_1} \wedge m'_{P_3} \wedge \cdots \wedge m'_{P_8} \wedge m'_Q \wedge m'_P,$$

where $m'_{(f,g)} \in \mathbb{C}^9$ is a vector of monomials $f^i g^j$ $(0 \le i, j \le 2)$. Then we have

$$A + 2By + Cy^2 = \lim_{\tilde{y} \to y} \frac{\phi_{22}(x, \tilde{y}; x, y)}{y - \tilde{y}} = m'_{P_1} \wedge m'_{P_3} \wedge \cdots \wedge m'_{P_8} \wedge m'_Q \wedge \frac{\partial m'_Q}{\partial y}.$$

The last determinant is of degree $(4, 2)$ in (x, y) and satisfy the vanishing properties for G in Lemma 7. ■

Lemma 9. *For the polynomial $G = A + 2By + Cy^2$ and $\tilde{y} = \tilde{y}(x, y)$, $\tilde{g} = \tilde{g}(f, g)$, we have*

$$\frac{G(x, \tilde{y})}{G(x, y)} = \frac{AC - B^2}{(B + Cy)^2} = \frac{\partial \tilde{y}}{\partial y} = -\frac{(\tilde{g} - \tilde{y})(g - \tilde{y})}{(\tilde{g} - y)(g - y)}\Big|_{f=x}.$$

Proof. All equalities follow from direct computation by using the transformation (22):

$$\tilde{y}(x, y) = -\frac{A + By}{B + Cy} \qquad \text{and} \qquad \tilde{g}(f, g)\Big|_{f=x} = -\frac{A + Bg}{B + Cg}.$$ ■

Lemma 10. *The following relation holds:*

$$\frac{\mathcal{F}(f, g : x, \tilde{y})}{\mathcal{F}(f, g; x, y)}\Big|_{f=f_1} = -\frac{(\tilde{g} - \tilde{y})(g - \tilde{y})}{(\tilde{g} - y)(g - y)}\Big|_{f=x},$$

and both sides of this equation are actually independent of g.

Proof. This is a corollary of the Lemmas 7, 8 and 9. ■

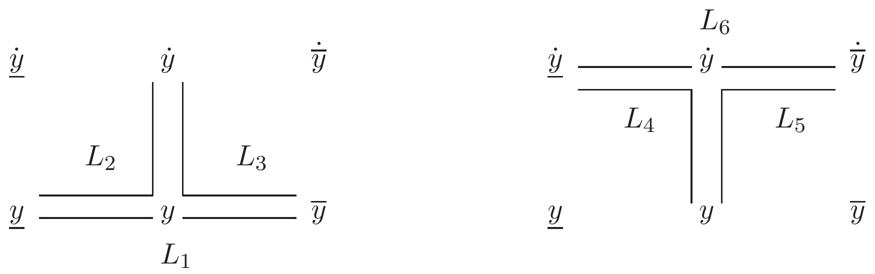

Figure 1. Lax equations.

5 The compatibility

The compatibility of the Lax pair (L1), (L2) in equation (11) is analyzed through the following four steps (Fig. 1).

1. Eliminating $\underline{y}$ from (L1) and (L2) $\rightarrow$ equation (L3) between y, $\overline{y}$, $\dot{y}$.
2. Eliminating $\underline{y}$ from (L2) and $\underline{\text{(L3)}}$ $\rightarrow$ equation (L4) between y, $\dot{y}$, $\underline{\dot{y}}$.
3. Eliminating $\overline{y}$ from $\overline{\text{(L2)}}$ and (L3) $\rightarrow$ equation (L5) between y, $\dot{y}$, $\overline{\dot{y}}$.
4. Eliminating y from (L4) and (L5) $\rightarrow$ equation (L6) between $\dot{y}$, $\underline{\dot{y}}$, $\overline{\dot{y}}$.

Then the compatibility means the equivalence (L6) $\Leftrightarrow T_{E_2-E_1}$(L1) which is the main result of this paper (Theorem 1).

