Local Quasitriangular Hopf Algebras

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Abstract. We find a new class of Hopf algebras, local quasitriangular Hopf algebras, which generalize quasitriangular Hopf algebras. Using these Hopf algebras, we obtain solutions of the Yang–Baxter equation in a systematic way. The category of modules with finite cycles over a local quasitriangular Hopf algebra is a braided tensor category.

Key words: Hopf algebra; braided category

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1 Introduction

The Yang–Baxter equation first came up in the paper by Yang as factorization condition of the scattering S-matrix in the many-body problem in one dimension and in the work by Baxter on exactly solvable models in statistical mechanics. It has been playing an important role in mathematics and physics (see [2, 16]). Attempts to find solutions of the Yang–Baxter equation in a systematic way have led to the theory of quantum groups and quasitriangular Hopf algebras (see [6, 9]).

Since the category of modules with finite cycles over a local quasitriangular Hopf algebra is a braided tensor category, we may also find solutions of the Yang–Baxter equation in a systematic way.

The main results in this paper are summarized in the following statement.

Theorem 1. *(i) Assume that $(H, \{R_n\})$ is a local quasitriangular Hopf algebra. Then $({}_H\mathcal{M}^{\rm cf}, C^{\{R_n\}})$, $({}_H\mathcal{M}^{\rm dcf}, C^{\{R_n\}})$ and $({}_H\mathcal{M}^{\rm df}, C^{\{R_n\}})$ are braided tensor categories. Furthermore, if (M, α^-) is an H-module with finite cycles and $R_{n+1} = R_n + W_n$ with $W_n \in H_{n+1} \otimes H_{(n+1)}$, then (M, α^-, δ^-) is a Yetter–Drinfeld H-module.*

(ii) Assume that B is a finite dimensional Hopf algebra and M is a finite dimensional B-Hopf bimodule. Then $((T_B(M))^{\rm cop} \bowtie_\tau T^c_{B^*}(M^*), \{R_n\})$ is a local quasitriangular Hopf algebra. Furthermore, if (kQ^a, kQ^c) and (kQ^s, kQ^{sc}) are arrow dual pairings with finite Hopf quiver Q, then both $((kQ^a)^{\rm cop} \bowtie_\tau kQ^c, \{R_n\})$ and $((kQ^s)^{\rm cop} \bowtie_\tau kQ^{sc}, \{R_n\})$ are local quasitriangular Hopf algebras.

2 Preliminaries

Throughout, we work over a fixed field k. All algebras, coalgebras, Hopf algebras, and so on, are defined over k. Books [7, 11, 15, 13] provide the necessary background for Hopf algebras and book [1] provides a nice description of the path algebra approach.

Let V and W be two vector spaces. σ_V denotes the map from V to V^{**} by defining $\langle \sigma_V(x), f\rangle = \langle f, x\rangle$ for any $f \in V^*$, $x \in V$. $C_{V,W}$ denotes the map from $V \otimes W$ to $W \otimes V$

by defining $C_{V,W}(x\otimes y)=y\otimes x$ for any $x\in V$, $y\in W$. Denote P by $\sum P'\otimes P''$ for $P\in V\otimes W$. If V is a finite-dimensional vector space over field k with $V^*=\mathrm{Hom}_k(V,k)$. Define maps $b_V: k\to V\otimes V^*$ and $d_V: V^*\otimes V\to k$ by

$$b_V(1)=\sum_i v_i\otimes v_i^* \qquad \text{and} \qquad \sum_{i,j} d_V(v_i^*\otimes v_j)=\langle v_i^*, v_j\rangle,$$

where $\{v_i \mid i=1,2,\ldots,n\}$ is any basis of V and $\{v_i^* \mid i=1,2,\ldots,n\}$ is its dual basis in V^*. d_V and b_V are called evaluation and coevaluation of V, respectively. It is clear $(d_V\otimes id_U)(id_U\otimes b_V)=id_U$ and $(id_V\otimes d_V)(b_V\otimes id_V)=id_V$. ξ_V denotes the linear isomorphism from V to V^* by sending v_i to v_i^* for $i=1,2,\ldots,n$. Note that we can define evaluation d_V when V is infinite.

We will use μ to denote the multiplication map of an algebra and use Δ to denote the comultiplication of a coalgebra. For a (left or right) module and a (left or right) comodule, denote by α^-, α^+, δ^- and δ^+ the left module, right module, left comodule and right comodule structure maps, respectively. The Sweedler's sigma notations for coalgebras and comodules are $\Delta(x)=\sum x_1\otimes x_2$, $\delta^-(x)=\sum x_{(-1)}\otimes x_{(0)}$, $\delta^+(x)=\sum x_{(0)}\otimes x_{(1)}$. Let $(H,\mu,\eta,\Delta,\epsilon)$ be a bialgebra and let $\Delta^{\mathrm{cop}}:=C_{H,H}\Delta$ and $\mu^{\mathrm{op}}:=\mu C_{H,H}$. We denote $(H,\mu,\eta,\Delta^{\mathrm{cop}},\epsilon)$ by H^{cop} and $(H,\mu^{\mathrm{op}},\eta,\Delta,\epsilon)$ by H^{op}. Sometimes, we also denote the unit element of H by 1_H.

Let A and H be two bialgebras with $\varnothing\neq X\subseteq A$, $\varnothing\neq Y\subseteq H$ and $P\in Y\otimes X$, $R\in Y\otimes Y$. Assume that τ is a linear map from $H\otimes A$ to k. We give the following notations.

(Y,R) is called almost cocommutative if the following condition satisfied:

$$(\mathrm{ACO}): \quad \sum y_2R'\otimes y_1R''=\sum R'y_1\otimes R''y_2 \quad \text{for any} \quad y\in Y.$$

τ is called a skew pairing on $H\otimes A$ if for any $x,u\in H$, $y,z\in A$ the following conditions are satisfied:

$$\begin{aligned}
&(\mathrm{SP1}): \quad \tau(x,yz)=\sum \tau(x_1,y)\tau(x_2,z);\\
&(\mathrm{SP2}): \quad \tau(xu,z)=\sum \tau(x,z_2)\tau(u,z_1);\\
&(\mathrm{SP3}): \quad \tau(x,\eta)=\epsilon_H(x);\\
&(\mathrm{SP4}): \quad \tau(\eta,y)=\epsilon_A(y).
\end{aligned}$$

P is called a copairing of $Y\otimes X$ if for any $x,u\in H$, $y,z\in A$ the following conditions are satisfied:

$$\begin{aligned}
&(\mathrm{CP1}): \quad \sum P'\otimes P''_1\otimes P''_2=\sum P'Q'\otimes Q''\otimes P'' \quad \text{with} \quad P=Q;\\
&(\mathrm{CP1}): \quad \sum P'_1\otimes P'_2\otimes P''=\sum P'\otimes Q'\otimes P''Q'' \quad \text{with} \quad P=Q;\\
&(\mathrm{CP3}): \quad \sum P'\otimes \epsilon_A(P'')=\eta_H;\\
&(\mathrm{CP4}): \quad \sum \epsilon_H(P')\otimes P''=\eta_A.
\end{aligned}$$

For $R\in H\otimes H$ and two H-modules U and V, define a linear map $C^R_{U,V}$ from $U\otimes V$ to $V\otimes U$ by sending $(x\otimes y)$ to $\sum R''y\otimes R'x$ for any $x\in U$, $y\in V$.

If $V=\oplus_{i=0}^{\infty}V_i$ is a graded vector space, let $V_{>n}$ and $V_{\leq n}$ denote $\oplus_{i=n+1}^{\infty}V_i$ and $\oplus_{i=0}^{n}V_i$, respectively. We usually denote $\oplus_{i=0}^{n}V_i$ by $V_{(n)}$. If $\dim V_i<\infty$ for any natural number i, then V is called a local finite graded vector space. We denote by ι_i the natural injection from V_i to V and by π_i the corresponding projection from V to V_i.

Let H be a bialgebra and a graded coalgebra with an invertible element R_n in $H_{(n)}\otimes H_{(n)}$ for any natural n. Assume $R_{n+1}=R_n+W_n$ with $W_n\in H_{(n+1)}\otimes H_{n+1}+H_{n+1}\otimes H_{(n+1)}$. $(H,\{R_n\})$ is called a local quasitriangular bialgebra if R_n is a copairing on $H_{(n)}\otimes H_{(n)}$, and

$(H_{(n)}, R_n)$ is almost cocommutative for any natural number n. In this case, $\{R_n\}$ is called a local quasitriangular structure of H. Obviously, if (H, R) is a quasitriangular bialgebra, then $(H, \{R_n\})$ is a local quasitriangular bialgebra with $R_0 = R$, $R_i = 0$, $H_0 = H$, $H_i = 0$ for $i > 0$.

