Vector Fields on the Space of Functions Univalent Inside the Unit Disk via Faber Polynomials*

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Abstract. We obtain the Kirillov vector fields on the set of functions f univalent inside the unit disk, in terms of the Faber polynomials of $1/f(1/z)$. Our construction relies on the generating function for Faber polynomials.

Key words: vector fields; univalent functions; Faber polynomials

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1 Introduction

The Virasoro algebra has a representation in the tangent bundle over the space of functions univalent in the unit disk which are smooth on its boundary. This realization was obtained by A.A. Kirillov and D.V. Yur'ev [4, 5] as first-order differential operators. Following [4], consider $f(z)$ a holomorphic function univalent in the unit disc $D = \{z \in \mathbb{C}; |z| \leq 1\}$, smooth up to the boundary of the disc and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, thus

$$f(z) = z\left(1 + \sum_{n\geq 1} c_n z^n\right) \tag{1.1}$$

and the series (1.1) converges to $f(z)$ on D. By De Branges theorem proving the Bieberbach conjecture, the coefficients $(c_n)_{n\geq 1}$ lie in the infinite-dimensional domain $|c_n| < n+1$, for $n \geq 1$. In the following, we shall call this infinite domain the manifold of coefficients. On the other hand, we denote

$$g(z) = b_0 z + b_1 + \frac{b_2}{z} + \cdots + \frac{b_n}{z} + \cdots$$

a function univalent outside the unit disc. In order to study the representations of the Virasoro algebra [6], A.A. Kirillov considered the action of vector fields on the set of the diffeomorphisms of the circle by perturbing the equation $f \circ \gamma = g$, where γ is a diffeomorphism of the circle. He obtained a sequence of vector fields L_p, (p being positive or negative integer) acting on the set of functions univalent inside the unit disk. These vector fields are expressed as

$$L_{-p}f(z) = \frac{f(z)^2}{2i\pi} \int_{\partial D} \frac{t^2 f'(t)^2}{f(t)^2} \frac{1}{f(t) - f(z)} \frac{dt}{t^{p+1}} = \phi_p(z) + z^{1-p} f'(z) \qquad \forall\, p \in \mathbb{Z}, \tag{1.2}$$

where z is inside the unit disk and the integral is a contour integral over the unit circle. When $n > 0$ is a positive integer, the action of L_n is given by $L_n f(z) = z^{n+1} f'(z)$ since by evaluation

*This paper is a contribution to the Special Issue on Kac–Moody Algebras and Applications. The full collection is available at http://www.emis.de/journals/SIGMA/Kac-Moody_algebras.html of the contour integral, it is not difficult to see that $\phi_{-n}(z)$ vanishes in (1.2). The term $\phi_p(z)$ comes from the residue at the pole $t = 0$ and $z^{1-p}f'(z)$ comes from the residue at $t = z$. We have $\phi_0(z) = -f(z)$ and $L_0 f(z) = zf'(z) - f(z)$. When $p > 0$ the evaluation of the integral (1.2) is more delicate since we have a non-vanishing residue at zero. For $p \geq 0$, we find

$$
\begin{aligned}
L_{-1}f(z) &= f'(z) - 1 - 2c_1 f(z),\\
L_{-2}f(z) &= \frac{f'(z)}{z} - \frac{1}{f(z)} - 3c_1 - \left(4c_2 - c_1^2\right) f(z),\\
L_{-3}f(z) &= \frac{f'(z)}{z^2} - \frac{1}{f(z)^2} - \frac{4c_1}{f(z)} - \left(c_1^2 + 5c_2\right) - (6c_3 - 2c_1c_2) f(z),
\end{aligned}
$$

where the coefficients $(c_j)_{j\geq 1}$ are the coefficients of $f(z)$ given by (1.1). With the residue calculus, it has been made explicit in [1] that $\phi_p(z) = \Lambda_p(f(z))$ where $u \to \Lambda_p(u)$ is a function of the form

$$
\Lambda_p(u) = -\frac{1}{u^{p-1}} - \frac{(p+1)c_1}{u^{p-2}} - \cdots - a_p^p u.
$$

