Symmetry, Integrability and Geometry: Methods and Applications SIGMA **5** (2009), 024, 11 pages

Bäcklund Transformations for First and Second Painlevé Hierarchies

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Received November 25, 2008, in final form February 24, 2009; Published online March 02, 2009
doi:10.3842/SIGMA.2009.024

Abstract. We give Bäcklund transformations for first and second Painlevé hierarchies. These Bäcklund transformations are generalization of known Bäcklund transformations of the first and second Painlevé equations and they relate the considered hierarchies to new hierarchies of Painlevé-type equations.

Key words: Painlevé hierarchies; Bäcklund transformations

2000 Mathematics Subject Classification: 34M55; 33E17

1 Introduction

One century ago Painlevé and Gambier have discovered the six Painlevé equations, PI–PVI. These equations are the only second-order ordinary differential equations whose general solutions can not be expressed in terms of elementary and classical special functions; thus they define new transcendental functions. Painlevé transcendental functions appear in many areas of modern mathematics and physics and they paly the same role in nonlinear problems as the classical special functions play in linear problems.

In recent years there is a considerable interest in studying hierarchies of Painlevé equations. This interest is due to the connection between these hierarchies of Painlevé equations and completely integrable partial differential equations. A Painlevé hierarchy is an infinite sequence of nonlinear ordinary differential equations whose first member is a Painlevé equation. Airault [1] was the first to derive a Painlevé hierarchy, namely a second Painlevé hierarchy, as the similarity reduction of the modified Korteweg–de Vries (mKdV) hierarchy. A first Painlevé hierarchy was given by Kudryashov [2]. Later on several hierarchies of Painlevé equations were introduced [3, 4, 5, 6, 7, 8, 9, 10, 11].

As it is well known, Painlevé equations possess Bäcklund transformations; that is, mappings between solutions of the same Painlevé equation or between solutions of a particular Painlevé equation and other second-order Painlevé-type equations. Various methods to derive these Bäcklund transformations can be found for example in [12, 13, 14, 15]. Bäcklund transformations are nowadays considered to be one of the main properties of integrable nonlinear ordinary differential equations, and there is much interest in their derivation.

In the present article, we generalize known Bäcklund transformations of the first and second Painlevé equations to the first and second Painlevé hierarchies given in [6, 11]. We give a Bäcklund transformation between the considered first Painlevé hierarchy and a new hierarchy of Painlevé-type equations. In addition, we give two new hierarchies of Painlevé-type equations related, via Bäcklund transformations, to the considered second Painlevé hierarchy. Then we derive auto-Bäcklund transformations for this second Painlevé hierarchy. Bäcklund transformations of the second Painlevé hierarchy have been studied in [6, 16].

2 Bäcklund transformations for PI hierarchy

In this section, we will derive a Bäcklund transformation for the first Painlevé hierarchy (PI hierarchy) [6]

$$\sum_{j=2}^{n+1} \gamma_j L^j[u] = \gamma x, \tag{2.1}$$

where the operator $L^j[u]$ satisfies the Lenard recursion relation

$$D_x L^{j+1}[u] = \big(D_x^3 - 4uD_x - 2u_x\big) L^j[u], \qquad L^1[u] = u. \tag{2.2}$$

The special case $\gamma_j = 0$, $2 \le j \le n$, of this hierarchy is a similarity reduction of the Schwarz–Korteweg–de Vries hierarchy [2, 4]. Moreover its members may define new transcendental functions.

The PI hierarchy (2.1) can be written in the following form [11]

$$\mathcal{R}_{\mathrm{I}}^n u + \sum_{j=2}^{n} \kappa_j \mathcal{R}_{\mathrm{I}}^{n-j} u = x, \tag{2.3}$$

where $\mathcal{R}_{\mathrm{I}}$ is the recursion operator

$$\mathcal{R}_{\mathrm{I}} = D_x^2 - 8u + 4D_x^{-1} u_x.$$

In [17, 18], it is shown that the Bäcklund transformation

$$u = -y_x, \qquad y = \tfrac{1}{2}\big(u_x^2 - 4u^3 - 2xu\big), \tag{2.4}$$

defines a one-to-one correspondence between the first Painlevé equation

$$u_{xx} = 6u^2 + x. \tag{2.5}$$

and the SD-I.e equation of Cosgrove and Scoufis [17]

$$y_{xx}^2 = -4y_x^3 - 2(xy_x - y). \tag{2.6}$$

We will show that there is a generalization of this Bäcklund transformation to all members of the PI hierarchy (2.3). Let

$$y = -xu + D_x^{-1} u_x \left[\mathcal{R}_{\mathrm{I}}^n u + \sum_{j=2}^{n} \kappa_j \mathcal{R}_{\mathrm{I}}^{n-j} u\right]. \tag{2.7}$$

