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Geometric Realizations of Bi-Hamiltonian Completely Integrable Systems*

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Abstract. In this paper we present an overview of the connection between completely integrable systems and the background geometry of the flow. This relation is better seen when using a group-based concept of moving frame introduced by Fels and Olver in [*Acta Appl. Math.* **51** (1998), 161–213; **55** (1999), 127–208]. The paper discusses the close connection between different types of geometries and the type of equations they realize. In particular, we describe the direct relation between symmetric spaces and equations of KdV-type, and the possible geometric origins of this connection.

Key words: invariant evolutions of curves; Hermitian symmetric spaces; Poisson brackets; differential invariants; projective differential invariants; equations of KdV type; completely integrable PDEs; moving frames; geometric realizations

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1 Introduction

Example 1. One of the simplest examples of a geometric realization of a completely integrable system is that of the nonlinear Schrödinger equation (NLS) realized by the self-induction Vortex Filament flow (VF). The VF flow is a flow in the Euclidean space $\mathrm{SO}(3) \ltimes \mathbb{R}^3/\mathrm{SO}(3)$ (see [3]). In [21] Hasimoto showed that, if $u(x,t)$ is a flow solution of the VF equation

$$u_t = \kappa B,$$

where κ is the Euclidean curvature of the curve $u(\cdot, x) \in \mathbb{R}^3$, x is the arc-length and B is the binormal, then the evolution of the curvature and torsion of u is equivalent to the NLS equation via the Hasimoto transformation $\Phi = \kappa e^{i\int\tau}$. The Hasimoto transformation $(\kappa,\tau) \to (\nu,\eta)$, with $\Phi = \nu + i\eta$, is, in fact, induced by a change from classical Euclidean moving frame to the *natural moving frame* (ν and η are the *natural curvatures*, see [28]). Thus, VF is a *Euclidean geometric realization* of NLS if we use natural moving frames. Equivalently, NLS is the *invariantization* of VF. The relation to the Euclidean geometry of the flow goes further; consider the evolution

$$u_t = hT + \frac{h'}{\kappa}N + gB, \tag{1}$$

where $\{T, N, B\}$ is the classical Euclidean moving frame and h and g are arbitrary smooth functions of the curvature, torsion and their derivatives. Equation (1) is the general form of

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an arc-length preserving evolution of space curves, invariant under the action of the Euclidean group (i.e., $E(n)$ takes solutions to solutions). Its invariantization can be written as

$$\begin{pmatrix}\kappa\\ \tau\end{pmatrix}_t = \mathcal{P}\begin{pmatrix}g\\ h\end{pmatrix}, \tag{2}$$

where $\mathcal{P}$ defines a Poisson bracket generated by the *second Hamiltonian structure* for NLS via the Hasimoto transformation, i.e. the Hasimoto transformation is a *Poisson map* (see [28, 38, 39]). (For more information on infinite dimensional Poisson brackets see [40], and for more information on $\mathcal{P}$ see [38].) Clearly, $\mathcal{P}$ can be generated using a classical Euclidean moving frame and the invariants κ and τ. The NLS equation is a *bi-Hamiltonian system*, i.e., Hamiltonian with respect to two compatible Hamiltonian structures. One of the structures is invertible and a recursion operator can be constructed to generate integrals of the system (see [30] or [40]). The first Hamiltonian structure for NLS is invertible and can also be proved to be generated by the geometry of the flow, although it is of a different character as we will see below.

Example 2. A second example is that of the Korteweg–de Vries (KdV) equation. If a 1-parameter family of functions $u(t,\cdot)\in\mathbb{R}$ evolves following the *Schwarzian KdV equation*

$$u_t = u' S(u) = u''' - \frac{3}{2}\frac{(u'')^2}{u'},$$

where $S(u) = \dfrac{u'''}{u'} - \dfrac{3}{2}\left(\dfrac{u''}{u'}\right)^2$ is the *Schwarzian derivative* of u, then $k = S(u)$ itself evolves following the KdV equation

$$k_t = k''' + 3kk',$$

one of the best known completely integrable nonlinear PDEs. The KdV equation is Hamiltonian with respect to two compatible Hamiltonian structures, namely $D=\frac{d}{dx}$ and $D^3+2kD+k'$ (called respectively first and second KdV Hamiltonian structures). As before, if we consider the general curve evolution given by

$$u_t = u'h, \tag{3}$$

where h is any smooth function depending on $S(u)$ and its derivatives with respect to the parameter x, then $k = S(u)$ evolves following the evolution