The coefficients of $\Lambda_p(u)$ depend on the $(c_j)_{j\geq 1}$ and have been calculated in [3, Proposition 3.2]. We note that $\phi_p(z)$ is obtained by eliminating the powers of z^n, $n \leq 1$ in $z^{1-p}f'(z)$; the elimination is done by expanding $z^{1-p}f'(z)$ in powers of $f(z)$, then substracting the part $\phi_p(z)$ of the series with powers $f(z)^n$, $n \leq 1$. This method is analogous to the elimination of terms in power series developed by Schiffer for Faber polynomials [8]. Let $h(z)$ be a univalent function holomorphic outside the unit disc except for a pole with residue equal to 1 at infinity, thus for $|z| > 1$,

$$
h(z) = z + b_1 + b_2\frac{1}{z} + \cdots + b_n\frac{1}{z^{n-1}} + \cdots .
$$

Let $t \in \mathbb{C}$, at a neighborhood of $z = \infty$, we have the expansion

$$
\frac{zh'(z)}{h(z) - t} = \sum_{n=0}^{\infty} F_n(t) z^{-n}.
$$

The function $F_n(t)$ is a polynomial of degree n in the variable t and is called the n^{th} Faber polynomial of the function h. Schiffer showed in [8] that the polynomial $F_n(t)$ is the unique polynomial in t of degree m such that

$$
F_m(h(z)) = z^m + \sum_{n=1}^{\infty} a_{mn} z^{-n}.
$$

The objective of this paper is to show that with a method analogous to that of M. Schiffer for obtaining Faber polynomials, we can recover Kirillov vector fields L_{-p} when $p \geq 0$. For this, let $f(z)$ be a univalent function as in (1.1), for $p \geq 0$, we start from $z^{1-p}f'(z)$, it expands in D as

$$
z^{1-p}f'(z) = z^{1-p} + \sum_{1\leq n\leq p+1} nc_n z^{n-p} + \text{terms in } z^n \qquad (n \geq 2).
$$

In Section 2, the function $\Lambda_p(u)$ for $p \geq 0$, is constructed in such a way that

$$
z^{1-p}f'(z) + \Lambda_p(f(z)) \tag{1.3}
$$

expands in powers z^n with $n \geq 2$. In fact, the function $\Lambda_p(t)$ with respect to $z^{1-p}f'(z)$ plays the same role as the Faber polynomials $F_m(t)$ with respect to $h(z)^m$. Then we prove that

$z^{1-p}f'(z)+\Lambda_p(f(z))$ is equal to the expression (1.2) of the vector field $L_{-p}f(z)$ found by Kirillov and Yur'ev.

Note that the method of elimination of terms in power series as developed in [8] for Faber polynomials is a formal calculation on series and does not require smoothness assumptions for $f(z)$ at the boundary of the unit disk. Thus, it is conceivable to extend the calculations of the vector fields for functions $f(z)$ which present a singularity at the boundary of D. This is our main motivation for adapting Schiffer's method to the formal series $z^{1-p}f'(z)$. The regularity assumptions on $f(z)$ are stronger in the case of the variational approaches developed in [4] or [7] since in the variational case, it is assumed that $f(z)$ is smooth up to the boundary of the unit disc.

Let $\Lambda_p(u)$ as in (1.3), we give an expression of $\Lambda_p(u)$, $p>0$, in terms of the Faber polynomials $F_n(w)$ of the function $h(z)=1/f(1/z)$. We have

$$\frac{\xi h'(\xi)}{h(\xi)-w}=\sum_{n=0}^{\infty}F_n(w)\xi^{-n},$$

where $F_n(w)$ are the Faber polynomials associated to the function h. In terms of f,

$$\frac{zf'(z)}{f(z)-wf(z)^2}=1+\sum_{n\geq 1}F_n(w)z^n, \tag{1.4}$$

$$F_1(w)=w+c_1, \qquad F_2(w)=w^2+2c_1w+2c_2-c_1^2,$$
$$F_3(w)=w^3+3c_1w^2+3c_2w+c_1^3-3c_1c_2+3c_3.$$

We find that the functions $u\to\Lambda_p(u)$ are determined by the expansion

$$\frac{\xi^2f'(\xi)^2}{f(\xi)^2}\frac{u^2}{f(\xi)-u}=\sum_{p\geq 0}\Lambda_p(u)\xi^p \tag{1.5}$$

at a neighborhood of $\xi=0$, and u is a complex number. For $p\geq 2$, we show that we can calculate the coefficient $\Lambda_p(u)$ of ξ^p in the expansion (1.5) as follows,

$$\Lambda_p(u)+a_p^pu=-T_{p-1}\left(\frac{1}{u}\right),$$

where

$$T_{p-1}(w)=F_{p-1}(w)+2c_1F_{p-2}(w)+3c_2F_{p-3}(w)+\cdots+(p-1)c_{p-2}F_1(w)+pc_{p-1}$$

is determined by

$$\frac{zf'(z)^2}{f(z)-wf(z)^2}=1+\sum_{n\geq 1}T_n(w)z^n$$

and a_p^p,

$$\frac{z^2f'(z)^2}{f(z)^2}=1+\sum_{p\geq 1}a_p^pz^p. \tag{1.6}$$

Then, we recover (1.2) as Corollary, see (2.21). In Section 3, we put

$$z^{1-p}f'(z)+\Lambda_p(f(z))=\sum_{n\geq 1}A_n^pz^{n+1}. \tag{1.7}$$

We prove that for any u and v in the unit disc, there holds

$$\sum_{k\geq 1}\sum_{p\geq 0} A_k^p u^p v^k = \frac{u^2 f'(u)^2}{f(u)^2}\frac{f(v)^2}{v[f(u)-f(v)]} + \frac{vf'(v)}{v-u}. \tag{1.8}$$