Differentiating (2.7) and using (2.3), we find

$$u = -y_x. \tag{2.8}$$

Substituting $u = -y_x$ into (2.7), we obtain the following hierarchy of differential equation for y

$$D_x^{-1} y_{xx} \left[\mathcal{S}_{\mathrm{I}}^n y_x + \sum_{j=2}^{n} \kappa_j \mathcal{S}_{\mathrm{I}}^{n-j} y_x\right] + (xy_x - y) = 0, \tag{2.9}$$

where $\mathcal{S}_{\mathrm{I}}$ is the recursion operator

$$\mathcal{S}_{\mathrm{I}} = D_x^2 + 8y_x - 4D_x^{-1} y_{xx}.$$

The first member of the hierarchy (2.9) is the SD-I.e equation (2.6). Thus we shall call this hierarchy SD-I.e hierarchy.

Therefor we have derived the Bäcklund transformation (2.7)–(2.8) between solutions u of the first Painlevé hierarchy (2.3) and solutions y of the SD-I.e hierarchy (2.9).

When $n = 1$, the Bäcklund transformation (2.7)–(2.8) gives the Bäcklund transformation (2.4) between the first Painlevé equation (2.5) and the SD-I.e equation (2.6). Next we will consider the cases $n = 2$ and $n = 3$.

Example 1 ($n = 2$). The second member of the PI hierarchy (2.3) is the fourth-order equation

$$u_{xxxx} = 20uu_{xx} + 10u_x^2 - 40u^3 - \kappa_2 u + x. \tag{2.10}$$

In this case, the Bäcklund transformation (2.7) reads

$$y = \tfrac{1}{2}\big(2u_x u_{xxx} - u_{xx}^2 - 20uu_x^2 + 20u^4 + \kappa_2 u^2 - 2xu\big). \tag{2.11}$$

Equations (2.11) and (2.8) give one-to-one correspondence between (2.10) and the following equation

$$2y_{xx}y_{xxxx} - y_{xxx}^2 + 20y_x y_{xx}^2 + 20y_x^4 + \kappa_2 y_x^2 + 2(xy_x - y) = 0. \tag{2.12}$$

Equation (2.12) and the Bäcklund transformation (2.8) and (2.11) were given before [19].

Example 2 ($n = 3$). The third member of the PI hierarchy (2.3) reads

$$\begin{aligned} u_{xxxxxx} &= 28uu_{xxxx} + 56u_x u_{xxx} + 42u_{xx}^2 - 280u^2 u_{xx} \\ &\quad - 280uu_x^2 + 280u^4 - \kappa_2\big(u_{xx} - 6u^2\big) - \kappa_3 u + x. \end{aligned} \tag{2.13}$$

In this case, the Bäcklund transformation (2.7) has the form

$$\begin{aligned} y = \tfrac{1}{2}\big[&2u_x u_{xxxxx} - 2u_{xx}u_{xxxx} + u_{xxx}^2 - 56uu_x u_{xxx} + 28uu_{xx}^2 \\ &- 56u_x^2 u_{xx} + 280u^2u_x^2 - 112u^5 + \kappa_2\big(u_x^2 - 4u^3\big) + \kappa_3 u^2 - 2xu\big]. \end{aligned} \tag{2.14}$$

Equations (2.8) and (2.14) give one-to-one correspondence between solutions u of (2.13) and solutions y of the following equation

$$\begin{aligned} &2y_{xx}y_{xxxxxx} - 2y_{xxx}y_{xxxxx} + y_{xxxx}^2 + 56y_x y_{xx} y_{xxxx} - 28y_x y_{xxx}^2 \\ &\quad + 56y_{xx}^2 y_{xxx} + 280y_x^2 y_{xxx}^2 + 112y_x^5 + \kappa_2\big(y_{xx}^2 + 4y_x^3\big) + \kappa_3 y_x^2 + 2(xy_x - y) = 0. \end{aligned} \tag{2.15}$$

Equation (2.15) is a new sixth-order Painlevé-type equation.

3 Bäcklund transformations for second Painlevé hierarchy

In the present section, we will study Bäcklund transformations of the second Painlevé hierarchy (PII hierarchy) [6]

$$(D_x - 2u)\sum_{j=1}^{n} \gamma_j L^j\left[u_x + u^2\right] + 2\gamma xu - \gamma - 4\delta = 0,$$

where the operator $L^j[u]$ is defined by (2.2). The special case $\gamma_j = 0$, $1 \le j \le n-1$, of this hierarchy is a similarity reduction of the modified Korteweg–de Vries hierarchy [2, 4]. The members of this hierarchy may define new transcendental functions.

