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Middle Convolution and Heun's Equation*

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Abstract. Heun's equation naturally appears as special cases of Fuchsian system of differential equations of rank two with four singularities by introducing the space of initial conditions of the sixth Painlevé equation. Middle convolutions of the Fuchsian system are related with an integral transformation of Heun's equation.

Key words: Heun's equation; the space of initial conditions; the sixth Painlevé equation; middle convolution

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1 Introduction

Heun's equation is a standard form of a second-order Fuchsian differential equation with four singularities, and it is given by

$$\frac{d^2y}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-t}\right)\frac{dy}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-t)}y=0, \tag{1.1}$$

with the condition

$$\gamma+\delta+\epsilon=\alpha+\beta+1.$$

The parameter q is called an accessory parameter. Although the local monodromy (local exponent) is independent of q, the global monodromy (e.g. the monodromy on the cycle enclosing two singularities) depends on q. Some properties of Heun's equation are written in the books [21, 23], but an important feature related with the theory of finite-gap potential for the case $\gamma,\delta,\epsilon,\alpha-\beta\in\mathbb{Z}+\frac{1}{2}$ (see [6, 24, 25, 26, 27, 28, 29, 31] etc.), which leads to an algorithm to calculate the global monodromy explicitly for all q, is not written in these books.

The sixth Painlevé equation is a non-linear ordinary differential equation written as

$$\begin{aligned}\frac{d^2\lambda}{dt^2}=&\frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}\\&+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left\{\frac{(1-\theta_\infty)^2}{2}-\frac{\theta_0^2}{2}\frac{t}{\lambda^2}+\frac{\theta_1^2}{2}\frac{(t-1)}{(\lambda-1)^2}+\frac{(1-\theta_t^2)}{2}\frac{t(t-1)}{(\lambda-t)^2}\right\}.\end{aligned} \tag{1.2}$$

A remarkable property of this differential equation is that the solutions do not have movable singularities other than poles. It is known that the sixth Painlevé equation is obtained by monodromy preserving deformation of Fuchsian system of differential equations,

$$\frac{d}{dz}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\left(\frac{A_0}{z}+\frac{A_1}{z-1}+\frac{A_t}{z-t}\right)\begin{pmatrix}y_1\\y_2\end{pmatrix},\qquad A_0,A_1,A_t\in\mathbb{C}^{2\times 2}.$$

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See Section 2 for expressions of the elements of the matrices A_0, A_1, A_t. By eliminating y_2 we have second-order differential equation for y_1, which have an additional apparent singularity $z = \lambda$ other than $\{0, 1, t, \infty\}$ for generic cases, and the point λ corresponds to the variable of the sixth Painlevé equation. For details of monodromy preserving deformation, see [10]. In this paper we investigate the condition that the second-order differential equation for y_1 is written as Heun's equation. To get a preferable answer, we introduce the space of initial conditions for the sixth Painlevé equation which was discovered by Okamoto [18] to construct a suitable defining variety for the set of solutions to the (sixth) Painlevé equation.

For Fuchsian systems of differential equations and local systems on a punctured Riemann sphere, Dettweiler and Reiter [2, 3] gave an algebraic analogue of Katz' middle convolution functor [12]. Filipuk [5] applied them for the Fuchsian systems with four singularities, obtained an explicit relationship with the symmetry of the sixth Painlevé equation, and the author [30] calculated the corresponding integral transformation for the Fuchsian systems with four singularities. The middle convolution is labeled by a parameter ν, and we have two values which leads to non-trivial transformation on 2×2 Fuchsian system with four singularities (see Section 4). In this paper we consider the middle convolution which is a different value of the parameter ν from the one discussed in [5, 30]. We will also study the relationship between middle convolution and Heun's equation. For special cases, the integral transformation raised by the middle convolution turns out to be a transformation on Heun's equation, and we investigate these cases. Note that the description by the space of initial conditions for the sixth Painlevé equation is favorable. The integral transformation of Heun's equation is applied for the study of novel solutions, which we will discuss in a separated publication. If the parameter of the middle convolution is a negative integer, then the integral transformation changes to a successive differential, and a transformation defined by a differential operator on Heun's equation was found in [29] as a generalized Darboux transformation (Crum–Darboux transformation). Hence the integral transformation on Heun's equation can be regarded as a generalization of the generalized Darboux transformation, which is related with the conjectual duality by Khare and Sukhatme [15].

Special functions of the isomonodromy type including special solutions to the sixth Painlevé equation have been studied actively and they are related with various objects in mathematics and physics [16, 32]. On the other hand, special functions of Fuchsian type including special solutions to Heun's equation are also interesting objects which are related with general relativity and so on. This paper is devoted to an attempt to clarify both sides of viewpoints.

This paper is organized as follows: In Section 2, we fix notations for the Fuchsian system with four singularities. In Section 3, we define the space of initial conditions for the sixth Painlevé equation and observe that Heun's equation is obtained from the Fuchsian equation by restricting to certain lines in the space of initial conditions. In Section 4, we review results on the middle convolution and construct integral transformations. In Section 5, we investigating relationship among the middle convolution, integral transformations of Heun's equation and the space of initial conditions. In Section 6, we consider the case that the parameter on the middle convolution is integer. In the appendix, we describe topics which was put off in the text.

2 Fuchsian system of rank two with four singularities

We consider a system of ordinary differential equations,

$$\frac{dY}{dz} = A(z)Y, \qquad A(z) = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}, \qquad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad (2.1)$$

where $t \neq 0, 1$, A_0, A_1, A_t are 2×2 matrix with constant elements. Then equation (2.1) is Fuchsian, i.e., any singularities on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ are regular, and it may have regular singularities at $z = 0, 1, t, \infty$ on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Exponents of equation (2.1) at $z = 0$ (resp. $z = 1$, $z = t$, $z = \infty$) are described by eigenvalues of the matrix A_0 (resp. A_1, A_t, $-(A_0 + A_1 + A_t)$). By the transformation $Y \to z^{n_0}(z-1)^{n_1}(z-t)^{n_t} Y$, the system of differential equations (2.1) is replaced as $A(z) \to A(z) + (n_0/z + n_1/(z-1) + n_2/(z-t))I$ (I: unit matrix), and we can transform equation (2.1) to the one where one of the eigenvalues of A_i is zero for $i \in \{0, 1, t\}$ by putting $-n_i$ to be one of the eigenvalues of the original A_i. If the exponents at $z = \infty$ are distinct, then we can normalize the matrix $-(A_0 + A_1 + A_t)$ to be diagonal by a suitable gauge transformation $Y \to GY$, $A(z) \to GA(z)G^{-1}$. In this paper we assume that one of the eigenvalues of A_i is zero for $i = 0, 1, t$ and the matrix $-(A_0 + A_1 + A_t)$ is diagonal, and we set

$$A_\infty = -(A_0 + A_1 + A_t) = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}. \tag{2.2}$$

By eliminating y_2 in equation (2.1), we have a second-order linear differential equation,

$$\frac{d^2y_1}{dz^2} + p_1(z)\frac{dy_1}{dz} + p_2(z)y_1 = 0, \qquad p_1(z) = -a_{11}(z) - a_{22}(z) - \frac{\frac{d}{dz}a_{12}(z)}{a_{12}(z)},$$
$$p_2(z) = a_{11}(z)a_{22}(z) - a_{12}(z)a_{21}(z) - \frac{d}{dz}a_{11}(z) + \frac{a_{11}(z)\frac{d}{dz}a_{12}(z)}{a_{12}(z)}. \tag{2.3}$$

Set

$$A_i = \begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} \\ a_{21}^{(i)} & a_{22}^{(i)} \end{pmatrix}, \qquad (i = 0, 1, t). \tag{2.4}$$

It follows from equation (2.2) that $a_{12}^{(0)} + a_{12}^{(1)} + a_{12}^{(t)} = 0$, $a_{21}^{(0)} + a_{21}^{(1)} + a_{21}^{(t)} = 0$. Hence $a_{12}(z)$ and $a_{21}(z)$ are expressed as

$$a_{12}(z) = \frac{k_1 z + k_2}{z(z-1)(z-t)}, \qquad a_{21}(z) = \frac{\tilde{k}_1 z + \tilde{k}_2}{z(z-1)(z-t)},$$

and we have

$$a_{12}^{(0)} + a_{12}^{(1)} + a_{12}^{(t)} = 0, \qquad (t+1)a_{12}^{(0)} + ta_{12}^{(1)} + a_{12}^{(t)} = -k_1, \qquad ta_{12}^{(0)} = k_2,$$
$$a_{21}^{(0)} + a_{21}^{(1)} + a_{21}^{(t)} = 0, \qquad (t+1)a_{21}^{(0)} + ta_{21}^{(1)} + a_{21}^{(t)} = -\tilde{k}_1, \qquad ta_{21}^{(0)} = \tilde{k}_2.$$

If $k_1 = k_2 = 0$, then y_1 satisfies a first-order linear differential equation, and it is integrated easily. Hence we assume that $(k_1, k_2) \neq (0, 0)$. Then it is shown that two of $a_{12}^{(0)}$, $a_{12}^{(1)}$, $a_{12}^{(t)}$, $(t+1)a_{12}^{(0)} + ta_{12}^{(1)} + a_{12}^{(t)}$ cannot be zero. We set $\lambda = -k_2/k_1$ ($k_1 \neq 0$) and $\lambda = \infty$ ($k_1 = 0$). The condition that none of $a_{12}^{(0)}$, $a_{12}^{(1)}$, $a_{12}^{(t)}$ nor $(t+1)a_{12}^{(0)} + ta_{12}^{(1)} + a_{12}^{(t)}$ is zero is equivalent to that $\lambda \neq 0, 1, t, \infty$, and the condition $a_{12}^{(0)} = 0$ (resp. $a_{12}^{(1)} = 0$, $a_{12}^{(t)} = 0$, $(t+1)a_{12}^{(0)} + ta_{12}^{(1)} + a_{12}^{(t)} = 0$) is equivalent to $\lambda = 0$ (resp. $\lambda = 1$, $\lambda = t$, $\lambda = \infty$).

We consider the case $\lambda \neq 0, 1, t, \infty$, i.e., the case $a_{12}^{(0)} \neq 0$, $a_{12}^{(1)} \neq 0$, $a_{12}^{(t)} \neq 0$, $(t+1)a_{12}^{(0)} + ta_{12}^{(1)} + a_{12}^{(t)} \neq 0$. Let θ_0 (resp. θ_1, θ_t) and 0 be the eigenvalues of A_0 (resp. A_1, A_t). Then we can set A_0, A_1, A_t as

$$A_0 = \begin{pmatrix} u_0 + \theta_0 & -w_0 \\ u_0(u_0 + \theta_0)/w_0 & -u_0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} u_1 + \theta_1 & -w_1 \\ u_1(u_1 + \theta_1)/w_1 & -u_1 \end{pmatrix},$$
$$A_t = \begin{pmatrix} u_t + \theta_t & -w_t \\ u_t(u_t + \theta_t)/w_t & -u_t \end{pmatrix}, \tag{2.5}$$

by introducing variables u_0, w_0, u_1, w_1, u_t, w_t. By taking trace of equation (2.2), we have the relation $\theta_0+\theta_1+\theta_t+\kappa_1+\kappa_2=0$. We set $\theta_\infty=\kappa_1-\kappa_2$, then we have $\kappa_1=(\theta_\infty-\theta_0-\theta_1-\theta_t)/2$, $\kappa_2=-(\theta_\infty+\theta_0+\theta_1+\theta_t)/2$.