We say that the right hand side of (1.8) is a generating function for the homogeneous polynomials A_k^p. Note that the right hand side in (1.8) has a meaning when $u \to v$. All series obtained as expansions of a function are convergent inside their disc of convergence which is determined by the singularities of the function. In Section 4, we identify as in [4] the vector fields $(L_k)_{k\geq 1}$ with first order differential operators on the manifold of coefficients of functions univalent on D; as quoted before, this manifold comes from De Branges theorem, the coefficients $(c_n)_{n\geq 1}$ lie in the infinite-dimensional domain $|c_n| < n+1$, for $n \geq 1$. Some properties of this infinite-dimensional manifold have been investigated in [4, 5], for example Kähler structure or Ricci curvature. Here, we shall not develop the properties of this manifold. We only examine the action of the $(L_k)_{k\geq 1}$ on the functions $\Lambda_p(u)$. The functions $\Lambda_p(u)$ have their coefficients in this manifold. We find that $L_k(\Lambda_{p+k}(u)) = (2k+p)\Lambda_p(u)$ for $p \geq 1$. In Section 5, we consider the reverse series $f^{-1}(z)$ of $f(z)$, i.e. $f^{-1}\circ f(z) = z$. We prove that for $k > 0$, $L_k[f^{-1}(z)] = -[f^{-1}(z)]^{k+1}$ and

$$L_k\big[\big(f^{-1}(z)\big)^{-k}\big] = k \qquad \text{if} \quad k \geq 1,$$

thus the coefficients in the expansion of $1/[f^{-1}(z)]^k$ in powers z^n, are vectors $v(c_1, c_2, \dots)$ solution of $L_k(v) = 0$. On the other hand for $p \geq 1$, there holds

$$L_{-p}\big[\big(f^{-1}(z)\big)^p\big] = -p - \Lambda_p(z)\frac{d}{dz}\big[f^{-1}(z)\big]^p.$$

2 Elimination of terms in power series and the vector fields $L_{-p}f(z)$ in terms of the Faber polynomials of $1/f(1/z)$

Let $f(z)$ as in (1.1), there exists a unique sequence of rational functions $(\Lambda_p)_{p\geq 0}$ of the form

$$\Lambda_p(u) = \alpha_0 u + \alpha_1 + \frac{\alpha_2}{u} + \cdots - \frac{1}{u^{p-1}}, \tag{2.1}$$

such that

$$z^{1-p}f'(z) = -\Lambda_p[f(z)] + \text{series of terms in } z^k, \qquad k \geq 2. \tag{2.2}$$

To prove the existence of $\Lambda_p(u)$, we expand $z^{1-p}f'(z)$ in powers of $f(z)$. Then $-\Lambda_p(f(z))$ is the sum of terms with powers of $f(z)^n$ such that $n \leq 1$. The unicity of the function $\Lambda_p(u)$ satisfying (2.1), (2.2) results from matching equal powers of z in the expansions (2.1), (2.2). We can calculate directly

$$\Lambda_0(w) = -w, \qquad \Lambda_1(w) = -1 - 2c_1 w, \qquad \Lambda_2(w) = -\frac{1}{w} - 3c_1 - \big(4c_2 - c_1^2\big)w,$$
$$\Lambda_3(w) = -\frac{1}{w^2} - 4c_1\frac{1}{w} - \big(c_1^2 + 5c_2\big) - (6c_3 - 2c_1c_2)w, \qquad \dots.$$

For $p = 0$, we have

$$f(z) = z + c_1 z^2 + \cdots, \qquad zf'(z) = z + 2c_1 z^2 + 3c_2 z^3 + \cdots,$$

thus

$$zf'(z) - f(z) = c_1 z^2 + 2c_2 z^3 + \cdots = L_0[f(z)]$$

and $\Lambda_0(u) = -u$.

For $p = 1$,

$$f'(z) = 1 + 2c_1 z + 3c_2 z^2 + \cdots ,$$

then

$$f'(z) - 1 - 2c_1 f(z) = \big(3c_2 - 2c_1^2\big) z^2 + \cdots = L_{-1}[f(z)].$$

We obtain $\Lambda_1(u) = -1 - 2c_1 u$.