This hierarchy can be written in the following alternative form [11]

$$\mathcal{R}_{\text{II}}^n u + \sum_{j=1}^{n-1} \kappa_j \mathcal{R}_{\text{II}}^j u - (xu + \alpha) = 0, \tag{3.1}$$

where $\mathcal{R}_{\text{II}}$ is the recursion operator

$$\mathcal{R}_{\text{II}} = D_x^2 - 4u^2 + 4uD_x^{-1}u_x.$$

3.1 A hierarchy of SD-I.d equation

As a first Bäcklund transformation for the PII hierarchy (3.1), we will generalize the Bäcklund transformation between the second Painlevé equation and the SD-I.d equation of Cosgrove and Scoufis [17, 18].

Let

$$y = D_x^{-1}\left[u_x\left(\mathcal{R}_{\text{II}}^n u + \sum_{j=1}^{n-1}\kappa_j \mathcal{R}_{\text{II}}^j u\right)\right] - \tfrac{1}{2}xu^2 - \tfrac{1}{2}(2\alpha - \epsilon)u, \tag{3.2}$$

where $\epsilon = \pm 1$. Differentiating (3.2) and using (3.1), we find

$$u_x = \epsilon\big(u^2 + 2y_x\big). \tag{3.3}$$

Now we will show that

$$D_x^{-1}\big(u_x \mathcal{R}_{\text{II}}^j u\big) = \tfrac{1}{2}\big(u^2 H^j[y_x] + D_x^{-1} y_x H_x^j[y_x]\big), \tag{3.4}$$

where the operator $H^j[p]$ satisfies the Lenard recursion relation

$$D_x H^{j+1}[p] = \big(D_x^3 + 8pD_x + 4p_x\big)H^j[p], \qquad H^1[p] = 4p. \tag{3.5}$$

Firstly, we will use induction to show that for any $j = 1, 2, \ldots,$

$$\mathcal{R}_{\text{II}}^j u = \tfrac{1}{2}(\epsilon D_x + 2u)H^j[y_x]. \tag{3.6}$$

For $j = 1$, $\mathcal{R}_{\text{II}} u = u_{xx} - 2u^3$. Using (3.3), we find that

$$u_{xx} = 2u^3 + 4y_x u + 2\epsilon y_{xx}. \tag{3.7}$$

Thus

$$\mathcal{R}_{\text{II}} u = 4uy_x + 2\epsilon y_{xx} = \tfrac{1}{2}(\epsilon D_x + 2u)H^1[y_x].$$

Assume that it is true for $j = k$. Then

$$\begin{aligned} 2\mathcal{R}_{\text{II}}^{k+1} u = \mathcal{R}_{\text{II}}(\epsilon D_x + 2u)H^k[y_x] &= \epsilon H_{xxx}^k[y_x] + 2uH_{xx}^k[y_x] + 4u_x H_x^k[y_x] + 2u_{xx}H^k[y_x] \\ &\quad - 4u^2\big(\epsilon H_x^k[y_x] + 2uH^k[y_x]\big) + 4uD_x^{-1}\big(\epsilon u_x H_x^k[y_x] + 2uu_x H^k[y_x]\big). \end{aligned} \tag{3.8}$$

Integration by parts gives

$$D_x^{-1}\big(\epsilon u_x H_x^k[y_x] + 2uu_x H^k[y_x]\big) = u^2 H^k[y_x] + D_x^{-1}\big[\big(\epsilon u_x - u^2\big)H_x^k[y_x]\big].$$

Hence (3.8) can be written as

$$\begin{aligned} 2\mathcal{R}_{\text{II}}^{k+1} u &= \epsilon H_{xxx}^k[y_x] + 2uH_{xx}^k[y_x] + 4u_x H_x^k[y_x] + 2u_{xx}H^k[y_x] - 4u^2\big(\epsilon H_x^k[y_x] + 2uH^k[y_x]\big) \\ &\quad + 4u\big(u^2 H^k[y_x] + D_x^{-1}\big[\big(\epsilon u_x - u^2\big)H_x^k[y_x]\big]\big). \end{aligned} \tag{3.9}$$