The following facts are obvious: $\tau^{-1} = \tau(id_H \otimes S)$ (or $= \tau(S^{-1} \otimes id_A)$) if A is a Hopf algebra (or H is a Hopf algebra with invertible antipode) and τ is a skew pairing; $P^{-1} = (S \otimes id_A)P$ (or $= (id_H \otimes S^{-1})P$) if H is a Hopf algebra (or A is a Hopf algebra with invertible antipode) and P is a copairing.

Let A be an algebra and M be an A-bimodule. Then the tensor algebra $T_A(M)$ of M over A is a graded algebra with $T_A(M)_0 = A$, $T_A(M)_1 = M$ and $T_A(M)_n = \otimes_A^n M$ for $n > 1$. That is, $T_A(M) = A \oplus (\bigoplus_{n>0} \otimes_A^n M)$ (see [12]). Let D be another algebra. If h is an algebra map from A to D and f is an A-bimodule map from M to D, then by the universal property of $T_A(M)$ (see [12, Proposition 1.4.1]) there is a unique algebra map $T_A(h, f) : T_A(M) \to D$ such that $T_A(h, f)\iota_0 = h$ and $T_A(h, f)\iota_1 = f$. One can easily see that $T_A(h, f) = h + \sum_{n>0} \mu^{n-1} T_n(f)$, where $T_n(f)$ is the map from $\otimes_A^n M$ to $\otimes_A^n D$ given by $T_n(f)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = f(x_1) \otimes f(x_2) \otimes \cdots \otimes f(x_n)$, i.e., $T_n(f) = f \otimes_A f \otimes_A \cdots \otimes_A f$. Note that μ can be viewed as a map from $D \otimes_A D$ to D. For the details, the reader is directed to [12, Section 1.4].

Dually, let C be a coalgebra and let M be a C-bicomodule. Then the cotensor coalgebra $T_C^c(M)$ of M over C is a graded coalgebra with $T_C^c(M)_0 = C$, $T_C^c(M)_1 = M$ and $T_C^c(M)_n = \Box_C^n M$ for $n > 1$. That is, $T_C^c(M) = C \oplus (\bigoplus_{n>0} \Box_C^n M)$ (see [12]). Let D be another coalgebra. If h is a coalgebra map from D to C and f is a C-bicomodule map from D to M such that $f(\mathrm{corad}(D)) = 0$, then by the universal property of $T_C^c(M)$ (see [12, Proposition 1.4.2]) there is a unique coalgebra map $T_C^c(h, f)$ from D to $T_C^c(M)$ such that $\pi_0 T_C^c(h, f) = h$ and $\pi_1 T_C^c(h, f) = f$. It is not difficult to see that $T_C^c(h, f) = h + \sum_{n>0} T_n^c(f)\Delta_{n-1}$, where $T_n^c(f)$ is the map from $\Box_C^n D$ to $\Box_C^n M$ induced by $T_n(f)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = f(x_1) \otimes f(x_2) \otimes \cdots \otimes f(x_n)$, i.e., $T_n^c(f) = f \otimes f \otimes \cdots \otimes f$.

Furthermore, if B is a Hopf algebra and M is a B-Hopf bimodule, then $T_B(M)$ and $T_B^c(M)$ are two graded Hopf algebra. Indeed, by [12, Section 1.4] and [12, Proposition 1.5.1], $T_B(M)$ is a graded Hopf algebra with the counit $\varepsilon = \varepsilon_B \pi_0$ and the comultiplication $\Delta = (\iota_0 \otimes \iota_0)\Delta_B + \sum_{n>0} \mu^{n-1} T_n(\Delta_M)$, where $\Delta_M = (\iota_0 \otimes \iota_1)\delta_M^- + (\iota_1 \otimes \iota_0)\delta_M^+$. Dually, $T_B^c(M)$ is a graded Hopf algebra with multiplication $\mu = \mu_B(\pi_0 \otimes \pi_0) + \sum_{n>0} T_n^c(\mu_M)\Delta_{n-1}$, where $\mu_M = \alpha_M^-(\pi_0 \otimes \pi_1) + \alpha_M^+(\pi_1 \otimes \pi_0)$.

3 Yang–Baxter equations

Assume that H is a bialgebra and a graded coalgebra with an invertible element R_n in $H_{(n)} \otimes H_{(n)}$ for any natural n. For convenience, let (LQT1), (LQT2) and (LQT3) denote (CP1), (CP2) and (ACO), respectively;

$$\text{(LQT4)}: \quad R_{n+1} = R_n + W_n \quad \text{with} \quad W_n \in H_{(n+1)} \otimes H_{n+1} + H_{n+1} \otimes H_{(n+1)};$$
$$\text{(LQT4}'\text{)}: \quad R_{n+1} = R_n + W_n \quad \text{with} \quad W_n \in H_{n+1} \otimes H_{n+1}.$$

Then $(H, \{R_n\})$ is a local quasitriangular bialgebra if and only if (LQT1), (LQT2), (LQT3) and (LQT4) hold for any natural number n.

Let H be a graded coalgebra and a bialgebra. A left H-module M is called an H-module with finite cycles if, for any $x \in M$, there exists a natural number n_x such that $H_i x = 0$ when $i > n_x$. Let ${}_H\mathcal{M}^{\mathrm{cf}}$ denote the category of all left H-modules with finite cycles.

Lemma 1. *Let H be a graded coalgebra and a bialgebra. If U and V are left H-modules with finite cycles, so is $U \otimes V$.*

Proof. For any $x \in U$, $y \in V$, there exist two natural numbers n_x and n_y, such that $H_{>n_x}x = 0$ and $H_{>n_y}y = 0$. Set $n_{x\otimes y} = 2n_x + 2n_y$. It is clear that $H_{>n_{x\otimes y}}(x \otimes y) = 0$. Indeed, for any $h \in H_i$ with $i > n_{x\otimes y}$, we see

$$h(x \otimes y) = \sum h_1 x \otimes h_2 y = 0 \qquad \text{(since } H \text{ is graded coalgebra).} \qquad \blacksquare$$

Lemma 2. *Assume that $(H, \{R_n\})$ is a local quasitriangular Hopf algebra. Then for any left H-modules U and V with finite cycles, there exists an invertible linear map $C^{\{R_n\}}_{U,V} : U \otimes V \to V \otimes U$ such that*

$$C^{\{R_n\}}_{U,V}(x \otimes y) := C^{R_n}(x \otimes y) = \sum R''_n y \otimes R'_n x$$

with $n > 2n_x + 2n_y$, for $x \in U$, $y \in V$.

Proof. We first define a map f from $U \times V$ to $V \otimes U$ by sending (x, y) to $\sum R''_n y \otimes R'_n x$ with $n > 2n_x + 2n_y$ for any $x \in U$, $y \in V$. It is clear that f is well defined. Indeed, if $n > 2n_x + 2n_y$, then $C^{R_{n+1}}(x \otimes y) = C^{R_n}(x \otimes y)$ since $R_{n+1} = R_n + W_n$ with $W_n \in H_{(n+1)} \otimes H_{n+1} + H_{n+1} \otimes H_{(n+1)}$. f is a k-balanced function. Indeed, for $x, y \in U$, $z, w \in V$, $\alpha \in k$, let $n > 2n_x + 2n_y + 2n_z$. See

$$\begin{aligned} f(x + y, z) &= \sum R''_n z \otimes R'_n(x + y) = \sum R''_n z \otimes R'_n x + \sum R''_n z \otimes R'_n y \\ &= f(x, z) + f(y, z). \end{aligned}$$

Similarly, we can show that $f(x, z + w) = f(x, z) + f(x, w)$, $f(x\alpha, z) = f(x, \alpha z)$. Consequently, there exists a linear map $C^{\{R_n\}}_{U,V} : U \otimes V \to V \otimes U$ such that

$$C^{\{R_n\}}_{U,V}(x \otimes y) = C^{R_n}(x \otimes y)$$

with $n > 2n_x + 2n_y$, for $x \in U$, $y \in V$.

