Dunkl Hyperbolic Equations*

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Abstract. We introduce and study the Dunkl symmetric systems. We prove the well-posedness results for the Cauchy problem for these systems. Eventually we describe the finite speed of it. Next the semi-linear Dunkl-wave equations are also studied.

Key words: Dunkl operators; Dunkl symmetric systems; energy estimates; finite speed of propagation; Dunkl-wave equations with variable coefficients

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1 Introduction

We consider the differential-difference operators T_j, $j = 1,\dots,d$, on $\mathbb{R}^d$ introduced by C.F. Dunkl in [7] and called Dunkl operators in the literature. These operators are very important in pure mathematics and in physics. They provide a useful tool in the study of special functions with root systems [8].

In this paper, we are interested in studying two types of Dunkl hyperbolic equations. The first one is the Dunkl-linear symmetric system

$$\begin{cases} \partial_t u - \displaystyle\sum_{j=1}^{d} A_j T_j u - A_0 u = f, \\ u|_{t=0} = v, \end{cases} \tag{1.1}$$

where the A_p are square matrices $m \times m$ which satisfy some hypotheses (see Section 3), the initial data belong to Dunkl–Sobolev spaces $[H^s_k(\mathbb{R}^d)]^m$ (see [22]) and f is a continuous function on an interval I with value in $[H^s_k(\mathbb{R}^d)]^m$. In the classical case the Cauchy problem for symmetric hyperbolic systems of first order has been introduced and studied by Friedrichs [13]. The Cauchy problem will be solved with the aid of energy integral inequalities, developed for this purpose by Friedrichs. Such energy inequalities have been employed by H. Weber [33], Hadamard [17], Zaremba [34] to derive various uniqueness theorems, and by Courant–Friedrichs–Lewy [6], Friedrichs [13], Schauder [27] to derive existence theorems. In all these treatments the energy inequality is used to show that the solution, at some later time, depends boundedly on the initial values in an appropriate norm. However, to derive an existence theorem one needs, in addition to the a priori energy estimates, some auxiliary constructions. Thus, motivated by these methods we will prove by energy methods and Friedrichs approach local well-posedness and principle of finite speed of propagation for the system (1.1).

Let us first summarize our well-posedness results and finite speed of propagation (Theorems 3.1 and 3.2).

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Well-posedness for DLS. For all given $f \in [C(I, H_k^s(\mathbb{R}^d))]^m$ and $v \in [H_k^s(\mathbb{R}^d)]^m$, there exists a unique solution u of the system (1.1) in the space

$$[C(I, H_k^s(\mathbb{R}^d))]^m \bigcap [C^1(I, H_k^{s-1}(\mathbb{R}^d))]^m.$$

In the classical case, a similar result can be found in [5], where the authors used another method based on the symbolic calculations for the pseudo-differential operators that we cannot adapt for the system (1.1) at the moment. Our method uses some ideas inspired by the works [5, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 24]. We note that K. Friedrichs has solved the Cauchy problem in a lens-shaped domain [13]. He proved existence of extended solutions by Hilbert space method and showed the differentiability of these solutions using complicated calculations. A similar problem is that of a symmetric hyperbolic system studied by P. Lax, who gives a method offering both the existence and the differentiability of solutions at once [19]. He reduced the problem to the case where all functions are periodic in every independent variable.

Finite speed of propagation. Let (1.1) be as above. We assume that $f \in [C(I, L_k^2(\mathbb{R}^d))]^m$ and $v \in [L_k^2(\mathbb{R}^d)]^m$.

- There exists a positive constant C_0 such that, for any positive real R satisfying

$$\begin{cases} f(t,x) \equiv 0 & \text{for} \quad \|x\| < R - C_0 t, \\ v(x) \equiv 0 & \text{for} \quad \|x\| < R, \end{cases}$$

the unique solution u of the system (1.1) satisfies

$$u(t,x) \equiv 0 \qquad \text{for} \quad \|x\| < R - C_0 t.$$

- If given f and v are such that

$$\begin{cases} f(t,x) \equiv 0 & \text{for} \quad \|x\| > R + C_0 t, \\ v(x) \equiv 0 & \text{for} \quad \|x\| > R, \end{cases}$$

then the unique solution u of the system (1.1) satisfies

$$u(t,x) \equiv 0 \qquad \text{for} \quad \|x\| > R + C_0 t.$$

In the classical case, similar results can be found in [5] (see also [28]).

A standard example of the Dunkl linear symmetric system is the Dunkl-wave equations with variable coefficients defined by

$$\partial_t^2 u - \mathrm{div}_k[A \cdot \nabla_{k,x} u] + Q(t, x, \partial_t u, T_x u), \qquad t \in \mathbb{R}, \qquad x \in \mathbb{R}^d,$$

where

$$\nabla_{k,x} u = (T_1 u, \dots, T_d u), \qquad \mathrm{div}_k (v_1, \dots, v_d) = \sum_{i=1}^d T_i v_i,$$

A is a real symmetric matrix which satisfies some hypotheses (see Subsection 3.2) and $Q(t, x, \partial_t u, T_x u)$ is differential-difference operator of degree 1 such that these coefficients are C^∞, and all derivatives are bounded.

From the previous results we deduce the well-posedness of the generalized Dunkl-wave equations (Theorem 3.3).

Well-posedness for GDW. For all $s \in \mathbb{N}$, $u_0 \in H_k^{s+1}(\mathbb{R}^d)$, $u_1 \in H_k^s(\mathbb{R}^d)$ and f in $C(\mathbb{R}, H_k^s(\mathbb{R}^d))$, there exists a unique $u \in C^1(\mathbb{R}, H_k^s(\mathbb{R}^d)) \cap C(\mathbb{R}, H_k^{s+1}(\mathbb{R}^d))$ such that

$$\begin{cases} \partial_t^2 u - \mathrm{div}_k[A \cdot \nabla_{k,x} u] + Q(t, x, \partial_t u, T_x u) = f, \\ u|_{t=0} = u_0, \\ \partial_t u|_{t=0} = u_1. \end{cases}$$

The second type of Dunkl hyperbolic equations that we are interested is the semi-linear Dunkl-wave equation

$$\begin{cases} \partial_t^2 u - \triangle_k u = Q(\Lambda_k u, \Lambda_k u), \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \tag{1.2}$$

where

$$\triangle_k = \sum_{j=1}^{d} T_j^2, \qquad \Lambda_k u = (\partial_t u, T_1 u, \dots, T_d u),$$

and Q is a quadratic form on $\mathbb{R}^{d+1}$.

Our main result for this type of Dunkl hyperbolic equations is the following.

Well-posedness for SLDW. Let (u_0, u_1) be in $H_k^s(\mathbb{R}^d) \times H_k^{s-1}(\mathbb{R}^d)$ for $s > \gamma + \frac{d}{2} + 1$. Then there exists a positive time T such that the problem (1.2) has a unique solution u belonging to

$$C([0, T], H_k^s(\mathbb{R}^d)) \cap C^1([0, T], H_k^{s-1}(\mathbb{R}^d))$$

and satisfying the blow up criteria (Theorem 4.1).

In the classical case see [2, 3, 4, 29]. We note that the Huygens' problem for the homogeneous Dunkl-wave equation is studied by S. Ben Saïd and B. Ørsted [1].

The paper is organized as follows. In Section 2 we recall the main results about the harmonic analysis associated with the Dunkl operators. We study in Section 3 the generalized Cauchy problem of the Dunkl linear symmetric systems, and we prove the principle of finite speed of propagation of these systems. In the last section we study a semi-linear Dunkl-wave equation and we prove the well-posedness of this equation.

Throughout this paper by C we always represent a positive constant not necessarily the same in each occurrence.

2 Preliminaries

This section gives an introduction to the theory of Dunkl operators, Dunkl transform, Dunkl convolution and to the Dunkl–Sobolev spaces. Main references are [7, 8, 9, 10, 22, 23, 25, 26, 30, 31, 32].

2.1 Reflection groups, root systems and multiplicity functions

The basic ingredient in the theory of Dunkl operators are root systems and finite reflection groups, acting on $\mathbb{R}^d$ with the standard Euclidean scalar product $\langle \cdot, \cdot \rangle$ and $||x|| = \sqrt{\langle x, x \rangle}$. On $\mathbb{C}^d$, $\| \cdot \|$ denotes also the standard Hermitian norm, while $\langle z, w \rangle = \sum_{j=1}^{d} z_j \overline{w}_j$.