For $p = 2$,

$$\frac{f'(z)}{z} = \frac{1}{z} + 2c_1 + 3c_2 z + 4c_3 z^2 + \cdots ,$$

then

$$\frac{f'(z)}{z} - \frac{1}{f(z)} = 3c_1 + (3c_2 - G_2) z + \cdots \qquad \text{with} \quad G_2 = c_1^2 - c_2,$$

$$\frac{f'(z)}{z} - \frac{1}{f(z)} - 3c_1 - (3c_2 - G_2) f(z) = \text{coefficients} \cdot z^2 + \cdots .$$

This gives $\Lambda_2(u) = -\frac{1}{u} - 3c_1 - (3c_2 - G_2)u$. In the following theorem, we prove (1.5).

Theorem 1. *$\Lambda_p(w)$ is expressed in terms of the Faber polynomials of $h(z) = \frac{1}{f(1/z)}$; we have*

$$\Lambda_p(u) + a_p^p u = -T_{p-1}\left(\frac{1}{u}\right), \tag{2.3}$$

$$T_{p-1}(w) = F_{p-1}(w) + 2c_1 F_{p-2}(w) + 3c_2 F_{p-3}(w) + \cdots + (p-1)c_{p-2} F_1(w) + p c_{p-1}, \tag{2.4}$$

$$\frac{z f'(z)^2}{f(z) - w f(z)^2} = 1 + \sum_{n \geq 1} T_n(w) z^n, \tag{2.5}$$

$$\frac{z^2 f'(z)^2}{f(z)^2} = 1 + \sum_{p \geq 1} a_p^p z^p. \tag{2.6}$$

The functions $(\Lambda_p(w))_{p \geq 0}$ are given by

$$\frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \frac{u^2}{f(\xi) - u} = \sum_{p \geq 0} \Lambda_p(u) \xi^p. \tag{2.7}$$

Proof. Consider the function

$$h(z) = \frac{1}{f(\frac{1}{z})} = z - c_1 + \big(c_1^2 - c_2\big)\frac{1}{z} + \big(2c_1c_2 - c_3 - c_1^3\big)\frac{1}{z^2}$$
$$+ \big(2c_1c_3 - c_4 + c_2^2 - 3c_1^2 c_2 + c_1^4\big)\frac{1}{z^3} + \cdots .$$

As in [8], we have

$$\frac{\xi h'(\xi)}{h(\xi) - w} = \sum_{n=0}^{\infty} F_n(w) \xi^{-n},$$

where $F_n(w)$ are the Faber polynomials associated to the function h. Since $h(z) = 1/f(1/z)$, we have (1.4). If we take the derivative of (1.4) with respect to w and then integrate with respect to z, we obtain

$$\frac{f(z)}{(1 - w f(z))} = \sum_{n \geq 1} F_n'(w) \frac{z^n}{n}.$$

Moreover

$$F_n(h(z)) = z^n + \sum_{k=1}^{\infty} \beta_{n,k} z^{-k},$$

where the $\beta_{n,k}$ are the Grunsky coefficients of h, see [8]. In terms of $f(z)$,

$$K(u,v) = \log \frac{\frac{1}{f(u)} - \frac{1}{f(v)}}{\frac{1}{u} - \frac{1}{v}} = -\sum_{n\geq 1}\sum_{k\geq 1} \frac{1}{n}\beta_{n,k} u^n v^k. \tag{2.8}$$

Because of the symmetry in (u,v) of the left hand side in (2.8), we see that

$$\frac{1}{n}\beta_{n,k} = \frac{1}{k}\beta_{k,n}. \tag{2.9}$$

Thus for $n \geq 1$,

$$F_n\left(\frac{1}{f(z)}\right) = z^{-n} + \sum_{k=1}^{\infty} \beta_{n,k} z^k. \tag{2.10}$$

We rewrite (2.10) as

$$z^{-n} = F_n\left(\frac{1}{f(z)}\right) - \sum_{k=1}^{\infty} \beta_{n,k} z^k.$$

On the other hand, if $p > 1$,

$$\begin{aligned} z^{1-p} f'(z) &= z^{1-p}(1 + 2c_1 z + 3c_2 z^2 + \cdots + (n+1)c_n z^n + \cdots \\ &= \frac{1}{z^{p-1}} + \frac{2c_1}{z^{p-2}} + \frac{3c_2}{z^{p-3}} + \cdots + \frac{(p-1)c_{p-2}}{z} \\ &\quad + pc_{p-1} + (p+1)c_p z + \sum_{k\geq 1} (p+k+1)c_{p+k} z^{k+1}. \end{aligned} \tag{2.11}$$