The inverse $(C^{\{R_n\}}_{U,V})^{-1}$ of $C^{\{R_n\}}_{U,V}$ is defined by sending $(y \otimes x)$ to $\sum (R_n^{-1})' x \otimes (R_n^{-1})'' y$ with $n > 2n_x + 2n_y$ for any $x \in U$, $y \in V$. $\blacksquare$

Theorem 2. *Assume that $(H, \{R_n\})$ is a local quasitriangular Hopf algebra. Then $({}_H\mathcal{M}^{\mathrm{cf}}, C^{\{R_n\}})$ is a braided tensor category.*

Proof. Since H is a bialgebra, we have that $({}_H\mathcal{M}, \otimes, I, a, r, l)$ is a tensor category by [14, Proposition XI.3.1]. It follows from Lemma 1 that $({}_H\mathcal{M}^{\mathrm{cf}}, \otimes, I, a, r, l)$ is a tensor subcategory of $({}_H\mathcal{M}, \otimes, I, a, r, l)$. $C^{\{R_n\}}$ is a braiding of ${}_H\mathcal{M}^{\mathrm{cf}}$, which can be shown by the way similar to the proof of [14, Proposition VIII.3.1, Proposition XIII.1.4]. $\blacksquare$

An H-module M is called a graded H-module if $M = \oplus_{i=0}^{\infty} M_i$ is a graded vector space and $H_i M_j \subseteq M_{i+j}$ for any natural number i and j.

Lemma 3. *Assume that H is a local finite graded coalgebra and bialgebra. If $M = \oplus_{i=0}^{\infty} M_i$ is a graded H-module, then the following conditions are equivalent:*

(i) M is an H-module with finite cycles;

(ii) Hx is finite dimensional for any $x \in M$;

(iii) Hx is finite dimensional for any homogeneous element x in M.

Proof. (i) $\Rightarrow$ (ii). For any $x \in M$, there exists a natural number n_x such that $H_i x = 0$ with $i > n_x$. Since $H/(0 : x)_H \cong Hx$, where $(0 : x)_H := \{h \in H \mid h \cdot x = 0\}$, we have that Hx is finite dimensional.

(ii) $\Rightarrow$ (iii). It is clear.

(iii) $\Rightarrow$ (i). We first show that, for any homogeneous element $x \in M_i$, there exists a natural number n_x such that $H_j x = 0$ with $j > n_x$. In fact, if the above does not hold, then there exists $h_j \in H_{n_j}$ such that $h_j x \neq 0$ with $n_1 < n_2 < \cdots$. Considering $h_j x \in M_{n_j+i}$ we have that $\{h_j x \mid j = 1, 2, \dots\}$ is linear independent in Hx, which contradicts to that Hx is finite dimensional.

For any $x \in M$, then $x = \sum_{i=1}^{l} x_i$ and x_i is a homogeneous element for $i = 1, 2, \dots, l$. There exists a natural number n_{x_i} such that $H_j x_i = 0$ with $j > n_{x_i}$. Set $n_x = \sum_{s=1}^{l} n_{x_s}$. Then $H_j x = 0$ with $j > n_x$. Consequently, M is an H-module with finite cycles. ■

Note that if $M = \oplus_{i=0}^{\infty} M_i$ is a graded H-module, then both (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) hold in Lemma 3.

Let ${}_H\mathcal{M}^{\mathrm{gf}}$ and ${}_H\mathcal{M}^{\mathrm{gcf}}$ denote the category of all finite dimensional graded left H-modules and the category of all graded left H-modules with finite cycles. Obviously, they are two tensor subcategories of ${}_H\mathcal{M}^{\mathrm{cf}}$. Therefore we have

Theorem 3. *Assume that $(H, \{R_n\})$ is a local quasitriangular Hopf algebra. Then $({}_H\mathcal{M}^{\mathrm{gf}}, C^{\{R_n\}})$ and $({}_H\mathcal{M}^{\mathrm{gcf}}, C^{\{R_n\}})$ are two braided tensor categories.*

Therefore, if M is a finite dimensional graded H-module (or H-module with finite cycles) over local quasitriangular Hopf algebra $(H, \{R_n\})$, then $C^{\{R_n\}}_{M,M}$ is a solution of Yang–Baxter equation on M.

It is easy to prove the following.

Theorem 4. *Assume that $(H, \{R_n\})$ is a local quasitriangular Hopf algebra and $R_{n+1} = R_n + W_n$ with $W_n \in H_{n+1} \otimes H_{(n+1)}$. If (M, α^-) is an H-modules with finite cycles then (M, α^-, δ^-) is a Yetter–Drinfeld H-module, where $\delta^-(x) = \sum R_n'' \otimes R_n' x$ for any $x \in M$ and $n \geq n_x$.*

4 Relation between tensor algebras and co-tensor coalgebras

Lemma 4 (See [3, 7]). *Let A, B and C be finite dimensional coalgebras, $(M, \delta_M^-, \delta_M^+)$ and $(N, \delta_N^-, \delta_N^+)$ be respectively a finite dimensional A-B-bicomodule and a finite dimensional B-C-bicomodule. Then*

(i) $(M^*, \delta_M^{-*}, \delta_M^{+*})$ is a finite dimensional A^*-B^*-bimodule;

(ii) $(M \square_B N, \delta^-_{M \square_B N}, \delta^+_{M \square_B N})$ is an A-C-bicomodule with structure maps $\delta^-_{M \square_B N} = \delta^-_M \otimes id_N$ and $\delta^+_{M \square_B N} = id_M \otimes \delta^+_N$;

(iii) $M^* \otimes_{B^*} N^* \cong (M \square_B N)^*$ (as A^*-C^*-bimodules).

Lemma 5 (See [3, 7]). *Let A, B and C be finite dimensional algebras, $(M, \alpha_M^-, \alpha_M^+)$ and $(N, \alpha_N^-, \alpha_N^+)$ be respectively a finite dimensional A-B-bimodule and a finite dimensional B-C-bimodule. Then*

(i) $(M^*, \alpha_M^{-*}, \alpha_M^{+*})$ is a finite dimensional A^*-B^*-bicomodule;

(ii) $(M \otimes_B N, \alpha^-_{M \otimes_B N}, \alpha^+_{M \otimes_B N})$ is an A-C-bimodule with structure maps $\alpha^-_{M \otimes_B N} = \alpha^-_M \otimes id_N$ and $\alpha^+_{M \otimes_B N} = id_M \otimes \alpha^+_N$;

(iii) $M^* \square_{B^*} N^* \cong (M \otimes_B N)^*$ (as A^*-C^*-bicomodules).

Proof. (i) and (ii) are easy.

(iii) Consider

$$M^* \square_{B^*} N^* \cong (M^* \square_{B^*} N^*)^{**} \cong (M \otimes_B N)^* \quad \text{(by Lemma 4)}.$$

Of course, we can also prove it in the dual way of the proof of Lemma 4, by sending $f \otimes_k g$ to $f \otimes_{B^*} g$ for any $f \in M^*$, $g \in N^*$ with $f \otimes g \in M^* \square_{B^*} N^*$. ■

Theorem 5. *If A is a finite dimensional algebra and M is a finite dimensional A-bimodule, then $T_A(M)$ is isomorphic to subalgebra $\sum_{n=0}^{\infty}(\Box_{A^*}^n M^*)^*$ of $(T_{A^*}^c(M^*))^0$ under map $\sigma_{T_A(M)}$ and $\sigma_{T_A(M)} = \sigma_A + \sum_{n>0}\mu^{n-1}T_n(\sigma_M)$ with $\mu^{n-1}T_n(\sigma_M) = \sigma_{\otimes_A^n M}$.*

Proof. We view $\oplus_{n=0}^{\infty}(\Box_{A^*}^n M^*)^*$ as inner direct sum of vector spaces. It is clear that σ_A is algebra homomorphism from A to $A^{**} \subseteq (T_{A^*}^c(M^*))^*$ and σ_M is a A-bimodule homomorphism from M to $M^{**} \subseteq (T_{A^*}^c(M^*))^*$. Thus it follows from [12, Proposition 1.4.1] that $\phi = \sigma_A + \sum_{n>0}\mu^{n-1}T_n(\sigma_M)$ is an algebra homomorphism from $T_A(M)$ to $(T_{A^*}^c(M^*))^*$.