For $\alpha \in \mathbb{R}^d \backslash \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x \rangle}{||\alpha||^2}\alpha.$$

A finite set $R \subset \mathbb{R}^d \backslash \{0\}$ is called a root system if $R \cap \mathbb{R} \cdot \alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflections σ_α, $\alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group associated with R. All reflections in W correspond to suitable pairs of roots. We fix a positive root system $R_+ = \{\alpha \in R / \langle \alpha, \beta \rangle > 0\}$ for some $\beta \in \mathbb{R}^d \backslash \bigcup_{\alpha \in R} H_\alpha$. We will assume that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R_+$.

A function $k : R \longrightarrow \mathbb{C}$ is called a multiplicity function if it is invariant under the action of the associated reflection group W. For abbreviation, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha \in R_+} k(\alpha).$$

Throughout this paper, we will assume that the multiplicity is non-negative, that is $k(\alpha) \geq 0$ for all $\alpha \in R$. We write $k \geq 0$ for short. Moreover, let ω_k denote the weight function

$$\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)},$$

which is invariant and homogeneous of degree 2γ. We introduce the Mehta-type constant

$$c_k = \left(\int_{\mathbb{R}^d} \exp(-||x||^2)\omega_k(x)\, dx \right)^{-1}.$$

2.2 The Dunkl operators and the Dunkl kernel

We denote by

- $C(\mathbb{R}^d)$ the space of continuous functions on $\mathbb{R}^d$;
- $C^p(\mathbb{R}^d)$ the space of functions of class C^p on $\mathbb{R}^d$;
- $C_b^p(\mathbb{R}^d)$ the space of bounded functions of class C^p;
- $\mathcal{E}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$;
- $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$;
- $D(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$ which are of compact support;
- $\mathcal{S}'(\mathbb{R}^d)$ the space of temperate distributions on $\mathbb{R}^d$. It is the topological dual of $\mathcal{S}(\mathbb{R}^d)$.

In this subsection we collect some notations and results on the Dunkl operators (see [7, 8] and [9]). The Dunkl operators T_j, $j = 1, \dots, d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \qquad f \in C^1(\mathbb{R}^d).$$

Some properties of the T_j, $j = 1, \dots, d$, are given in the following:

For all f and g in $C^1(\mathbb{R}^d)$ with at least one of them is W-invariant, we have

$$T_j(fg) = (T_j f)g + f(T_j g), \qquad j = 1, \dots, d. \tag{2.1}$$

For f in $C_b^1(\mathbb{R}^d)$ and g in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} T_j f(x) g(x) \omega_k(x)\, dx = - \int_{\mathbb{R}^d} f(x) T_j g(x) \omega_k(x)\, dx, \qquad j = 1, \dots, d. \tag{2.2}$$

We define the Dunkl–Laplace operator on $\mathbb{R}^d$ by

$$\triangle_k f(x) = \sum_{j=1}^d T_j^2 f(x) = \triangle f(x) + 2 \sum_{\alpha \in R^+} k(\alpha) \left[\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2} \right].$$

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x,y) = y_j u(x,y), & j = 1,\dots,d, \\ u(0,y) = 1, \end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $K(x,y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

The Dunkl kernel possesses the following properties:

i) For $z, t \in \mathbb{C}^d$, we have $K(z,t) = K(t,z)$; $K(z,0) = 1$ and $K(\lambda z, t) = K(z, \lambda t)$ for all $\lambda \in \mathbb{C}$.

ii) For all $\nu \in \mathbb{N}^d$, $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ we have

$$|D_z^\nu K(x,z)| \le ||x||^{|\nu|} \exp(||x||\,||\mathrm{Re}\, z||),$$

with

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \cdots \partial z_d^{\nu_d}} \qquad \text{and} \qquad |\nu| = \nu_1 + \cdots + \nu_d.$$

In particular for all $x, y \in \mathbb{R}^d$:

$$|K(-ix, y)| \le 1.$$

iii) The function $K(x,z)$ admits for all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ the following Laplace type integral representation

$$K(x,z) = \int_{\mathbb{R}^d} e^{\langle y,z\rangle} d\mu_x(y), \tag{2.3}$$

where μ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, ||x||)$ of center 0 and radius $\|x\|$ (see [25]).

The Dunkl intertwining operator V_k is the operator from $C(\mathbb{R}^d)$ into itself given by

$$V_k f(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \qquad \text{for all } \; x \in \mathbb{R}^d,$$

where μ_x is the measure given by the relation (2.3) (see [25]). In particular, we have

$$K(x,z) = V(e^{\langle \cdot, z\rangle})(x), \qquad \text{for all } \; x \in \mathbb{R}^d \quad \text{and} \quad z \in \mathbb{C}^d.$$

In [8] C.F. Dunkl proved that V_k is a linear isomorphism from the space of homogeneous polynomial $\mathcal{P}_n$ on $\mathbb{R}^d$ of degree n into itself satisfying the relations

$$\begin{cases} T_j V_k = V_k \dfrac{\partial}{\partial x_j}, & j = 1,\dots,d, \\ V_k(1) = 1. \end{cases} \tag{2.4}$$

K. Trimèche has proved in [31] that the operator V_k can be extended to a topological isomorphism from $\mathcal{E}(\mathbb{R}^d)$ into itself satisfying the relations (2.4).

2.3 The Dunkl transform

We denote by $L_k^p(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$||f||_{L_k^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p \omega_k(x)\, dx \right)^{\frac{1}{p}} < +\infty \qquad \text{if} \quad 1 \le p < +\infty,$$

$$||f||_{L_k^\infty(\mathbb{R}^d)} := \operatorname{ess} \sup_{x \in \mathbb{R}^d} |f(x)| < +\infty.$$

The Dunkl transform of a function f in $L^1_k(\mathbb{R}^d)$ is given by

$$\mathcal{F}_D(f)(y) = \int_{\mathbb{R}^d} f(x)K(-iy,x)\omega_k(x)dx, \qquad \text{for all} \ \ y \in \mathbb{R}^d.$$

In the following we give some properties of this transform (see [9, 10]).

i) For f in $L^1_k(\mathbb{R}^d)$ we have

$$||\mathcal{F}_D(f)||_{L^\infty_k(\mathbb{R}^d)} \leq ||f||_{L^1_k(\mathbb{R}^d)}.$$

ii) For f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\mathcal{F}_D(T_j f)(y) = iy_j \mathcal{F}_D(f)(y), \qquad \text{for all} \ \ j = 1, \dots, d \quad \text{and} \quad y \in \mathbb{R}^d.$$

2.4 The Dunkl convolution

Definition 2.1. Let y be in $\mathbb{R}^d$. The Dunkl translation operator $f \mapsto \tau_y f$ is defined on $\mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{F}_D(\tau_y f)(x) = K(ix,y)\mathcal{F}_D(f)(x), \qquad \text{for all} \ \ x \in \mathbb{R}^d.$$

Proposition 2.1. i) *The operator τ_y, $y \in \mathbb{R}^d$, can also be defined on $\mathcal{E}(\mathbb{R}^d)$ by*

$$\tau_y f(x) = (V_k)_x (V_k)_y [(V_k)^{-1}(f)(x+y)], \qquad \text{for all} \ \ x \in \mathbb{R}^d$$

(see [32]*).*

ii) *If $f(x) = F(||x||)$ in $\mathcal{E}(\mathbb{R}^d)$, then we have*

$$\tau_y f(x) = V_k \left[F(\sqrt{||x||^2 + ||y||^2 + 2\langle x, \cdot \rangle}) \right](x), \qquad \text{for all} \ \ x \in \mathbb{R}^d$$

(see [26]*).*

Using the Dunkl translation operator, we define the Dunkl convolution product of functions as follows (see [30, 32]).

Definition 2.2. The Dunkl convolution product of f and g in $\mathcal{S}(\mathbb{R}^d)$ is the function $f *_D g$ defined by

$$f *_D g(x) = \int_{\mathbb{R}^d} \tau_x f(-y) g(y) \omega_k(y) dy, \qquad \text{for all} \ \ x \in \mathbb{R}^d.$$

Definition 2.3. The Dunkl transform of a distribution τ in $\mathcal{S}'(\mathbb{R}^d)$ is defined by

$$\langle \mathcal{F}_D(\tau), \phi \rangle = \langle \tau, \mathcal{F}_D(\phi) \rangle, \qquad \text{for all} \ \ \phi \in \mathcal{S}(\mathbb{R}^d).$$

Theorem 2.1. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{S}'(\mathbb{R}^d)$ onto itself.*

2.5 The Dunkl–Sobolev spaces

In this subsection we recall some definitions and results on Dunkl–Sobolev spaces (see [22, 23]).