We replace in (2.11) the negative powers of z by their expressions given in (2.10). We obtain

$$\begin{aligned} z^{1-p} f'(z) &= F_{p-1}\left(\frac{1}{f(z)}\right) + 2c_1 F_{p-2}\left(\frac{1}{f(z)}\right) + 3c_2 F_{p-3}\left(\frac{1}{f(z)}\right) + \cdots \\ &\quad + (p-1)c_{p-2} F_1\left(\frac{1}{f(z)}\right) + pc_{p-1} + (p+1)c_p z \\ &\quad - \left[\beta_{p-1,1} + 2c_1\beta_{p-2,1} + \cdots + (p-1)c_{p-2}\beta_{1,1}\right] z \\ &\quad + \sum_{k\geq 1} \left[(p+k+1)c_{p+k} - \left[\beta_{p-1,k+1} + 2c_1\beta_{p-2,k+1} + \cdots \right.\right. \\ &\quad \left.\left. + (p-1)c_{p-2}\beta_{1,k+1}\right]\right] z^{k+1}. \end{aligned} \tag{2.12}$$

For $p \geq 2$, we consider

$$T_{p-1}(w) = F_{p-1}(w) + 2c_1 F_{p-2}(w) + 3c_2 F_{p-3}(w) + \cdots + (p-1)c_{p-2} F_1(w) + pc_{p-1}.$$

From the expansion of $f'(z)$ and (1.4), we obtain

$$\frac{z f'(z)^2}{f(z) - w f(z)^2} = 1 + \sum_{n\geq 1} T_n(w) z^n,$$

$$T_0(w) = 1, \qquad T_1(w) = w + 3c_1, \qquad T_2(w) = w^2 + 4c_1 w + \left(c_1^2 + 5c_2\right),$$
$$T_3(w) = w^3 + 5c_1 w^2 + \left(4c_1^2 + 6c_2\right)w - c_1^3 + 4c_1c_2 + 7c_3, \qquad \ldots.$$

We write (2.12) as

$$\begin{aligned} z^{1-p} f'(z) - T_{p-1}\left(\frac{1}{f(z)}\right) &= \sum_{k\geq 1} B^p_{k-1} z^k \\ &= (p+1)c_p z - \left[\beta_{p-1,1} + 2c_1\beta_{p-2,1} + \cdots + (p-1)c_{p-2}\beta_{1,1}\right] z + \sum_{k\geq 1} B^p_k z^{k+1} \end{aligned} \tag{2.13}$$

with

$$B^p_k = (p+k+1)c_{p+k} - \left[\beta_{p-1,k+1} + 2c_1\beta_{p-2,k+1} + \cdots + (p-1)c_{p-2}\beta_{1,k+1}\right]. \tag{2.14}$$

At this step we have eliminated all the negative powers and the constant term in such a way that the series on the right side of (2.13) has only terms in z^n with $n \geq 1$. To eliminate the term in z in order to have only terms in z^n, $n \geq 2$, we put for $p \geq 1$,

$$z^{1-p} f'(z) - T_{p-1}\left(\frac{1}{f(z)}\right) - a^p_p f(z) = \sum_{k\geq 1} A^p_k z^{k+1}. \tag{2.15}$$

From (2.13), we see that the coefficients of z, a^p_p are determined with $a^1_1 = 2c_1$ and if $p > 1$,

$$a^p_p = -\left[\beta_{p-1,1} + 2c_1\beta_{p-2,1} + \cdots + (p-1)c_{p-2}\beta_{1,1}\right] + (p+1)c_p. \tag{2.16}$$

For $p \geq 1$, $k \geq 1$, we put

$$A^p_k = B^p_k - a^p_p c_k. \tag{2.17}$$

With the convention $c_0 = 1$, $B^p_0 = a^p_p$, we have $A^p_0 = 0$. Now, we prove that (a^p_p) is given by (1.6) or (2.6). For this, we consider (2.10) with $n = 1$, it gives

$$F_1\left(\frac{1}{f(z)}\right) = \frac{1}{z} + \sum_{k\geq 1} \beta_{1,k} z^k = \frac{1}{f(z)} + c_1. \tag{2.18}$$

Because of the symmetry (2.9) in the Grunsky coefficients, taking the derivative with respect to z in (2.18), we obtain,

$$-\frac{1}{z^2} + \sum_{k\geq 1} \beta_{k,1} z^{k-1} = -\frac{f'(z)}{f(z)^2}. \tag{2.19}$$