It follows from Lemma 4 (iii) that $\mu^{n-1}T_n(\sigma_M) = \sigma_{\otimes_A^n M}$. Indeed, we use induction on $n > 0$. Obviously, the conclusion holds when $n = 1$. Let $n > 1$, $N = \otimes_A^{n-1}M$, $L = (\Box_{A^*}^{n-1}M^*)^*$ and $\zeta = \mu^{n-2}T_{n-1}(\sigma_M)$. Obviously, $\mu^{n-1}T_n(\sigma_M) = \mu(\zeta \otimes \sigma_M)$. By inductive assumption, $\zeta = \sigma_N$ is an A-bimodule isomorphism from N to L. See

$$\begin{aligned}
\otimes_A^n M = N \otimes_A M &\overset{\nu_1}{\cong} L \otimes_A M^{**} \quad \text{(by inductive assumption)}\\
&\overset{\nu_2}{\cong} (\Box_{A^*}^{n-1}M^*)^* \otimes_{A^{**}} M^{**}\\
&\overset{\nu_3}{\cong} ((\Box_{A^*}^{n-1}M^*)\Box_{A^*}M^*)^* \quad \text{(by Lemma 4 (iii))}\\
&= (\Box_{A^*}^n M^*)^*,
\end{aligned}$$

where $\nu_1 = \sigma_N \otimes_A \sigma_M$, $\nu_2(f^{**} \otimes_A g^{**}) = f^{**} \otimes_{A^{**}} g^{**}$ and $\nu_3(f^{**} \otimes_{A^{**}} g^{**}) = f^{**} \otimes_k g^{**}$ for any $f^{**} \in (\Box_{A^*}^{n-1}M^*)^*$, $g^{**} \in M^{**}$. Now we have to show $\nu_3\nu_2\nu_1 = \sigma_{\otimes_A^n M} = \mu^{n-1}T_n(\sigma_M)$. For any $f^* \in \Box_{A^*}^{n-1}M^*$, $g^* \in M^*$, $x \in \otimes_A^{n-1}M$, $y \in M$, on the one hand

$$\langle \sigma_{\otimes_A^n M}(x \otimes_A y), f^* \otimes_k g^* \rangle = \langle f^*, x\rangle\langle g^*, y\rangle.$$

On the other hand,

$$\begin{aligned}
\langle \nu_3\nu_2\nu_1(x \otimes_A y), f^* \otimes_k g^* \rangle &= \langle \nu_3\nu_2(\sigma_N(x) \otimes_A \sigma_M(y)), f^* \otimes_k g^* \rangle\\
&= \langle \sigma_M(x) \otimes_k \sigma_N(y), f^* \otimes_k g^* \rangle = \langle f^*, x\rangle\langle g^*, y\rangle.
\end{aligned}$$

Thus $\nu_3\nu_2\nu_1 = \sigma_{\otimes_A^n M}$. See

$$\begin{aligned}
\langle \mu(\zeta \otimes_A \sigma_M)(x \otimes_A y), f^* \otimes_k g^* \rangle &= \langle \zeta(x) \otimes_A \sigma_M(y), \Delta(f^* \otimes_k g^*)\rangle\\
&= \langle \zeta(x), f^*\rangle\langle \sigma_M(y), g^*\rangle = \langle f^*, x\rangle\langle g^*, y\rangle.
\end{aligned}$$

Thus $\sigma_{\otimes_A^n M} = \mu^{n-1}T_n(\sigma_M)$.

Finally, for any $x \in T_A(M)$ with $x = x^{(1)} + x^{(2)} + \cdots + x^{(n)}$ and $x^{(i)} \in \otimes_A^i M$,

$$\phi(x) = \sum_{i=1}^{n}\phi(x^{(i)}) = \sum_{i=1}^{n}\sigma_{\otimes_A^i M}(x^{(i)}) = \sigma_{T_A(M)}(x). \qquad \blacksquare$$

5 (Co-)tensor Hopf algebras

Lemma 6. *Assume that B is a finite dimensional Hopf algebra and M is a finite dimensional B-Hopf bimodule. Let $A := T_B(M)^{\text{cop}}$, $H := T_{B^*}^c(M^*)$. Then*

(i) $\phi := \sigma_B + \sum_{i>0}\mu^{i-1}T_i(\sigma_M)$ is a Hopf algebra isomorphism from $T_B(M)$ to the Hopf subalgebra $\sum_{i=0}^{\infty}(\Box_{B^*}^i M^*)^*$ of $(T_{B^*}^c(M^*))^0$;

(ii) Let $\phi_n := \phi \mid_{A_{(n)}}$ for any natural number $n \geq 0$. Then there exists $\psi_n : (H_{(n)})^* \to A_{(n)}$ such that $\phi_n\psi_n = id_{(H_{(n)})^*}$ and $\psi_n\phi_n = id_{A_{(n)}}$, and $\psi_{n+1}(x) = \psi_n(x)$ for any $x \in (H_{(n)})^*$. Furthermore, ϕ_n and ψ_n preserve the (co)multiplication operations of $T_B(M)$ and $(T_{B^*}^c(M^*))^0$, respectively.

Proof. (i) We first show that $(\square^n_{B^*} M^*)^* \subseteq (T^c_{B^*}(M^*))^0$. For any $f \in (\square^n_{B^*} M^*)^*$, $\sum_{i=n+1}^{\infty} \square^i_{B^*} M^* \subseteq \ker f$ and $\sum_{i=n+1}^{\infty} \square^i_{B^*} M^*$ is a finite codimensional ideal of $T^c_{B^*}(M^*)$. Consequently, $f \in (T^c_{B^*}(M^*))^0$.

Next we show that $\phi := \sigma_B + \sum_{n>0} \mu^{n-1} T_n(\sigma_M)$ (see the proof of Theorem 5) is a coalgebra homomorphism from $T_B(M)$ to $\sum_{i=0}^{\infty} (\square^c_{B^*} M^*)^*$. For any $x \in \otimes^n_B M$, $f, g \in T^c_{B^*}(M^*)$, on the one hand

$$\begin{aligned} \langle \phi(x), f * g\rangle &= \langle f * g, x\rangle \quad \text{(by Theorem 5)} \\ &= \sum_x \langle f, x_1\rangle \langle g, x_2\rangle = \sum_x \langle \phi(x_1), f\rangle \langle \phi(x_2), g\rangle. \end{aligned}$$

On the other hand

$$\langle \phi(x), f * g\rangle = \sum \langle (\phi(x))_1, f\rangle \langle (\phi(x))_2, g\rangle,$$

since $\phi(x) \in (T^c_{B^*}(M^*))^0$. Considering $T^c_{B^*}(M^*) = \oplus_{n\geq 0} \square^n_{B^*} M^* \cong \oplus_{n\geq 0} (\otimes^n_B M)^*$ as vector spaces, we have that $T_{B^*}(M^*)$ is dense in $(T_B(M))^*$. Consequently, $\sum \phi(x_1) \otimes \phi(x_2) = \sum (\phi(x))_1 \otimes (\phi(x))_2$, i.e. ϕ is a coalgebra homomorphism.

(ii) It follows from Theorem 5. ■

Recall the double cross product $A_\alpha \bowtie_\beta H$, defined in [17, p. 36]) and [14, Definition IX.2.2]. Assume that H and A are two bialgebras; (A, α) is a left H-module coalgebra and (H, β) is a right A-module coalgebra. We define the multiplication m_D, unit η_D, comultiplication Δ_D and counit ϵ_D on $A \otimes H$ as follows:

$$\begin{aligned} &\mu_D((a \otimes h) \otimes (b \otimes g)) = \sum a\alpha(h_1, b_1) \otimes \beta(h_2, b_2) g, \\ &\Delta_D(a \otimes h) = \sum (a_1 \otimes h_1 \otimes a_2 \otimes h_2), \end{aligned}$$

$\epsilon_D = \epsilon_A \otimes \epsilon_H$, $\eta_D = \eta_A \otimes \eta_H$ for any $a, b \in A$, $h, g \in H$. We denote $(A \otimes H, \mu_D, \eta_D, \Delta_D, \epsilon_D)$ by $A_\alpha \bowtie_\beta H$, which is called the double cross product of A and H.

Lemma 7 (See [8]). *Let H and A be two bialgebras. Assume that τ is an invertible skew pairing on $H \otimes A$. If we define $\alpha(h, a) = \sum \tau(h_1, a_1) a_2 \tau^{-1}(h_2, a_3)$ and $\beta(h, a) = \sum \tau(h_1, a_1) h_2 \tau^{-1}(h_3, a_2)$ then the double cross product $A_\alpha \bowtie_\beta H$ of A and H is a bialgebra. Furthermore, if A and H are two Hopf algebras, then so is $A_\alpha \bowtie_\beta H$.*

Proof. We can check that (A, α) is an H-module coalgebra and (H, β) is an A-module coalgebra step by step. We can also check that (M1)–(M4) in [17, pp. 36–37] hold step by step. Consequently, it follows from [17, Corollary 1.8, Theorem 1.5] or [14, Theorem IX.2.3] that $A_\alpha \bowtie_\beta H$ is a Hopf algebra. ■

In this case, $A_\alpha \bowtie_\beta H$ can be written as $A \bowtie_\tau H$.

Lemma 8. *Let H and A be two Hopf algebra. Assume that there exists a Hopf algebra monomorphism $\phi : A^{\mathrm{cop}} \to H^0$. Set $\tau = d_H(\phi \otimes id_H) C_{H,A}$. Then $A \bowtie_\tau H$ is Hopf algebra.*

Proof. Using [10, Proposition 2.4] or the definition of the evaluation and coevaluation on tensor product, we can obtain that τ is a skew pairing on $H \otimes A$. Considering Lemma 7, we complete the proof. ■

Lemma 9. *(i) If $H = \oplus_{n=0}^{\infty} H_n$ is a graded bialgebra and H_0 has an invertible antipode, then H has an invertible antipode.*

(ii) Assume that B is a finite dimensional Hopf algebra and M is a B-Hopf bimodule. Then both $T_B(M)$ and $T^c_B(M)$ have invertible antipodes.