Let τ be in $\mathcal{S}'(\mathbb{R}^d)$. We define the distributions $T_j \tau$, $j = 1, \dots, d$, by

$$\langle T_j \tau, \psi \rangle = -\langle \tau, T_j \psi \rangle, \qquad \text{for all} \ \ \psi \in \mathcal{S}(\mathbb{R}^d).$$

These distributions satisfy the following property

$$\mathcal{F}_D(T_j \tau) = iy_j \mathcal{F}_D(\tau), \qquad j = 1, \dots, d.$$

Definition 2.4. Let $s \in \mathbb{R}$, we define the space $H_k^s(\mathbb{R}^d)$ as the set of distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ satisfying $(1+||\xi||^2)^{\frac{s}{2}}\mathcal{F}_D(u) \in L_k^2(\mathbb{R}^d)$.

We provide this space with the scalar product

$$\langle u, v\rangle_{H_k^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d}(1+||\xi||^2)^s\mathcal{F}_D(u)(\xi)\overline{\mathcal{F}_D(v)(\xi)}\omega_k(\xi)d\xi$$

and the norm

$$||u||^2_{H_k^s(\mathbb{R}^d)} = \langle u, u\rangle_{H_k^s(\mathbb{R}^d)}.$$

Proposition 2.2. i) *For $s \in \mathbb{R}$ and $\mu \in \mathbb{N}^d$, the Dunkl operator T^μ is continuous from $H_k^s(\mathbb{R}^d)$ into $H_k^{s-|\mu|}(\mathbb{R}^d)$.*

ii) *Let $p \in \mathbb{N}$. An element u is in $H_k^s(\mathbb{R}^d)$ if and only if for all $\mu \in \mathbb{N}^d$, with $|\mu| \le p$, $T^\mu u$ belongs to $H_k^{s-p}(\mathbb{R}^d)$, and we have*

$$||u||_{H_k^s(\mathbb{R}^d)} \sim \sum_{|\mu|\le p} ||T^\mu u||_{H_k^{s-p}(\mathbb{R}^d)}.$$

Theorem 2.2. i) *Let u and $v \in H_k^s(\mathbb{R}^d)\bigcap L_k^\infty(\mathbb{R}^d)$, $s > 0$, then $uv \in H_k^s(\mathbb{R}^d)$ and*

$$||uv||_{H_k^s(\mathbb{R}^d)} \le C(k,s)\big[||u||_{L_k^\infty(\mathbb{R}^d)}||v||_{H_k^s(\mathbb{R}^d)} + ||v||_{L_k^\infty(\mathbb{R}^d)}||u||_{H_k^s(\mathbb{R}^d)}\big].$$

ii) *For $s > \frac{d}{2}+\gamma$, $H_k^s(\mathbb{R}^d)$ is an algebra with respect to pointwise multiplications.*

3 Dunkl linear symmetric systems

For any interval I of $\mathbb{R}$ we define the mixed space-time spaces $C(I, H_k^s(\mathbb{R}^d))$, for $s \in \mathbb{R}$, as the spaces of functions from I into $H_k^s(\mathbb{R}^d)$ such that the map

$$t \mapsto ||u(t,\cdot)||_{H_k^s(\mathbb{R}^d)}$$

is continuous. In this section, I designates the interval $[0, T[$, $T > 0$ and

$$u = (u_1, \dots, u_m), \qquad u_p \in C(I, H_k^s(\mathbb{R}^d)),$$

a vector with m components elements of $C(I, H_k^s(\mathbb{R}^d))$. Let $(A_p)_{0\le p\le d}$ be a family of functions from $I \times \mathbb{R}^d$ into the space of $m \times m$ matrices with real coefficients $a_{p,i,j}(t,x)$ which are W-invariant with respect to x and whose all derivatives in $x \in \mathbb{R}^d$ are bounded and continuous functions of (t,x).

For a given $f \in [C(I, H_k^s(\mathbb{R}^d))]^m$ and $v \in [H_k^s(\mathbb{R}^d)]^m$, we find $u \in [C(I, H_k^s(\mathbb{R}^d))]^m$ satisfying the system (1.1).

We shall first define the notion of symmetric systems.

Definition 3.1. The system (1.1) is symmetric, if and only if, for any $p \in \{1, \dots, d\}$ and any $(t,x) \in I \times \mathbb{R}^d$ the matrices $A_p(t,x)$ are symmetric, i.e. $a_{p,i,j}(t,x) = a_{p,j,i}(t,x)$.

In this section, we shall assume $s \in \mathbb{N}$ and denote by $||u(t)||_{s,k}$ the norm defined by

$$||u(t)||^2_{s,k} = \sum_{\substack{1\le p\le m \\ 1\le|\mu|\le s}} ||T_x^\mu u_p(t)||^2_{L_k^2(\mathbb{R}^d)}.$$

3.1 Solvability for Dunkl linear symmetric systems

The aim of this subsection is to prove the following theorem.

Theorem 3.1. *Let* (1.1) *be a symmetric system. Assume that* f *in* $[C(I, H_k^s(\mathbb{R}^d))]^m$ *and* v *in* $[H_k^s(\mathbb{R}^d)]^m$*, then there exists a unique solution* u *of* (1.1) *in* $[C(I, H_k^s(\mathbb{R}^d))]^m \bigcap [C^1(I, H_k^{s-1}(\mathbb{R}^d))]^m$.

The proof of this theorem will be made in several steps:

A. We prove a priori estimates for the regular solutions of the system (1.1).

B. We apply the Friedrichs method.

C. We pass to the limit for regular solutions and we obtain the existence in all cases by the regularization of the Cauchy data.

D. We prove the uniqueness using the existence result on the adjoint system.

A. Energy estimates. The symmetric hypothesis is crucial for the energy estimates which are only valid for regular solutions. More precisely we have:

Lemma 3.1. (Energy Estimate in $[H_s^k(\mathbb{R}^d)]^m$). *For any positive integer* s*, there exists a positive constant* λ_s *such that, for any function* u *in* $[C^1(I, H_k^s(\mathbb{R}^d))]^m \bigcap [C(I, H_k^{s+1}(\mathbb{R}^d))]^m$*, we have*

$$||u(t)||_{s,k} \leq e^{\lambda_s t}||u(0)||_{s,k} + \int_0^t e^{\lambda_s(t-t')}||f(t')||_{s,k}dt', \qquad \text{for all} \quad t \in I, \tag{3.1}$$

with

$$f = \partial_t u - \sum_{p=1}^{d} A_p T_p u - A_0 u.$$

To prove Lemma 3.1, we need the following lemma.

Lemma 3.2. *Let* g *be a* C^1*-function on* $[0, T[$*,* a *and* b *two positive continuous functions. We assume*

$$\frac{d}{dt} g^2(t) \leq 2a(t)g^2(t) + 2b(t)|g(t)|. \tag{3.2}$$

Then, for $t \in [0, T[$*, we have*

$$|g(t)| \leq |g(0)| \exp \int_0^t a(s)ds + \int_0^t b(s) \exp\left(\int_s^t a(\tau)d\tau\right) ds.$$

Proof. To prove this lemma, let us set for $\varepsilon > 0$, $g_\varepsilon(t) = \left(g^2(t) + \varepsilon\right)^{\frac{1}{2}}$; the function g_ε is C^1, and we have $|g(t)| \leq g_\varepsilon(t)$. Thanks to the inequality (3.2), we have

$$\frac{d}{dt}(g^2)(t) \leq 2a(t)g_\varepsilon^2(t) + 2b(t)g_\varepsilon(t).$$

As $\frac{d}{dt}(g^2)(t) = \frac{d}{dt}(g_\varepsilon^2)(t)$. Then

$$\frac{d}{dt}(g_\varepsilon^2)(t) = 2g_\varepsilon(t)\frac{dg_\varepsilon}{dt}(t) \leq 2a(t)g_\varepsilon^2(t) + 2b(t)g_\varepsilon(t).$$

Since for all $t \in [0, T[$ $g_\varepsilon(t) > 0$, we deduce then

$$\frac{dg_\varepsilon}{dt}(t) \leq a(t)g_\varepsilon(t) + b(t).$$

Thus

$$\frac{d}{dt}\left[g_\varepsilon(t)\exp\left(-\int_0^t a(s)ds\right)\right] \leq b(t)\exp\left(-\int_0^t a(s)ds\right).$$

So, for $t \in [0, T[$,

$$g_\varepsilon(t) \leq g_\varepsilon(0)\exp\int_0^t a(s)ds + \int_0^t b(s)\exp\left(\int_s^t a(\tau)d\tau\right)ds.$$

Thus, we obtain the conclusion of the lemma by tending ε to zero. ■

Proof of Lemma 3.1. We prove this estimate by induction on s. We firstly assume that u belongs to $[C^1(I, L_k^2(\mathbb{R}^d))]^m \bigcap [C(I, H_k^1(\mathbb{R}^d))]^m$. We then have $f \in [C(I, L_k^2(\mathbb{R}^d))]^m$, and the function $t \mapsto ||u(t)||_{0,k}^2$ is C^1 on the interval I.