Then we multiply the power series $f'(z) = 1 + 2c_1 z + 3c_2 z^2 + \cdots$ by the power series in (2.19), it gives (2.16) on one side and (1.6) or (2.6) on the other side. To prove (1.5) or (2.7), we put

$$M_p(w) = -T_{p-1}(w) - a^p_p \frac{1}{w} \qquad \text{for} \quad p \geq 1 \qquad \text{and} \qquad M_0(w) = -\frac{1}{w},$$

then with (2.5) and (2.6),

$$\sum_{p\geq 0} M_p(w)\xi^p = -\frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \frac{1}{w(1 - wf(\xi))}$$

and from (2.15), we see that

$$z^{1-p} f'(z) + M_p\left(\frac{1}{f(z)}\right) = \sum_{n=1}^{\infty} A^p_n z^{n+1}.$$

We put $\Lambda_p(w) = M_p(\frac{1}{w})$. We obtain (2.7) or (1.5). ∎

Corollary 1. *Let $\phi_p(z) = \Lambda_p(f(z))$, then*

$$\sum_{p\geq 0} \phi_p(z)\xi^p = \frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \frac{f(z)^2}{(f(\xi) - f(z))} \tag{2.20}$$

and

$$\frac{f(z)^2}{2i\pi} \int \frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \frac{1}{(f(\xi) - f(z))} \frac{d\xi}{\xi^{p+1}} = \phi_p(z) + z^{1-p} f'(z). \tag{2.21}$$

Proof. (2.20) is the immediate consequence of (1.5). To prove (2.21), we see that $\phi_p(z)$ is the coefficient of ξ^p in the expansion of $\frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \frac{f(z)^2}{(f(\xi)-f(z))}$ in powers of ξ. We calculate the contour integral in (2.21) with the residue method. The term $z^{1-p} f'(z)$ comes from the residue at $\xi = z$. ■

3 The generating function for the A_n^p

We put, see (1.7),

$$z^{1-p} f'(z) + \Lambda_p(f(z)) = \sum_{n\geq 1} A_n^p z^{n+1}.$$

With (2.17), (2.14) and (1.6), we have obtained A_n^p explicitly in terms of the Grunsky coefficients $\beta_{n,k}$ of $h(z) = 1/f(1/z)$ and in terms of the coefficients $(c_j)_{j\geq 1}$ of $f(z)$. In this section, we prove that A_n^p are given by (1.8). We consider for $|\xi| < |z|$ the series

$$\sum_{p\geq 0} z^{1-p} f'(z)\xi^p = z f'(z) \sum_{p\geq 0} \left(\frac{\xi}{z}\right)^p = \frac{z^2 f'(z)}{z - \xi} \tag{3.1}$$

and, see (1.5),

$$\sum_{p\geq 0} \Lambda_p(f(z))\xi^p = \frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \frac{f(z)^2}{f(\xi) - f(z)}. \tag{3.2}$$

Adding (3.1) and (3.2), then dividing by z, we find for $|\xi| < |z|$,

$$\sum_{p\geq 0} \sum_{k\geq 1} A_k^p z^k \xi^p = \frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \frac{f(z)^2}{z(f(\xi) - f(z))} + \frac{z f'(z)}{z - \xi},$$

which is (1.8) for $|u| < |v|$. Below, we prove that (1.8) is true for any u and v in the unit disk.

Theorem 2. *The polynomials A_k^p defined by* (1.7) *satisfy* (1.8).

Proof. Taking the derivative of (2.8) with respect to u yields

$$\frac{u f'(u)}{f(u)} \frac{f(v)}{f(v) - f(u)} - \frac{v}{v - u} = \sum_{n\geq 1} \sum_{k\geq 1} \beta_{n,k} u^n v^k. \tag{3.3}$$

Multiplying (3.3) by $f'(u)$, we deduce that

$$\frac{u f'(u)^2}{f(u)} \frac{f(v)}{f(v) - f(u)} - \frac{v f'(u)}{v - u} = \sum_{n\geq 1} \sum_{k\geq 1} \left[\beta_{n,k} + 2c_1\beta_{n-1,k} + \cdots + nc_{n-1}\beta_{1,k}\right] u^n v^k. \tag{3.4}$$

Consider the homogeneous polynomials B_k^p given by (2.14) for $p>1$ and $B_k^1=(k+2)c_{k+1}$. We rewrite (3.4) as

$$\sum_{k\geq 0}\sum_{p\geq 2}\left(B_k^p-(p+k+1)c_{p+k}\right)u^{p-1}v^{k+1}=-\frac{uf'(u)^2}{f(u)}\frac{f(v)}{f(v)-f(u)}+\frac{vf'(u)}{v-u}. \tag{3.5}$$