Using (3.3) to substitute u_x and (3.7) to substitute u_{xx}, (3.9) becomes

$$\begin{aligned} 2\mathcal{R}_{\text{II}}^{k+1} u &= \epsilon\big(H_{xxx}^k[y_x] + 8y_x H_x^k[y_x] + 4y_{xx}H^k[y_x]\big) \\ &\quad + 2u\big(H_{xx}^k[y_x] + 4y_x H^k[y_x] + 4D_x^{-1} y_x H_x^k[y_x]\big) \\ &= (\epsilon D_x + 2u)\big(H_{xx}^k[y_x] + 4y_x H^k[y_x] + 4D_x^{-1} y_x H_x^k[y_x]\big). \end{aligned}$$

Since

$$D_x\big(H^k_{xx}[y_x] + 4y_xH^k[y_x] + 4D_x^{-1}y_xH^k_x[y_x]\big) = H^k_{xxx}[y_x] + 8y_xH^k_x[y_x] + 4y_{xx}H^k[y_x],$$

we have $H^k_{xx}[y_x] + 4y_xH^k[y_x] + 4D_x^{-1}y_xH^k_x[y_x] = H^{k+1}[y_x]$, see (3.5), and hence the proof by induction is finished.

Now using (3.6) we find

$$2u_x\mathcal{R}^k_{\scriptscriptstyle\rm II}(u) = \big(\epsilon u_x - u^2\big)H^k_x[y_x] + D_x\big(u^2H^k[y_x]\big). \tag{3.10}$$

Using (3.3) to substitute u_x into (3.10) and then integrating, we obtain (3.4).

Therefore (3.2) can be used to obtain the following quadratic equation for u

$$\left(-x + H^n[y_x] + \sum_{j=1}^{n-1}\kappa_jH^j[y_x]\right)u^2 - (2\alpha-\epsilon)u$$
$$+ 2D_x^{-1}y_x\left(H^n_x[y_x] + \sum_{j=1}^{n-1}\kappa_jH^j_x[y_x]\right) - 2y = 0. \tag{3.11}$$

Eliminating u between (3.3) and (3.11) gives a one-to-one correspondence between the second Painlevé hierarchy (3.1) and the following hierarchy of second-degree equations

$$\left(H^n_x[y_x] + \sum_{j=1}^{n-1}\kappa_jH^j_x[y_x] - 1\right)^2 + 8\left(H^n[y_x] + \sum_{j=1}^{n-1}\kappa_jH^j[y_x] - x\right)$$
$$\times\left(D_x^{-1}y_xH^n_x[y_x] + \sum_{j=1}^{n-1}\kappa_jD_x^{-1}y_xH^j_x[y_x] - y\right) = (2\alpha-\epsilon)^2. \tag{3.12}$$

Therefore we have derived the Bäcklund transformation (3.2) and (3.11) between the PII hierarchy (3.1) and the new hierarchy (3.12).

Next we will give the explicit forms of the above results when $n = 1, 2, 3$.

Example 3 ($n = 1$). The first member of the second Painlevé hierarchy (3.1) is the second Painlevé equation

$$u_{xx} = 2u^3 + xu + \alpha.$$

In this case, (3.2) and (3.11) read

$$y = \tfrac{1}{2}\big[u_x^2 - u^4 - xu^2 - (2\alpha-\epsilon)u\big]$$

and

$$(4y_x - x)u^2 - (2\alpha-\epsilon)u + 4y_x^2 - 2y = 0,$$

respectively. The second-degree equation for y is

$$(4y_{xx} - 1)^2 + 8(4y_x - x)\big(2y_x^2 - y\big) = (2\alpha-\epsilon)^2. \tag{3.13}$$

The change of variables $w = y - \frac{1}{8}x^2$ transforms (3.13) into the SD-I.d equation of Cosgrove and Scoufis [17]

$$w_{xx}^2 + 4w_x^3 + 2w_x(xw_x - w) = \tfrac{1}{16}(2\alpha-\epsilon)^2.$$

Thus when $n = 1$, the Bäcklund transformation (3.2) and (3.11) is the known Bäcklund transformation between the second Painlevé equation and the SD-I.d equation (3.12). Since the first member of the hierarchy (3.12) is the SD-I.d equation, we shall call it SD-I.d hierarchy.

Example 4 ($n = 2$). The second member of the second Painlevé hierarchy (3.1) reads

$$u_{xxxx} = 10u^2 u_{xx} + 10uu_x^2 - 6u^5 - \kappa_1\big(u_{xx} - 2u^3\big) + xu + \alpha. \tag{3.14}$$

Equation (3.14) is labelled in [20, 21] as F-XVII.