By definition of f we have

$$\frac{d}{dt}||u(t)||_{0,k}^2 = 2\langle \partial_t u, u\rangle_{L_k^2(\mathbb{R}^d)} = 2\langle f, u\rangle_{L_k^2(\mathbb{R}^d)} + 2\langle A_0 u, u\rangle_{L_k^2(\mathbb{R}^d)} + 2\sum_{p=1}^d \langle A_p T_p u, u\rangle_{L_k^2(\mathbb{R}^d)}.$$

We will estimate the third term of the sum above by using the symmetric hypothesis of the matrix A_p. In fact from (2.1) and (2.2) we have

$$\begin{aligned}\langle A_p T_p u, u\rangle_{L_k^2(\mathbb{R}^d)} &= \sum_{1\leq i,j\leq m}\int_{\mathbb{R}^d} a_{p,i,j}(t,x)[(T_p)_x\, u_j(t,x)]u_i(t,x)\omega_k(x)dx\\ &= -\sum_{1\leq i,j\leq m}\int_{\mathbb{R}^d} a_{p,i,j}(t,x)[(T_p)_x\, u_i(t,x)]u_j(t,x)\omega_k(x)dx\\ &\quad -\sum_{1\leq i,j\leq m}\int_{\mathbb{R}^d}[(T_p)_x\, a_{p,i,j}(t,x)]u_j(t,x)u_i(t,x)\omega_k(x)dx.\end{aligned}$$

The matrix A_p being symmetric, we have

$$-\sum_{1\leq i,j\leq m}\int_{\mathbb{R}^d} a_{p,i,j}(t,x)T_p u_i(t,x)u_j(t,x)\omega_k(x)dx = -\langle A_p T_p u, u\rangle_{L_k^2(\mathbb{R}^d)}.$$

Thus

$$\langle A_p T_p u, u\rangle_{L_k^2(\mathbb{R}^d)} = -\frac{1}{2}\sum_{1\leq i,j\leq m}\int_{\mathbb{R}^d} T_p\, a_{p,i,j}(t,x)u_i(t,x)u_j(t,x)\omega_k(x)dx.$$

Since the coefficients of the matrix A_p, as well as their derivatives are bounded on $\mathbb{R}^d$ and continuous on $I \times \mathbb{R}^d$, there exists a positive constant λ_0 such that

$$\frac{d}{dt}||u(t)||_{0,k}^2 \leq 2||f(t)||_{0,k}||u(t)||_{0,k} + 2\lambda_0||u(t)||_{0,k}^2.$$

To complete the proof of Lemma 3.1 in the case $s = 0$ it suffices to apply Lemma 3.2. We assume now that Lemma 3.1 is proved for s.

Let u be the function of $[C^1(I, H_k^{s+1}(\mathbb{R}^d))]^m \cap [C(I, H_k^{s+2}(\mathbb{R}^d))]^m$, we now introduce the function (with $m(d+1)$ components) U defined by

$$U = (u, T_1 u, \ldots, T_d u).$$

Since

$$\partial_t u = f + \sum_{p=1}^{d} A_p T_p u + A_0 u,$$

for any $j \in \{1, \dots, d\}$, applying the operator T_j on the last equation we obtain

$$\partial_t (T_j u) = \sum_{p=1}^{d} A_p T_p (T_j u) + \sum_{p=1}^{d} (T_j A_p) T_p u + T_j (A_0 u) + T_j f.$$

We can then write

$$\partial_t U = \sum_{p=1}^{d} B_p T_p U + B_0 U + F,$$

with

$$F = (f, T_1 f, \dots, T_d f),$$

and

$$B_p = \begin{pmatrix} A_p & 0 & \cdot & \cdot & 0 \\ 0 & A_p & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & A_p \end{pmatrix}, \qquad p = 1, \dots, d,$$

and the coefficients of B_0 can be calculated from the coefficients of A_p and from $T_j A_p$ with $(p = 0, \dots, d)$ and $(j = 1, \dots, d)$. Using the induction hypothesis we then deduce the result, and the proof of Lemma 3.1 is finished. ■

B. Estimate about the approximated solution. We notice that the necessary hypothesis to the proof of the inequalities of Lemma 3.1 require exactly one more derivative than the regularity which appears in the statement of the theorem that we have to prove. We then have to regularize the system (1.1) by adapting the Friedrichs method. More precisely we consider the system

$$\begin{cases} \partial_t u_n - \displaystyle\sum_{p=1}^{d} J_n (A_p T_p (J_n u_n)) - J_n (A_0 J_n u_n) = J_n f, \\ u_n|_{t=0} = J_n u_0, \end{cases} \tag{3.3}$$

with J_n is the cut off operator given by

$$J_n w = (J_n w_1, \dots, J_n w_m) \qquad \text{and} \qquad J_n w_j := \mathcal{F}_D^{-1}(1_{B(0,n)}(\xi) \mathcal{F}_D(w_j)), \qquad j = 1, \dots, m.$$

Now we state the following proposition (see [5, p. 389]) which we need in the sequel of this subsection.

Proposition 3.1. *Let E be a Banach space, I an open interval of $\mathbb{R}$, $f \in C(I, E)$, $u_0 \in E$ and M be a continuous map from I into $\mathcal{L}(E)$, the set of linear continuous applications from E into itself. There exists a unique solution $u \in C^1(I, E)$ satisfying*

$$\begin{cases} \dfrac{du}{dt} = M(t) u + f, \\ u|_{t=0} = u_0. \end{cases}$$

By taking $E = [L_k^2(\mathbb{R}^d)]^m$, and using the continuity of the operators T_pJ_n on $[L_k^2(\mathbb{R}^d)]^m$, we reduce the system (3.3) to an evolution equation

$$\begin{cases} \dfrac{du_n}{dt} = M_n(t)u_n + J_nf, \\ u_n|_{t=0} = J_nu_0 \end{cases}$$

on $[L_k^2(\mathbb{R}^d)]^m$, where

$$t \mapsto M_n(t) = \sum_{p=1}^{d} J_nA_p(t,\cdot)T_pJ_n + J_nA_0(t,\cdot)J_n,$$

is a continuous application from I into $\mathcal{L}([L_k^2(\mathbb{R}^d)]^m)$. Then from Proposition 3.1 there exists a unique function u_n continuous on I with values in $[L_k^2(\mathbb{R}^d)]^m$. Moreover, as the matrices A_p are C^∞ functions of t, $J_nf \in [C(I, L_k^2(\mathbb{R}^d))]^m$ and u_n satisfy

$$\frac{du_n}{dt} = M_n(t)u_n + J_nf.$$

Then $\frac{du_n}{dt} \in [C(I, L_k^2(\mathbb{R}^d))]^m$ which implies that $u_n \in [C^1(I, L_k^2(\mathbb{R}^d))]^m$. Moreover, as $J_n^2 = J_n$, it is obvious that J_nu_n is also a solution of (3.3). We apply Proposition 3.1 we deduce that $J_nu_n = u_n$. The function u_n is then belongs to $[C^1(I, H_k^s(\mathbb{R}^d))]^m$ for any integer s and so (3.3) can be written as

$$\begin{cases} \partial_t u_n - \displaystyle\sum_{p=1}^{d} J_n(A_pT_pu_n) - J_n(A_0u_n) = J_nf, \\ u_n|_{t=0} = J_nu_0. \end{cases}$$

Now, let us estimate the evolution of $\|u_n(t)\|_{s,k}$.