We determine u_0, u_1, u_t, w_0, w_1, w_t so as to satisfy equation (2.2) and the following relations:

$$\begin{aligned}
&a_{12}(z)=-\frac{w_0}{z}-\frac{w_1}{z-1}-\frac{w_t}{z-t}=\frac{k(z-\lambda)}{z(z-1)(z-t)},\\
&a_{11}(\lambda)=\frac{u_0+\theta_0}{\lambda}+\frac{u_1+\theta_1}{\lambda-1}+\frac{u_t+\theta_t}{\lambda-t}=\mu,
\end{aligned}$$

(see [11]). Namely, we solve the following equations for u_0, u_1, u_t, w_0, w_1, w_t:

$$\begin{aligned}
&-w_0-w_1-w_t=0, \qquad w_0(t+1)+w_1t+w_t=k, \qquad -w_0t=-k\lambda,\\
&u_0(u_0+\theta_0)/w_0+u_1(u_1+\theta_1)/w_1+u_t(u_t+\theta_t)/w_t=0,\\
&u_0+\theta_0+u_1+\theta_1+u_t+\theta_t=-\kappa_1, \qquad -u_0-u_1-u_t=-\kappa_2,\\
&(u_0+\theta_0)/\lambda+(u_1+\theta_1)/(\lambda-1)+(u_t+\theta_t)/(\lambda-t)=\mu.
\end{aligned} \tag{2.6}$$

The linear equations for w_0, w_1, w_t are solved as

$$w_0=\frac{k\lambda}{t}, \qquad w_1=-\frac{k(\lambda-1)}{t-1}, \qquad w_t=\frac{k(\lambda-t)}{t(t-1)}. \tag{2.7}$$

By the equations which are linear in u_0, u_1 and u_t, we can express $u_1+\theta_1$ and $u_t+\theta_t$ as linear functions in u_0. We substitute $u_1+\theta_1$ and $u_t+\theta_t$ into a quadratic equation in u_0, u_1 and u_t. Then the coefficient of u_0^2 disappears, and u_0, u_1, u_t are solved as

$$\begin{aligned}
u_0=&-\theta_0+\frac{\lambda}{t\theta_\infty}[\lambda(\lambda-1)(\lambda-t)\mu^2+\{2\kappa_1(\lambda-1)(\lambda-t)-\theta_1(\lambda-t)\\
&-t\theta_t(\lambda-1)\}\mu+\kappa_1\{\kappa_1(\lambda-t-1)-\theta_1-t\theta_t\}],\\
u_1=&-\theta_1-\frac{\lambda-1}{(t-1)\theta_\infty}[\lambda(\lambda-1)(\lambda-t)\mu^2+\{2\kappa_1(\lambda-1)(\lambda-t)+(\theta_\infty-\theta_1)(\lambda-t)\\
&-t\theta_t(\lambda-1)\}\mu+\kappa_1\{\kappa_1(\lambda-t+1)+\theta_0-(t-1)\theta_t\}],\\
u_t=&-\theta_t+\frac{\lambda-t}{t(t-1)\theta_\infty}[\lambda(\lambda-1)(\lambda-t)\mu^2+\{2\kappa_1(\lambda-1)(\lambda-t)-\theta_1(\lambda-t)\\
&+t(\theta_\infty-\theta_t)(\lambda-1)\}\mu+\kappa_1\{\kappa_1(\lambda-t+1)+\theta_0+(t-1)(\theta_\infty-\theta_t)\}].
\end{aligned} \tag{2.8}$$

We denote the Fuchsian system of differential equations

$$\frac{dY}{dz}=\left(\frac{A_0}{z}+\frac{A_1}{z-1}+\frac{A_t}{z-t}\right)Y, \qquad Y=\begin{pmatrix} y_1\\ y_2\end{pmatrix}, \tag{2.9}$$

with equations (2.5), (2.7), (2.8) by $D_Y(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu;k)$. Then the second-order differential equation (2.3) is written as

$$\begin{aligned}
&\frac{d^2y_1}{dz^2}+\left(\frac{1-\theta_0}{z}+\frac{1-\theta_1}{z-1}+\frac{1-\theta_t}{z-t}-\frac{1}{z-\lambda}\right)\frac{dy_1}{dz}\\
&\quad+\left(\frac{\kappa_1(\kappa_2+1)}{z(z-1)}+\frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}-\frac{t(t-1)H}{z(z-1)(z-t)}\right)y_1=0,\\
&H=\frac{1}{t(t-1)}[\lambda(\lambda-1)(\lambda-t)\mu^2-\{\theta_0(\lambda-1)(\lambda-t)+\theta_1\lambda(\lambda-t)\\
&\quad+(\theta_t-1)\lambda(\lambda-1)\}\mu+\kappa_1(\kappa_2+1)(\lambda-t)],
\end{aligned} \tag{2.10}$$

which we denote by $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$. This equation has regular singularities at $z = 0, 1, t, \lambda, \infty$. Exponents of the singularity $z = \lambda$ are 0, 2, and it is apparent (non-logarithmic) singularity. Note that the differential equations

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \qquad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda} \tag{2.11}$$

describe the condition for monodromy preserving deformation of equation (2.3) with respect to the variable t. By eliminating the variable μ in equation (2.11), we have the sixth Painlevé equation on the variable λ (see equation (1.2)). See [20] on equations (2.3), (2.10) and (2.11).

We consider realization of the Fuchsian system (equation (2.1)) for the case $\lambda = 0, 1, t, \infty$ in the appendix.

3 The space of initial conditions for the sixth Painlevé equation and Heun's equation

In this section, we introduce the space of initial conditions for the sixth Painlevé equation, restrict the variables of the space of initial conditions $E(t)$ to certain lines, and we obtain Heun's equation.

The space of initial conditions was introduced by Okamoto [18], which is a suitable defining variety for the set of solutions to the Painlevé system. In [22], Shioda and Takano studied the space of initial conditions further for the sixth Painlevé system (equation (2.11)) to study roles of holomorphy on the Hamiltonian. It was also constructed as a moduli space of parabolic connections by Inaba, Iwasaki and Saito [8, 9]. Here we adopt the coordinate of initial coordinate by Shioda and Takano [22] (see also [33]). The space of initial condition $E(t)$ is defined by patching six copies

$$\begin{aligned} &U_0 = \{(q_0, p_0)\}, \qquad U_1 = \{(q_1, p_1)\}, \qquad U_2 = \{(q_2, p_2)\}, \\ &U_3 = \{(q_3, p_3)\}, \qquad U_4 = \{(q_4, p_4)\}, \qquad U_\infty = \{(q_\infty, p_\infty)\}, \end{aligned} \tag{3.1}$$

of $\mathbb{C}^2$ for fixed $(t; \theta_0, \theta_1, \theta_t, \theta_\infty)$, and the rule of patching is defined by

$$\begin{aligned} &q_0 q_\infty = 1, && q_0 p_0 + q_\infty p_\infty = -\kappa_1, && (U_0 \cap U_\infty), \\ &q_0 p_0 + q_1 p_1 = \theta_0, && p_0 p_1 = 1, && (U_0 \cap U_1), \\ &(q_0 - 1) p_0 + q_2 p_2 = \theta_1, && p_0 p_2 = 1, && (U_0 \cap U_2), \\ &(q_0 - t) p_0 + q_3 p_3 = \theta_t, && p_0 p_3 = 1, && (U_0 \cap U_3), \\ &q_\infty p_\infty + q_4 p_4 = 1 - \theta_\infty, && p_\infty p_4 = 1, && (U_\infty \cap U_4). \end{aligned} \tag{3.2}$$

The variables (λ, μ) of the sixth Painlevé system (see equation (2.11)) are realized as $q_0 = \lambda$, $p_0 = \mu$ in U_0.

We define complex lines in the space of initial conditions as follows:

$$\begin{aligned} &L_0 = \{(0, p_0)\} \subset U_0, && L_1 = \{(1, p_0)\} \subset U_0, \\ &L_t = \{(t, p_0)\} \subset U_0, && L_\infty = \{(0, p_\infty)\} \subset U_\infty, \\ &L_0^* = \{(q_1, 0)\} \subset U_1, && L_1^* = \{(q_2, 0)\} \subset U_2, \\ &L_t^* = \{(q_3, 0)\} \subset U_3, && L_\infty^* = \{(q_4, 0)\} \subset U_4. \end{aligned} \tag{3.3}$$

Set

$$U_0^{q_0 \neq 0,1,t} = U_0 \setminus (L_0 \cup L_1 \cup L_t).$$

Then the space of initial conditions $E(t)$ is a direct sum of the sets $U_0^{q_0\neq 0,1,t}$, L_0, L_1, L_t, L_∞, L_0^*, L_1^*, L_t^*, L_∞^*. If $(\lambda,\mu)\in U_0^{q_0\neq 0,1,t}$, then $\lambda\neq 0,1,t,\infty$ and equation (2.10) has five regular singularities $\{0,1,t,\lambda,\infty\}$.

Although equation (2.6) was considered on the set $U_0^{q_0\neq 0,1,t}$, we may consider realization of a second-order differential equation as equation (2.10) on the space of initial conditions $E(t)$. On the lines L_0, L_1, L_t, equation (2.10) is realized by setting $\lambda=0,1,t$, and the equation is written in the form of Heun's equation

$$\frac{d^2y_1}{dz^2}+\left(\frac{-\theta_0}{z}+\frac{1-\theta_1}{z-1}+\frac{1-\theta_t}{z-t}\right)\frac{dy_1}{dz}+\frac{\kappa_1(\kappa_2+1)z+t\theta_0\mu}{z(z-1)(z-t)}y_1=0, \tag{3.4}$$

$$\frac{d^2y_1}{dz^2}+\left(\frac{1-\theta_0}{z}+\frac{-\theta_1}{z-1}+\frac{1-\theta_t}{z-t}\right)\frac{dy_1}{dz}+\frac{\kappa_1(\kappa_2+1)(z-1)+(1-t)\theta_1\mu}{z(z-1)(z-t)}y_1=0, \tag{3.5}$$

$$\frac{d^2y_1}{dz^2}+\left(\frac{1-\theta_0}{z}+\frac{1-\theta_1}{z-1}+\frac{-\theta_t}{z-t}\right)\frac{dy_1}{dz}+\frac{\kappa_1(\kappa_2+1)(z-t)+t(t-1)\theta_t\mu}{z(z-1)(z-t)}y_1=0, \tag{3.6}$$

respectively. Note that if $\theta_0\theta_1\theta_t\neq 0$ then we can realize all values of accessory parameter as varying μ. For the case $\theta_0\theta_1\theta_t=0$, we should consider other realizations.