Proof. (i) It is clear that $H^{\rm op}$ is a graded bialgebra with $(H^{\rm op})_0 = (H_0)^{(\rm op)}$. Thus $H^{\rm op}$ has an antipode by [12, Proposition 1.5.1]. However, the antipode of $H^{\rm op}$ is the inverse of antipode of H.

(ii) It follows from (i). ■

Lemma 10. *Let $A = \oplus_{n=0}^{\infty} A_n$ and $H = \oplus_{n=0}^{\infty} H_n$ be two graded Hopf algebras with invertible antipodes. Let τ be a skew pairing on $(H \otimes A)$ and P_n be a copairing of $H_{(n)} \otimes A_{(n)}$ for any natural number n. Set $D = A \bowtie_\tau H$ and $[P_n] = 1_A \otimes P_n \otimes 1_H$. Then $(D, \{[P_n]\})$ is almost cocommutative on $D_{(n)}$ if and only if*

$$\text{(ACO1)}: \sum P' y_1 \otimes P'' \otimes y_2 = \sum y_4 P' \otimes P_2'' \otimes y_2 \tau(y_1, P_1'') \tau^{-1}(y_3, P_3'') \quad \text{for any } y \in H_{(n)};$$

$$\text{(ACO2)}: \sum x_2 \otimes P' \otimes x_1 P'' = \sum x_2 \otimes P_2' \otimes P'' x_4 \tau(P_1', x_1) \tau^{-1}(P_3', x_3) \quad \text{for any } x \in A_{(n)}.$$

Proof. It is clear that $(D, \{[P_n]\})$ is almost cocommutative on $D_{(n)}$ if and only if the following holds:

$$\begin{aligned}&\sum x_2 \otimes y_4 P' \otimes x_1 P_2'' \otimes y_2 \tau(y_1, P_1'') \tau^{-1}(y_3, P_3'') \\ &\quad = \sum x_2 \otimes P_2' y_1 \otimes P'' x_4 \otimes y_2 \tau(P_1', x_1) \tau^{-1}(P_3', x_3) \end{aligned} \tag{1}$$

for any $x \in A_{(n)}$, $y \in H_{(n)}$.

Assume that both (ACO1) and (ACO2) hold. See that

$$\text{the left hand of (1)} \overset{\text{by (ACO1)}}{=} \sum x_2 \otimes P' y_1 \otimes x_1 P'' \otimes y_2 \overset{\text{by (ACO2)}}{=} \text{the right hand of (1)}$$

for any $x \in A_{(n)}$, $y \in H_{(n)}$. That is, (1) holds.

Conversely, assume that (1) holds. Thus we have that

$$\begin{aligned}&\sum x_2 \otimes P' \otimes x_1 P_2'' \otimes \epsilon_H(1_H) \tau(1_H, P_1'') \tau^{-1}(1_H, P_3'') \\ &\quad = \sum x_2 \otimes P_2' \otimes P'' x_4 \otimes \epsilon_H(1_H) \tau(P_1', x_1) \tau^{-1}(P_3', x_3)\end{aligned}$$

and

$$\begin{aligned}&\sum \epsilon_A(1_A) \otimes y_4 P' \otimes P_2'' \otimes y_2 \tau(y_1, P_1'') \tau^{-1}(y_3, P_3'') \\ &\quad = \sum \epsilon_A(1_A) \otimes P_2' y_1 \otimes P'' \otimes y_2 \tau(P_1', 1_A) \tau^{-1}(P_3', 1_A)\end{aligned}$$

for any $x \in A_{(n)}$, $y \in H_{(n)}$. Consequently, (ACO1) and (ACO2) hold. ■

Lemma 11. *Let $A = \oplus_{n=0}^{\infty} A_n$ and $H = \oplus_{n=0}^{\infty} H_n$ be two graded Hopf algebras with invertible antipodes. Let τ be a skew pairing on $(H \otimes A)$ and P_n be a copairing of $(H_{(n)} \otimes A_{(n)})$ with $P_{n+1} = P_n + W_n$ and $W_n \in H_{(n+1)} \otimes A_{n+1} + H_{n+1} \otimes A_{(n+1)}$ for any natural number n. Set $D = A \bowtie_\tau H$. If $\tau(P_n', x) P_n'' = x$ and $\tau(y, P_n'') P_n' = y$ for any $x \in A_{(n)}$, $y \in H_{(n)}$, then $(D, \{[P_n]\})$ is a local quasitriangular Hopf algebra.*

Proof. It follows from Lemma 7 that $D = \oplus_{n=0}^{\infty} D_n$ is a Hopf algebra. Let $D_n = \sum_{i+j=n} A_i \otimes H_j$. It is clear that $D = \oplus_{n=0}^{\infty} D_n$ is a graded coalgebra. We only need to show that $(D, \{[P_n]\})$ is almost cocommutative on $D_{(n)}$. Now fix n. For convenience, we denote P_n by P and Q in the following formulae. For any $x \in A_i$, $y \in H_j$ with $i + j \le n$,

$$\text{the right hand of (ACO1)} \overset{\text{by (CP1)}}{=} \sum y_4 Q' P' \otimes Q_1'' \otimes y_2 \tau(y_1, P'') \tau^{-1}(y_3, Q_2'')$$

$$\overset{\text{by assumption}}{=} \sum y_4 Q' y_1 \otimes Q''_1 \otimes y_2 \tau^{-1}(y_3, Q''_2) \overset{\text{by (CP1)}}{=} \sum y_4 Q' P' y_1 \otimes P'' \otimes y_2 \tau^{-1}(y_3, Q'')$$
$$= \sum y_4 Q' P' y_1 \otimes P'' \otimes y_2 \tau(S^{-1}(y_3), Q'') \overset{\text{by assumption}}{=} \sum y_4 S^{-1}(y_3) P' y_1 \otimes P'' \otimes y_2$$
$$= \sum P' y_1 \otimes P'' \otimes y_2 = \text{the left hand of (ACO1)}.$$

Similarly, we can show that (ACO2) holds on $A_{(n)}$. ■

Theorem 6. *Assume that B is a finite dimensional Hopf algebra and M is a finite dimenaional B-Hopf bimodule. Let $A := T_B(M)^{\rm cop}$, $H := T^c_{B^*}(M^*)$ and $D = A \bowtie_\tau H$ with $\tau := d_H(\phi \otimes id)C_{H,A}$. Then $((T_B(M))^{\rm cop} \bowtie_\tau T^c_{B^*}(M^*), \{R_n\})$ is a local quasitriangular Hopf algebra. Here $P_n = (id \otimes \psi_n) b_{H_{(n)}}$, $R_n = [P_n] = 1_B \otimes (id \otimes \psi_n) b_{H_{(n)}} \otimes 1_{B^*}$, ϕ and ψ_n are defined in Lemma 6.*

Proof. By Lemma 9 (ii), A and H have invertible antipodes. Assume that $e^{(i)}_1, e^{(i)}_2, \ldots, e^{(i)}_{n_i}$ is a basis of H_i and $e^{(i)*}_1, e^{(i)*}_2, \ldots, e^{(i)*}_{n_i}$ is an its dual basis in $(H_i)^*$. Then $\{e^{(i)}_j \mid i = 0, 1, 2, \ldots, n; j = 1, 2, \ldots, n_i\}$ is a basis of $H_{(n)}$ and $\{e^{(i)*}_j \mid i = 0, 1, 2, \ldots, n; j = 1, 2, \ldots, n_i\}$ is its dual basis in $(H_{(n)})^*$. Thus $b_{H_{(n)}} = \sum_{i=0}^{n} \sum_{j=1}^{n_i} e^{(i)}_j \otimes e^{(i)*}_j$. See that

$$\begin{aligned} P_{n+1} &= \sum_{i=0}^{n} \sum_{j=1}^{n_i} e^{(i)}_j \otimes \psi_{n+1}(e^{(i)*}_j) + \sum_{j=1}^{n_{n+1}} e^{(n+1)}_j \otimes \psi_{n+1}(e^{(n+1)*}_j) \\ &= P_n + \sum_{j=1}^{n_{n+1}} e^{(n+1)}_j \otimes \psi_{n+1}(e^{(n+1)*}_j). \end{aligned}$$

Obviously, $\sum_{j=1}^{n_{n+1}} e^{(n+1)}_j \otimes \psi_{n+1}(e^{(n+1)*}_j) \in H_{n+1} \otimes A_{n+1}$. It is clear that P_n is a copairing on $H_{(n)} \otimes A_{(n)}$ and τ is a skew pairing on $H \otimes A$ with $\sum \tau(P'_n, x) P''_n = x$ and $\sum \tau(y, P''_n) P'_n = y$ for any $x \in A_{(n)}$, $y \in H_{(n)}$. We complete the proof by Lemma 11. ■

Note that $R_{n+1} = R_n + W_n$ with $W_n \in D_{n+1} \otimes D_{n+1}$ in the above theorem.