Moreover, since $B_k^1=(k+2)c_{k+1}$, we can write the sum in (3.5), starting from $p=1$. On the other hand,

$$-\frac{vf'(u)}{v-u}+\frac{vf'(v)}{v-u}=\frac{v}{v-u}(f'(v)-f'(u))=v\sum_{k\geq 1}(k+1)c_k\frac{v^k-u^k}{v-u}$$

thus

$$\sum_{k\geq 0}\sum_{p\geq 1}(p+k+1)c_{p+k}u^{p-1}v^{k+1}=-\frac{vf'(u)}{v-u}+\frac{vf'(v)}{v-u}.$$

Finally, we obtain

$$\sum_{k\geq 0}\sum_{p\geq 1}B_k^pu^pv^{k+1}=-\frac{u^2f'(u)^2}{f(u)}\frac{f(v)}{f(v)-f(u)}+\frac{uvf'(v)}{v-u}.$$

Since $A_k^p=B_k^p-a_p^pc_k$ as in (2.17), $A_0^p=0$ and since (2.6) is true, we have

$$\sum_{k\geq 1}\sum_{p\geq 1}A_k^pu^pv^{k+1}=\frac{u^2f'(u)^2}{f(u)^2}\frac{f(v)^2}{f(u)-f(v)}+f(v)+\frac{uv}{v-u}f'(v).$$

We divide by v. Since

$$\sum_{k\geq 1}\sum_{p\geq 0}A_k^pu^pv^k=\sum_{k\geq 1}\sum_{p\geq 1}A_k^pu^pv^k+\sum_{k\geq 1}A_k^0v^k$$

and

$$\sum_{k\geq 1}A_k^0v^k=\sum_{k\geq 0}kc_kv^k=f'(v)-\frac{f(v)}{v},$$

we obtain (1.8). ■

4 The differential operators $(L_k)_{k\geq 1}$ on the functions $\Lambda_p(u)$

We identify the set of functions univalent on the unit disk with the set of their coefficients via the map

$$f(z)=z\left(1+\sum_{n\geq 1}c_nz^n\right)\quad\rightarrow\quad(c_1,c_2,\ldots,c_n,\ldots).$$

For $k\geq 1$, we put $\partial_k=\frac{\partial}{\partial c_k}$. Following [4], we consider the partial differential operators

$$L_k=\partial_k+\sum_{n=1}^{\infty}(n+1)c_n\partial_{n+k}. \tag{4.1}$$

We have $z^{1+k}f'(z)=L_k[f(z)]$ and $\partial_nL_j=L_j\partial_n+(n+1)\partial_{n+j}$. We put

$$L_0=\sum_{n\geq 1}nc_n\partial_n\qquad\text{and for}\quad k\geq 1\qquad L_{-k}=\sum_{n\geq 1}A_n^k\partial_n.$$

The operators ∂_k are related to the $(L_{k+p})_{p\geq 0}$ as follows, see [2],

Lemma 1. *For $k \geq 1$,*

$$\partial_k = L_k - 2c_1 L_{k+1} + \left(4c_1^2 - 3c_2\right) L_{k+2} + \cdots + B_n L_{k+n} + \cdots, \tag{4.2}$$

where the $(B_n)_{n\geq 1}$ are independent of k. We calculate the B_n, $n \geq 0$, with

$$\frac{1}{f'(z)} = 1 + \sum_{n\geq 1} B_n z^n.$$

Proof. We verify (4.2) on $f(z)$. We have $L_k[f(z)] = z^{k+1} f'(z)$. Since

$$\partial_k[f(z)] = z^{k+1} \qquad \text{and} \qquad \partial_k[f'(z)] = (k+1)z^k,$$

we have to prove

$$z^{k+1} = z^{k+1} f'(z) - 2c_1 z^{k+2} f'(z) + \cdots + B_n z^{k+n} f'(z).$$

We divide by z^{k+1} and we obtain (4.2). ■

By considering expansions as in [1], we obtain with (1.5), the action of $(L_k)_{k\geq 1}$ on the functions $(\Lambda_p(u))_{p\geq 0}$.