$$k_t = (D^3 + 2kD + k')h. \tag{4}$$

This time $S(u)$ is the generating *differential invariant* associated to the action of PSL(2) on $\mathbb{RP}^1$. That is, any projective differential invariant of curves $u(x)$ is a function of $S(u)$ and its derivatives. Equation (3) is the most general form for evolutions of reparametrizations of $\mathbb{RP}^1$ (or parametrized "curves") invariant under the action of PSL(2). Evolution (4) can be viewed as the invariantization of (3). Equivalently, the family of evolutions in (3) provides $\mathbb{RP}^1$ geometric realizations for the Hamiltonian evolutions defined by (4). Thus, one can obtain geometric realizations in $\mathbb{RP}^1$ not only for KdV, but also for any system which is Hamiltonian with respect to the second KdV Hamiltonian structure. For example, the Sawada–Koterra equation

$$k_t = (D^3 + 2kD + k')\left(2k'' + \frac{1}{2}k^2\right) = 2k^{(5)} + 5kk''' + 5k'k'' + \frac{5}{2}k'k^2$$

is bi-Hamiltonian with respect to the same Hamiltonian structures as KdV is. Its Hamiltonian functional (4) is $h(k) = \int(\frac{1}{6}k^3 - (k')^2)dx$. Therefore, the Sawada–Koterra equation has

$$u_t = u'\left(2S(u)'' + \frac{1}{2}S(u)^2\right)$$

as $\mathbb{RP}^1$ realization. (Incidentally, Sawada–Koterra has a second realization as an equi-affine flow, see [41].) The manifold $\mathbb{RP}^1$ is an example of a *parabolic homogeneous space*, i.e., a manifold of the form G/P with G semisimple and P a *parabolic subgroup*. From (3) and (4) we can see that the second Hamiltonian structure for KdV can be generated with the sole knowledge of u' (a classical projective moving frame along u) and $S(u)$, its projective differential invariant. The first KdV structure is also similarly generated, although, again, it is of a different nature.

These two simple examples illustrate the close relationship between the classical geometry of curves and bi-Hamiltonian completely integrable PDEs. In the last years many examples of geometric realizations for most known completely integrable systems have been appearing in the literature. Some are linked to the geometric invariants of the flow (see for example [1, 2, 11, 13, 17, 25, 26, 28, 29, 31, 36, 38, 39, 46, 47, 48, 49, 51]). This list is, by no means, exhaustive as this paper is not meant to be an exhaustive review of the subject.

Perhaps the simplest way to understand the close relationship between differential invariants and integrable systems is through the AKNS representation on one hand and group-based moving frames on the other. If G is a Lie group, a G-AKNS representation of a nonlinear PDE

$$k_t = F(k, k_x, k_{xx}, \dots) \tag{5}$$

is a linear system of equations

$$\begin{aligned} &\varphi_x = A(t, x, \lambda)\varphi, \\ &\varphi_t = B(x, t, \lambda)\varphi, \qquad \varphi(t, x, \lambda) \in G, \qquad A(x, t, \lambda), B(x, t, \lambda) \in \mathfrak{g} \end{aligned}$$

such that the compatibility condition for the existence of a solution,

$$A_t = B_x + [B, A],$$

is independent of λ and equivalent to the nonlinear PDE (5). Such a representation is a basis for generating solutions and integrating the system. Indeed, most integrable systems have an AKNS representation. Geometrically, this is thought of as having a 2-parameter flat connection defined by $-\frac{d}{dx} + A$ and $-\frac{d}{dt} + B$ along the flow (see [22]). The bridge to differential invariants and differential geometry a-la-Cartan appears when one realizes that this 2-parameter connection is a reduction of the *Maurer–Cartan connection* of G along the flow φ and the AKNS system could be interpreted as the Serret–Frenet equations and the t-evolution of a group-based (right) moving frame (φ) along a flow $u : \mathbb{R}^2 \to G/H$ in a certain homogeneous space. This is explained in the next section.

In this paper we describe how the background geometry of *affine* and some *symmetric manifolds* generates Hamiltonian structures and geometric realizations for some completely integrable systems. Our affine manifolds will be homogeneous manifolds of the form $G \ltimes \mathbb{R}^n/G$ with G semisimple. They include Euclidean, Minkowski, affine, equi-affine and symplectic geometry among others. We will also discuss related geometries, like the centro-affine or geometry of star-shaped curves, for which the action of the group is linear instead of affine. On the other hand, our symmetric manifolds are locally equivalent to a homogeneous manifold of the form G/H where $\mathfrak{g}$, the Lie algebra associated to G, has a gradation of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and where $g_0 \oplus \mathfrak{g}_1 = \mathfrak{h}$ is the Lie algebra of H. These includes projective geometry ($G = \mathrm{PSL}(n+1)$), the Grassmannian ($G = \mathrm{SL}(p+q)$), the conformally flat Möbius sphere ($G = O(n+1, 1)$), the Lagrangian Grassmannian ($G = \mathrm{Sp}(2n)$), the manifold of reduced pure spinors ($G = O(n, n)$) and more. We will see how the Cartan geometry of curves in these manifolds induces a Hamiltonian structure on the space of differential invariants. In the last part of the paper we look closely at the case of symmetric spaces. We define *differential invariants of projective type* as those generated by the action of the group on second order frames. We then describe how, in most cases, the reduced Hamiltonian structure can be further reduced or restricted to the space of curves with vanishing non-projective differential invariants. On this manifold the Hamiltonian structure has a (geometrically defined) compatible Poisson companion. They define a bi-Hamiltonian pencil for some integrable equations of KdV-type, and they provide geometric realizations for them. We finally state a conjecture by M. Eastwood on what the presence of these flows might say about the geometry of curves in symmetric spaces.