To realize equation (2.10) on the line L_0^*, we change the variables (λ,μ) into the ones (q_1,p_1) on equation (2.10) by applying relations $\lambda\mu+q_1p_1=\theta_0$, $\mu p_1=1$. Then we have

$$\begin{aligned}&\frac{d^2y_1}{dz^2}+\left(\frac{1-\theta_0}{z}+\frac{1-\theta_1}{z-1}+\frac{1-\theta_t}{z-t}-\frac{1}{z+p_1(p_1q_1-\theta_0)}\right)\frac{dy_1}{dz}\\ &\quad+\frac{\kappa_1(\kappa_2+1)z^2+(tq_1-\theta_0(\theta_t+t\theta_1+p_1\,\mathrm{pol}_1)z+(p_1q_1-\theta_0)(-t-p_1\,\mathrm{pol}_2)}{z(z-1)(z-t)(z+p_1(p_1q_1-\theta_0))}y_1=0,\end{aligned}$$

where pol_1 and pol_2 are polynomials in p_1, q_1, t, θ_0, θ_1, θ_t, θ_∞. By setting $p_1=0$, we obtain

$$\begin{aligned}&\frac{d^2y_1}{dz^2}+\left(\frac{-\theta_0}{z}+\frac{1-\theta_1}{z-1}+\frac{1-\theta_t}{z-t}\right)\frac{dy_1}{dz}\\ &\quad+\frac{\kappa_1(\kappa_2+1)z^2+(tq_1-\theta_0(\theta_t+t\theta_1))z+t\theta_0}{z^2(z-1)(z-t)}y_1=0.\end{aligned} \tag{3.7}$$

Since the exponents of equation (3.7) at $z=0$ are 1 and θ_0, we consider gauge-transformation $v_1=z^{-1}y_1$ to obtain Heun's equation, and we have

$$\begin{aligned}&\frac{d^2v_1}{dz^2}+\left(\frac{2-\theta_0}{z}+\frac{1-\theta_1}{z-1}+\frac{1-\theta_t}{z-t}\right)\frac{dv_1}{dz}+\frac{(\kappa_1+1)(\kappa_2+2)z-q}{z(z-1)(z-t)}v_1=0,\\ &q=-tq_1+(\theta_0-1)\{t(\theta_1-1)+\theta_t-1\}.\end{aligned} \tag{3.8}$$

To realize the second-order Fuchsian equation on the line L_1^*, we change the variables (λ,μ) into the ones (q_2,p_2), substitute $p_2=0$ into equation (2.10) and set $v_1=(z-1)^{-1}y_1$. Then v_1 satisfies the following equation;

$$\begin{aligned}&\frac{d^2v_1}{dz^2}+\left(\frac{1-\theta_0}{z}+\frac{2-\theta_1}{z-1}+\frac{1-\theta_t}{z-t}\right)\frac{dv_1}{dz}+\frac{(\kappa_1+1)(\kappa_2+2)(z-1)-q}{z(z-1)(z-t)}v_1=0,\\ &q=(t-1)q_2-(\theta_1-1)\{(1-t)(\theta_0-1)+\theta_t-1\}.\end{aligned} \tag{3.9}$$

The second-order Fuchsian equation on the line L_t^* is realized as

$$\frac{d^2v_1}{dz^2}+\left(\frac{1-\theta_0}{z}+\frac{1-\theta_1}{z-1}+\frac{2-\theta_t}{z-t}\right)\frac{dv_1}{dz}+\frac{(\kappa_1+1)(\kappa_2+2)(z-t)-q}{z(z-1)(z-t)}v_1=0,$$

$$q = t(1-t)q_3 - (\theta_t - 1)((t-1)(\theta_0 - 1) + t(\theta_1 - 1)), \tag{3.10}$$

by setting $p_3 = 0$ and $v_1 = (z-t)^{-1}y_1$.

We investigate equation (2.10) on the line L_∞. We change the variables (λ, μ) into the ones (q_∞, p_∞) on equation (2.10) by applying relations $\lambda q_\infty = 1$, $\lambda\mu + q_\infty p_\infty = -\kappa_1$, and substitute $q_\infty = 0$. Then we have

$$\frac{d^2y_1}{dz^2} + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t}\right)\frac{dy_1}{dz} + \frac{\kappa_1(\kappa_2+2)z - q}{z(z-1)(z-t)}y_1 = 0,$$
$$q = (\theta_\infty - 1)p_\infty + \kappa_1(t(\kappa_2 + \theta_t + 1) + \kappa_2 + \theta_1 + 1).$$

Note that the exponents at $z = \infty$ are κ_1 and $\kappa_2 + 2$.

To realize equation (2.10) on the line L_∞^*, we change the variables (λ, μ) into the ones (q_4, p_4) on equation (2.10) by applying relations $\lambda q_\infty = 1$, $\lambda\mu + q_\infty p_\infty = -\kappa_1$, $q_\infty p_\infty + q_4 p_4 = 1 - \theta_\infty$, $p_\infty p_4 = 1$, substitute $p_4 = 0$. We obtain

$$\frac{d^2y_1}{dz^2} + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t}\right)\frac{dy_1}{dz} + \frac{(\kappa_1+1)(\kappa_2+1)z - q}{z(z-1)(z-t)}y_1 = 0,$$
$$q = -q_4 + (\kappa_2 + 1)(t(\kappa_1 + \theta_t) + \kappa_1 + \theta_1). \tag{3.11}$$

The exponents at $z = \infty$ are $\kappa_1 + 1$ and $\kappa_2 + 1$, which are different from the case of the line L_∞.

The Fuchsian system $D_Y(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu; k)$ is originally defined on the set $U_0^{q_0 \neq 0,1,t}$. We try to consider realization of Fuchsian system (equation (2.1)) on the lines L_0, L_0^*, L_1, L_1^*, L_t, L_t^*, L_∞, L_∞^* in the appendix.

4 Middle convolution

First, we review an algebraic analogue of Katz' middle convolution functor developed by Dettweiler and Reiter [2, 3], which we restrict to the present setting. Let A_0, A_1, A_t be matrices in $\mathbb{C}^{2\times 2}$. For $\nu \in \mathbb{C}$, we define the convolution matrices $B_0, B_1, B_t \in \mathbb{C}^{6\times 6}$ as follows:

$$B_0 = \begin{pmatrix} A_0 + \nu & A_1 & A_t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ A_0 & A_1 + \nu & A_t \\ 0 & 0 & 0 \end{pmatrix},$$
$$B_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_0 & A_1 & A_t + \nu \end{pmatrix}. \tag{4.1}$$

Let $z \in \mathbb{C}\setminus\{0, 1, t\}$, γ_p $(p \in \mathbb{C})$ be a cycle in $\mathbb{C}\setminus\{0, 1, t, z\}$ turning the point $w = p$ anti-clockwise whose fixed base point is $o \in \mathbb{C} \setminus \{0, 1, t, z\}$, and $[\gamma_p, \gamma_{p'}] = \gamma_p\gamma_{p'}\gamma_p^{-1}\gamma_{p'}^{-1}$ be the Pochhammer contour.

Proposition 1 ([3]). *Assume that $Y = {}^t(y_1(z), y_2(z))$ is a solution to the system of differential equations*

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}\right)Y.$$

For $p \in \{0,1,t,\infty\}$, the function

$$U = \begin{pmatrix} \int_{[\gamma_z,\gamma_p]} w^{-1} y_1(w)(z-w)^{\nu} dw \\ \int_{[\gamma_z,\gamma_p]} w^{-1} y_2(w)(z-w)^{\nu} dw \\ \int_{[\gamma_z,\gamma_p]} (w-1)^{-1} y_1(w)(z-w)^{\nu} dw \\ \int_{[\gamma_z,\gamma_p]} (w-1)^{-1} y_2(w)(z-w)^{\nu} dw \\ \int_{[\gamma_z,\gamma_p]} (w-t)^{-1} y_1(w)(z-w)^{\nu} dw \\ \int_{[\gamma_z,\gamma_p]} (w-t)^{-1} y_2(w)(z-w)^{\nu} dw \end{pmatrix},$$

satisfies the system of differential equations

$$\frac{dU}{dz} = \left(\frac{B_0}{z} + \frac{B_1}{z-1} + \frac{B_t}{z-t}\right) U. \tag{4.2}$$

We set

$$\mathcal{L}_0 = \begin{pmatrix} \mathrm{Ker}(A_0) \\ 0 \\ 0 \end{pmatrix}, \qquad \mathcal{L}_1 = \begin{pmatrix} 0 \\ \mathrm{Ker}(A_1) \\ 0 \end{pmatrix}, \qquad \mathcal{L}_t = \begin{pmatrix} 0 \\ 0 \\ \mathrm{Ker}(A_t) \end{pmatrix},$$
$$\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_t, \qquad \mathcal{K} = \mathrm{Ker}(B_0) \cap \mathrm{Ker}(B_1) \cap \mathrm{Ker}(B_t), \tag{4.3}$$

where $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_t$, $\mathcal{K} \subset \mathbb{C}^6$ and 0 in equation (4.3) means the zero vector in $\mathbb{C}^2$. We fix an isomorphism between $\mathbb{C}^6/(\mathcal{K}+\mathcal{L})$ and $\mathbb{C}^m$ for some m. A tuple of matrices $mc_{\nu}(A) = (\tilde{B}_0, \tilde{B}_1, \tilde{B}_t)$, where $\tilde{B}_p$ ($p = 0,1,t$) is induced by the action of B_p on $\mathbb{C}^m \simeq \mathbb{C}^6/(\mathcal{K}+\mathcal{L})$, is called an additive version of the middle convolution of (A_0, A_1, A_t) with the parameter ν. Let A_0, A_1, A_t be the matrices defined by equation (2.5). Then it is shown that if $\nu = 0, \kappa_1, \kappa_2$ (resp. $\nu \neq 0, \kappa_1, \kappa_2$) then $\dim \mathbb{C}^6/(\mathcal{K}+\mathcal{L}) = 2$ (resp. $\dim \mathbb{C}^6/(\mathcal{K}+\mathcal{L}) = 3$). If $\nu = 0$, then the middle convolution is identity (see [3]). Hence the middle convolutions for two cases $\nu = \kappa_1, \kappa_2$ may lead to non-trivial transformations on the 2×2 Fuchsian system with four singularities $\{0,1,t,\infty\}$. Filipuk [5] obtained that the middle convolution for the case $\nu = \kappa_1$ induce an Okamoto's transformation of the sixth Painlevé system.

We now calculate explicitly the Fuchsian system of differential equations determined by the middle convolution for the case $\nu = \kappa_2$. Note that the following calculation is analogous to the one in [30] for the case $\nu = \kappa_1$. If $\nu = \kappa_2$, then the spaces $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_t$, $\mathcal{K}$ are written as

$$\mathcal{L}_0 = \mathbb{C}\begin{pmatrix} w_0 \\ u_0+\theta_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathcal{L}_1 = \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ w_1 \\ u_1+\theta_1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathcal{L}_t = \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ w_t \\ u_t+\theta_t \end{pmatrix}, \qquad \mathcal{K} = \mathbb{C}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Set

$$S=\begin{pmatrix} 0 & 0 & 0 & w_0 & 0 & 0 \\ 0 & 0 & 1 & u_0+\theta_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_1 & 0 \\ s_{41} & s_{42} & 1 & 0 & u_1+\theta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_t \\ s_{61} & s_{62} & 1 & 0 & 0 & u_t+\theta_t \end{pmatrix},$$

$$s_{41}=\frac{\mu(\lambda-t)+\kappa_1}{k\kappa_1}, \qquad s_{61}=\frac{t(\mu(\lambda-1)+\kappa_1)}{k\kappa_1}, \qquad s_{42}=\frac{\tilde{\lambda}-\lambda}{\lambda(\lambda-1)\kappa_2},$$
$$s_{62}=\frac{t(\tilde{\lambda}-\lambda)}{\lambda(\lambda-t)\kappa_2}, \qquad \tilde{\lambda}=\lambda-\frac{\kappa_2}{\mu-\frac{\theta_0}{\lambda}-\frac{\theta_1}{(\lambda-1)}-\frac{\theta_t}{(\lambda-t)}}, \tag{4.4}$$

and $\tilde{U}=S^{-1}U$, where U is a solution to equation (4.2). Then $\det U=k^2(\tilde{\lambda}-\lambda)/(t(1-t)\kappa_2)$ and $\tilde{U}$ satisfies

$$\frac{d\tilde{U}}{dz}=\begin{pmatrix} b_{11}(z) & b_{12}(z) & 0 & 0 & 0 & 0 \\ b_{21}(z) & b_{22}(z) & 0 & 0 & 0 & 0 \\ \frac{-(u_0+\theta_0)\theta_\infty t}{k\kappa_1\lambda z} & \frac{\tilde{\lambda}}{\lambda z} & 0 & 0 & 0 & 0 \\ \frac{t}{k\lambda z} & 0 & 0 & \frac{\kappa_2}{z} & 0 & 0 \\ \frac{1-t}{k(\lambda-1)(z-1)} & 0 & 0 & 0 & \frac{\kappa_2}{z-1} & 0 \\ \frac{t(1-t)}{k(\lambda-t)(z-t)} & 0 & 0 & 0 & 0 & \frac{\kappa_2}{z-t} \end{pmatrix}\tilde{U},$$