Theorem 3. *Let $\Lambda_p(u)$ as in* (1.5)*, then $L_k(\Lambda_n(u)) = 0$ if $1 \leq n < k$, $L_k(\Lambda_k(u)) = -2ku$ for $k \geq 1$,*

$$L_k(\Lambda_{p+k}(u)) = (2k+p)\Lambda_p(u) \qquad \text{for} \quad p \geq 0 \quad \text{and} \quad k \geq 1. \tag{4.3}$$

Rremark. The functions $\Lambda_p(u)$ are of the form (see also [1, (A.1.7)] and [3, Proposition 3.2])

$$\Lambda_p(u) = -\left[\frac{1}{u^{p-1}} + (p+1)c_1 \frac{1}{u^{p-2}} + \cdots + \frac{(p+n)c_n + \gamma_n(p)}{u^{p-n-1}} + \cdots + (2pc_p + \gamma_p(p))u\right],$$

where for $2 \leq n \leq p$, $\gamma_n(p)(c_1, c_2, \ldots, c_{n-1})$ are homogeneous polynomials of degree n in the variables $(c_1, c_2, \ldots, c_{n-1})$ with c_j having weight j. With $k = 1$ in (4.3), we have

$$\left[\frac{\partial}{\partial c_1} + 2c_1 \frac{\partial}{\partial c_2} + 3c_2 \frac{\partial}{\partial c_3} + \cdots\right] (\Lambda_{p+1}(u)) = (p+2)\Lambda_p(u). \tag{4.4}$$

Since $\Lambda_0(u) = -u$, one can calculate the sequence $(\Lambda_p(u))_{p\geq 1}$ recursively by identifying equal powers of u in (4.4).

5 The $(L_k)_{k\geq 1}$ and the reverse series of $f(z)$

As in Section 4, we consider the differential operators $(L_k)_{k\geq 1}$ given by (4.1). We denote $f^{-1}(z)$ the inverse function of $f(z)$, ($f^{-1} \circ f = \text{Identity}$), we say also reverse series of $f(z)$. For any integer q, consider the series

$$\left(f^{-1}(z)\right)^q = z^q \left(1 + \sum_{n\geq 1} \delta_n^q z^n\right),$$

where δ_n^q are homogeneous polynomials in the variables $(c_1, c_2, \ldots, c_n, \ldots)$, the coefficients of $f(z)$. Then

$$L_p\left[\left(f^{-1}(z)\right)^q\right] = z^q \sum_{n\geq 1} L_p\left[\delta_n^q\right] z^n.$$

Theorem 4. *Let $f^{-1}(z)$ be the reverse series of $f(z)$, then*

$$L_k\big[f^{-1}(z)\big] = -\big[f^{-1}(z)\big]^{k+1} \qquad \text{for} \quad k \geq 1, \tag{5.1}$$

$$L_0\big[f^{-1}(z)\big] = -f^{-1}(z) + z\big(f^{-1}\big)'(z),$$

$$L_{-1}\big[f^{-1}(z)\big] = -1 + (1 + 2c_1 z)\big(f^{-1}\big)'(z),$$

$$L_{-p}\big[f^{-1}(z)\big] = -\big[f^{-1}(z)\big]^{1-p} - \Lambda_p(z)\big(f^{-1}\big)'(z) \qquad \text{for} \quad p \geq 2. \tag{5.2}$$

In particular there exists a unique rational function of z, which is $\Lambda_p(z)$, such that $[f^{-1}(z)]^{1-p} + \Lambda_p(z)(f^{-1})'(z)$ expands in a Taylor series $\sum_{n\geq 2} a_n z^n$ with powers z^n, $n \geq 2$. Moreover

$$L_k\big[\big(f^{-1}(z)\big)^{-k}\big] = k \qquad \text{for} \quad k \geq 1, \tag{5.3}$$

$$L_{-p}\big[\big(f^{-1}(z)\big)^{p}\big] = -p - \Lambda_p(z) \times \frac{d}{dz}\big[f^{-1}(z)\big]^p \qquad \text{for} \quad p \geq 1. \tag{5.4}$$

Proof. $f \circ f^{-1}(z) = z$, differentiating with the vector field L_k,

$$(L_k f)\big(f^{-1}(z)\big) + f'\big(f^{-1}(z)\big) L_k\big[f^{-1}(z)\big] = 0.$$

For $k \geq 1$,

$$L_k[f(z)] = z^{1+k} f'(z),$$

thus

$$L_k f\big(f^{-1}(z)\big) = f'\big(f^{-1}(z)\big)\big[f^{-1}(z)\big]^{1+k}$$

and this gives (5.1). Then we obtain (5.3) because

$$L_k\big[\big(f^{-1}(z)\big)^{-k}\big] = -k f^{-1}(z)^{-k-1} L_k\big[f^{-1}(z)\big].$$

Similarly,

$$L_{-p} f(z) = z^{1-p} f'(z) + \Lambda_p(f(z))$$

for $p \geq 0$, we find (5.2) and we deduce (5.4) from

$$L_{-k}\big[\big(f^{-1}(z)\big)^p\big] = -p\big(f^{-1}(z)\big)^{p-k} - \Lambda_k(z)\frac{d}{dz}\big(f^{-1}(z)\big)^p. \qquad \blacksquare$$

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