As group based moving frames are relatively new, our next section will describe them in detail. We will also describe their role in AKNS representations.

2 Moving frames

The classical concept of moving frame was developed by Élie Cartan [8, 9]. A classical moving frame along a curve in a manifold M is a curve in the frame bundle of the manifold over the curve, invariant under the action of the transformation group under consideration. This method is a very powerful tool, but its explicit application relied on intuitive choices that were not clear on a general setting. Some ideas in Cartan's work and later work of Griffiths [19], Green [20] and others laid the foundation for the concept of a group-based moving frame, that is, an equivariant map between the jet space of curves in the manifold and the group of transformations. Recent work by Fels and Olver [14, 15] finally gave the precise definition of the group-based moving frame and extended its application beyond its original geometric picture to an astonishingly large group of applications. In this section we will describe Fels and Olver's moving frame and its relation to the classical moving frame. We will also introduce some definitions that are useful to the study of Poisson brackets and bi-Hamiltonian nonlinear PDEs. From now on we will assume $M = G/H$ with G acting on M via left multiplication on representatives of a class. We will also assume that curves in M are *parametrized* and, therefore, the group G does not act on the parameter.

Definition 1. Let $J^k(\mathbb{R}, M)$ the space of k-jets of curves, that is, the set of equivalence classes of curves in M up to k^{th} order of contact. If we denote by $u(x)$ a curve in M and by u_r the r derivative of u with respect to the parameter x, $u_r = \frac{d^r u}{dx^r}$, the jet space has local coordinates that can be represented by $u^{(k)} = (x, u, u_1, u_2, \dots, u_k)$. The group G acts naturally on parametrized curves, therefore it acts naturally on the jet space via the formula

$$g \cdot u^{(k)} = (x, g \cdot u, (g \cdot u)_1, (g \cdot u)_2, \dots),$$

where by $(g \cdot u)_k$ we mean the formula obtained when one differentiates $g \cdot u$ and then writes the result in terms of g, u, u_1, etc. This is usually called the *prolonged* action of G on $J^k(\mathbb{R}, M)$.

Definition 2. A function

$$I : J^k(\mathbb{R}, M) \to \mathbb{R}$$

is called a kth order *differential invariant* if it is invariant with respect to the prolonged action of G.

Definition 3. A map

$$\rho : J^k(\mathbb{R}, M) \to G$$

is called a left (resp. right) *moving frame* if it is equivariant with respect to the prolonged action of G on $J^k(\mathbb{R}, M)$ and the left (resp. right) action of G on itself.

If a group acts (locally) *effectively on subsets*, then for k large enough the prolonged action is locally free on regular jets. This guarantees the existence of a moving frame on a neighborhood of a regular jet (for example, on a neighborhood of a generic curve, see [14, 15]).

The group-based moving frame already appears in a familiar method for calculating the curvature of a curve $u(s)$ in the Euclidean plane. In this method one uses a translation to take $u(s)$ to the origin, and a rotation to make one of the axes tangent to the curve. The curvature can classically be found as the coefficient of the second order term in the expansion of the curve around $u(s)$. The crucial observation made by Fels and Olver is that *the element of the group* carrying out the translation and rotation depends on u and its derivatives and so it defines a map from the jet space to the group. *This map is a right moving frame*, and it carries all the geometric information of the curve. In fact, Fels and Olver developed a similar normalization process to find right moving frames (see [14, 15] and our next theorem).

Theorem 1 ([14, 15]). *Let $\cdot$ denote the prolonged action of the group on $u^{(k)}$ and assume we have **normalization equations** of the form*

$$g \cdot u^{(k)} = c_k,$$

*where c_k are constants (they are called **normalization constants**). Assume we have enough normalization equations so as to determine g as a function of $u, u_1, \dots$. Then $g = \rho$ is a **right invariant moving frame**.*

The direct relation between classical moving frames and group-based moving frames is stated in the following theorem.

Theorem 2 ([32]). *Let $\Phi_g : G/H \to G/H$ be defined by multiplication by g. That is $\Phi_g([x]) = [gx]$. Let ρ be a group-based left moving frame with $\rho \cdot o = u$ where $o = [H] \in G/H$. Identify $d\Phi_\rho(o)$ with an element of $GL(n)$, where n is the dimension of M.*

Then, the matrix $d\Phi_\rho(o)$ contains in its columns a classical moving frame.