where $b_{11}(z),\dots,b_{22}(z)$ are calculated such that the system of differential equation

$$\frac{d\tilde{Y}}{dz}=\begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix}\tilde{Y}, \qquad \tilde{Y}=\begin{pmatrix} \tilde{u}_1(z) \\ \tilde{u}_2(z) \end{pmatrix},$$

coincides with the Fuchsian system $D_Y(\tilde{\theta}_0,\tilde{\theta}_1,\tilde{\theta}_t,\tilde{\theta}_\infty;\tilde{\lambda},\tilde{\mu};\tilde{k})$ (see equation (2.9)), where

$$\tilde{\theta}_0=\frac{\theta_0-\theta_1-\theta_t-\theta_\infty}{2}, \qquad \tilde{\theta}_1=\frac{-\theta_0+\theta_1-\theta_t-\theta_\infty}{2}, \qquad \tilde{\theta}_t=\frac{-\theta_0-\theta_1+\theta_t-\theta_\infty}{2},$$
$$\tilde{\theta}_\infty=\frac{-\theta_0-\theta_1-\theta_t+\theta_\infty}{2}, \qquad \tilde{\lambda}=\lambda-\frac{\kappa_2}{\mu-\frac{\theta_0}{\lambda}-\frac{\theta_1}{\lambda-1}-\frac{\theta_t}{\lambda-t}},$$
$$\tilde{\mu}=\frac{\kappa_2+\theta_0}{\tilde{\lambda}}+\frac{\kappa_2+\theta_1}{\tilde{\lambda}-1}+\frac{\kappa_2+\theta_t}{\tilde{\lambda}-t}+\frac{\kappa_2}{\lambda-\tilde{\lambda}}, \qquad \tilde{k}=k. \tag{4.5}$$

The functions $\tilde{u}_1(z)$ and $\tilde{u}_2(z)$ are expressed as

$$\begin{aligned} \tilde{u}_1(z)&=(u_0+\theta_0)u_1(z)-\frac{k\lambda}{t}u_2(z)+(u_1+\theta_1)u_3(z) \\ &\quad+\frac{k(\lambda-1)}{t-1}u_4(z)+(u_t+\theta_t)u_5(z)+\frac{k(\lambda-t)}{t(1-t)}u_6(z), \\ \tilde{u}_2(z)&=\frac{\kappa_2\lambda(\lambda-1)(\lambda-t)}{\kappa_1(\lambda-\tilde{\lambda})}\left(\frac{(\lambda\mu+\kappa_1)(u_0+\theta_0)}{k\lambda}u_1(z)-\frac{\lambda\mu+\kappa_1}{t}u_2(z)\right. \\ &\quad+\frac{((\lambda-1)\mu+\kappa_1)(u_1+\theta_1)}{k(\lambda-1)}u_3(z)+\frac{(\lambda-1)\mu+\kappa_1}{t-1}u_4(z) \\ &\quad\left.+\frac{((\lambda-t)\mu+\kappa_1)(u_t+\theta_t)}{k(\lambda-t)}u_5(z)+\frac{(\lambda-t)\mu+\kappa_1}{t(1-t)}u_6(z)\right). \end{aligned} \tag{4.6}$$

Combining Proposition 1 with equation (4.6) and setting $\tilde{y}_1(z)=\tilde{u}_1(z)$, $\tilde{y}_2(z)=\tilde{u}_2(z)$, we have the following theorem by means of a straightforward calculation:

Theorem 1. *Set $\kappa_1 = (\theta_\infty - \theta_0 - \theta_1 - \theta_t)/2$ and $\kappa_2 = -(\theta_\infty + \theta_0 + \theta_1 + \theta_t)/2$. If $y_1(z)$ is a solution to the Fuchsian equation $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$, then the function $\tilde{Y} = {}^t(\tilde{y}_1(z), \tilde{y}_2(z))$ defined by*

$$\tilde{y}_1(z) = \int_{[\gamma_z, \gamma_p]} \frac{dy_1(w)}{dw}(z-w)^{\kappa_2} dw, \tag{4.7}$$
$$\tilde{y}_2(z) = \frac{\kappa_2 \lambda(\lambda-1)(\lambda-t)}{k(\lambda - \tilde{\lambda})} \int_{[\gamma_z, \gamma_p]} \left\{ \left(\frac{dy_1(w)}{dw} - \mu y_1(w) \right) \frac{1}{\lambda - w} + \frac{\mu}{\kappa_1} \frac{dy_1(w)}{dw} \right\} (z-w)^{\kappa_2} dw,$$

satisfies the Fuchsian system $D_Y(\kappa_2 + \theta_0, \kappa_2 + \theta_1, \kappa_2 + \theta_t, \kappa_2 + \theta_\infty; \tilde{\lambda}, \tilde{\mu}; k)$ for $p \in \{0, 1, t, \infty\}$, where

$$\tilde{\lambda} = \lambda - \frac{\kappa_2}{\mu - \frac{\theta_0}{\lambda} - \frac{\theta_1}{\lambda-1} - \frac{\theta_t}{\lambda-t}}, \qquad \tilde{\mu} = \frac{\kappa_2 + \theta_0}{\tilde{\lambda}} + \frac{\kappa_2 + \theta_1}{\tilde{\lambda} - 1} + \frac{\kappa_2 + \theta_t}{\tilde{\lambda} - t} + \frac{\kappa_2}{\lambda - \tilde{\lambda}}. \tag{4.8}$$

Since

$$0 = \int_{[\gamma_z, \gamma_p]} \frac{d}{dw} \left(y_1(w)(z-w)^{\kappa_2} \right) dw$$
$$= \int_{[\gamma_z, \gamma_p]} \frac{dy_1(w)}{dw}(z-w)^{\kappa_2} dw + \kappa_2 \int_{[\gamma_z, \gamma_p]} y_1(w)(z-w)^{\kappa_2 - 1} dw, \tag{4.9}$$

we have

Proposition 2 ([17]). *If $y_1(z)$ is a solution to $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$, then the function*

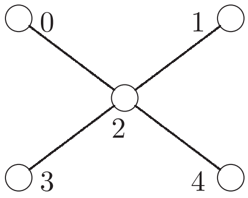
$$\tilde{y}(z) = \int_{[\gamma_z, \gamma_p]} y_1(w)(z-w)^{\kappa_2 - 1} dw, \tag{4.10}$$

satisfies $D_{y_1}(\kappa_2 + \theta_0, \kappa_2 + \theta_1, \kappa_2 + \theta_t, \kappa_2 + \theta_\infty; \tilde{\lambda}, \tilde{\mu})$ for $p \in \{0, 1, t, \infty\}$, where $\tilde{\lambda}$ and $\tilde{\mu}$ are defined in equation (4.8).

Note that this proposition was obtained by Novikov [17] by another method. Kazakov and Slavyanov [14] essentially obtained this proposition by investigating Euler transformation of 2×2 Fuchsian systems with singularities $\{0, 1, t, \infty\}$ which are realized differently from $D_Y(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu, k)$.

Let us recall the symmetry of the sixth Painlevé equation. It was essentially established by Okamoto [19] that the sixth Painlevé equation has symmetry of the affine Weyl group $W(D_4^{(1)})$. More precisely, the sixth Painlevé system is invariant under the following transformations, which are involutive and satisfy Coxeter relations attached to the Dynkin diagram of type $D_4^{(1)}$, i.e. $(s_i)^2 = 1$ $(i = 0, 1, 2, 3, 4)$, $s_j s_k = s_k s_j$ $(j, k \in \{0, 1, 3, 4\})$, $s_j s_2 s_j = s_2 s_j s_2$ $(j = 0, 1, 3, 4)$:

	θ_t	θ_∞	θ_1	θ_0	λ	μ	t
s_0	$-\theta_t$	θ_∞	θ_1	θ_0	λ	$\mu - \frac{\theta_t}{\lambda - t}$	t
s_1	θ_t	$2 - \theta_\infty$	θ_1	θ_0	λ	μ	t
s_2	$\kappa_1 + \theta_t$	$-\kappa_2$	$\kappa_1 + \theta_1$	$\kappa_1 + \theta_0$	$\lambda + \frac{\kappa_1}{\mu}$	μ	t
s_3	θ_t	θ_∞	$-\theta_1$	θ_0	λ	$\mu - \frac{\theta_1}{\lambda - 1}$	t
s_4	θ_t	θ_∞	θ_1	$-\theta_0$	λ	$\mu - \frac{\theta_0}{\lambda}$	t



The map $(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu) \mapsto (\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ determined by equation (4.5) coincides with the composition map $s_0 s_3 s_4 s_2 s_0 s_3 s_4$, because

$$(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu) \overset{s_0 s_3 s_4}{\mapsto} \left(-\theta_0, -\theta_1, -\theta_t, \theta_\infty; \lambda, \mu - \tfrac{\theta_0}{\lambda} - \tfrac{\theta_1}{\lambda-1} - \tfrac{\theta_t}{\lambda-t} \right)$$

$$\overset{s_2}{\mapsto} \big(-\kappa_2-\theta_0, -\kappa_2-\theta_1, -\kappa_2-\theta_t, \kappa_1; \tilde{\lambda}, \mu - \tfrac{\theta_0}{\lambda} - \tfrac{\theta_1}{\lambda-1} - \tfrac{\theta_t}{\lambda-t}\big)$$
$$\overset{s_0 s_3 s_4}{\mapsto} \big(\kappa_2+\theta_0, \kappa_2+\theta_1, \kappa_2+\theta_t, \kappa_2+\theta_\infty; \tilde{\lambda}, \tilde{\mu}\big).$$

Therefore, if we know a solution to the Fuchsian system $D_Y(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu;k)$, then we have integral representations of solutions to the Fuchsian system $D_Y(\tilde{\theta}_0,\tilde{\theta}_1,\tilde{\theta}_t,\tilde{\theta}_\infty;\tilde{\lambda},\tilde{\mu};k)$ obtained by the transformation $s_0s_3s_4s_2s_4s_3s_0$. Note that the transformations s_i $(i=0,1,2,3,4)$ are extended to isomorphisms of the space of initial conditions $E(t)$.

We recall the middle convolution for the case $\nu=\kappa_1$.

Proposition 3 ([30, Proposition 3.2]). *If $Y = {}^t(y_1(z), y_2(z))$ is a solution to the Fuchsian system $D_Y(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu;k)$ (see equation (2.9)), then the function $\tilde{Y} = {}^t(\tilde{y}_1(z),\tilde{y}_2(z))$ defined by*

$$\begin{aligned}
&\tilde{y}_1(z) = \int_{[\gamma_z,\gamma_p]} \left\{\kappa_1 y_1(w) + (w-\tilde{\lambda})\frac{dy_1(w)}{dw}\right\} \frac{(z-w)^{\kappa_1}}{w-\lambda} dw, \\
&\tilde{y}_2(z) = \frac{-\theta_\infty}{\kappa_2}\int_{[\gamma_z,\gamma_p]} \frac{dy_2(w)}{dw}(z-w)^{\kappa_1} dw,
\end{aligned} \tag{4.11}$$

satisfies the Fuchsian system $D_Y(\kappa_1+\theta_0,\kappa_1+\theta_1,\kappa_1+\theta_t,-\kappa_2;\lambda+\kappa_1/\mu,\mu;k)$ for $p\in\{0,1,t,\infty\}$.

The parameters $(\kappa_1+\theta_0,\kappa_1+\theta_1,\kappa_1+\theta_t,-\kappa_2;\lambda+\kappa_1/\mu,\mu)$ are obtained from the parameters $(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$ by applying the transformation s_2. Note that the relationship the transformation s_2 was obtained by Filipuk [5] explicitly (see also [7]), and Boalch [1] and Dettweiler and Reiter [4] also obtained results on the symmetry of the sixth Painlevé equation and the middle convolution.