6 Quiver Hopf algebras

A quiver $Q = (Q_0, Q_1, s, t)$ is an oriented graph, where Q_0 and Q_1 are the sets of vertices and arrows, respectively; s and t are two maps from Q_1 to Q_0. For any arrow $a \in Q_1$, $s(a)$ and $t(a)$ are called its start vertex and end vertex, respectively, and a is called an arrow from $s(a)$ to $t(a)$. For any $n \geq 0$, an n-path or a path of length n in the quiver Q is an ordered sequence of arrows $p = a_n a_{n-1} \cdots a_1$ with $t(a_i) = s(a_{i+1})$ for all $1 \leq i \leq n-1$. Note that a 0-path is exactly a vertex and a 1-path is exactly an arrow. In this case, we define $s(p) = s(a_1)$, the start vertex of p, and $t(p) = t(a_n)$, the end vertex of p. For a 0-path x, we have $s(x) = t(x) = x$. Let Q_n be the set of n-paths, $Q_{(n)}$ be the set of i-paths with $i \leq n$ and Q_∞ be the set of all paths in Q. Let ${}^yQ^x_n$ denote the set of all n-paths from x to y, $x, y \in Q_0$. That is, ${}^yQ^x_n = \{p \in Q_n \mid s(p) = x, t(p) = y\}$. A quiver Q is *finite* if Q_0 and Q_1 are finite sets.

Let G be a group. Let $\mathcal{K}(G)$ denote the set of conjugate classes in G. $r = \sum_{C \in \mathcal{K}(G)} r_C C$ is called a *ramification* (or *ramification data*) of G, if r_C is the cardinal number of a set for any $C \in \mathcal{K}(G)$. We always assume that the cardinal number of the set $I_C(r)$ is r_C. Let $\mathcal{K}_r(G) := \{C \in \mathcal{K}(G) \mid r_C \neq 0\} = \{C \in \mathcal{K}(G) \mid I_C(r) \neq \varnothing\}$.

Let G be a group. A quiver Q is called *a quiver of G* if $Q_0 = G$ (i.e., $Q = (G, Q_1, s, t)$). If, in addition, there exists a ramification r of G such that the cardinal number of ${}^yQ^x_1$ is equal to r_C for any $x, y \in G$ with $x^{-1}y \in C \in \mathcal{K}(G)$, then Q is called a *Hopf quiver with respect to the ramification data r*. In this case, there is a bijection from $I_C(r)$ to ${}^yQ^x_1$. Denote by (Q, G, r) the

Hopf quiver of G with respect to r. e denotes the unit element of G. $\{p_g \mid g \in G\}$ denotes the dual basis of $\{g \mid g \in G\}$ of finite group algebra kG.

Let $Q = (G, Q_1, s, t)$ be a quiver of a group G. Then kQ_1 becomes a kG-bicomodule under the natural comodule structures:

$$\delta^-(a) = t(a) \otimes a, \qquad \delta^+(a) = a \otimes s(a), \qquad a \in Q_1, \tag{2}$$

called an *arrow comodule*, written as kQ_1^c. In this case, the path coalgebra kQ^c is exactly isomorphic to the cotensor coalgebra $T^c_{kG}(kQ_1^c)$ over kG in a natural way (see [3] and [4]). We will regard $kQ^c = T^c_{kG}(kQ_1^c)$ in the following. Moreover, kQ_1 becomes a $(kG)^*$-bimodule with the module structures defined by

$$p \cdot a := \langle p, t(a)\rangle a, \qquad a \cdot p := \langle p, s(a)\rangle a, \qquad p \in (kG)^*, \qquad a \in Q_1, \tag{3}$$

written as kQ_1^a, called an *arrow module*. Therefore, we have a tensor algebra $T_{(kG)^*}(kQ_1^a)$. Note that the tensor algebra $T_{(kG)^*}(kQ_1^a)$ of kQ_1^a over $(kG)^*$ is exactly isomorphic to the path algebra kQ^a. We will regard $kQ^a = T_{(kG)^*}(kQ_1^a)$ in the following.

Lemma 12 (See [4], Theorem 3.3, and [5], Theorem 3.1). *Let Q be a quiver over group G. Then the following statements are equivalent:*

(i) Q is a Hopf quiver.

(ii) Arrow comodule kQ_1^c admits a kG-Hopf bimodule structure.

If Q is finite, then the above statements are also equivalent to the following:

(iii) Arrow module kQ_1^a admits a $(kG)^*$-Hopf bimodule structure.

Assume that Q is a Hopf quiver. It follows from Lemma 12 that there exist a left kG-module structure α^- and a right kG- module structure α^+ on arrow comodule $(kQ_1^c, \delta^-, \delta^+)$ such that $(kQ_1^c, \alpha^-, \alpha^+, \delta^-, \delta^+)$ becomes a kG-Hopf bimodule, called a co-arrow Hopf bimodule. We obtain two graded Hopf algebras $T_{kG}(kQ_1^c)$ and $T^c_{kG}(kQ_1^c)$, called semi-path Hopf algebra and co-path Hopf algebra, written as kQ^s and kQ^c, respectively.

Assume that Q is a finite Hopf quiver. Dually, it follows from Lemma 12 that there exist a left $(kG)^*$-comodule structure δ^- and a right $(kG)^*$-comodule structure δ^+ on arrow module $(kQ_1^a, \alpha^-, \alpha^+)$ such that $(kQ_1^a, \alpha^-, \alpha^+, \delta^-, \delta^+)$ becomes a $(kG)^*$-Hopf bimodule, called an arrow Hopf bimodule. We obtain two graded Hopf algebras $T_{(kG)^*}(kQ_1^a)$ and $T^c_{(kG)^*}(kQ_1^a)$, called path Hopf algebra and semi-co-path Hopf algebra, written as kQ^a and kQ^{sc}, respectively.

From now on, we assume that Q is a finite Hopf quiver on finite group G. Let $\xi_{kQ_1^a}$ denote the linear map from kQ_1^a to $(kQ_1^c)^*$ by sending a to a^* for any $a \in Q_1$ and $\xi_{kQ_1^c}$ denote the linear map from kQ_1^c to $(kQ_1^a)^*$ by sending a to a^* for any $a \in Q_1$. It is easy to check the following.

Lemma 13. *(i) If $(M, \alpha^-, \alpha^+, \delta^-, \delta^+)$ is a finite dimensional B-Hopf bimodule and B is a finite dimensional Hopf algebra, then $(M^*, \delta^{-*}, \delta^{+*}, \alpha^{-*}, \alpha^{+*})$ is a B^*-Hopf bimodule.*

(ii) If $(kQ_1^c, \alpha^-, \alpha^+, \delta^-, \delta^+)$ is a co-arrow Hopf bimodule, then there exist unique left $(kG)^*$-comodule operation $\delta^-_{kQ_1^a}$ and right $(kG)^*$-comodule operation $\delta^+_{kQ_1^a}$ such that $(kQ_1^a, \alpha^-_{kQ_1^a}, \alpha^+_{kQ_1^a}, \delta^-_{kQ_1^a}, \delta^+_{kQ_1^a})$ becomes a $(kG)^*$-Hopf bimodule and $\xi_{kQ_1^a}$ becomes a $(kG)^*$-Hopf bimodule isomorphism from $(kQ_1^a, \alpha^-_{kQ_1^a}, \alpha^+_{kQ_1^a}, \delta^-_{kQ_1^a}, \delta^+_{kQ_1^a})$ to $((kQ_1^c)^*, \delta^{-*}, \delta^{+*}, \alpha^{-*}, \alpha^{+*})$.

(iii) If $(kQ_1^a, \alpha^-, \alpha^+, \delta^-, \delta^+)$ is an arrow Hopf bimodule, then there exist unique left kG-module operation $\alpha^-_{kQ_1^c}$ and right kG-module $\alpha^+_{kQ_1^c}$ such that $(kQ_1^c, \alpha^-_{kQ_1^c}, \alpha^+_{kQ_1^c}, \delta^-_{kQ_1^c}, \delta^+_{kQ_1^c})$ become a kG-Hopf bimodule and $\xi_{kQ_1^c}$ becomes a kG-Hopf bimodule isomorphism from $(kQ_1^c, \alpha^-_{kQ_1^c}, \alpha^+_{kQ_1^c}, \delta^-_{kQ_1^c}, \delta^+_{kQ_1^c})$ to $((kQ_1^a)^*, \delta^{-*}, \delta^{+*}, \alpha^{-*}, \alpha^{+*})$.