This theorem illustrates how classical moving frames are described only by the action of the group-based moving frame on first order frames, while the action on higher order frames is left out. Accordingly, those invariants determined by the action on higher order frames will be not be found with the use of a classical moving frame.

Next is the equivalent to the classical Serret–Frenet equations. This concept if fundamental in our Poisson geometry study.

Definition 4. Consider Kdx to be the horizontal component of the pullback of the left (resp. right)-invariant Maurer–Cartan form of the group G via a group-based left (resp. right) moving frame ρ. That is

$$K = \rho^{-1}\rho_x \in \mathfrak{g} \qquad (\text{resp.} \quad K = \rho_x\rho^{-1})$$

(K is the coefficient matrix of the first order differential equation satisfied by ρ). We call K the *left (resp. right) Serret–Frenet equations* for the moving frame ρ.

Notice that, if ρ is a left moving frame, then ρ^{-1} is a right moving frame and their Serret–Frenet equations are the negative of each other. A complete set of generating differential invariants can always be found among the coefficients of group-based Serret–Frenet equations, a crucial difference with the classical picture. The following theorem is a direct consequence of the results in [14, 15]. A more general result can be found in [23].

Theorem 3. *Let ρ be a (left or right) moving frame along a curve u. Then, the coefficients of the (left or right) Serret–Frenet equations for ρ contain a basis for the space of differential invariants of the curve. That is, any other differential invariant for the curve is a function of the entries of K and their derivatives with respect to x.*

Example 3. Assume $G = \mathrm{PSL}(2)$ so that $M = \mathbb{RP}^1$. The action of G on $\mathbb{RP}^1$ is given by fractional transformations. Assume $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ is a (right) moving frame satisfying the normalization equations

$$\rho \cdot u = \frac{au+b}{cu+d} = 0,$$
$$\rho \cdot u_1 = \frac{au_1}{cu+d} - \frac{(au+b)cu_1}{(cu+d)^2} = 1,$$
$$\rho \cdot u_2 = \frac{au_2}{cu+d} - 2\frac{acu_1^2}{(cu+d)^2} + \frac{au+b}{(cu+d)^3}\left(cu_2(cu+d) + 2\frac{c^2u_1^2}{(cu+d)^3}\right) = 2\lambda.$$

Then it is straightforward to check that ρ is completely determined to be

$$\rho = \begin{pmatrix} 1 & 0 \\ \frac{1}{2}\frac{u_2}{u_1} - \lambda & 1 \end{pmatrix} \begin{pmatrix} u_1^{-1/2} & 0 \\ 0 & u_1^{1/2} \end{pmatrix} \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}.$$

A moving frame satisfying this normalization will have the following right Serret–Frenet equation

$$\rho_x = \begin{pmatrix} -\lambda & -1 \\ \frac{1}{2}S(u) + \lambda^2 & \lambda \end{pmatrix} \rho. \tag{6}$$

This equation is *gauge* equivalent to the $\lambda = 0$ equation via the constant *gauge*

$$g = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}. \tag{7}$$

This gauge g will take the second normalization constant to zero.

Furthermore, if u is a solution of (3), it is known (see [31]) that the t-evolution induced on ρ is given by

$$\rho_t = \begin{pmatrix} -\frac{1}{2}h_x - \lambda h & -h \\ \frac{1}{2}h_{xx} + \lambda h_x + \lambda^2 h + \frac{1}{2}S(u)h & \frac{1}{2}h_x + \lambda h \end{pmatrix} \rho. \tag{8}$$

We can now see the link between the AKNS representation of KdV and the evolution of a right moving frame. Assume a completely integrable system (5) has a geometric realization which is invariant under the action of the geometric group G

$$u_t = f(\lambda, u, u_1, u_2, \dots). \tag{9}$$

Then, under regularity assumptions of the flow, the invariantization of (9) is the integrable system (5). A right moving frame along u will be a solution of its Serret–Frenet equation

$$\rho_x = K(t, x, \lambda)\varphi$$

and the time evolution will induce a time evolution on ρ of the form

$$\rho_t = N(t, x, \lambda)\varphi.$$

Furthermore, since (9) is invariant under the group, both K and N will depend on the differential invariants of the flow. These equations are defined by the horizontal component of the pullback of the Maurer–Cartan form of the group, $\omega = dgg^{-1}$ by the moving frame ρ, that is $Kdx + Ndt$.