We will track down these equations step by step. The resulting properties are summarized as follows.

equation	term	coefficient	divisors	additional zeros
L_1	$\underline{y}$	$(f-f_z)\varphi_{22}$	$(H_1)+(\delta)$	P_z, P_{h_1-z}
	y	φ_{32}	$(H_1+\delta)$	$P_z, P_{h_1+\delta-z}$
	$\overline{y}$	$(f-f_{\underline{z}})\varphi_{22}$	$(H_1)+(\delta)$	$P_{z-\delta}, P_{h_1+\delta-z}$
L_2	$\underline{y}$	$(f-f_1)\varphi_{22}$	$(H_1-E_1)+(\delta)$	$-$
	y	$F_{32}(h_1-\underline{z})$	$(H_1+\delta-E_1+E_2)$	$P_{z-u_1+u_2}, P_{h_1+\delta-z}$
	$\dot{y}$	$(f-f_{\underline{z}})\varphi_{22}$	$(H_1)+(\delta)$	$P_{z-\delta}, P_{h_1+\delta-z}$
L_3	$\dot{y}$	$(f-f_z)\varphi_{22}$	$(H_1)+(\delta)$	P_z, P_{h_1-z}
	y	$F_{32}(z)$	$(H_1+\delta-E_1+E_2)$	$P_z, P_{h_1+\delta-u_1+u_2-z}$
	$\overline{y}$	$(f-f_1)\varphi_{22}$	$(H_1-E_1)+(\delta)$	$-$
L_4	y	$\varphi_{54}(\underline{z})$	$(H_1+2\delta-2E_1+2E_2)$	$P_{z-u_1+u_2}, P_{h_1+2\delta-u_1+u_2-z}$
	$\dot{y}$	$F_{32}(\underline{z})\varphi_{22}$	$(H_1+\delta-E_1+E_2)+(\delta)$	$P_{z-\delta}, P_{h_1+2\delta-u_1+u_2-z}$
	$\underline{\dot{y}}$	$(f-f_1)(\varphi_{22})^2$	$(H_1-E_1)+2(\delta)$	$-$
L_5	y	$\varphi_{54}(z)$	$(H_1+2\delta-2E_1+2E_2)$	$P_{z+\delta-u_1+u_2}, P_{h_1+\delta-u_1+u_2-z}$
	$\dot{y}$	$F_{32}(h_1-z)\varphi_{22}$	$(H_1+\delta-E_1+E_2)+(\delta)$	$P_{z+\delta-u_1+u_2}, P_{h_1-z}$
	$\overline{\dot{y}}$	$(f-f_1)(\varphi_{22})^2$	$(H_1-E_1)+2(\delta)$	$-$
L_6	$\underline{\dot{y}}$	$\varphi_{54}(z)\varphi_{22}$	$(H_1+2\delta-2E_1+2E_2)+(\delta)$	$P_{z+\delta-u_1+u_2}, P_{h_1+\delta-u_1+u_2-z}$
	$\dot{y}$	φ_{76}	$(H_1+3\delta-2E_1+2E_2)$	$P_{z+\delta-u_1+u_2}, P_{h_1+2\delta-u_1+u_2-z}$
	$\overline{\dot{y}}$	$\varphi_{54}(\underline{z})\varphi_{22}$	$(H_1+2\delta-2E_1+2E_2)+(\delta)$	$P_{z-u_1+u_2}, P_{h_1+2\delta-u_1+u_2-z}$

Step 1:

Lemma 11. *The Lax equation $L_3=0$ is uniquely characterised by the following properties:*

(L3a) $L_3 \in \Phi^3_{32}(P_1P_3\cdots P_8P_zP_{h_1+\delta-z-u_1+u_2}\,|\,P_1)$,

(L3b) *passing through* 2 *more points:* Q_{u_1} *in* (13) *and* Q_z *in* (12).

Proof. The property (L3b) follows directly from the corresponding conditions in (L1b) and (L2b). Let us consider the property (L3a). We know that the Lax equations (L1), (L2) have the following form:

$$\text{(L1)}\quad L_1 = (f - f_z)\varphi_{22}\underline{y} + Fy + *(f - f_{\underline{z}})\varphi_{22}\overline{y} = 0,$$
$$\text{(L2)}\quad L_2 = (f - f_1)\varphi_{22}\underline{y} + F'y + *(f - f_{\underline{z}})\varphi_{22}\dot{y} = 0.$$

Here F, F' are some polynomials of degree $(3,2)$ and $*$ represent some constant independent of (f,g). From the equation $(f - f_1)L_1 - (f - f_z)L_2 = 0$, we have three term relation between y, $\overline{y}$, $\dot{y}$. This relation is apparently of degree $(4,2)$, however, it is divisible by $f - f_{\underline{z}}$. Since if it is not so, then it follows that $y = 0$ for $f = f_{\underline{z}}$ and for any g, which contradict the 2nd conditions of (L1b), (L2b). Then the quotient should belong to $\Phi^3_{32}(P_1^2 P_3 \cdots P_8 P_z \,|\, P_{h_1+\delta-z-u_1+u_2})$ as desired. Uniqueness follows by a simple dimensional argument as before. ■

The coefficients in (L2), (L3) are related as follows.