Lemma 3.3. *For any positive integer s, there exists a positive constant λ_s such that for any integer n and any t in the interval I, we have*

$$\|u_n(t)\|_{s,k} \le e^{\lambda_s t}\|J_nu(0)\|_{s,k} + \int_0^t e^{\lambda_s(t-t')}\|J_nf(t')\|_{s,k}dt'.$$

Proof. The proof uses the same ideas as in Lemma 3.1. ∎

C. Construction of solution. This step consists on the proof of the following existence and uniqueness result:

Proposition 3.2. *For $s \ge 0$, we consider the symmetric system* (1.1) *with f in $[C(I, H_k^{s+3}(\mathbb{R}^d))]^m$ and v in $[H_k^{s+3}(\mathbb{R}^d)]^m$. There exists a unique solution u belonging to the space $[C^1(I, H_k^s(\mathbb{R}^d))]^m \cap [C(I, H_k^{s+1}(\mathbb{R}^d))]^m$ and satisfying the energy estimate*

$$\|u(t)\|_{\sigma,k} \le e^{\lambda_s t}\|v\|_{\sigma,k} + \int_0^t e^{\lambda_s(t-\tau)}\|f(\tau)\|_{\sigma,k}d\tau, \qquad \text{for all } \ \sigma \le s+3 \ \text{ and } \ t \in I. \quad (3.4)$$

Proof. Us consider the sequence $(u_n)_n$ defined by the Friedrichs method and let us prove that this sequence is a Cauchy one in $[L^\infty(I, H_k^{s+1}(\mathbb{R}^d))]^m$. We put $V_{n,p} = u_{n+p} - u_n$, we have

$$\begin{cases} \partial_t V_{n,p} - \displaystyle\sum_{j=1}^{d} J_{n+p}(A_jT_jV_{n,p}) - J_{n+p}(A_0V_{n,p}) = f_{n,p}, \\ V_{n,p}|_{t=0} = (J_{n+p} - J_n)v \end{cases}$$

with

$$f_{n,p} = -\sum_{j=1}^{d}(J_{n+p} - J_n)(A_j T_j V_{n,p}) - (J_{n+p} - J_n)(A_0 V_{n,p}) + (J_{n+p} - J_n)f.$$

From Lemma 3.3, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $[L^\infty(I, H_k^{s+3}(\mathbb{R}^d))]^m$. Moreover, by a simple calculation we find

$$\|(J_{n+p} - J_n)(A_j T_j V_{n,p})\|_{s+1,k} \le \frac{C}{n}\|A_j T_j V_{n,p}\|_{s+2,k} \le \frac{C}{n}\|u_n(t)\|_{s+3,k}.$$

Similarly, we have

$$\|(J_{n+p} - J_n)(A_0 V_{n,p}) + (J_{n+p} - J_n)f\|_{s+1,k} \le \frac{C}{n}\big(\|u_n(t)\|_{s+3,k} + \|f(t)\|_{s+3,k}\big).$$

By Lemma 3.3 we deduce that

$$\|V_{n,p}(t)\|_{s+1,k} \le \frac{C}{n}e^{\lambda_s t}.$$

Then $(u_n)_n$ is a Cauchy sequence in $[L^\infty(I, H_k^{s+1}(\mathbb{R}^d))]^m$. We then have the existence of a solution u of (1.1) in $[C(I, H_k^{s+1}(\mathbb{R}^d))]^m$. Moreover by the equation stated in (1.1) we deduce that $\partial_t u$ is in $[C(I, H_k^s(\mathbb{R}^d))]^m$, and so u is in $[C^1(I, H_k^s(\mathbb{R}^d))]^m$. The uniqueness follows immediately from Lemma 3.3.

Finally we will prove the inequality (3.4). From Lemma 3.3 we have

$$\|u_n(t)\|_{s+3,k} \le e^{\lambda_s t}\|J_n u(0)\|_{s+3,k} + \int_0^t e^{\lambda_s(t-\tau)}\|J_n f(\tau)\|_{s+3,k}d\tau.$$

Thus

$$\limsup_{n\to\infty}\|u_n(t)\|_{s+3,k} \le e^{\lambda_s t}\|v\|_{s+3,k} + \int_0^t e^{\lambda_s(t-\tau)}\|f(\tau)\|_{s+3,k}d\tau.$$

Since for any $t \in I$, the sequence $(u_n(t))_{n\in\mathbb{N}}$ tends to $u(t)$ in $[H_k^{s+1}(\mathbb{R}^d)]^m$, $(u_n(t))_{n\in\mathbb{N}}$ converge weakly to $u(t)$ in $[H_k^{s+3}(\mathbb{R}^d)]^m$, and then

$$u(t) \in [H_k^{s+3}(\mathbb{R}^d)]^m \text{ and } \|u(t)\|_{s+3,k} \le \limsup_{n\to\infty}\|u_n(t)\|_{s+3,k}.$$

The Proposition 3.2 is thus proved. ■

Now we will prove the existence part of Theorem 3.1.

Proposition 3.3. *Let s be an integer. If v is in $[H_k^s(\mathbb{R}^d)]^m$ and f is in $[C(I, H_k^s(\mathbb{R}^d))]^m$, then there exists a solution of a symmetric system (1.1) in the space*

$$[C(I, H_k^s(\mathbb{R}^d))]^m \cap [C^1(I, H_k^{s-1}(\mathbb{R}^d))]^m.$$

Proof. We consider the sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ of solutions of

$$\begin{cases} \partial_t \tilde{u}_n - \displaystyle\sum_{j=1}^{d}(A_j T_j \tilde{u}_n) - (A_0 \tilde{u}_n) = J_n f, \\ \tilde{u}_n|_{t=0} = J_n v. \end{cases}$$

From Proposition 3.2 $(\tilde{u}_n)_n$ is in $[C^1(I, H^s_k(\mathbb{R}^d))]^m$. We will prove that $(\tilde{u}_n)_n$ is a Cauchy sequence in $[L^\infty(I, H^s_k(\mathbb{R}^d))]^m$. We put $\tilde{V}_{n,p} = \tilde{u}_{n+p} - \tilde{u}_n$. By difference, we find

$$\begin{cases} \partial_t \tilde{V}_{n,p} - \displaystyle\sum_{j=1}^d A_j T_j \tilde{V}_{n,p} - A_0 \tilde{V}_{n,p} = (J_{n+p} - J_n)f, \\ \widetilde{V_{n,p}}|_{t=0} = (J_{n+p} - J_n)v. \end{cases}$$

By Lemma 3.3 we deduce that

$$\|\tilde{V}_{n,p}\|_{s,k} \leq e^{\lambda_s t} \|(J_{n+p} - J_n)v\|_{s,k} + \int_0^t e^{\lambda_s (t-\tau)} \|(J_{n+p} - J_n) f(\tau)\|_{s,k} d\tau.$$

Since f is in $[C(I, H^s_k(\mathbb{R}^d))]^m$, the sequence $(J_n f)_n$ converges to f in $[L^\infty([0,T], H^s_k(\mathbb{R}^d))]^m$, and since v is in $[H^s_k(\mathbb{R}^d)]^m$, the sequence $(J_n v)_n$ converge to v in $[H^s_k(\mathbb{R}^d)]^m$ and so $(\tilde{u}_n)_n$ is a Cauchy sequence in $[L^\infty(I, H^s_k(\mathbb{R}^d))]^m$. Hence it converges to a function u of $[C(I, H^s_k(\mathbb{R}^d))]^m$, solution of the system (1.1). Thus $\partial_t u$ is in $[C(I, H^{s-1}_k(\mathbb{R}^d))]^m$ and the proposition is proved. ■

The existence in Theorem 3.1 is then proved as well as the uniqueness, when $s \geq 1$.

D. Uniqueness of solutions. In the following we give the result of uniqueness for $s = 0$ and hence Theorem 3.1 is proved.

Proposition 3.4. *Let u be a solution in $[C(I, L^2_k(\mathbb{R}^d))]^m$ of the symmetric system*

$$\begin{cases} \partial_t u - \displaystyle\sum_{j=1}^d A_j T_j u - A_0 u = 0, \\ u|_{t=0} = 0. \end{cases}$$

Then $u \equiv 0$.

Proof. Let ψ be a function in $[D(]0,T[\times\mathbb{R}^d)]^m$; we consider the following system

$$\begin{cases} -\partial_t \varphi + \displaystyle\sum_{j=1}^d T_j(A_j \varphi) - {}^t A_0 \varphi = \psi, \\ \varphi|_{t=T} = 0. \end{cases} \tag{3.5}$$

Since

$$T_j(A_j\varphi) = A_j T_j \varphi + (T_j A_j)\varphi,$$

the system (3.5) can be written

$$\begin{cases} -\partial_t \varphi + \displaystyle\sum_{j=1}^d A_j T_j \varphi - \tilde{A}_0 \varphi = \psi, \\ \varphi|_{t=T} = 0 \end{cases} \tag{3.6}$$

with

$$\tilde{A}_0 = {}^t A_0 - \sum_{j=1}^d T_j A_j.$$

Due to Proposition 3.2, for any integer s there exists a solution φ of (3.6) in $[C^1([0,T],H_k^s(\mathbb{R}))]^m$. We then have

$$\begin{aligned}\langle u,\psi\rangle_k &= \langle u, -\partial_t\varphi + \sum_{j=1}^d A_jT_j\varphi - \tilde{A}_0\varphi\rangle_k \\ &= -\int_I \langle u(t,\cdot),\partial_t\varphi(t,\cdot)\rangle_k dt + \sum_{j=1}^d \int_{I\times\mathbb{R}^d} u(t,x)T_j(A_j\varphi)(t,x)\omega_k(x)dtdx \\ &\quad - \int_{I\times\mathbb{R}^d} u(t,x)\,{}^tA_0\varphi(t,x)\omega_k(x)dtdx\end{aligned}$$

with $\langle\cdot,\cdot\rangle_k$ defined by

$$\langle u,\chi\rangle_k = \int_I \langle u(t,\cdot),\chi(t,\cdot)\rangle_k dt = \int_{I\times\mathbb{R}^d} u(t,x)\chi(t,x)\omega_k(x)dxdt, \qquad \chi\in\mathcal{S}(\mathbb{R}^{d+1}).$$