If we now evaluate the *structure equation* for the Maurer–Cartan form, i.e., $d\omega + \frac{1}{2}[\omega, \omega] = 0$, along ρ_x and ρ_t, we get

$$K_t = N_x + [N, K]$$

which is exactly the invariantization of the flow (9); therefore it is independent of λ. Hence, a λ-dependent geometric realization of an integrable system provides an AKNS representation of the system. See [7] for more information.

Example 4. The AKNS representation for KdV is very well known. It is given by the system

$$\varphi_x = \begin{pmatrix} -\lambda & -1 \\ -q & \lambda \end{pmatrix} \varphi,$$
$$\varphi_t = \begin{pmatrix} -\frac{1}{2}q_x - \lambda q + 2\lambda^3 & -q + 2\lambda^2 \\ \frac{1}{2}q_{xx} + \lambda q_x + q(-q + 2\lambda^2) & \frac{1}{2}q_x + \lambda q - 2\lambda^3 \end{pmatrix} \varphi.$$

Comparing it to (6) and (8) we see that $q = \frac{1}{2}S(u) + \lambda^2$ and hence u will depend on λ. Furthermore, $h = q - 2\lambda^2$ provides λ-dependent $\mathbb{RP}^1$ geometric realizations for KdV, namely

$$u_t = u_x \left(-\frac{1}{2}S(u) - 3\lambda^3 \right).$$

A complete description of this example can be found in [7]. (Notice that the KdV equation they represent is different, but equivalent, to our introductory example. This is merely due to a different choice of invariant.)

In this paper we will not focus on the study of solutions (see [7] instead) but rather on the interaction between geometry and integrable systems. Hence we will largely ignore the spectral parameter λ and its role.

3 Hamiltonian structures generated by group-based moving frames

Consider the group of loops $\mathcal{L}G = C^\infty(S^1, G)$ and its Lie algebra $\mathcal{L}\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$. Assume $\mathfrak{g}$ is semisimple. One can define two natural Poisson brackets on $\mathcal{L}\mathfrak{g}^*$ (see [45] for more information), namely, if $\mathcal{H}, \mathcal{F} : \mathcal{L}\mathfrak{g}^* \to \mathbb{R}$ are two functionals defined on $\mathcal{L}\mathfrak{g}^*$ and if $L \in \mathcal{L}\mathfrak{g}^*$, we define

$$\{\mathcal{H}, \mathcal{F}\}_1(L) = \int_{S^1} \left\langle \left(\frac{\delta \mathcal{H}}{\delta L}(L) \right)_x + \mathrm{ad}^* \left(\frac{\delta \mathcal{H}}{\delta L}(L) \right)(L), \frac{\delta \mathcal{F}}{\delta L}(L) \right\rangle dx, \tag{10}$$

where $\langle\ ,\ \rangle$ is the natural coupling between $\mathfrak{g}^*$ and $\mathfrak{g}$, and where $\frac{\delta \mathcal{H}}{\delta L}(L)$ is the variational derivative of $\mathcal{H}$ at L identified, as usual, with an element of $\mathcal{L}\mathfrak{g}$.

One also has a compatible family of second brackets, namely

$$\{\mathcal{H}, \mathcal{F}\}_2(L) = \int_{S^1} \left\langle \mathrm{ad}^* \left(\frac{\delta \mathcal{H}}{\delta L}(L) \right)(L_0), \frac{\delta \mathcal{F}}{\delta L}(L) \right\rangle dx, \tag{11}$$

where $L_0 \in \mathfrak{g}^*$ is any constant element. Since $\mathfrak{g}$ is semisimple we can identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ and we will do so from now on.

From now on we will also assume that our curves on homogeneous manifolds have a *group monodromy*, i.e., there exists $m \in G$ such that

$$u(t + T) = m \cdot u(t),$$

where T is the period. Under these assumptions, the Serret–Frenet equations will be periodic. One could, instead, assume that u is asymptotic at $\pm\infty$, so that the invariants will vanish at infinity. We would then work with the analogous of (10) and (11).

The question we would like to investigate next is whether or not these two brackets can be reduced to the space of differential invariants, or the space of differential invariants associated to special types of flows. We will describe affine and symmetric cases separately.

3.1 Affine manifolds

Assume $M = (G \ltimes \mathbb{R}^n)/G$ is an affine manifold, G semisimple. In this case a moving frame can be represented as

$$\rho = \begin{pmatrix} 1 & 0 \\ \rho_u & \rho_G \end{pmatrix} \tag{12}$$

acting on $\mathbb{R}^n$ as

$$\rho \cdot u = \rho_G u + \rho_u.$$

A left invariant moving frame with $\rho \cdot o = u$ will hold $\rho_u = u$ and, in view of Theorem 2, ρ_G will have in its columns a classical moving frame. In this case $K = \rho^{-1}\rho_x$ is given by

$$K = \begin{pmatrix} 0 & 0 \\ \rho_G^{-1}(\rho_u)_x & \rho_G^{-1}(\rho_G)_x \end{pmatrix}.$$

In [32] it was shown that $\rho_G^{-1}(\rho_u)_x$ contains all first order differential invariants. It was also explained how one could make this term constant by choosing a special parametrization if necessary. Let's call that constant $\rho_G^{-1}(\rho_u)_x = \Lambda$. Our main tool to find Poisson brackets is via reduction, and as a previous step, we need to write the space of differential invariants as a quotient in $\mathcal{L}\mathfrak{g}^*$. The proof of the following Theorem can be found in [32].