(iv) $\xi_{kQ_1^a}$ is a $(kG)^*$-Hopf bimodule isomorphism from$(kQ_1^a, \alpha^-_{kQ_1^a}, \alpha^+_{kQ_1^a}, \delta^-_{kQ_1^a}, \delta^+_{kQ_1^a})$ to $((kQ_1^c)^*, {\delta^-_{kQ_1^c}}^*, {\delta^+_{kQ_1^c}}^*, {\alpha^-_{kQ_1^c}}^*, {\alpha^+_{kQ_1^c}}^*)$ if and only if $\xi_{kQ_1^c}$ becomes a kG-Hopf bimodule isomorphism from $(kQ_1^c, \alpha^-_{kQ_1^c}, \alpha^+_{kQ_1^c}, \delta^-_{kQ_1^c}, \delta^+_{kQ_1^c})$ to $((kQ_1^a)^*, {\delta^-_{kQ_1^a}}^*, {\delta^+_{kQ_1^a}}^*, {\alpha^-_{kQ_1^a}}^*, {\alpha^+_{kQ_1^a}}^*)$.

Let B be a Hopf algebra and ${}^B_B\mathcal{M}^B_B$ denote the category of B-Hopf bimodules. Let $G\ \mathcal{H}$opf denote the category of graded Hopf algebras. Define $T_B(\psi) =: T_B(\iota_0, \iota_1\psi)$ and $T^c_B(\psi) := T^c_B(\pi_0, \psi\pi_1)$ for any B-Hopf bimodule homomorphism ψ.

Lemma 14. *Let B be a Hopf algebra. Then T_B and T^c_B are two functors from ${}^B_B\mathcal{M}^B_B$ to $G\ \mathcal{H}opf$.*

Proof. (i) If ψ is a B-Hopf bimodule homomorphism from M to M', then $T_B(\iota_0, \iota_1\psi)$ is a graded Hopf algebra homomorphism from $T_B(M)$ to $T_B(M')$. Indeed, let $\Phi := T_B(\iota_0, \iota_1\psi)$. Then both $\Delta_{T_{B'}(M')}\Phi$ and $(\Phi\otimes\Phi)\Delta_{T_B(M)}$ are graded algebra maps from $T_B(M)$ to $T_B(M')\otimes T_B(M')$. Now we show that $\Delta_{T_B(M')}\Phi = (\Phi\otimes\Phi)\Delta_{T_B(M)}$. Considering that Φ is an algebra homomorphism, we only have to show that $\Delta_{T_B(M')}\Phi\iota_0 = (\Phi\otimes\Phi)\Delta_{T_B(M)}\iota_0$ and $\Delta_{T_B(M')}\Phi\iota_1 = (\Phi\otimes\Phi)\Delta_{T_B(M)}\iota_1$. Obviously, the first equation holds. For the second equation, see

$$\begin{aligned}
\Delta_{T_{B'}(M')}\Phi\iota_1 &= \Delta_{T_{B'}(M')}\iota_1\psi = (\iota_0\otimes\iota_1)\delta^-_{M'}\psi + (\iota_1\otimes\iota_0)\delta^+_{M'}\psi\\
&= (\iota_0\otimes\iota_1)(id\otimes\psi)\delta^-_M + (\iota_1\otimes\iota_0)(\psi\otimes id)\delta^+_M\\
&= (\iota_0\otimes\iota_1\psi)\delta^-_M + (\iota_1\psi\otimes\iota_0)\delta^+_M\\
&= (\Phi\otimes\Phi)(\iota_0\otimes\iota_1)\delta^-_M + (\Phi\otimes\Phi)(\iota_1\otimes\iota_0)\delta^+_M\\
&= (\Phi\otimes\Phi)[(\iota_0\otimes\iota_1)\delta^-_M + (\iota_1\otimes\iota_0)\delta^+_M]\\
&= (\Phi\otimes\Phi)\Delta_{T_B(M)}\iota_1.
\end{aligned}$$

Consequently, $T_B(\iota_0, \iota_1\psi)$ is a graded Hopf algebra homomorphisms.

(ii) If ψ is a B-Hopf bimodule homomorphism from M to M', then $T^c_B(\pi_0, \psi\pi_1)$ is a graded Hopf algebra homomorphism from $T^c_B(M)$ to $T^c_B(M')$. Indeed, let $\Psi := T^c_B(\pi_0, \psi\pi_1)$. Then both $\Psi\mu_{T^c_B(M)}$ and $\mu_{T^c_B(M')}(\Psi\otimes\Psi)$ are graded coalgebra maps from $T^c_B(M)\otimes T^c_B(M)$ to $T^c_B(M')$. Since $T^c_B(M)\otimes T^c_B(M)$ is a graded coalgebra, $\mathrm{corad}(T^c_B(M)\otimes T^c_B(M)) \subseteq (T^c_B(M)\otimes T^c_B(M))_0 = \iota_0(B)\otimes\iota_0(B)$. It follows that $(\pi_1\Psi\mu_{T^c_B(M)})(\mathrm{corad}(T^c_B(M)\otimes T^c_B(M))) = 0$. Thus by the universal property of $T^c_B(M')$, in order to prove $\Psi\mu_{T^c_B(M)} = \mu_{T^c_B(M')}(\Psi\otimes\Psi)$, we only need to show $\pi_n\Psi\mu_{T^c_B(M)} = \pi_n\mu_{T^c_B(M')}(\Psi\otimes\Psi)$ for $n = 0, 1$. However, this follows from a straightforward computation dual to part (i). Furthermore, one can see $\Psi(1) = 1$. Hence Ψ is an algebra map, and so a Hopf algebra map.

(iii) It is straightforward to check $T_B(\psi)T_B(\psi') = T_B(\psi\psi')$ and $T^c_B(\psi)T^c_B(\psi') = T^c_B(\psi\psi')$ for B-Hopf bimodule homomorphisms $\psi : M' \to M''$ and $\psi' : M \to M'$.

(iv) $T_B(id_M) = id_{T_B(M)}$ and $T^c_B(id_M) = id_{T^c_B(M)}$. ■

Lemma 15. *If ψ is a Hopf algebra isomorphism from B to B' and $(M, \alpha^-, \alpha^+, \delta^-, \delta^+)$ is a B-Hopf bimodule, then $(M, \alpha^-(\psi^{-1}\otimes id_M), \alpha^+(id_M\otimes\psi^{-1}), (\psi\otimes id_M)\delta^-, (id_M\otimes\psi)\delta^+)$ is a B'-Hopf bimodule. Furthermore $T_B(\iota_0\psi, \iota_1)$ and $T^c_B(\psi\pi_0, \pi_1)$ are graded Hopf algebra isomorphisms from $T_B(M)$ to $T_{B'}(M)$ and from $T^c_B(M)$ to $T^c_{B'}(M)$, respectively.*

By Lemma 14 and Lemma 13 (ii) and (iii), $T_{(kG)^*}(\iota_0, \iota_1\xi_{kQ^a_1})$ and $T^c_{kG}(\pi_0, \xi_{kQ^c_1}\pi_1)$ are graded Hopf algebra isomorphisms from $T_{(kG)^*}(kQ^a_1)$ to $T_{(kG)^*}((kQ^c_1)^*)$ and from $T^c_{kG}(kQ^c_1)$ to $T^c_{kG}((kQ^a_1)^*)$, respectively. $T^c_{(kG)^*}(\pi_0, \xi_{kQ^a_1}\pi_1)$ and $T_{kG}(\iota_0, \iota_1\xi_{kQ^c_1})$ are graded Hopf algebra isomorphisms from $T^c_{(kG)^*}(kQ^a_1)$ to $T^c_{(kG)^*}((kQ^c_1)^*)$ and from $T_{kG}(kQ^c_1)$ to $T_{kG}((kQ^a_1)^*)$, respectively. Furthermore, (kQ^a_1, kQ^c_1), (kQ^a, kQ^c) and (kQ^s, kQ^{sc}) are said to be arrow dual pairings.

Theorem 7. *Assume that (Q, G, r) is a finite Hopf quiver on finite group G. If (kQ^a, kQ^c) and (kQ^s, kQ^{sc}) are said to be arrow dual pairings, then*

(i) $((kQ^a)^{\mathrm{cop}} \bowtie_\tau kQ^c, \{R_n\})$ is a local quasitriangular Hopf algebra. Here

$$R_n = \sum_{g\in G} p_e\otimes g\otimes p_g\otimes e + \sum_{q\in Q_{(n)},\ q\notin G} p_e\otimes q\otimes q\otimes e$$

and $\tau(a, b) = \delta_{a,b}$, for any two paths a and b in Q, where $\delta_{a,b}$ is the Kronecker symbol.