By using that $u(t,\cdot)$ is in $[L_k^2(\mathbb{R}^d)]^m$ for any t in I and the fact that A_j is symmetric we obtain

$$\int_{I\times\mathbb{R}^d} u(t,x)T_j(A_j\varphi)(t,x)\omega_k(x)dtdx = -\int_I \langle A_jT_ju(t,\cdot),\varphi(t,\cdot)\rangle_k dt.$$

So

$$\langle u,\psi\rangle_k = -\int_I \langle u(t,\cdot),\partial_t\varphi(t,\cdot)\rangle_k dt - \sum_{j=1}^d \langle A_jT_ju + A_0u,\varphi\rangle_k.$$

As u is not very regular, we have to justify the integration by parts in time on the quantity $\int_I \langle u(t,\cdot),\partial_t\varphi(t,\cdot)\rangle_k dt$. Since $J_nu(\cdot,x)$ are C^1 functions on I, then by integration by parts, we obtain, for any $x\in\mathbb{R}^d$,

$$-\int_I J_nu(t,x)\partial_t\varphi(t,x)dt = -J_nu(T,x)\varphi(T,x) + J_nu(0,x)\varphi(0,x) + \int_I \partial_tJ_nu(t,x)\varphi(t,x)dt.$$

Since $u(0,\cdot)=\varphi(T,\cdot)=0$, we have

$$-\int_I J_nu(t,x)\partial_t\varphi(t,x)dt = \int_I \partial_t(J_nu)(t,x)\varphi(t,x)dt.$$

Integrating with respect to $\omega_k(x)dx$ we obtain

$$-\int_{I\times\mathbb{R}^d} J_nu(t,x)\partial_t\varphi(t,x)\omega_k(x)dtdx = \int_I \langle\partial_t(J_nu)(t,\cdot),\varphi(t,\cdot)\rangle_k dt. \tag{3.7}$$

Since u is in $[C(I,L_k^2(\mathbb{R}^d))]^m \cap [C^1(I,H_k^{-1}(\mathbb{R}^d))]^m$, we have

$$\lim_{n\to\infty} J_nu = u \text{ in } [L^\infty(I,L_k^2(\mathbb{R}^d))]^m \qquad \text{and} \qquad \lim_{n\to\infty} J_n\partial_tu = \partial_tu \quad \text{in} \quad [L^\infty(I,H_k^{-1}(\mathbb{R}^d))]^m.$$

By passing to the limit in (3.7) we obtain

$$-\int_I \langle u(t,\cdot),\partial_t\varphi(t,\cdot)\rangle_k dt = \int_I \langle\partial_tu(t,\cdot),\varphi(t,\cdot)\rangle_k dt.$$

Hence

$$\langle u,\psi\rangle_k = \int_I \langle\partial_tu(t,\cdot) - \sum_{j=1}^d \langle A_jT_ju(t,\cdot) - A_0u(t,\cdot),\varphi(t,\cdot)\rangle_k dt.$$

However since u is a solution of (1.1) with $f\equiv 0$, then $u\equiv 0$. This ends the proof. ■

3.2 The Dunkl-wave equations with variable coefficients

For $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, let $P(t, x, \partial_t, T_x)$ be a differential-difference operator of degree 2 defined by

$$P(u) = \partial_t^2 u - \mathrm{div}_k[A \cdot \nabla_{k,x} u] + Q(t, x, \partial_t u, T_x u),$$

where

$$\nabla_{k,x} u := (T_1 u, \dots, T_d\, u)\,, \qquad \mathrm{div}_k\,(v_1, \dots, v_d) := \sum_{i=1}^{d} T_i v_i,$$

A is a real symmetric matrix such that there exists $m > 0$ satisfying

$$\langle A(t,x)\xi, \xi\rangle \geq m\|\xi\|^2, \qquad \text{for all } \ (t,x) \in \mathbb{R} \times \mathbb{R}^d \ \text{ and } \ \xi \in \mathbb{R}^d$$

and $Q(t, x, \partial_t u, T_x u)$ is differential-difference operator of degree 1, and the matrix A is W-invariant with respect to x; the coefficients of A and Q are C^∞ and all derivatives are bounded. If we put $B = \sqrt{A}$ it is easy to see that the coefficients of B are C^∞ and all derivatives are bounded.

We introduce the vector U with $d + 2$ components

$$U = (u, \partial_t u, B\nabla_{k,x} u)\,.$$

Then, the equation $P(u) = f$ can be written as

$$\partial_t U = \left(\sum_{p=1}^{d} A_p T_p\right) U + A_0 U + (0, f, 0) \tag{3.8}$$

with

$$A_p = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & b_{p\,1} & \cdot & \cdot & \cdot & b_{p\,d} \\ \cdot & b_{1\,p} & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & b_{d\,p} & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

and $B = (b_{ij})$. Thus the system (3.8) is symmetric and from Theorem 3.1 we deduce the following.

Theorem 3.2. *For all $s \in \mathbb{N}$ and $u_0 \in H_k^{s+1}(\mathbb{R}^d)$, $u_1 \in H_k^s(\mathbb{R}^d)$ and $f \in C(\mathbb{R}, H_k^s(\mathbb{R}^d))$, there exists a unique $u \in C^1(\mathbb{R}, H_k^s(\mathbb{R}^d)) \cap C(\mathbb{R}, H_k^{s+1}(\mathbb{R}^d))$ such that*

$$\begin{cases} \partial_t^2 u - \mathrm{div}_k[A \cdot \nabla_{k,x} u] + Q(t, x, \partial_t u, T_x u) = f, \\ u|_{t=0} = u_0, \\ \partial_t u|_{t=0} = u_1. \end{cases}$$

3.3 Finite speed of propagation

Theorem 3.3. *Let (1.1) be a symmetric system. There exists a positive constant C_0 such that, for any positive real R, any function $f \in [C(I, L_k^2(\mathbb{R}^d))]^m$ and any $v \in [L_k^2(\mathbb{R}^d)]^m$ satisfying*

$$f(t,x) \equiv 0 \qquad \text{for} \quad \|x\| < R - C_0 t, \tag{3.9}$$

$$v(x) \equiv 0 \qquad \text{for} \quad \|x\| < R, \tag{3.10}$$

the unique solution u of system (1.1) belongs to $[C(I, L_k^2(\mathbb{R}^d))]^m$ with

$$u(t,x) \equiv 0 \qquad \text{for} \quad \|x\| < R - C_0 t.$$

Proof. If $f_\varepsilon \in [C(I, H^1_k(\mathbb{R}^d))]^m$, $v_\varepsilon \in [H^1_k(\mathbb{R}^d))]^m$ are given such that $f_\varepsilon \to f$ in $[C(I, L^2_k(\mathbb{R}^d))]^m$ and $v_\varepsilon \to v$ in $[L^2_k(\mathbb{R}^d)]^m$, we know by Subsection 3.1 that the solution u_ε belongs to $[C(I, H^1_k(\mathbb{R}^d))]^m$ and satisfies $u_\varepsilon \to u$ in $[C(I, L^2_k(\mathbb{R}^d))]^m$. Therefore, if we construct f_ε and v_ε satisfying (3.9) and (3.10) with R replaced by $R - \varepsilon$, it suffices to prove Theorem 3.3 for $f \in [C(I, H^1_k(\mathbb{R}^d))]^m$ and $v \in [H^1_k(\mathbb{R}^d))]^m$. We have then $u \in [C^1(I, L^2_k(\mathbb{R}^d))]^m \bigcap [C(I, H^1_k(\mathbb{R}^d))]^m$. To this end let us consider $\chi \in D(\mathbb{R}^d)$ radial with $\operatorname{supp}\chi \subset B(0,1)$ and

$$\int_{\mathbb{R}^d} \chi(x)\omega_k(x)dx = 1.$$

For $\varepsilon > 0$, we put

$$u_{0,\varepsilon} = \chi_\varepsilon *_D v := (\chi_\varepsilon *_D v_1, \dots, \chi_\varepsilon *_D v_d),$$
$$f_\varepsilon(t,\cdot) = \chi_\varepsilon *_D f(t,\cdot) := (\chi_\varepsilon *_D f_1(t,\cdot), \dots, \chi_\varepsilon *_D f_d(t,\cdot)),$$

with

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon^{d+2\gamma}}\chi\left(\frac{x}{\varepsilon}\right).$$

The hypothesis (3.9) and (3.10) are then satisfied by f_ε and $u_{0,\varepsilon}$ if we replace R by $R-\varepsilon$. On the other hand the solution u_ε associated with f_ε and $u_{0,\varepsilon}$ is $[C^1(I, H^s_k(\mathbb{R}^d))]^m$ for any integer s. For $\tau \geq 1$, we put

$$u_\tau(t,x) = \exp\big(\tau(-t + \psi(x))\big)u(t,x),$$

where the function $\psi \in C^\infty(\mathbb{R}^d)$ will be chosen later.