Theorem 4 ([32]). *Let $N \subset G$ be the isotropy subgroup of Λ. Assume that we choose moving frames as above and let $\mathcal{K}$ be the space of Serret–Frenet equations determined by these moving frames for curves in a neighborhood of a generic curve u. Then, there exists an open set of $\mathcal{L}\mathfrak{g}^*$, let's call it U, such that $U/\mathcal{L}N \cong \mathcal{K}$, where $\mathcal{L}N$ acts on $\mathcal{L}\mathfrak{g}^*$ using the gauge (or Kac–Moody) transformation*

$$a^*(n)(L) = n^{-1}n_x + n^{-1}Ln. \tag{13}$$

In view of this theorem, our next theorem comes as no surprise.

Theorem 5 ([32]). *The Hamiltonian structure* (10) *reduces to $U/\mathcal{L}N \cong \mathcal{K}$ to define a Poisson bracket in the space of differential invariants of curves.*

Example 5. If we choose $G = SO(3)$ and M the Euclidean space, for appropriate choice of normalization constants our left moving frame is given by

$$\rho = \begin{pmatrix} 1 & 0 \\ u & T\ N\ B \end{pmatrix},$$

where $\{T, N, B\}$ is the classical Euclidean Serret–Frenet frame. Its Serret–Frenet equations will look like

$$K = \rho^{-1}\rho_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ (u_1 \cdot u_1) & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & -\tau \\ 0 & 0 & \tau & 0 \end{pmatrix}.$$

In this case, if we choose to parametrize our curve by arc-length, $\Lambda = e_1$ where, as usual, we denote by e_k the standard basis of $\mathbb{R}^n$. The matrix K can be clearly identified with its $\mathfrak{o}(3)$ block and hence $\mathcal{K}$ can be considered as a subspace of $\mathcal{L}\mathfrak{o}(3)^*$. The isotropy subgroup N is given by matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & \Theta \end{pmatrix}$ with $\Theta \in SO(2)$.

Using this information we can find the reduced bracket algebraically. For this we take any functional $h : \mathcal{K} \to \mathbb{R}$. Let's call $\mathcal{H}$ an extension of h to $\mathcal{L}\mathfrak{o}(3)^*$, constant on the gauge leaves of $\mathcal{L}N$. Its variational derivative at K needs to look like

$$\frac{\delta \mathcal{H}}{\delta L}(K) = \begin{pmatrix} 0 & \frac{\delta h}{\delta \kappa} & \alpha \\ -\frac{\delta h}{\delta \kappa} & 0 & \frac{\delta h}{\delta \tau} \\ -\alpha & -\frac{\delta h}{\delta \tau} & 0 \end{pmatrix}$$

for some α to be determined. Since $\mathcal{H}$ is constant on the gauge leaves of $\mathcal{L}N$

$$\left\langle n^{-1} n_x + n^{-1} K n, \frac{\delta \mathcal{H}}{\delta L}(K) \right\rangle = 0,$$

for any $n \in \mathcal{L}N$. This is equivalent to

$$\left(\frac{\delta \mathcal{H}}{\delta L}(K)\right)_x + \left[K, \frac{\delta \mathcal{H}}{\delta L}(K)\right] \in \mathcal{L}\mathfrak{n}^o,$$

where $\mathfrak{n}$ is the Lie algebra of N and $\mathfrak{n}^o$ is its annihilator. From here

$$\begin{pmatrix} 0 & (\frac{\delta h}{\delta \kappa})_x - \alpha\tau & \alpha_x - \kappa\frac{\delta h}{\delta \tau} + \tau\frac{\delta h}{\delta \kappa} \\ * & 0 & (\frac{\delta h}{\delta \tau})_x + \kappa\alpha \\ * & * & 0 \end{pmatrix} = \begin{pmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix},$$

where $*$ indicates entries that are not, at least for now, relevant. Hence $\alpha = -\frac{1}{\kappa}(\frac{\delta h}{\delta \tau})_x$. The bracket is thus given by