Lemma 12. *For the normalized equations*

$$\text{(L2)}\quad \underline{y} - A_2(z)y + B_2(z)\dot{y} = 0,$$
$$\text{(L3)}\quad \overline{y} - A_3(z)y + B_3(z)\dot{y} = 0,$$

we have

$$A_3(h_1 + \delta - z) = A_2(z), \qquad B_3(h_1 + \delta - z) = B_2(z).$$

Proof. This is because that the characterization properties (L2) and (L3) are related by $\underline{y} \leftrightarrow \overline{y}$ and $z \leftrightarrow h_1 + \delta - z$. ■

Step 2:

Lemma 13. *The Lax equation $L_4 = 0$ has the following characterizing properties:*

(L4a) $L_4 \in \Phi^3_{54}(P_1^3 P_3^2 \cdots P_8^2 P_{z-u_1+u_2} P_{h_1+2\delta-z-u_1+u_2} \,|\, P_1)$,

(L4b) *passing through 2 more points: Q_{u_1} in (13) and $\dot{Q}_{\underline{z}}$ defined by*

$$\dot{Q}_{\underline{z}} = \{\dot{f} = f_{\underline{z}}\} \cap \{(\dot{g} - g_{h_1-\underline{z}})\dot{y} = (\dot{g} - g_{\underline{z}})\underline{\dot{y}}\}. \tag{23}$$

Proof. Eliminating $\underline{y}$ from (L2) and $\underline{\text{(L3)}}$, one get three term relation between y, $\underline{\dot{y}}$, $\dot{y}$. It is apparently of degree $(6,4)$ but divisible by $f - f_{\underline{z}}$. It is easy to check that quotient L_4 belongs to $\Phi^3_{54}(P_1^3 P_3^2 \cdots P_8^2 P_{z-u_1+u_2} P_{h_1+2\delta-z-u_1+u_2} \,|\, P_1)$.

The first condition in (L4b) is the direct consequence of (L2b) or (L3b).

We will show the second condition in (L4b). Using the Lemma 12, the (L4) equation can be written as

$$Ky + A_2(z')B_2(z)\dot{y} + B_2(z')\underline{\dot{y}} = 0,$$

where $z' = h_1 + 2\delta - z$ (i.e. $\underline{z} + \underline{z'} = h_1$) and the coefficient of y is $K = 1 - A_2(z)A_2(z')$. By tracing the zeros, we see that the numerator of K is proportional to

$$\varphi_{54}(\underline{z}) \in \Phi^1_{54}\left(P_1^4 P_3^2 \cdots P_8^2 P_{z-u_1+u_2} P_{h_1+2\delta-z-u_1+u_2}\right).$$

Hence, by Lemma 3, we have $K = 0$ when $\dot{f} = f_{\underline{z}}$. Thus, we have

$$\frac{\underline{\dot{y}}}{\dot{y}} = -b(z)A_2(z') \qquad \text{for} \quad \dot{f} = f_{\underline{z}}.$$

Here, we put $b(z) = \frac{B_2(z)}{B_2(z')}$ which is independent of (f,g).

$A_2(z')|_{\dot f=f_{\underline{z}}}$ is evaluated as follows. By Lemma 5, we have $A_2(z;f,g)=A_2(z;\dot f,\tilde{\dot g})$. Hence, by using the condition (L2b), we have

$$A_2(z)|_{\dot f=f_{\underline{z}}}=\frac{\tilde{\dot g}-g_{\underline{z'}}}{\tilde{\dot g}-g_{\underline{z}}},\qquad \text{i.e.}\qquad A_2(z')|_{\dot f=f_{\underline{z}}}=\frac{\tilde{\dot g}-g_{\underline{z}}}{\tilde{\dot g}-g_{\underline{z'}}}. \tag{24}$$

Next, let us compute the factor $b(z)$. Since

$$\lim_{f\to f_1}\frac{A_2(z)}{B_2(z)}=\frac{\dot y}{y}=\frac{g-g_{u_1}}{g-g_{h_1-u_1}}$$

is independent of z, and $A_2(z)=\frac{F_{32}(h_1-\underline{z})}{(f-f_1)\varphi_{22}}$, we have

$$b(z)=\frac{B_2(z)}{B_2(z')}=\frac{A_2(z)}{A_2(z')}\Big|_{f=f_1}=\frac{F_{32}(\underline{z'})}{F_{32}(\underline{z})}\Big|_{f=f_1}. \tag{25}$$