In this case, (3.2) and (3.11) read

$$y = \tfrac{1}{2}\big[2u_x u_{xxx} - u_{xx}^2 - 10u^2u_x^2 + 2u^6 + \kappa_1\big(u_x^2 - u^4\big) - xu^2 - (2\alpha - \epsilon)u\big] \tag{3.15}$$

and

$$\big(4y_{xxx} + 24y_x^2 + 4\kappa_1 y_x - x\big)u^2 - (2\alpha - \epsilon)u + 8y_x y_{xxx} - 4y_{xx}^2 + 32y_x^3 + 4\kappa_1 y_x^2 - 2y = 0, \tag{3.16}$$

respectively. Equations (3.15) and (3.16) give one-to-one correspondence between (3.14) and the following fourth-order second-degree equation

$$\begin{aligned}&[4y_{xxxx} + 48y_x y_{xx} + 4\kappa_1 y_{xx} - 1]^2 \\ &\quad + 8\big[4y_{xxx} + 24y_x^2 + 4\kappa_1 y_x - x\big]\big[4y_x y_{xxx} - 2y_{xx}^2 + 16y_x^3 + 2\kappa_1 y_x^2 - y\big] = (2\alpha - \epsilon)^2.\end{aligned} \tag{3.17}$$

Equation (3.17) is a first integral of the following fifth-order equation

$$y_{xxxxx} = -20y_x y_{xxx} - 10y_{xx}^2 - 40y_x^3 - \kappa_1 y_{xxx} - 6\kappa_1 y_x^2 + xy_x + y. \tag{3.18}$$

The transformation $y = -(w + \frac{1}{2}\gamma z + 5\gamma^3)$, $z = x + 30\gamma^2$ transforms (3.18) into the equation

$$w_{zzzzz} = 20w_z w_{zzz} + 10w_{zz}^2 - 40w_z^3 + zw_z + w + \gamma z. \tag{3.19}$$

The Bäcklund transformation [22]

$$v = w_z, \qquad w = v_{zzzz} - 20vv_{zz} - 10v_z^2 + 40v^3 - zv - \gamma z, \tag{3.20}$$

gives a one-to-one correspondence between (3.19) and Cosgrove's Fif-III equation [20]

$$v_{zzzzz} = 20vv_{zzz} + 40v_z v_{zz} - 120v^2 v_z + zv_z + 2v + \gamma. \tag{3.21}$$

Therefore we have rederived the known relation

$$v = -\frac{1}{2}\big(\epsilon u_x - u^2 + \gamma\big), \qquad u = \frac{-\epsilon[v_{zzz} - 12vv_z + 4\gamma v_z + \frac{\epsilon}{2}\alpha]}{2[v_{zz} - 6v^2 + 4\gamma v + \frac{1}{4}z - 4\gamma^2]}.$$

between Cosgrove's equations Fif-III (3.21) and F-XVII (3.14) [20].

Example 5 ($n = 3$). The third member of the second Painlevé hierarchy (3.1) reads

$$\begin{aligned}u_{xxxxxx} &= 14u^2 u_{xxxx} + 56uu_x u_{xxx} + 42uu_{xx}^2 + 70u_x^2 u_{xx} - 70u^4 u_{xx} - 140u^3u_x^2 + 20u^7 \\ &\quad - \kappa_2(u_{xxxx} - 10u^2u_{xx} - 10uu_x^2 + 6u^5) - \kappa_1(u_{xx} - 2u^3) + xu + \alpha.\end{aligned} \tag{3.22}$$

In this case, (3.2) and (3.11) have the following forms respectively

$$\begin{aligned}2y &= 2u_x u_{xxxxx} - 2u_{xx}u_{xxxx} + u_{xxx}^2 - 28u^2u_xu_{xxx} + 14u^2u_{xx}^2 - 56uu_x^2u_{xx} - 21u_x^4 + 70u^4u_x^2 \\ &\quad - 5u^8 + \kappa_2(2u_xu_{xxx} - u_{xx}^2 - 10u^2u_x^2 + 2u^6) + \kappa_1(u_x^2 - u^4) - xu^2 - (2\alpha - \epsilon)u\end{aligned} \tag{3.23}$$

and

$$4\left[y_{xxxxx} + 20y_x y_{xx} + 10y_{xx}^2 + 40y_x^3 + \kappa_2\big(y_{xxx} + 6y_x^2\big) + \kappa_1 y_x - \tfrac{1}{4}x\right]u^2$$

$$- (2\alpha - \epsilon)u + 4\big(2y_x y_{xxxxx} - 2y_{xx}y_{xxxx} + y_{xxx}^2 + 40y_x^2 y_{xxx} + 60y_x^4\big)$$
$$+ 4\kappa_2\big(2y_x y_{xxx} - y_{xx}^2 + 8y_x^3\big) + 4\kappa_1 y_x^2 - 2y = 0. \quad (3.24)$$