$$\begin{aligned} \{f, h\}_1^R(K) &= \int_{S^1} \operatorname{tr}\left(\begin{pmatrix} 0 & * & * \\ -(\frac{\delta h}{\delta \kappa})_x - \frac{\tau}{\kappa}(\frac{\delta h}{\delta \tau})_x & 0 & 0 \\ \left(\frac{1}{\kappa}\left(\frac{\delta h}{\delta \tau}\right)_x\right)_x + \kappa\frac{\delta h}{\delta \tau} - \tau\frac{\delta h}{\delta \kappa} & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{\delta f}{\delta \kappa} & -\frac{1}{\kappa}(\frac{\delta f}{\delta \tau})_x \\ -\frac{\delta f}{\delta \kappa} & 0 & \frac{\delta f}{\delta \tau} \\ \frac{1}{\kappa}(\frac{\delta f}{\delta \tau})_x & -\frac{\delta f}{\delta \tau} & 0 \end{pmatrix}\right) dx \\ &= -2\int_{S^1} \begin{pmatrix} \frac{\delta f}{\delta \kappa} & \frac{\delta f}{\delta \tau} \end{pmatrix} \mathcal{R} \begin{pmatrix} \frac{\delta h}{\delta \kappa} \\ \frac{\delta h}{\delta \tau} \end{pmatrix} dx, \end{aligned}$$

where $\mathcal{R}$ is

$$\mathcal{R} = \begin{pmatrix} D & \frac{\tau}{\kappa} D \\ D\frac{\tau}{\kappa} & -D - D\frac{1}{\kappa}D\frac{1}{\kappa}D \end{pmatrix}.$$

The second Hamiltonian structure (11) can also be reduced to $\mathcal{K}$ with the general choice $L_0 = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$. The reduced bracket is found when applying (11) to the variational derivatives of extensions that, as before, are constant on the $\mathcal{L}N$ leaves. Thus, it is straightforward to check that the second reduced bracket is given by

$$\{f, h\}_2^R(K) = 2\int_{S^1} \begin{pmatrix} \frac{\delta f}{\delta \kappa} & \frac{\delta f}{\delta \tau} \end{pmatrix} (a\mathcal{A} + b\mathcal{B} + c\mathcal{C}) \begin{pmatrix} \frac{\delta h}{\delta \kappa} \\ \frac{\delta h}{\delta \tau} \end{pmatrix} dx,$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\kappa}D - D\frac{1}{\kappa} \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 0 & \frac{1}{\kappa}D \\ D\frac{1}{\kappa} & 0 \end{pmatrix}.$$

These are all Hamiltonian structures and they appeared in [38]. In fact, the structure $\mathcal{P}$ shown in the introduction can be written as $\mathcal{P} = -\mathcal{R}\mathcal{C}^{-1}\mathcal{R}$ and, hence, $\mathcal{P}$ is in the Hamiltonian hierarchy generated by $\mathcal{R}$ and $\mathcal{C}$. A study of integrable systems associated to these brackets, and their geometric realizations, was done in [38]. See also [24].

Our first reduced bracket is directly related to geometric realizations. In fact, by choosing a special parameter x we can obtain geometric realizations of systems that are Hamiltonian with respect to the reduced bracket. We explain this next. Let ρ be given as in (12) and assume u is a solution of the invariant equation

$$u_t = \rho_G \mathbf{r}, \tag{14}$$

where $\mathbf{r} = (r_i)$ is a differential invariant vector, that is, r_i are all functions of the entries of K and its derivatives. If the parameter has been fixed so as to guarantee that $\rho_G^{-1}\rho_u = \Lambda$ is constant, then $\mathbf{r}$ has to be modified to guarantee that the evolutions (14) preserve the parameter. (In the running example $\rho_G = (T, N, B)$ and $\rho_G\mathbf{r} = r_1T + r_2N + r_3B$ with $r_2 = \frac{r_1'}{\kappa}$ once the arc-length is chosen as parameter.)

Theorem 6 ([32]). *If there exists a Hamiltonian $h : \mathcal{K} \to \mathbb{R}$ and a local extension $\mathcal{H}$ constant on the leaves of N such that*

$$\frac{\delta\mathcal{H}}{\delta L}(K)\Lambda = \mathbf{r}_x + K\mathbf{r}, \tag{15}$$

then the invariantization of evolution (14) *is Hamiltonian with respect to the reduced bracket $\{\ ,\ \}_1^R$ and its associated Hamiltonian is h.*

If (after choosing a special parameter if necessary) relation (15) can be solved for $\mathbf{r}$ given a certain Hamiltonian h, then the Theorem guarantees a geometric realization for the reduced Hamiltonian system. Such is the case for the VF flow, Sawada–Koterra [32, 41], modified KdV and others [38].