5 Middle convolution, integral transformations of Heun's equation and the space of initial conditions

In this section, we investigating relationship among the middle convolution, integral transformations of Heun's equation and the space of initial conditions.

Kazakov and Slavyanov established an integral transformation on solutions to Heun's equation in [13], which we express in a slightly different form.

Theorem 2 ([13]). *Set*

$$\begin{aligned}
&(\eta-\alpha)(\eta-\beta)=0, \qquad \gamma'=\gamma+1-\eta, \qquad \delta'=\delta+1-\eta, \qquad \epsilon'=\epsilon+1-\eta, \\
&\{\alpha',\beta'\} = \{2-\eta, -2\eta+\alpha+\beta+1\}, \\
&q' = q+(1-\eta)(\epsilon+\delta t+(\gamma-\eta)(t+1)).
\end{aligned} \tag{5.1}$$

Let $v(w)$ be a solution to

$$\frac{d^2v}{dw^2} + \left(\frac{\gamma'}{w}+\frac{\delta'}{w-1}+\frac{\epsilon'}{w-t}\right)\frac{dv}{dw} + \frac{\alpha'\beta' w - q'}{w(w-1)(w-t)} v = 0. \tag{5.2}$$

Then the function

$$y(z) = \int_{[\gamma_z,\gamma_p]} v(w)(z-w)^{-\eta} dw$$

is a solution to

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-t}\right)\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)} y = 0, \tag{5.3}$$

for $p\in\{0,1,t,\infty\}$.

Here we derive Theorem 2 by considering the limit $\lambda \to 0$ in Proposition 2. Let us recall notations in Proposition 2. We consider the limit $\lambda \to 0$ while fixing μ for the case $\theta_0 \neq 0$ and $\theta_0 + \kappa_2 \neq 0$. Then we have $\tilde{\lambda} \to 0$ and $\tilde{\mu} \to (t\theta_0\mu + \kappa_2(t(\kappa_1 + \theta_t) + \kappa_1 + \theta_1))/(t(\kappa_2 + \theta_0))$. Hence it follows from Proposition 2 and equation (3.4) that, if $y(z)$ satisfies

$$\frac{d^2y(z)}{dz^2} + \left(\frac{-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t}\right)\frac{dy(z)}{dz} + \frac{\kappa_1(\kappa_2+1)z + t\theta_0\mu}{z(z-1)(z-t)}y(z) = 0, \tag{5.4}$$

then the function

$$\tilde{y}(z) = \int_{[\gamma_z,\gamma_p]} y(w)(z-w)^{\kappa_2-1}dw, \tag{5.5}$$

satisfies

$$\frac{d^2\tilde{y}(z)}{dz^2} + \left(\frac{-\kappa_2-\theta_0}{z} + \frac{1-\kappa_2-\theta_1}{z-1} + \frac{1-\kappa_2-\theta_t}{z-t}\right)\frac{d\tilde{y}(z)}{dz} + \left(\frac{\theta_\infty(1-\kappa_2)z + t(\kappa_2+\theta_0)\frac{t\theta_0\mu+\kappa_2(t(\kappa_1+\theta_t)+\kappa_1+\theta_1))}{t(\kappa_2+\theta_0)}}{z(z-1)(z-t)}\right)\tilde{y}(z) = 0. \tag{5.6}$$

By setting $\gamma = -\kappa_2 - \theta_0$, $\delta = 1 - \kappa_2 - \theta_1$, $\epsilon = 1 - \kappa_2 - \theta_t$, $\alpha = \eta = 1 - \kappa_2$, $\beta = \theta_\infty$, $q = -\{t\theta_0\mu + \kappa_2(t(\kappa_1+\theta_t)+\kappa_1+\theta_1))\}$ and comparing with the standard form of Heun's equation (equation (1.1)), we recover Theorem 2. Note that we can obtain the formula corresponding to the case $\theta_0 = 0$ (resp. $\theta_0 + \kappa_2 = 0$) by considering the limit $\theta_0 \to 0$ (resp. $\theta_0 + \kappa_2 \to 0$).

The limit $\lambda \to 0$ while fixing μ implies the restriction of the coordinate (λ, μ) to the line L_0 in the space of initial conditions $E(t)$, and the line L_0 with the parameter $(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ is mapped to the line L_0 in the space of initial conditions with the parameter $(\kappa_2+\theta_0, \kappa_2+\theta_1, \kappa_2+\theta_t, \kappa_2 + \theta_\infty; \tilde{\lambda}, \tilde{\mu})$ where $\tilde{\lambda}$ and $\tilde{\mu}$ are defined in equation (4.8), because $\tilde{\lambda} \to 0$ and $\tilde{\mu}$ converges by the limit. It follows from equations (5.4), (5.5), (5.6) that the integral transformation in Proposition 2 reproduces the integral transformation on Heun's equations in Theorem 2 by restricting to the line L_0. We can also establish that the line L_1 (resp. L_t) in the space of initial conditions with the parameter $(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ is mapped to the line L_1 (resp. L_t) with the parameter $(\kappa_2 + \theta_0, \kappa_2 + \theta_1, \kappa_2 + \theta_t, \kappa_2 + \theta_\infty; \tilde{\lambda}, \tilde{\mu})$ by taking the limit $\lambda \to 1$ (resp. $\lambda \to t$), and the integral transformation in Proposition 2 reproduces the integral transformation on Heun's equations in Theorem 2. We discuss the restriction of the map $(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu) \mapsto (\kappa_2+\theta_0, \kappa_2+\theta_1, \kappa_2+\theta_t, \kappa_2+\theta_\infty; \tilde{\lambda}, \tilde{\mu})$ to the line L^*_∞. Let (q_4, p_4) (resp. $(\tilde{q}_4, \tilde{p}_4)$) be the coordinate of U_4 for the parameters $(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. $(\kappa_2 + \theta_0, \kappa_2 + \theta_1, \kappa_2 + \theta_t, \kappa_2 + \theta_\infty; \tilde{\lambda}, \tilde{\mu})$ (see equations (3.1), (3.2)). Then we can express $\tilde{q}_4$ and $\tilde{p}_4$ by the variables q_4 and p_4. By setting $p_4 = 0$, we have $\tilde{p}_4 = 0$ and $\tilde{q}_4 = q_4 - \kappa_2(t(\kappa_1 + \theta_t - 1) + \kappa_1 + \theta_1 - 1)$. Hence the line L^*_∞ with the parameter $(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ is mapped to the line L^*_∞ with the parameter $(\kappa_2 + \theta_0, \kappa_2 + \theta_1, \kappa_2 + \theta_t, \kappa_2 + \theta_\infty; \tilde{\lambda}, \tilde{\mu})$. It follows from Proposition 2 that if $y_1(z)$ satisfies equation (3.11) then the function $\tilde{y}(z)$ defined by Proposition 2 satisfies Heun's equation with the parameters $\gamma = 1 - \theta_0 - \kappa_2$, $\delta = 1 - \theta_1 - \kappa_2$, $\epsilon = 1 - \theta_t - \kappa_2$, $\alpha = 1 - \kappa_2$, $\beta = 1 + \theta_\infty$, $q = -q_4 - (1+t)\kappa_2 + (t(\kappa_1+\theta_t)+\kappa_1+\theta_1))$, and the integral representation reproduces Theorem 2 by setting $\eta = \alpha = 1 - \kappa_2$. Therefore we have the following theorem:

Theorem 3. *Let $X = L_0$, L_1, L_t or L^*_∞. By the map $s_0s_3s_4s_2s_4s_3s_0 : (\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu) \mapsto (\kappa_2 + \theta_0, \kappa_2 + \theta_1, \kappa_2 + \theta_t, \kappa_2 + \theta_\infty; \tilde{\lambda}, \tilde{\mu})$ where $\tilde{\lambda}$ and $\tilde{\mu}$ are defined in equation* (4.8)*, the line X in the space of initial conditions with the parameter $(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ is mapped to the line X in the space of initial conditions with the parameter $(\kappa_2 + \theta_0, \kappa_2 + \theta_1, \kappa_2 + \theta_t, \kappa_2 + \theta_\infty; \tilde{\lambda}, \tilde{\mu})$ where $\tilde{\lambda}$ and $\tilde{\mu}$ are defined in equation* (4.8)*, and the integral transformation in Proposition* 2 *determined by the middle convolution reproduces the integral transformation on Heun's equations in Theorem* 2 *by the restriction to the line X.*

Note that if $X = L_0^*$, L_1^*, L_t^* or L_∞ then the image of the line X by the map $s_0s_3s_4s_2s_4s_3s_0$ may not included in X with the parameter $(\kappa_2+\theta_0,\kappa_2+\theta_1,\kappa_2+\theta_t,\kappa_2+\theta_\infty;\tilde{\lambda},\tilde{\mu})$.

We consider restrictions of the middle convolution for the case $\nu=\kappa_1$ (see Proposition 3) to lines in the space of initial conditions. We discuss the restriction of the map $(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$ $\mapsto (\kappa_1+\theta_0,\kappa_1+\theta_1,\kappa_1+\theta_t,-\kappa_2;\lambda+\kappa_1/\mu,\mu)$ to the line L_0^*. Let (q_1,p_1) (resp. $(\tilde{q}_1,\tilde{p}_1)$) be the coordinate of U_1 for the parameters $(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$ (resp. $(\kappa_1+\theta_0,\kappa_1+\theta_1,\kappa_1+\theta_t,-\kappa_2;\lambda+\kappa_1/\mu,\mu)$). Then we can express $\tilde{q}_1$ and $\tilde{p}_1$ by the variables q_1 and p_1, and by setting $p_1=0$ we have $\tilde{p}_1=0$ and $\tilde{q}_1=q_1$. Let $y_1(z)$ be a solution to the Fuchsian equation $D_{y_1}(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$ for the case $p_1=0$ and set $v_1(z)=z^{-1}y_1(z)$. Then $v_1(z)$ satisfies Heun's equation written as equation (3.8). On the case $p_1=0$, the integral representation (equation (4.11)) is written as

$$\tilde{y}_1(z)=\int_{[\gamma_z,\gamma_p]}\left\{\kappa_1 y_1(w)+w\frac{dy_1(w)}{dw}\right\}\frac{(z-w)^{\kappa_1}}{w}dw, \qquad (p\in\{0,1,t,\infty\}).$$

We set $\tilde{v}_1(z)=z^{-1}\tilde{y}_1(z)$. By integration by parts we have

$$\begin{aligned}\tilde{v}_1(z)&=\frac{1}{z}\int_{[\gamma_z,\gamma_p]}\left\{\kappa_1 v_1(w)(z-w)^{\kappa_1}+\frac{d(wv_1(w))}{dw}(z-w)^{\kappa_1}\right\}dw\\&=\frac{1}{z}\int_{[\gamma_z,\gamma_p]}\left\{\kappa_1(z-w)v_1(w)(z-w)^{\kappa_1-1}+\kappa_1 wv_1(w)(z-w)^{\kappa_1-1}\right\}dw\\&=\kappa_1\int_{[\gamma_z,\gamma_p]}v_1(w)(z-w)^{\kappa_1-1}dw. \end{aligned}\tag{5.7}$$

On the other hand, it follows from Proposition 3 and equation (3.8) that $\tilde{v}_1(z)$ satisfies Heun's equation with the parameters $\gamma=2-\theta_0-\kappa_1$, $\delta=1-\theta_1-\kappa_1$, $\epsilon=1-\theta_t-\kappa_1$, $\alpha=1-\kappa_1$, $\beta=2-\theta_\infty$, $q=-tq_1+(\kappa_1+\theta_0-1)\{t(\kappa_1+\theta_1-1)+\kappa_1+\theta_t-1\}$. Hence equation (5.7) reproduces Theorem 2 by setting $\eta=\alpha=1-\kappa_1$, We can also obtain similar results for L_1^*, L_t^* and L_∞^*. Therefore we have the following theorem:

Theorem 4. *Let $X=L_0^*$, L_1^*, L_t^* or L_∞^*. By the map $s_2:\ (\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)\mapsto(\kappa_1+\theta_0,\kappa_1+\theta_1,\kappa_1+\theta_t,-\kappa_2;\lambda+\kappa_1/\mu,\mu)$, the line X in the space of initial conditions with the parameter $(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$ is mapped to the line X in the space of initial conditions with the parameter $(\kappa_1+\theta_0,\kappa_1+\theta_1,\kappa_1+\theta_t,-\kappa_2;\lambda+\kappa_1/\mu,\mu)$, and the integral transformation in Proposition 3 determined by the middle convolution reproduces the integral transformation on Heun's equations in Theorem 2 by the restriction to the line X.*

Note that if $X=L_0$, L_1, L_t or L_∞ then the image of the line X by the map s_2 may not included in X with the parameter $(\kappa_1+\theta_0,\kappa_1+\theta_1,\kappa_1+\theta_t,-\kappa_2;\lambda+\kappa_1/\mu,\mu)$.