Equations (3.23) and (3.24) give one-to-one correspondence between (3.22) and the following six-order second-degree equation

$$\big[y_{xxxxxxx} + 20y_x y_{xxxxx} + 40y_{xx}y_{xxxx} + 120y_x^2 y_{xxx} + \kappa_2(y_{xxxxx} + 12y_x y_{xxx}) + \kappa_1 y_{xxx} - \tfrac{1}{4}\big]^2$$
$$+ 2\left[y_{xxxxxx} + 20y_x y_{xxx} + 10y_{xx}^2 + 40y_x^3 + \kappa_2\big(y_{xxxx} + 6y_x^2\big) + \kappa_1 y_x - \tfrac{1}{4}x\right]$$
$$\times \big[4y_x y_{xxxxxx} - 4y_{xx}y_{xxxxx} + 2y_{xxxx}^2 + 80y_x^2 y_{xxxx} + 120y_x^4$$
$$+ 2\kappa_2\big(2y_x y_{xxx} - y_{xx}^2 + 8y_x^3\big) + 2\kappa_1 y_x^2 - y\big] = \tfrac{1}{16}(2\alpha - \epsilon)^2. \quad (3.25)$$

The Bäcklund transformation (3.23), (3.24) and the equation (3.25) are not given before.

3.2 A hierarchy of a second-order fourth-degree equation

In this subsection, we will generalize the Bäcklund transformation given in [23] between the second Painlevé equation and a second-order fourth-degree equation.

Let

$$y = D_x^{-1}\left[u_x\left(\mathcal{R}_{\mathrm{II}}^n u + \sum_{j=1}^{n-1}\kappa_j \mathcal{R}_{\mathrm{II}}^j u\right)\right] - \tfrac{1}{2}xu^2 - \alpha u. \quad (3.26)$$

Differentiating (3.26) and using (3.1), we find

$$u^2 + 2y_x = 0. \quad (3.27)$$

Equations (3.26) and (3.27) define a Bäcklund transformation between the second Painlevé hierarchy (3.1) and a new hierarchy of differential equations for y.

In order to obtain the new hierarchy, we will prove that

$$D_x^{-1}\big(u_x \mathcal{R}_{\mathrm{II}}^j u\big) = -D_x^{-1}\left(\frac{y_{xx}}{y_x}\mathcal{S}_{\mathrm{II}}^j y_x\right), \quad (3.28)$$

where $\mathcal{S}_{\mathrm{II}}$ is the recursion operator

$$\mathcal{S}_{\mathrm{II}} = D_x^2 - \frac{y_{xx}}{y_x}D_x - \frac{y_{xxx}}{2y_x} + \frac{3y_{xx}^2}{4y_x^2} + 8y_x - 4y_x D_x^{-1}\frac{y_{xx}}{y_x}.$$

First of all, we will use induction to prove that

$$\mathcal{R}_{\mathrm{II}}^j u = -\frac{2}{u}\mathcal{S}_{\mathrm{II}}^j y_x. \quad (3.29)$$

Using (3.27), we find

$$u_x = -\frac{y_{xx}}{u}, \qquad u_{xx} = -\frac{1}{u}\left(y_{xxx} - \frac{y_{xx}^2}{2y_x}\right). \quad (3.30)$$

Hence

$$\mathcal{R}_{\mathrm{II}} u = u_{xx} - 2u^3 = -\frac{1}{u}\left(y_{xxx} - \frac{y_{xx}^2}{2y_x} + 8y_x^2\right) = -\frac{2}{u}\mathcal{S}_{\mathrm{II}} y_x.$$

Thus (3.29) is true for $j = 1$.

Assume it is true for $j = k$. Then

$$\mathcal{R}_{\rm II}^{k+1}u = -2\mathcal{R}_{\rm II}\frac{1}{u}\mathcal{S}_{\rm II}^{k}y_x = -\frac{2}{u}\left\{D_x^2 - \frac{2u_x}{u}D_x - \frac{u_{xx}}{u} + \frac{2u_x^2}{u^2} - 4u^2 + 4u^2D_x^{-1}\frac{u_x}{u}\right\}\mathcal{S}_{\rm II}^{k}y_x.$$

Using (3.30) to substitute u_x and u_{xx} and using (3.27) to substitute u^2, we find the result.

As a second step, we use (3.29) to find

$$D_x^{-1}\big(u_x\mathcal{R}_{\rm II}^{k}u\big) = -2D_x^{-1}\left(\frac{u_x}{u}\mathcal{S}_{\rm II}^{k}y_x\right).$$

Thus using (3.30) to substitute u_x and using (3.27) to substitute u^2 we find (3.28).