Now, we apply the Lemma 10 in case of $Q=(x,y)=P_{\underline{z}}$. Then we have $x=f_{\underline{z}}=f_{\underline{z'}}$, $y=g_{\underline{z}}$, $\tilde y=g_{\underline{z'}}$, $\mathcal{F}|_{Q=P_{\underline{z}}}\propto F_{32}(\underline{z})$, and hence

$$\frac{F_{32}(\underline{z'})}{F_{32}(\underline{z})}\Big|_{f=f_1}=\frac{\mathcal{F}(f,g:x,\tilde y)}{\mathcal{F}(f,g;x,y)}\Big|_{f=f_1,Q=P_{\underline{z}}}=-\frac{(\tilde{\dot g}-g_{\underline{z'}})(\dot g-g_{\underline{z'}})}{(\tilde{\dot g}-g_{\underline{z}})(\dot g-g_{\underline{z}})}\Big|_{\dot f=f_{\underline{z}}}. \tag{26}$$

Here, in the last equation, variables (f,g) is replaced by $(\dot f,\dot g)$ using the g independence of the expression. It follows from (24), (25) and (26) that

$$\frac{\underline{\dot y}}{\dot y}=\frac{(\dot g-g_{\underline{z'}})}{(\dot g-g_{\underline{z}})}\qquad\text{at}\quad \dot f=f_{\underline{z}}.$$

This is the desired 2nd relation in (L4b). ■

Step 3:

Lemma 14. *The Lax equation $L_5=0$ has the following characterizing properties:*

(L5a) $L_5\in\Phi^3_{54}(P_1^3P_3^2\cdots P_8^2P_{z+\delta-u_1+u_2}P_{h_1+\delta-z-u_1+u_2}\,|\,P_1)$,

(L5b) *passing through* 2 *more points:* Q_{u_1} *in* (13) *and* $\dot Q_z$ *defined by*

$$\dot Q_z=\{\dot f=f_z\}\cap\{(\dot g-g_{h_1-z})\overline{\dot y}=(\dot g-g_z)\dot y\}. \tag{27}$$

The proof is omitted since it is almost the same as Step 2.

Step 4:

Lemma 15. *The Lax equation $L_6=0$ has the following characterizing properties:*

(L6a) $L_6\in\Phi^3_{76}(P_1^5P_2P_3^3\cdots P_8^3P_{z+\delta-u_1+u_2}\,|\,P_{h_1+2\delta-z-u_1+u_2})$,

(L6b) *passing through* 2 *more points:* $\dot Q_z$ *in* (27) *and* $\dot Q_{\underline{z}}$ *in* (23).

Proof. Eliminating y from equations

(L4) $\varphi_{54}(\underline{z})y+*A_{32}(\underline{z})\varphi_{22}\dot y+*(f-f_1)(\varphi_{22})^2\underline{\dot y}=0$,

(L5) $\varphi_{54}(z)y+*A_{32}(h_1-z)\varphi_{22}\dot y+*(f-f_1)(\varphi_{22})^2\overline{\dot y}=0$,

we have $\varphi_{54}(z)L_4-\varphi_{54}(\underline{z})L_5=0$. Which is apparently of degree $(10,8)$ but divisible by $(f-f_1)\varphi_{22}$, hence we have the equation of degree $(7,6)$. The vanishing conditions follows from that of L_4 and L_5. ■

Now we have the main result of this paper:

Theorem 1 (The compatibility). *The equation* (L6) *is equivalent with equation* (L1) *evolved by the translation* $T_{E_2-E_1}$*:*

$$u_i \mapsto \dot{u}_i, \qquad y \mapsto \dot{y}, \qquad (f,g) \mapsto (\dot{f},\dot{g}).$$

Namely, the Lax pair (L1), (L2) *is compatible if and only if the variables* (f,g) *solve the elliptic Painlevé equation for* $T = T_{E_2-E_1}$*.*

Proof. We have obtained the characterization properties of (L6), hence our task is to compare it with that for T(L1).

(1) From equation (7), we have

$$T(\mathrm{L1}) \in T(H_1+\delta) = H_1 + 3\delta - 2E_1 + 2E_2.$$

(2) Since (L1) has extra zeros at $P = P_z$ and $P = P_{h_1+\delta-z}$, T(L1) has zeros at $\dot{P} = P_z$ and $\dot{P} = P_{h_1+\delta-z}$. From the equation (15), these extra zeros of T(L1) are at $P = P_{z-u_1+u_2+\delta}$ and $P = P_{h_1+2\delta-z-u_1+u_2}$ in terms of original variable $P = (f,g)$.