6 Middle convolution for the case that the parameter is integer

On the case $\kappa_2\in\mathbb{Z}$, the function in equation (4.10) containing the Pochhammer contour may be vanished, and we propose other expressions of solutions to Fuchsian equation $D_{y_1}(\kappa_2+\theta_0,\kappa_2+\theta_1,\kappa_2+\theta_t,\kappa_2+\theta_\infty;\tilde{\lambda},\tilde{\mu})$ in use of solutions to $D_{y_1}(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$.

We have the following proposition for the case $\kappa_2\in\mathbb{Z}_{<0}$,

Proposition 4. *(i) Let A_0, A_1, A_t be matrices in $\mathbb{C}^{2\times 2}$, and let $B_0^{(\nu)},B_1^{(\nu)},B_t^{(\nu)}\in\mathbb{C}^{6\times 6}$ be the matrices defined in equation (4.1) for $\nu\in\mathbb{C}$. Assume that $\nu\in\mathbb{Z}_{<0}$ and $Y={}^t(y_1(z),y_2(z))$ is a solution to the system of differential equations*

$$\frac{dY}{dz}=\left(\frac{A_0}{z}+\frac{A_1}{z-1}+\frac{A_t}{z-t}\right)Y.$$

Write $\nu = -1-n$ $(n \in \mathbb{Z}_{\geq 0})$. Then the function

$$U = \begin{pmatrix} (d/dz)^n (z^{-1} y_1(z)) \\ (d/dz)^n (z^{-1} y_2(z)) \\ (d/dz)^n ((z-1)^{-1} y_1(z)) \\ (d/dz)^n ((z-1)^{-1} y_2(z)) \\ (d/dz)^n ((z-t)^{-1} y_1(z)) \\ (d/dz)^n ((z-t)^{-1} y_2(z)) \end{pmatrix},$$

satisfies the system of differential equations

$$\frac{dU}{dz} = \left(\frac{B_0^{(-1-n)}}{z} + \frac{B_1^{(-1-n)}}{z-1} + \frac{B_t^{(-1-n)}}{z-t} \right) U. \tag{6.1}$$

(ii) If $\kappa_2 \in \mathbb{Z}_{<0}$ and $Y = {}^t(y_1(z), y_2(z))$ is a solution to the Fuchsian system $D_Y(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu; k)$, then the function $\tilde{Y} = {}^t(\tilde{y}_1(z), \tilde{y}_2(z))$ defined by

$$\begin{aligned} \tilde{y}_1(z) &= \left(\frac{d}{dz}\right)^{-\kappa_2} y_1(z), \\ \tilde{y}_2(z) &= \frac{\kappa_2 \lambda(\lambda-1)(\lambda-t)}{k(\lambda-\tilde{\lambda})} \left[\left\{ \left(\frac{d}{dz}\right)^{-\kappa_2} y_1(z) - \mu \left(\frac{d}{dz}\right)^{-\kappa_2-1} y_1(z) \right\} \frac{1}{\lambda - z} \right. \\ &\quad \left. + \frac{\mu}{\kappa_1} \left(\frac{d}{dz}\right)^{-\kappa_2} y_1(z) \right], \end{aligned} \tag{6.2}$$

satisfies the Fuchsian system $D_Y(\kappa_2+\theta_0, \kappa_2+\theta_1, \kappa_2+\theta_t, \kappa_2+\theta_\infty; \tilde{\lambda}, \tilde{\mu}; k)$, where $\tilde{\lambda}$ and $\tilde{\mu}$ are defined in equation (4.8).

(iii) If $\kappa_2 \in \mathbb{Z}_{<0}$ and $y_1(z)$ is a solution to $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$, then the function

$$\tilde{y}(z) = \left(\frac{d}{dz}\right)^{-\kappa_2} y_1(z),$$

satisfies $D_{y_1}(\kappa_2+\theta_0, \kappa_2+\theta_1, \kappa_2+\theta_t, \kappa_2+\theta_\infty; \tilde{\lambda}, \tilde{\mu})$.

Proof. (i) If $\nu = -1$, then it follows immediately that the function $U = {}^t(z^{-1}y_1(z), z^{-1}y_2(z), (z-1)^{-1}y_1(z), (z-1)^{-1}y_2(z), (z-t)^{-1}y_1(z), (z-t)^{-1}y_2(z))$ satisfied equation (6.1) for $n = 0$. Assume now that the function $U = {}^t(u_1(z), u_2(z), u_3(z), u_4(z), u_5(z), u_6(z))$ satisfies equation (6.1). Set $V = dU/dz$. Since

$$\begin{aligned} &\frac{B_0^{(-1-n)}}{z^2} + \frac{B_1^{(-1-n)}}{(z-1)^2} + \frac{B_t^{(-1-n)}}{(z-t)^2} = \frac{1}{z}\frac{B_0^{(-1-n)}}{z} \begin{pmatrix} u_1(z) \\ u_2(z) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{z-1}\frac{B_0^{(-1-n)}}{z-1} \begin{pmatrix} 0 \\ 0 \\ u_3(z) \\ u_4(z) \\ 0 \\ 0 \end{pmatrix} \\ &\quad + \frac{1}{z-t}\frac{B_0^{(-1-n)}}{z-t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_5(z) \\ u_6(z) \end{pmatrix} = \frac{1}{z} \begin{pmatrix} u_1'(z) \\ u_2'(z) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{z-1} \begin{pmatrix} 0 \\ 0 \\ u_3'(z) \\ u_4'(z) \\ 0 \\ 0 \end{pmatrix} + \frac{1}{z-t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_5'(z) \\ u_6'(z) \end{pmatrix}, \end{aligned}$$

we have

$$\begin{aligned}\frac{dV}{dz} &= \frac{d}{dz}\left\{\left(\frac{B_0^{(-1-n)}}{z}+\frac{B_1^{(-1-n)}}{z-1}+\frac{B_t^{(-1-n)}}{z-t}\right)U\right\}\\ &= -\left(\frac{B_0^{(-1-n)}}{z^2}+\frac{B_1^{(-1-n)}}{(z-1)^2}+\frac{B_t^{(-1-n)}}{(z-t)^2}\right)U+\left(\frac{B_0^{(-1-n)}}{z}+\frac{B_1^{(-1-n)}}{z-1}+\frac{B_t^{(-1-n)}}{z-t}\right)V\\ &= \left(\frac{B_0^{(-2-n)}}{z}+\frac{B_1^{(-2-n)}}{z-1}+\frac{B_t^{(-2-n)}}{z-t}\right)V.\end{aligned}$$

Hence (i) is proved inductively.

(ii) Let A_0, A_1, A_t be the matrices defined by equation (2.5) and set $\nu = \kappa_2$. We define the matrix S by equation (4.4) and set $\tilde{U} = S^{-1}U$. Then $\tilde{Y} = {}^t(\tilde{u}_1(z), \tilde{u}_2(z))$ satisfies the Fuchsian differential equation $D_Y(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu}; \tilde{k})$ where the parameters are determined by equation (4.5), and $\tilde{u}_1(z)$, $\tilde{u}_2(z)$ are expressed as equation (4.6). By a straightforward calculation as obtaining equation (4.7), we have equation (6.2).

(iii) follows from (ii). ∎

Note that (i) is valid for Fuchsian differential systems of arbitrary size and arbitrary number of regular singularities. We have a similar statement for the case $\nu = \kappa_1$ and $\kappa_1 \in \mathbb{Z}_{<0}$. Namely, if $\kappa_1 \in \mathbb{Z}_{<0}$ and $Y = {}^t(y_1(z), y_2(z))$ is a solution to the Fuchsian differential equation $D_Y(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu; k)$ (see equation (2.9)), then the function $\tilde{Y} = {}^t(\tilde{y}_1(z), \tilde{y}_2(z))$ defined by

$$\begin{aligned}\tilde{y}_1(z) &= \left(\frac{d}{dz}\right)^{-\kappa_1-1}\left\{\frac{1}{z-\lambda}\left(\kappa_1 y_1(z)+(z-\tilde{\lambda})\frac{dy_1(z)}{dz}\right)\right\},\\ \tilde{y}_2(z) &= \left(\frac{d}{dz}\right)^{-\kappa_1} y_2(z),\end{aligned}$$

satisfies the Fuchsian system $D_Y(\kappa_1+\theta_0, \kappa_1+\theta_1, \kappa_1+\theta_t, -\kappa_2; \lambda+\kappa_1/\mu, \mu; k)$.

If $\kappa_2 = 0$ (resp. $\kappa_1 = 0$), then the Fuchsian system $D_Y(\kappa_2+\theta_0, \kappa_2+\theta_1, \kappa_2+\theta_t, \kappa_2+\theta_\infty; \tilde{\lambda}, \tilde{\mu}; k)$ (resp. $D_Y(\kappa_1+\theta_0, \kappa_1+\theta_1, \kappa_1+\theta_t, -\kappa_2; \lambda+\kappa_1/\mu, \mu; k)$) coincides with $D_Y(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu; k)$, and the function $\tilde{Y} = {}^t(\tilde{y}_1(z), \tilde{y}_2(z))$ just corresponds to ${}^t(y_1(z), y_2(z))$.

On the case $\kappa_2 \in \mathbb{Z}_{>0}$, we have the following proposition:

Proposition 5. *Let $p \in \{0, 1, t, \infty\}$ and C_p be the cycle starting from $w = z$, turning $w = p$ anti-clockwise and return to $w = z$.*

(i) If $\kappa_2 \in \mathbb{Z}_{>0}$ and $Y = {}^t(y_1(z), y_2(z))$ is a solution to the Fuchsian system $D_Y(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu; k)$, then the function $\tilde{Y} = {}^t(\tilde{y}_1(z), \tilde{y}_2(z))$ defined by

$$\begin{aligned}\tilde{y}_1(z) &= \int_{C_p} \frac{dy_1(w)}{dw}(z-w)^{\kappa_2}dw, && (6.3)\\ \tilde{y}_2(z) &= \frac{\kappa_2\lambda(\lambda-1)(\lambda-t)}{k(\lambda-\tilde{\lambda})}\int_{C_p}\left\{\left(\frac{dy_1(w)}{dw}-\mu y_1(w)\right)\frac{1}{\lambda-w}+\frac{\mu}{\kappa_1}\frac{dy_1(w)}{dw}\right\}(z-w)^{\kappa_2}dw,\end{aligned}$$

satisfies the Fuchsian system $D_Y(\kappa_2+\theta_0, \kappa_2+\theta_1, \kappa_2+\theta_t, \kappa_2+\theta_\infty; \tilde{\lambda}, \tilde{\mu}; k)$ for $p \in \{0, 1, t, \infty\}$ where $\tilde{\lambda}$ and $\tilde{\mu}$ are defined in equation (4.8).