(ii) There exist τ and $\{R_n\}$ such that $((kQ^s)^{\rm cop} \bowtie_\tau kQ^{sc}, \{R_n\})$ becomes a local quasitriangular Hopf algebra.

Proof. (i) Let $B := (kG)^*$ and $M := kQ_1^a$. Thus $T_B(M) = kQ^a$. Since (kQ_1^a, kQ_1^c) is an arrow dual pairing, $\xi_{kQ_1^c}$ is a kG-Hopf bimodule isomorphism from kQ_1^c to $(kQ_1^a)^*$ by Lemma 13. See that

$$\begin{aligned} T^c_{B^*}(M^*) &= T^c_{(kG)^{**}}((kQ_1^a)^*) \\ &\stackrel{\nu_1}{\cong} T^c_{kG}((kQ_1^a)^*) \qquad \text{(by Lemma 15)} \\ &\stackrel{\nu_2}{\cong} T^c_{kG}(kQ_1^c) \qquad \text{(by Lemma 13 and Lemma 14)} \\ &= kQ^c, \end{aligned}$$

where $\nu_1 = T^c_{kG}(\sigma^{-1}_{kG}\pi_0, \pi_1)$, $\nu_2 = T^c_{(kG)^{**}}(\pi_0, (\xi_{kQ_1^c})^{-1}\pi_1)$.

Let $H = T^c_{B^*}(M^*)$ and $A = T_B(M)^{\rm cop}$. By Theorem 6, $((kQ^a)^{\rm cop} \bowtie_\tau kQ^c, \{R_n\})$ is a local quasitriangular Hopf algebra. Here $\tau = d_H(\phi \otimes id)C_{H,A}((\nu_2\nu_1)^{-1} \otimes id_A)$; $R_n = (id_A \otimes \nu_2\nu_1 \otimes id_A \otimes \nu_2\nu_1)(1_B \otimes (id \otimes \psi_n)b_{H_{(n)}} \otimes 1_{B^*})$, ϕ and ψ_n are defined in Lemma 6. We have to show that they are the same as in this theorem. That is,

$$d_H(\phi \otimes id)C_{H,A}((\nu_2\nu_1)^{-1} \otimes id_A)(a \otimes b) = \delta_{a,b} \tag{4}$$

$$\begin{aligned} (1_B \otimes b_{H_{(n)}} \otimes 1_{B^*}) &= (id_A \otimes (\nu_2\nu_1)^{-1} \otimes \phi_n \otimes (\nu_2\nu_1)^{-1}) \\ &\quad \times \left(\sum_{g \in G} p_e \otimes g \otimes p_g \otimes e + \sum_{q \in Q_{(n)}, q \notin G} p_e \otimes q \otimes q \otimes e \right). \end{aligned} \tag{5}$$

If $b = b_n b_{n-1} \cdots b_1$ is a n-path in kQ^c with $b_i \in Q_1$ for $i = 1, 2, \ldots, n$, then

$$(\nu_2\nu_1)^{-1}(b) = b_n^* \otimes b_{n-1}^* \otimes \cdots \otimes b_1^*. \tag{6}$$

If $b \in G$, then $(\nu_2\nu_1)^{-1}(b) = \sigma_{kG}(b)$. Consequently, for any $a, b \in Q_\infty$ with $b \in kQ^c$ and $a \in kQ^a$, we have

$$\begin{aligned} &d_H(\phi \otimes id)C_{H,A}((\nu_2\nu_1)^{-1} \otimes id_A)(a \otimes b) = d_H(\phi(a) \otimes (\nu_2\nu_1)^{-1}(b)) \\ &= \begin{cases} \langle \phi(a), \sigma_{kG}(b) \rangle = \langle \sigma_{kG}(b), a \rangle = \delta_{a,b}, & \text{when } \ b \in Q_0, \\ \langle (\nu_2\nu_1)^{-1}, a \rangle \stackrel{\text{by (6)}}{=} \delta_{a,b}, & \text{when } \ b \notin Q_0. \end{cases} \end{aligned}$$

Thus (4) holds. Note that Q_n is a basis of not only $(kQ^a)_n$ but also $(kQ^c)_n$ for $n > 0$. By (6), $\{\phi(q) \mid q \in Q_n\}$ is the dual basis of $\{(\nu_2\nu_1)^{-1}(q) \mid q \in Q_n\}$ for any $n > 0$. Consequently, (5) holds.

(ii) Let $B := kG$ and $M := kQ_1^c$. Thus $T_B(M) = kQ^s$. Since (kQ_1^a, kQ_1^c) is an arrow dual pairing, $\xi_{kQ_1^a}$ is a $(kG)^*$-Hopf bimodule isomorphism from kQ_1^a to $(kQ_1^c)^*$ by Lemma 13. Thus

$$\begin{aligned} T^c_{B^*}(M^*) &= T^c_{(kG)^*}((kQ_1^c)^*) \\ &\stackrel{\nu_3}{\cong} T^c_{(kG)^*}(kQ_1^a) \qquad \text{(by Lemma 13 and Lemma 14)} \\ &= kQ^{sc}, \end{aligned}$$

where $\nu_3 = T^c_{(kG)^*}(\pi_0, (\xi_{kQ_1^a})^{-1}\pi_1)$. By Theorem 6, the double cross product of $T_B(M)^{\rm cop}$ and $T_{B^*}(M^*)$ is a local quasitriangular Hopf algebra. Consequently, so is the double cross product of $(kQ^s)^{\rm cop}$ and kQ^{sc}. ■

Note (LQT4') holds in the above theorem.

Example 1. Let $G = \mathbf{Z}_2 = (g) = \{e, g\}$ be the group of order 2 with $\operatorname{char} k \neq 2$, X and Y be respectively the set of arrows from g^0 to g^0 and the set of arrows from g to g, and $|X| = |Y| = 3$. The quiver Q is a Hopf quiver with respect to ramification $r = r_{\{e\}}\{e\}$ with $r_{\{e\}} = 3$. Let $\chi_e^{(i)} \in \mathbf{Z}_2$ and $a_{y,x}^{(i)}$ denote the arrow from x to y for $i = 1, 2, 3$. Define $\delta^-(a_{x,x}^{(i)}) = x \otimes a_{x,x}^{(i)}$, $\delta^+(a_{x,x}^{(i)}) = a_{x,x}^{(i)} \otimes x$, $g \cdot a_{x,x}^{(i)} = a_{gx,gx}^{(i)}$, $a_{e,e}^{(i)} \cdot g = \chi_e^{(i)}(g) a_{xg,xg}^{(i)}$ for $x \in G$, $i = 1, 2, 3$. By [4], kQ_1 is a kG-Hopf bialgebra. Therefore, it follows from Theorem 7 that $((kQ^a)^{\rm cop} \bowtie_\tau kQ^c, \{R_n\})$ is a local quasitriangular Hopf algebra and for every finite dimensional $(kQ^a)^{\rm cop} \bowtie_\tau kQ^c$-module M, $C^{\{R_n\}}$ is a solution of Yang–Baxter equations on M.

By the way, we obtain the relation between path algebras and path coalgebras by Theorem 5.

Corollary 1. *Let Q be a finite quiver over finite group G. Then Path algebra kQ^a is algebra isomorphic to subalgebra $\sum_{n=0}^{\infty}(\Box_{kG}^n kQ_1^c)^*$ of $(kQ^c)^*$.*

Proof. Let $A = (kG)^*$ and $M = kQ_1^c$. It is clear that $\xi_{kQ_1^c}$ is a kG-bicomodule isomorphism from kQ_1^c to $(kQ_1^a)^*$. See that

$$T_{B^*}^c(M^*) = T_{(kG)^{**}}^c((kQ_1^a)^*) \overset{\nu_1}{\cong} T_{kG}^c((kQ_1^a)^*) \overset{\nu_2}{\cong} T_{kG}^c(kQ_1^c) = kQ^c,$$

where $\nu_1 = T_{kG}^c(\sigma_{kG}^{-1}\pi_0, \pi_1)$, $\nu_2 = T_{kG}^c(\pi_0, (\xi_{kQ_1^c})^{-1}\pi_1)$. Obviously, ν_1 and ν_2 are coalgebra isomorphism. Consequently, it follows from Theorem 5 that kQ^a is algebra isomorphism to subalgebra $\sum_{n=0}^{\infty}(\Box_{kG}^n kQ_1^c)^*$ of $(kQ^c)^*$. ■

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