By a simple calculation we see that

$$\partial_t u_\tau - \sum_{j=1}^d A_j T_j u_\tau - B_\tau u_\tau = f_\tau$$

with

$$f_\tau(t,x) = \exp(\tau(-t+\psi(x)))f(t,x), \qquad B_\tau = A_0 + \tau\left(-\mathrm{Id} - \sum_{j=1}^d (T_j\psi)A_j\right).$$

There exists a positive constant K such that if $\|T_j\psi\|_{L^\infty_k(\mathbb{R}^d)} \leq K$ for any $j = 1, \dots, d$, we have for any (t,x)

$$\langle \mathrm{Re}(B_\tau y), \bar{y}\rangle \leq \langle \mathrm{Re}(A_0 y), \bar{y}\rangle \qquad \text{for all } \ \tau \geq 1 \ \text{ and } \ y \in \mathbb{C}^m.$$

We proceed as in the proof of energy estimate (3.1), we obtain the existence of positive constant δ_0, independent of τ, such that for any t in I, we have

$$\|u_\tau(t)\|_{0,k} \leq e^{\delta_0 t}\|u_\tau(0)\|_{0,k} + \int_0^t e^{\delta_0(t-t')}\|f_\tau(t')\|_{0,k}dt'. \tag{3.11}$$

We put $C_0 = \frac{1}{K}$ and choose $\psi = \psi(\|x\|)$ such that ψ is C^∞ and such that

$$-2\varepsilon + K(R - \|x\|) \leq \psi(x) \leq -\varepsilon + K(R - \|x\|).$$

There exists $\varepsilon > 0$ such that $\psi(x) \leq -\varepsilon + K(R - \|x\|)$. Hence

$$\|x\| \geq R - C_0 t, \qquad \text{for all } \ (t,x) \quad \Longrightarrow \quad -t + \psi(x) \leq -\varepsilon.$$

Let τ tend to $+\infty$ in (3.11), we deduce that

$$\lim_{\tau+\infty}\int_{\mathbb{R}^d}\exp(2\tau(-t+\psi(x)))||u(t,x)||^2\omega_k(x)dx=0, \qquad \text{for all } \ t\in I.$$

Then

$$u(t,x)=0 \qquad \text{on } \ \big\{(t,x)\in I\times\mathbb{R}^d;\ t<\psi(x)\big\}.$$

However if (t_0,x_0) satisfies $\|x_0\|<R-C_0t_0$, we can find a function ψ of previous type such that $t_0<\psi(x_0)$. Thus the theorem is proved. ■

Theorem 3.4. *Let* (1.1) *be a symmetric system. We assume that the functions* $f\in[C(I,L^2_k(\mathbb{R}^d))]^m$ *and* $v\in[L^2_k(\mathbb{R}^d)]^m$ *satisfy*

$$f(t,x)\equiv 0 \qquad \textit{for } \ \|x\|>R+C_0t,$$
$$v(x)\equiv 0 \qquad \textit{for } \ \|x\|>R.$$

Then the unique solution u *of system* (1.1) *belongs to* $[C(I,L^2_k(\mathbb{R}^d))]^m$ *with*

$$u(t,x)\equiv 0 \qquad \textit{for } \ \|x\|>R+C_0t.$$

Proof. The proof uses the same ideas as in Theorem 3.3. ■

4 Semi-linear Dunkl-wave equations

We consider the problem (1.2). We denote by $\|\Lambda_k u(t,\cdot)\|_{L^\infty_k}$ the norm defined by

$$\|\Lambda_k u(t,\cdot)\|_{L^\infty_k}=\|\partial_t u(t,\cdot)\|_{L^\infty_k(\mathbb{R}^d)}+\sum_{j=1}^d\|T_j u(t,\cdot)\|_{L^\infty_k(\mathbb{R}^d)}.$$

$\|\Lambda_k u(t,\cdot)\|_{s,k}$ the norm defined by

$$\|\Lambda_k u(t,\cdot)\|^2_{s,k}=\|\partial_t u(t,\cdot)\|^2_{H^s_k(\mathbb{R}^d)}+\sum_{j=1}^d\|T_j u(t,\cdot)\|^2_{H^s_k(\mathbb{R}^d)}.$$

The main result of this section is the following:

Theorem 4.1. *Let* (u_0,u_1) *be in* $H^s_k(\mathbb{R}^d)\times H^{s-1}_k(\mathbb{R}^d)$ *for* $s>\gamma+\frac{d}{2}+1$. *Then there exists a positive time* T *such that the problem* (1.2) *has a unique solution* u *belonging to*

$$C([0,T],H^s_k(\mathbb{R}^d))\cap C^1([0,T],H^{s-1}_k(\mathbb{R}^d))$$

and satisfying the following blow up criteria: if T^* *denotes the maximal time of existence of such a solution, we have:*

– *The existence of constant* C *depending only on* γ, d *and quadratic form* Q *such that*

$$T^*\geq\frac{C(\gamma,d,Q)}{\|\Lambda_k u(0,\cdot)\|_{s-1,k}}. \tag{4.1}$$

– *If* $T^*<\infty$, *then*

$$\int_0^{T^*}\|\Lambda_k u(t,\cdot)\|_{L^\infty_k}dt=+\infty. \tag{4.2}$$

To prove Theorem 4.1 we need the following lemmas.

Lemma 4.1. (Energy Estimate in $H^k_s(\mathbb{R}^d)$.) *If u belongs to $C^1(I, H^s_k(\mathbb{R}^d)) \cap C(I, H^{s+1}_k(\mathbb{R}^d))$ for an integer s and with f defined by*

$$f = \partial_t^2 u - \Delta_k u$$

then we have

$$\|\Lambda_k u(t,\cdot)\|_{s-1,k} \leq \|\Lambda_k u(0,\cdot)\|_{s-1,k} + \int_0^t \|f(t',\cdot)\|_{H_k^{s-1}(\mathbb{R}^d)} dt', \qquad \textit{for} \;\; t \in I. \tag{4.3}$$

Proof. We multiply the equation by $\partial_t u$ and we obtain

$$(\partial_t^2 u, \partial_t u)_{H_k^{s-1}(\mathbb{R}^d)} - (\Delta_k u, \partial_t u)_{H_k^{s-1}(\mathbb{R}^d)} = \langle f, \partial_t u\rangle_{H_k^{s-1}(\mathbb{R}^d)}.$$

A simple calculation yields that

$$-\langle \Delta_k u, \partial_t u\rangle_{H_k^{s-1}(\mathbb{R}^d)} = \langle \nabla_k u, \nabla_k \partial_t u\rangle_{H_k^{s-1}(\mathbb{R}^d)}.$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|\Lambda_k u\|^2_{s-1,k} = \langle f, \partial_t u\rangle_{H_k^{s-1}(\mathbb{R}^d)}.$$

If $f = 0$, we deduce the conservation of energy

$$\|\Lambda_k u(t,\cdot)\|^2_{s-1,k} = \|\Lambda_k u(0,\cdot)\|^2_{s-1,k}.$$

Otherwise, Lemma 3.2 gives

$$\|\Lambda_k u(t,\cdot)\|_{s-1,k} \leq \|\Lambda_k u(0,\cdot)\|_{s-1,k} + \int_0^t \|f(t',\cdot)\|_{H_k^{s-1}(\mathbb{R}^d)} dt'.$$

■

Lemma 4.2. *Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined by*

$$\partial_t^2 u_{n+1} - \Delta_k u_{n+1} = Q(\Lambda_k u_n, \Lambda_k u_n),$$
$$(u_{n+1}, \partial_t u_{n+1})|_{t=0} = (S_{n+1}u_0, S_{n+1}u_1),$$

where $u_0 = 0$ and $S_{n+1}u_j$ defined by

$$\mathcal{F}_D(S_{n+1}u_j)(\xi) = \psi(2^{-(n+1)}\xi)\mathcal{F}_D(u_j)(\xi),$$

with ψ a function of $D(\mathbb{R}^d)$ such that $0 \leq \psi \leq 1$ and $\operatorname{supp}\psi \subset B(0,1)$.