(ii) If $\kappa_2 \in \mathbb{Z}_{>0}$ and $y_1(z)$ is a solution to $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$, then the function

$$\tilde{y}(z) = \int_{C_p} y_1(w)(z-w)^{\kappa_2-1}dw, \qquad (6.4)$$

satisfies $D_{y_1}(\kappa_2+\theta_0, \kappa_2+\theta_1, \kappa_2+\theta_t, \kappa_2+\theta_\infty; \tilde{\lambda}, \tilde{\mu})$ for $p \in \{0, 1, t, \infty\}$.

Note that the functions $\tilde{y}_1(z)$, $\tilde{y}_2(z)$, $\tilde{y}(z)$ may not be polynomials although the integrands of equations (6.3) and (6.4) are polynomials in z.

Proof. Set

$$K_1(w) = \frac{dy_1(w)}{dw},$$
$$K_2(w) = \frac{\kappa_2 \lambda(\lambda-1)(\lambda-t)}{k(\lambda-\tilde{\lambda})}\left\{\left(\frac{dy_1(w)}{dw} - \mu y_1(w)\right)\frac{1}{\lambda - w} + \frac{\mu}{\kappa_1}\frac{dy_1(w)}{dw}\right\}.$$

It follows from Theorem 1 that the function $Y(z) = \begin{pmatrix} \tilde{y}_1(z) \\ \tilde{y}_2(z) \end{pmatrix}$ defined by

$$\tilde{y}_i(z) = \int_{\gamma_z\gamma_p\gamma_z^{-1}\gamma_p^{-1}} K_i(w)(z-w)^{\kappa_2}dw$$
$$= (1 - e^{2\pi\sqrt{-1}\kappa_2})\int_{\gamma_p} K_i(w)(z-w)^{\kappa_2}dw + \int_{\gamma_z}(K_i^{\gamma_p}(w) - K_i(w))(z-w)^{\kappa_2}dw \quad (i=1,2)$$

is a solution to $D_Y(\kappa_2+\theta_0, \kappa_2+\theta_1, \kappa_2+\theta_t, \kappa_2+\theta_\infty; \tilde{\lambda}, \tilde{\mu}; k)$ for $p \in \{0,1,t\}$, where $K_i^{\gamma_p}(w)$ is the function analytically continued along the cycle γ_p.

If $\kappa_2 > -1$, then the integrals $\int_{\gamma_z}(K_i^{\gamma_p}(w) - K_i(w))(z-w)^{\kappa_2}dw$ tends to zero as the base point o of the integral tends to z, and it follows that the function $\begin{pmatrix} \int_{C_p} K_1(w)(z-w)^{\kappa_2}dw \\ \int_{C_p} K_2(w)(z-w)^{\kappa_2}dw \end{pmatrix}$ is a solution to $D_Y(\kappa_2+\theta_0, \kappa_2+\theta_1, \kappa_2+\theta_t, \kappa_2+\theta_\infty; \tilde{\lambda}, \tilde{\mu}; k)$ for $\kappa_2 \in \mathbb{R}_{>-1} \setminus \mathbb{Z}_{>-1}$. By considering the limit $\kappa_2 \to n$, $n \in \mathbb{Z}_{>0}$, we obtain (i) for $p \in \{0,1,t\}$. The case $p=\infty$ follows from $C_\infty = C_t^{-1}C_1^{-1}C_0^{-1}$.

By integration by parts as equation (4.9) we obtain (ii). ■

We have similar proposition for the case $\nu = \kappa_1$ and $\kappa_1 \in \mathbb{Z}_{>0}$. On middle convolution mc_ν for Fuchsian differential systems of arbitrary size and arbitrary number of regular singularities whose parameter ν is positive integer, the contour $[\gamma_z, \gamma_p]$ can be replaced by C_p.

We can reformulate Theorem 3 (resp. Theorem 4) for the case $k_2 \in \mathbb{Z}_{<0}$ (resp. $k_1 \in \mathbb{Z}_{<0}$) by changing the integral to successive differential and for the case $k_2 \in \mathbb{Z}_{>0}$ (resp. $k_1 \in \mathbb{Z}_{>0}$) by changing the contour $[\gamma_z, \gamma_p]$ to C_p. The corresponding setting for Heun's equation is described as follows:

Proposition 6. *Let $v(w)$ be a solution to Heun's equation written as equation* (5.2) *with the parameters in equation* (5.1)*.*

(i) If $\eta \in \mathbb{Z}_{>1}$, then the function

$$y(z) = (d/dz)^{\eta-1}v(z)$$

is a solution to Heun's equation (5.3)*.*

(ii) If $\eta \in \mathbb{Z}_{<1}$, then the function

$$y(z) = \int_{C_p} v(w)(z-w)^{-\eta}dw$$

is a solution to Heun's equation (5.3) *for $p \in \{0,1,t,\infty\}$.*

The generalized Darboux transformation (Crum–Darboux transformation) for elliptical representation of Heun's equation was introduced in [29], and we can show that Proposition 6 (i)

gives another description of the generalized Darboux transformation. Hence the integral transformation given by Theorem 2 can be regarded as a generalization of the generalized Darboux transformation to non-integer cases. Khare and Sukhatme [15] conjectured a duality of quasi-exactly solvable (QES) eigenvalues for elliptical representation of Heun's equation. By rewriting parameters of the duality to Heun's equation on the Riemann sphere, we obtain a correspondence on the parameters α, β, γ, δ, ϵ, q and α', β', γ', δ', ϵ', q' on the integral transformation of Heun's equation in Theorem 2. We will report further from a viewpoint of monodromy in a separated paper.

A Appendix

We investigate the realization of the Fuchsian system (equation (2.1)) for the cases $\lambda = 0, 1, t, \infty$ in the setting of Section 2 and observe relationships with the lines L_0, L_0^*, L_1, L_1^*, L_t, L_t^*, L_∞, L_∞^* (see equation (3.3)) in the space of initial conditions $E(t)$.

We consider the case $\lambda = 0$, i.e., the case $a_{12}^{(0)} = 0$, $a_{12}^{(1)} \neq 0$, $a_{12}^{(t)} \neq 0$, $(t+1)a_{12}^{(0)} + ta_{12}^{(1)} + a_{12}^{(t)} \neq 0$ (see equation (2.4)). Since $a_{12}^{(0)} = 0$ and the eigenvalues of A_0 are θ_0 and 0, the matrix A_0 is written as

$$A_0 = \begin{pmatrix} \theta_0 & 0 \\ v & 0 \end{pmatrix} \qquad \text{or} \qquad A_0 = \begin{pmatrix} 0 & 0 \\ v & \theta_0 \end{pmatrix},$$

and it follows from $a_{12}^{(1)} \neq 0$, $a_{12}^{(t)} \neq 0$ that the matrices A_1, A_t may be expressed as equation (2.5). We determine w_1, w_t so as to satisfy $a_{12}(z) = -w_1(z-1) - w_t/(z-t) = k/(z(z-1))$. Then we have

$$w_1 = k/(t-1), \qquad w_t = -k/(t-1). \tag{A.1}$$

On the case

$$A_0 = \begin{pmatrix} 0 & 0 \\ v & \theta_0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} u_1 + \theta_1 & -w_1 \\ u_1(u_1+\theta_1)/w_1 & -u_1 \end{pmatrix},$$
$$A_t = \begin{pmatrix} u_t + \theta_t & -w_t \\ u_t(u_t+\theta_t)/w_t & -u_t \end{pmatrix}, \tag{A.2}$$

we determine u_1, u_t so as to satisfy equation (2.2), namely

$$v + u_1(u_1+\theta_1)/w_1 + u_t(u_t+\theta_t)/w_t = 0,$$
$$u_1 + \theta_1 + u_t + \theta_t = -\kappa_1, \quad \theta_0 - u_1 - u_t = -\kappa_2.$$

We have

$$u_1 = -\theta_1 + \frac{1}{\theta_\infty - \theta_0}\left(\frac{kv}{t-1} - \kappa_1(\kappa_1 + \theta_t)\right),$$
$$u_t = -\theta_t - \frac{1}{\theta_\infty - \theta_0}\left(\frac{kv}{t-1} + \kappa_1(\kappa_1 + \theta_1)\right). \tag{A.3}$$

Hence we have one-parameter realization of equation (A.2) with the prescribed condition. We discuss relationship with the Fuchsian system on the line L_0. For this purpose, we recall matrices A_0, A_1, A_t determined by equations (2.5), (2.7), (2.8) and restrict them to $\lambda = 0$. Then all elements in A_0, A_1 and A_t are well-defined and we have

$$A_0\,|_{\lambda=0} = \begin{pmatrix} 0 & 0 \\ \theta_0\{t(\theta_0 - \theta_\infty)\mu + \kappa_1(\kappa_1 + \theta_1 + t(\kappa_1 + \theta_t))\}/(k\theta_\infty) & \theta_0 \end{pmatrix}.$$

In fact the matrices A_0, A_1, A_t restricted to the line L_0 coincide with the ones determined by equations (A.2), (A.1), (A.3) and $v = \theta_0\{t(\theta_0-\theta_\infty)\mu + \kappa_1(\kappa_1+\theta_1+t(\kappa_1+\theta_t))\}/(k\theta_\infty)$. Note that the second-order differential equation for the function y_1 on the case of the matrices in equations (A.2), (A.1), (A.3) is obtained as equation (3.4) by substituting $\mu = (k\theta_\infty v - \kappa_1\theta_0(\kappa_1+\theta_1+t(\kappa_1+\theta_t)))/(t\theta_0(\theta_0-\theta_\infty))$.

On the case

$$A_0 = \begin{pmatrix} \theta_0 & 0 \\ v & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} u_1+\theta_1 & -w_1 \\ u_1(u_1+\theta_1)/w_1 & -u_1 \end{pmatrix},$$
$$A_t = \begin{pmatrix} u_t+\theta_t & -w_t \\ u_t(u_t+\theta_t)/w_t & -u_t \end{pmatrix}, \tag{A.4}$$

u_1, u_t are determined as

$$u_1 = \frac{1}{\theta_\infty+\theta_0}\left(\frac{kv}{t-1} - \kappa_2(\kappa_2+\theta_t)\right), \; u_t = \frac{-1}{\theta_\infty+\theta_0}\left(\frac{kv}{t-1} + \kappa_2(\kappa_2+\theta_1)\right), \tag{A.5}$$

to satisfy equation (2.2). To realize the Fuchsian system on the line L_0^*, we recall matrices A_0, A_1, A_t determined by equations (2.5), (2.7), (2.8), transform $(\lambda,\mu)(=(q_0,p_0))$ to (q_1,p_1) by equation (3.2) and restrict matrix elements to $q_1 = 0$. Then the matrices A_0, A_1 and A_t are determined as equations (A.4), (A.1), (A.5), where

$$v = \{-t(\theta_0+\theta_\infty)q_1 + \theta_0(\kappa_1+\theta_0)((\kappa_2+\theta_t)+t(\kappa_2+\theta_1))\}/(k\theta_\infty). \tag{A.6}$$

Note that the second-order differential equation for the function $\tilde{y}_1 = z^{-1}y_1$ on the case of the matrices in equations (A.4), (A.1), (A.5) is obtained as equation (3.8) by substituting equation (A.6).