From these two conditions, we see that T(L1) satisfy the condition (L6a) in Lemma 15. The condition (L6b) is exactly the condition (L1b) transformed by T. ∎

A Discussion on the differential case

In this appendix, we will discuss the differential case, taking the sixth Painlevé equation P_{VI} as an example. The P_{VI} equation has a Hamiltonian form

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{28}$$

with Hamiltonian

$$\begin{aligned} H = \frac{1}{t(t-1)}\Big[& q(q-1)(q-t)p^2 + \big\{(a_1+2a_2)(q-1)q + a_3(t-1)q + a_4 t(q-1)\big\}p \\ & + a_2(a_1+a_2)(q-t)\Big]. \end{aligned} \tag{29}$$

The equation (28) describes the iso-monodromy deformation of the Fuchsian differential equation on $\mathbb{P}^1 \setminus \{0,1,t,\infty\}$:

$$\begin{aligned} & \frac{\partial^2 y}{\partial z^2} + \left(\frac{1-a_4}{z} + \frac{1-a_3}{z-1} + \frac{1-a_0}{z-t} - \frac{1}{z-q}\right)\frac{\partial y}{\partial z} \\ & \quad + \left\{\frac{a_2(a_1+a_2)}{z(z-1)} - \frac{t(t-1)H}{z(z-1)(z-t)} + \frac{q(q-1)p}{z(z-1)(z-q)}\right\} y = 0, \end{aligned} \tag{30}$$

$(a_0 + a_1 + 2a_2 + a_3 + a_4 = 1)$ deformed by

$$\frac{\partial y}{\partial t} + \frac{z(z-1)(q-t)}{t(t-1)(q-z)}\frac{\partial y}{\partial z} + \frac{zp(q-1)(q-t)}{t(t-1)(z-q)} y = 0. \tag{31}$$

These equations (30), (31) can be viewed as a Lax pair for the P_{VI} equation. To see the geometric meaning of these Lax equations, let us introduce homogeneous coordinates $(X:Y:Z) \in \mathbb{P}^2$ by

$$q = \frac{Z}{Z-X}, \qquad p = \frac{Y(Z-X)}{XZ}.$$

Then we have

Proposition 2. *The equation* (30) *can be uniquely characterized as an algebraic curve* $F(X,Y,Z) = 0$ *of degree* 4 *in* $\mathbb{P}^2$*, satisfying the following vanishing conditions:*

$$
\begin{aligned}
&F(0,0,1) = F(1,-a_2,1) = F(1,0,0) = F(0,a_3,1) = F(1,-a_1-a_2,1) = F(1,a_4,0) = 0,\\
&F\left((t-1)\varepsilon, 1, t\varepsilon - a_0 t\varepsilon^2\right) = O\left(\varepsilon^3\right),\\
&F\left((z-1)\varepsilon, 1, z\varepsilon + z\varepsilon^2\right) = O\left(\varepsilon^4\right),\\
&F\left(\frac{1}{z}, \frac{1}{y}\frac{\partial y}{\partial z}, \frac{1}{z-1}\right)\Big|_{z\mapsto z+\varepsilon} = O\left(\varepsilon^2\right).
\end{aligned}
$$

Similarly the second Lax equation (31) *has also a similar characterization as an algebraic curve* $R(X,Y,Z) = 0$ *of degree* 2 *with the following conditions:*

$$
\begin{aligned}
&R(0,1,0) = R(1,0,0) = 0,\\
&R\left((t-1)\varepsilon, 1, t\varepsilon - \frac{t^2(t-z)}{z}\frac{1}{y}\frac{\partial y}{\partial t}\varepsilon^2\right) = O\left(\varepsilon^3\right),\\
&R\left(\frac{1}{z}, \frac{1}{y}\frac{\partial y}{\partial z}, \frac{1}{z}\right) = 0.
\end{aligned}
$$

This geometric characterization of the Lax equations for $P_{\rm VI}$ may be considered as a degenerate case of our construction. The above result bear resemblance to the geometric characterization of the Hamiltonian H (29) as a cubic pencil [9].

Finally, let us give a comment on the apparent singularity and the non-logarithmic property. The Hamiltonian H in equation (29) is usually fixed by non-logarithmic condition for equation (30) at the apparent singularity $z = q$. Namely, though the differential equation (30) has apparent singularity at $z = q$ with exponents 0 and 2, the solutions are actually holomorphic there. In the difference Lax equation (L1) defined in Definition 2, the factor $(f - f_z)$ or $(f - f_{\underline{z}})$ in its coefficients looks like an "apparent singularity". Since the non-logarithmicity is an essential property of the differential equation (30), it will be interesting if one can find the corresponding notion in difference cases.

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