Then there exists a positive time T such that, the sequence $(\Lambda_k u_n)_{n\in\mathbb{N}}$ is bounded in the space $[L^\infty([0,T], H_k^{s-1}(\mathbb{R}^d))]^{d+1}$.

Proof. First, from Theorem 3.2 the sequence $(u_n)_n$ is well defined. Moreover, due to the energy estimate,

$$\|\Lambda_k u_{n+1}(t,\cdot)\|_{s-1,k} \leq \|S_{n+1}\theta\|_{s-1,k} + \int_0^t \|Q(\Lambda_k u_n(t',\cdot), \Lambda_k u_n(t',\cdot))\|_{H_k^{s-1}(\mathbb{R}^d)} dt',$$

where

$$\theta = (u_1, \nabla_k u_0) = \Lambda_k u(0,\cdot).$$

Since $s-1>\gamma+\frac{d}{2}$, then from Theorem 2.2 ii, we have

$$\|\Lambda_k u_{n+1}(t,\cdot)\|_{s-1,k} \leq \|\theta\|_{s-1,k} + C\int_0^t \|\Lambda_k u_n(\tau,\cdot)\|^2_{s-1,k} d\tau. \tag{4.4}$$

Let T be a positive real such that

$$4CT\|\theta\|_{s-1,k} < 1. \tag{4.5}$$

We will prove by induction that, for any integer n

$$\|\Lambda_k u_{n+1}(t,\cdot)\|_{s-1,k} \leq 2\|\theta\|_{s-1,k}. \tag{4.6}$$

This property is true for $n=0$. We assume that it is true for n. With the inequalities (4.4) and (4.5), we deduce that, for all $t \leq T$, we have

$$\begin{aligned}
&\|\Lambda_k u_{n+1}(t,\cdot)\|_{s-1,k} \leq \|\theta\|_{s-1,k} + 4CT\|\theta\|^2_{s-1,k} \leq (1+4CT\|\theta\|_{s-1,k})\|\theta\|_{s-1,k},\\
&\|\Lambda_k u_{n+1}(t,\cdot)\|_{H^{s-1}_k(\mathbb{R}^d)} \leq 2\|\theta\|_{s-1,k}.
\end{aligned}$$

This gives (4.6) and the proof of lemma is established. ■

Lemma 4.3. *There exists a positive time T such that, $(\Lambda_k u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $[L^\infty([0,T], H^{s-1}_k(\mathbb{R}^d))]^{d+1}$.*

Proof. We put

$$V_{n,p} = u_{n+p} - u_n.$$

By difference, we see that

$$\begin{cases} \partial_t^2 V_{n+1,p} - \Delta_k V_{n+1,p} = Q(\Lambda_k V_{n,p}, \Lambda_k u_{n+p} + \Lambda_k u_n),\\ \Lambda_k V_{n+1,p}|_{t=0} = (S_{n+p+1} - S_{n+1})\theta. \end{cases}$$

By energy estimate, we establish from (4.6) that

$$\begin{aligned}
\|\Lambda_k V_{n+1,p}(t,\cdot)\|_{s-1,k} \leq\ & \|(S_{n+p+1} - S_{n+1})\theta\|_{s-1,k}\\
&+ 4CT\|\theta\|_{s-1,k}\|\Lambda_k V_{n,p}\|_{[L^\infty([0,T],H^{s-1}_k(\mathbb{R}^d))]^{d+1}}.
\end{aligned}$$

We put

$$\rho_n = \sup_{p\in\mathbb{N}} \|\Lambda_k V_{n,p}\|_{[L^\infty([0,T],H^{s-1}_k(\mathbb{R}^d))]^{d+1}} \qquad \text{and} \qquad \varepsilon_n = \sup_{p\in\mathbb{N}} \|(S_{n+p} - S_n)\theta\|_{s-1,k}.$$

From this and the last inequality, we have

$$\rho_{n+1} \leq \varepsilon_{n+1} + 4CT\rho_n\|\theta\|_{s-1,k}.$$

The sequence $(S_n\theta)_{n\in\mathbb{N}}$ converges to θ in $[H^{s-1}_s(\mathbb{R}^d)]^{d+1}$. By passing to the superior limit, we obtain

$$\limsup_{n\to\infty}(\rho_{n+1}) \leq 0 + 4CT\|\theta\|_{s-1,k} \limsup_{n\to\infty}(\rho_n).$$

However, since

$$\limsup_{n\to\infty}(\rho_{n+1}) = \limsup_{n\to\infty}(\rho_n),$$

we deduce

$$\limsup_{n\to\infty}(\rho_n) \le 4CT\|\theta\|_{s-1,k} \limsup_{n\to\infty}(\rho_n),$$

and the result holds by $4CT\|\theta\|_{s-1,k} < 1$.

Hence

$$\limsup_{n\to\infty}(\rho_n) = 0.$$

Then $(\Lambda_k u_n)$ is a Cauchy sequence in $[L^\infty([0,T], H_k^{s-1}(\mathbb{R}^d))]^{d+1}$, and so Lemma 4.3 is proved. ■

Proof of Theorem 4.1. In the following we will prove that the unique solution of (1.2) belongs to $C([0,T], H_k^{s-1}(\mathbb{R}^d))$. Indeed, Lemma 4.3 implies the existence of v in $[L^\infty([0,T], H_k^{s-1}(\mathbb{R}^d))]^{d+1}$ such that

$$\Lambda_k u_n \to v \qquad \text{in} \ \ [L^\infty([0,T], H_k^{s-1}(\mathbb{R}^d))]^{d+1}.$$

Moreover, Lemma 4.2 gives the existence of a positive time T such that the sequence $(u_n)_n$ is bounded in $L^\infty([0,T], H_k^s(\mathbb{R}^d))$. Thus there exists u such that the sequence $(u_n)_n$ converges weakly to u in $L^\infty([0,T], H_k^s(\mathbb{R}^d))$.

The uniqueness for the solution of (1.2) gives that $v = \Lambda_k u$ and that

$$u_n \to u \qquad \text{in} \ \ L^\infty([0,T], H_k^s(\mathbb{R}^d)).$$

Finally it is easy to see that u is the unique solution of (1.2) which belongs to $C([0,T], H_k^s(\mathbb{R}^d))$.

Now we are going to prove the inequalities (4.1). We have proved that if $T < \frac{1}{4C\|\theta\|_{s-1,k}}$, then $u \in C([0,T], H_k^s(\mathbb{R}^d))$. Hence, if T^* denote the maximal time of existence of such a solution we have $T^* > T$ and this gives that $T^* \ge \frac{1}{4C\|\theta\|_{s-1,k}}$ and $u \in C([0,T^*[, H_k^s(\mathbb{R}^d))$. Finally we will prove the condition (4.2). We assume that $T^* < \infty$ and $\int_0^{T^*} \|\Lambda_k u(t)\|_{s-1,k} dt < \infty$. Indeed, it is easy to see that the maximal time of solution of problem (1.2) with initial given $u(t)$ is $T^* - t$. Thus, from the relation (4.1) we deduce that

$$T^* - t \ge \frac{C}{\|\Lambda_k u(t,\cdot)\|_{s-1,k}}.$$

This implies that

$$\|\Lambda_k u(t,\cdot)\|_{s-1,k} \ge \frac{C}{T^* - t}, \qquad \text{for all} \ \ t \in [0, T^*[. \tag{4.7}$$

Hence $\|\Lambda_k u(t,\cdot)\|_{s-1,k}$ is not bounded if t tends to T^*.

On the other hand from (4.3) and Theorem 2.2 i there exists a positive constant C such that

$$\begin{aligned}\|\Lambda_k u(t)\|_{s-1,k} \le{}& \|\Lambda_k u(0)\|_{s-1,k} \\ &+ C\int_0^t \|\Lambda_k u(t',\cdot)\|_{L_k^\infty} \|\Lambda_k u(t',\cdot)\|_{s-1,k} dt', \qquad \text{for all} \ \ t \in [0,T^*[.\end{aligned}$$

Then from the usual Gronwall lemma we obtain

$$\begin{aligned}\|\Lambda_k u(t,\cdot)\|_{s-1,k} \le{}& C\|\Lambda_k u(0,\cdot)\|_{s-1,k} \\ &\times \exp\left(\int_0^t \|\Lambda_k u(t',\cdot)\|_{L_k^\infty} dt'\right), \qquad \text{for all} \ \ t \in [0,T^*[. \end{aligned}\tag{4.8}$$

Finally if we tend t to T^* in (4.8) we obtain that $\|\Lambda_k u(t)\|_{s-1,k}$ is bounded which is not true from (4.7). Thus we have proved (4.2) and the proof of Theorem 4.1 is finished. ■

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