We consider the case $\lambda = 1$, i.e., the case $a_{12}^{(1)} = 0, a_{12}^{(0)} \neq 0, a_{12}^{(t)} \neq 0, (t+1)a_{12}^{(0)} + ta_{12}^{(1)} + a_{12}^{(t)} \neq 0$. Since $a_{12}^{(1)} = 0$ and the eigenvalues of A_1 are θ_1 and 0, the matrix A_1 is written as

$$A_1 = \begin{pmatrix} \theta_1 & 0 \\ v & 0 \end{pmatrix} \qquad \text{or} \qquad A_1 = \begin{pmatrix} 0 & 0 \\ v & \theta_1 \end{pmatrix},$$

and the matrices A_0, A_t may be expressed as equation (2.5). To satisfy $a_{12}(z) = -w_0/z - w_t/(z-t) = k/(z(z-t))$, we have

$$w_0 = k/t, \qquad w_t = -k/t. \tag{A.7}$$

On the case

$$A_1 = \begin{pmatrix} 0 & 0 \\ v & \theta_1 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} u_0+\theta_0 & -w_0 \\ u_0(u_0+\theta_0)/w_0 & -u_0 \end{pmatrix},$$
$$A_t = \begin{pmatrix} u_t+\theta_t & -w_t \\ u_t(u_t+\theta_t)/w_t & -u_t \end{pmatrix}, \tag{A.8}$$

u_0, u_t are determined as

$$u_0 = -\theta_0 + \frac{1}{\theta_\infty-\theta_1}\left(\frac{kv}{t} - \kappa_1(\kappa_1+\theta_t)\right),$$
$$u_t = -\theta_t - \frac{1}{\theta_\infty-\theta_1}\left(\frac{kv}{t} + \kappa_1(\kappa_1+\theta_0)\right), \tag{A.9}$$

to satisfy equation (2.2). To realize the Fuchsian system on the line L_1, we recall matrices A_0, A_1, A_t determined by equations (2.5), (2.7), (2.8) and restrict matrix elements to $\lambda = 1$. Then the matrices A_0, A_1 and A_t are determined as equations (A.8), (A.7), (A.9), where

$$v = \theta_1\{(1-t)(\theta_1 - \theta_\infty)\mu - \kappa_1(\kappa_1 + \theta_0 + (1-t)(\kappa_1 + \theta_t))\}/(k\theta_\infty). \tag{A.10}$$

Note that the second-order differential equation for the function y_1 on the case of the matrices in equations (A.8), (A.7), (A.9) is obtained as equation (3.5) by substituting equation (A.10).

On the case

$$A_1 = \begin{pmatrix} \theta_1 & 0 \\ v & 0 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} u_0 + \theta_0 & -w_0 \\ u_0(u_0+\theta_0)/w_0 & -u_0 \end{pmatrix},$$
$$A_t = \begin{pmatrix} u_t + \theta_t & -w_t \\ u_t(u_t+\theta_t)/w_t & -u_t \end{pmatrix}, \tag{A.11}$$

u_0, u_t are determined as

$$u_0 = \frac{1}{\theta_\infty + \theta_1}\left(\frac{kv}{t} - \kappa_2(\kappa_2 + \theta_t)\right), \qquad u_t = \frac{-1}{\theta_\infty + \theta_1}\left(\frac{kv}{t} + \kappa_2(\kappa_2 + \theta_0)\right), \tag{A.12}$$

to satisfy equation (2.2). To realize the Fuchsian system on the line L_1^*, we recall matrices A_0, A_1, A_t determined by equations (2.5), (2.7), (2.8), transform $(\lambda,\mu)(= (q_0,p_0))$ to (q_2,p_2) by equation (3.2) and restrict matrix elements to $q_2 = 0$. Then the matrices A_0, A_1 and A_t are determined as equations (A.11), (A.7), (A.12), where

$$v = \{(t-1)(\theta_1 + \theta_\infty)q_2 - \theta_1(\kappa_1 + \theta_1)((\kappa_2 + \theta_t) + (1-t)(\kappa_2 + \theta_0))\}/(k\theta_\infty). \tag{A.13}$$

Note that the second-order differential equation for the function $\tilde{y}_1 = (z-1)^{-1}y_1$ on the case of the matrices in equations (A.11), (A.7), (A.12) is obtained as equation (3.9) by substituting equation (A.13).

We consider the case $\lambda = t$, i.e., the case $a_{12}^{(t)} = 0$, $a_{12}^{(0)} \neq 0$, $a_{12}^{(1)} \neq 0$, $(t+1)a_{12}^{(0)} + ta_{12}^{(1)} + a_{12}^{(t)} \neq 0$. Then the matrix A_t is written as

$$A_t = \begin{pmatrix} \theta_t & 0 \\ v & 0 \end{pmatrix} \qquad \text{or} \qquad A_t = \begin{pmatrix} 0 & 0 \\ v & \theta_t \end{pmatrix},$$

and the matrices A_0, A_1 may be expressed as equation (2.5). To satisfy $a_{12}(z) = -w_0/z - w_1/(z-1) = k/(z(z-1))$, we have

$$w_0 = k, \qquad w_1 = -k. \tag{A.14}$$

On the case

$$A_t = \begin{pmatrix} 0 & 0 \\ v & \theta_t \end{pmatrix}, \qquad A_0 = \begin{pmatrix} u_0 + \theta_0 & -w_0 \\ u_0(u_0+\theta_0)/w_0 & -u_0 \end{pmatrix},$$
$$A_1 = \begin{pmatrix} u_1 + \theta_1 & -w_1 \\ u_1(u_1+\theta_1)/w_1 & -u_1 \end{pmatrix}, \tag{A.15}$$

u_0, u_1 are determined as

$$u_0 = -\theta_0 + \frac{1}{\theta_\infty - \theta_t}\left(kv - \kappa_1(\kappa_1 + \theta_1)\right),$$
$$u_1 = -\theta_1 - \frac{1}{\theta_\infty - \theta_t}\left(kv + \kappa_1(\kappa_1 + \theta_0)\right), \tag{A.16}$$

to satisfy equation (2.2). To realize the Fuchsian system on the line L_t, we recall matrices A_0, A_1, A_t determined by equations (2.5), (2.7), (2.8) and restrict matrix elements to $\lambda = t$. Then the matrices A_0, A_1 and A_t are determined as equations (A.15), (A.14), (A.16), where

$$v = \theta_t\{t(t-1)(\theta_t - \theta_\infty)\mu - \kappa_1(t(\kappa_1+\theta_0) + (t-1)(\kappa_1+\theta_1))\}/(k\theta_\infty). \tag{A.17}$$

Note that the second-order differential equation for the function y_1 on the case of the matrices in equations (A.15), (A.14), (A.16) is obtained as equation (3.6) by substituting equation (A.17).

On the case

$$A_t = \begin{pmatrix} \theta_t & 0 \\ v & 0 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} u_0+\theta_0 & -w_0 \\ u_0(u_0+\theta_0)/w_0 & -u_0 \end{pmatrix},$$
$$A_1 = \begin{pmatrix} u_1+\theta_1 & -w_1 \\ u_1(u_1+\theta_1)/w_1 & -u_1 \end{pmatrix}, \tag{A.18}$$

u_0, u_t are determined as

$$u_0 = \frac{1}{\theta_\infty+\theta_t}\left(kv - \kappa_2(\kappa_2+\theta_1)\right), \qquad u_1 = \frac{-1}{\theta_\infty+\theta_t}\left(kv + \kappa_2(\kappa_2+\theta_0)\right), \tag{A.19}$$

to satisfy equation (2.2). To realize the Fuchsian system on the line L_t^*, we recall matrices A_0, A_1, A_t determined by equations (2.5), (2.7), (2.8), transform $(\lambda,\mu)(=(q_0,p_0))$ to (q_3,p_3) by equation (3.2) and restrict matrix elements to $q_3 = 0$. Then the matrices A_0, A_1 and A_t are determined as equations (A.18), (A.14), (A.19), where

$$v = \{t(1-t)(\theta_t+\theta_\infty)q_3 - \theta_t(\kappa_1+\theta_t)(t(\kappa_2+\theta_1)+(t-1)(\kappa_2+\theta_0))\}/(k\theta_\infty). \tag{A.20}$$

Note that the second-order differential equation for the function $\tilde{y}_1 = (z-1)^{-1}y_1$ on the case of the matrices in equations (A.18), (A.14), (A.19) is obtained as equation (3.10) by substituting equation (A.20).

We consider the case $\lambda = \infty$, i.e., the case $a_{12}^{(0)} \neq 0$, $a_{12}^{(1)} \neq 0$, $a_{12}^{(t)} \neq 0, (t+1)a_{12}^{(0)} + ta_{12}^{(1)} + a_{12}^{(t)} = 0$. We can set A_0, A_1, A_t as equation (2.5) and we determine u_0, u_1, u_t, w_0, w_1, w_t to satisfy equation (2.2) and

$$a_{12}(z) = -\frac{w_0}{z} - \frac{w_1}{z-1} - \frac{w_t}{z-t} = \frac{k}{z(z-1)(z-t)}.$$

Then we have

$$w_0 = -k/t, \qquad w_1 = k/(t-1), \qquad w_t = -k/(t(t-1)), \tag{A.21}$$

and other relations are written as

$$u_0(u_0+\theta_0)/w_0 + u_1(u_1+\theta_1)/w_1 + u_t(u_t+\theta_t)/w_t = 0,$$
$$u_0+\theta_0+u_1+\theta_1+u_t+\theta_t = -\kappa_1, \qquad -u_0-u_1-u_t = -\kappa_2.$$

We solve the equation for u_0, u_1, u_t by adding one more relation $-(u_1+\theta_1+t(u_t+\theta_t)) = l$. We have

$$u_0 = -\theta_0 - \kappa_1 + \frac{\tilde{l}}{t\theta_\infty}, \qquad u_1 = -\theta_1 + \frac{\tilde{l}-\theta_\infty l}{(1-t)\theta_\infty}, \qquad u_t = -\theta_t + \frac{\tilde{l}-t\theta_\infty l}{t(t-1)\theta_\infty},$$
$$\tilde{l} = l^2 + (\theta_1+t\theta_t)l + t\kappa_1(\kappa_1+\theta_0). \tag{A.22}$$

The second-order differential equation for the function y_1 is written as

$$\frac{d^2y_1}{dz^2}+\left(\frac{1-\theta_0}{z}+\frac{1-\theta_1}{z-1}+\frac{1-\theta_t}{z-t}\right)\frac{dy_1}{dz}+\frac{\kappa_1(\kappa_2+2)z-q}{z(z-1)(z-t)}y_1=0,$$
$$q=l(\theta_\infty-1)+\kappa_1(t(\kappa_2+\theta_t+1)+\kappa_2+\theta_1+1).$$

To realize the Fuchsian system on the line L_∞, we recall matrices A_0, A_1, A_t determined by equations (2.5), (2.7), (2.8), transform $(\lambda,\mu)(=(q_0,p_0))$ to (q_∞,p_∞) by equation (3.2), replace k by $-kq_\infty$ and restrict matrix elements to $q_\infty=0$. Then the matrices A_0, A_1 and A_t are determined as equations (2.5), (A.21), (A.22), where

$$l=p_\infty.$$

We have observed that Fuchsian systems on the lines L_0, L_0^*, L_1, L_1^*, L_t, L_t^*, L_∞ are realized by Fuchsian systems for the case $\lambda=0,1,t,\infty$. Here the case L_∞^* is missing. In fact this case does not simply correspond with the Fuchsian system as equation (2.1), because we have

$$u_0=\frac{1}{t\theta_\infty}\frac{1}{p_4^2}+O\left(p_4^{-1}\right),\qquad u_1=\frac{1}{(1-t)\theta_\infty}\frac{1}{p_4^2}+O\left(p_4^{-1}\right),$$
$$u_t=\frac{1}{t(t-1)\theta_\infty}\frac{1}{p_4^2}+O\left(p_4^{-1}\right),$$

as $p_4\to 0$ in the coordinate (q_4,p_4) in equation (3.2), although we can restrict the second-order differential equation for y_1 to $p_4=0$ as equation (3.11).

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