Therefore (3.26) implies

$$\alpha u = -y + xy_x - D_x^{-1}\left[\frac{y_{xx}}{y_x}\left(\mathcal{S}_{\rm II}^{n}y_x + \sum_{j=1}^{n-1}\kappa_j\mathcal{S}_{\rm II}^{j}y_x\right)\right]. \tag{3.31}$$

If $\alpha \neq 0$, then substituting u from (3.31) into (3.27) we obtain the following hierarchy of differential equations for y

$$\left(D_x^{-1}\left[\frac{y_{xx}}{y_x}\left(\mathcal{S}_{\rm II}^{n}y_x + \sum_{j=1}^{n-1}\kappa_j\mathcal{S}_{\rm II}^{j}y_x\right)\right] - xy_x + y\right)^2 + 2\alpha^2y_x = 0. \tag{3.32}$$

If $\alpha = 0$, then y satisfies the hierarchy

$$D_x^{-1}\left[\frac{y_{xx}}{y_x}\left(\mathcal{S}_{\rm II}^{n}y_x + \sum_{j=1}^{n-1}\kappa_j\mathcal{S}_{\rm II}^{j}y_x\right)\right] - xy_x + y = 0.$$

The first member of the hierarchy (3.32) is a fourth-degree equation, whereas the other members are second-degree equations. Now we give some examples.

Example 6 ($n = 1$). In the present case, (3.26) reads

$$2y = u_x^2 - u^4 - xu^2 - 2\alpha u. \tag{3.33}$$

Eliminating u between (3.27) and (3.33) yields the following second-order fourth-degree equation for y

$$\big[y_{xx}^2 + 8y_x^3 - 4y_x(xy_x - y)\big]^2 + 32\alpha^2y_x^3 = 0. \tag{3.34}$$

The change of variables $w = 2y$ transform (3.34) into the following equation

$$\big[w_{xx}^2 + 4w_x^3 - 4w_x(xw_x - w)\big]^2 + 16\alpha^2y_x^3 = 0. \tag{3.35}$$

Equation (3.35) was derived before [23].

Example 7 ($n = 2$). When $n = 2$, (3.26) reads

$$2y = 2u_xu_{xxx} - u_{xx}^2 - 10u^2u_x^2 + 2u^6 - xu^2 - 2\alpha u + \kappa_1\big(u_x^2 - u^4\big). \tag{3.36}$$

Equations (3.27) and (3.36) give a Bäcklund transformation between the second member of PII hierarchy (3.14) and the following fourth-order second-degree equation for y

$$\Bigg[y_{xx}y_{xxxx} - \frac{3y_{xx}^2}{2y_x}\left(y_{xxx} - \frac{y_{xx}^2}{2y_x}\right) - \frac{1}{2}\left(y_{xxx} - \frac{y_{xx}^2}{2y_x}\right)^2 + 10y_xy_{xx}^2 + 16y_x^4 - 2y_x(xy_x - y) + \frac{1}{2}\kappa_1\big(y_{xx}^2 + 8y_x^3\big)\Bigg]^2 + 8\alpha^2y_x^3 = 0. \tag{3.37}$$

Equation (3.37) was given before [19].

Example 8 ($n = 3$). In this case, (3.26) read

$$2y = 2u_x u_{xxxxx} - 2u_{xx}u_{xxxx} + u_{xxx}^2 - 28u^2 u_x u_{xxx} + 14u^2 u_{xx}^2 - 56uu_x^2 u_{xx} - 21u_x^4 + 70u^4 u_x^2 - 5u^8 + \kappa_2\left(2u_x u_{xxx} - u_{xx}^2 - 10u^2 u_x^2 + 2u^6\right) + \kappa_1\left(u_x^2 - u^4\right) - xu^2 - 2\alpha u, \tag{3.38}$$

and (3.32) has the form

$$\Bigg[2y_{xx}y_{xxxxxxx} - \left(2y_{xxx} + \frac{3y_{xx}^2}{y_x}\right)\left(y_{xxxxxx} + \frac{5y_{xx}y_{xxxxx}}{y_x}\right) + \left(y_{xxxx} - \frac{3y_{xx}y_{xxx}}{2y_x} + \frac{3y_{xx}^3}{4y_x^2}\right)^2 + \left(2y_{xxx} - \frac{y_{xx}^2}{y_x}\right)\left(\frac{2y_{xx}y_{xxxx}}{y_x} + \frac{3y_{xxx}^2}{2y_x} - \frac{9y_{xx}^2 y_{xxx}}{2y_x^2} + \frac{15y_{xx}^2}{8y_x^3} - 7y_{xx}^2 - 14y_x y_{xxx}\right) + \frac{15y_{xx}^2}{2y_x^2}\left(3y_{xxx}^2 - \frac{5y_{xx}^2 y_{xxx}}{y_x} + \frac{7y_{xx}^4}{4y_x^3}\right) + \frac{21y_{xx}^4}{2y_x} + 280y_x^2 y_{xx}^2 - 150y_x^5 - 4y_x(xy_x - y) + 2\kappa_2\left[y_{xx}y_{xxxxx} - \frac{3y_{xx}^2}{2y_x}\left(y_{xxx} - \frac{y_{xx}^2}{2y_x}\right) - \frac{1}{2}\left(y_{xxx} - \frac{y_{xx}^2}{2y_x}\right)^2 + 10y_x y_{xx}^2 + 16y_x^4\right] + \kappa_1\left(y_{xx}^2 + 8y_x^3\right)\Bigg]^2 + 32\alpha^2 y_x^3 = 0. \tag{3.39}$$

The Bäcklund transformation between the third member of PII hierarchy (3.22) and the new equation (3.39) is given by (3.27) and (3.38).

3.3 Auto-Bäcklund transformations for PII hierarchy

In this subsection, we will use the SD-I.d hierarchy (3.12) to derive auto-Bäcklund transformations for PII hierarchy (3.1).

Let u be solution of (3.1) with parameter α and let $\bar{u}$ be solution of (3.1) with parameter $\bar{\alpha}$. Since (3.12) is invariant under the transformation $2\alpha - \epsilon = -2\bar{\alpha} + \epsilon$, a solution y of (3.12) corresponds to two solutions u and $\bar{u}$ of (3.1). The relation between y and u is given by (3.11) and the relation between y and $\bar{u}$ is given by

$$\left(-x + H^n[y_x] + \sum_{j=1}^{n-1}\kappa_j H^j[y_x]\right)\bar{u}^2 - (2\bar{\alpha} - \epsilon)\bar{u} + 2D_x^{-1}y_x\left(H_x^n[y_x] + \sum_{j=1}^{n-1}\kappa_j H_x^j[y_x]\right) - 2y = 0. \tag{3.40}$$

Subtracting (3.11) from (3.40), we obtain

$$\left(-x + H^n[y_x] + \sum_{j=1}^{n-1}\kappa_j H^j[y_x]\right)\left(\bar{u}^2 - u^2\right) - (2\bar{\alpha} - \epsilon)\bar{u} + (2\alpha - \epsilon)u = 0. \tag{3.41}$$

Using $2\alpha - \epsilon = -2\bar{\alpha} + \epsilon$ and dividing by $\bar{u} + u$, (3.41) yields

$$\left(-x + H^n[y_x] + \sum_{j=1}^{n-1}\kappa_j H^j[y_x]\right)(\bar{u} - u) + (2\alpha - \epsilon) = 0.$$

Now using (3.3) to substitute y_x, we obtain the following two auto-Bäcklund transformations for PII hierarchy (3.1)

$$\bar{\alpha} = -\alpha + \epsilon, \qquad \epsilon = \pm 1,$$

$$\bar{u} = u - \frac{(2\alpha - \epsilon)}{\left(-x + H^n[\frac{1}{2}(\epsilon u_x - u^2)] + \sum_{j=1}^{n-1} \kappa_j H^j[\frac{1}{2}(\epsilon u_x - u^2)]\right)}. \tag{3.42}$$

These auto-Bäcklund transformations and the discrete symmetry $\bar{u} = -u$, $\bar{\alpha} = -\alpha$ can be used to derive the auto-Bäcklund transformations given in [6, 16].

The auto-Bäcklund transformations (3.42) can be used to obtain infinite hierarchies of solutions of the PII hierarchy (3.1). For example, starting by the solution $u = 0$, $\alpha = 0$ of (3.1), the auto-Bäcklund transformations (3.42) yields the new solution $\bar{u} = -\frac{\epsilon}{x}$, $\bar{\alpha} = \epsilon$. Now applying the auto-Bäcklund transformations (3.42) with $\epsilon = 1$ to the solution $\bar{u} = \frac{1}{x}$, $\bar{\alpha} = -1$, we obtain the new solution $\bar{\bar{u}} = \frac{-2(x^3 - 2\kappa_1)}{x(x^3+4\kappa_1)}$, $\bar{\bar{\alpha}} = 2$.

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