One can find many geometric realizations of integrable systems in affine manifolds (see, for example, [2, 11, 24, 25, 26, 28, 46, 47, 49, 51]). Many of these are realizations of modified KdV equations (or its generalizations), sine-Gordon and Schrödinger flows. These systems have geometric realizations also in non-affine manifolds (see [1, 25, 26, 47, 48, 49]). Nevertheless, a common feature to the generation of these realizations is the existence of a classical moving frame that resembles the classical *natural* moving frame, that is, the derivatives of the non-tangential vectors of the classical frame all have a tangential direction. Thus, it seem to be the case that the existence of geometric realizations for these systems is linked to the existence of a natural frame. This close relationship between geometry and the type of integrable system is perhaps clearer in our next study, that of symmetric manifolds.

Before moving on, we have one final comment in this line of thought. There are other manifolds whose geometry is given by a linear (rather than affine) action of the group. We can still follow a similar approach, reduce the brackets and study Hamiltonian structures on the space of differential invariants. For example, in the case of centro-affine geometry one considers the linear action of $\mathrm{SL}(n)$ on $\mathbb{R}^n$ and the associated geometry is that of star-shaped curves. If we assume that curves are parametrized by the centro-affine arc-length, we can reduce both brackets and obtain a pencil of Poisson brackets. This pencil coincides with the bi-Hamiltonian structure of KdV. Indeed, a geometric realization for KdV was found by Pinkall in [44]. This realization is the one guaranteed by the reduction, as explained in [7]. The interesting aspect of the centro-affine case is the following: there is a natural identification of a star-shaped curve with a projective curve. The identification is given by the intersection of the curve with the lines going through the origin. If the star-shaped curve is *nondegenerate* (that is, $\det(\gamma, \gamma_x, \dots, \gamma^{(n-1)}) \neq 0$. For example, in the planar case the curve is never in the radial direction), the identification is well-defined. Furthermore, if we parametrized star-shaped curves with centroaffine arc-length (that is, if $\det(\gamma, \gamma_x, \dots, \gamma^{(n-1)}) = 1$), the identification is 1-to-1 and the geometries are Poisson-equivalent, the Poisson isomorphism given by the identification. In fact, Pinkall's geometric realization is the star-shaped version of the Schwarzian KdV under this relation (see [7] for more details). The existence of a geometric realization for KdV seems to imply the existence of a background projective geometry.

3.2 Symmetric manifolds

Assume that M is a symmetric manifold which is locally equivalent to G/H with: (a) G semisimple; (b) its (Cartan) connection given by the Maurer–Cartan form (i.e. the manifold is flat); (c) the Lie algebra $\mathfrak{g}$ has a gradation of length 1, i.e.

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{16}$$

with $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{h}$ is the Lie algebra of H.

If M is a symmetric manifold of this type, G splits locally as $G_{-1} \cdot G_0 \cdot G_1$ with H given by $G_0 \cdot G_1$. The subgroup G_0 is called *the isotropic subgroup* of G and it is the component of G that acts linearly on G/H (for more information see [4] or [42]). That means G_0 is the component of the group acting on first order frames. According to Theorem 2, the ρ_0 factor of a left moving frame ρ will determine a classical moving frame (see also [31]).

As in the previous case, the basis for the definition of a Poisson bracket on the space of invariants is to express that space as a quotient in $\mathcal{L}\mathfrak{g}^*$. This is the result in the following Theorem. For a complete description and proofs see [31]. Notice that, if ρ is a (right) moving frame along a curve in a symmetric manifold with $\rho \cdot u = o$, then $\rho \cdot u_1$ is always constant. In general $\rho \cdot u_1$ is described by first order invariants, but curves in symmetric manifolds do not have non-constant first order differential invariants (invariants are third order or higher), and hence $\rho \cdot u_1$ must be constant.

Theorem 7 ([31]). *Let $M = G/H$ be a symmetric manifold as above. Assume that for every curve in a neighborhood of a generic curve u in M we choose a left moving frame ρ with $\rho \cdot o = u$ and $\rho^{-1} \cdot u_1 = \hat{\Lambda}$ constant. Assume that we choose a section of G/H so we can locally identify the manifold with G_{-1} and its tangent with $\mathfrak{g}_{-1}$. Let $\Lambda \in \mathfrak{g}_{-1}$ represent $\hat{\Lambda}$ and let $\mathcal{K}$ be the manifold of Serret–Frenet equations for ρ along curves in a neighborhood of u. Clearly $\mathcal{K} \subset \mathcal{L}\mathfrak{g}^*$. Denote by $\langle \Lambda \rangle$ the linear subspace of $C^\infty(S^1, \mathfrak{g}^*)$ given by $\langle \Lambda \rangle = \{\alpha\Lambda, \alpha(x) > 0\}$.*

Then the space $\mathcal{K}$ can be described as a quotient $U/\mathcal{N}$, where U is an open set of $\langle \Lambda \rangle \oplus \mathcal{L}\mathfrak{g}_0 \oplus \mathcal{L}\mathfrak{g}_1$ and where $\mathcal{N} = \mathcal{N}_0 \cdot \mathcal{L}G_1 \subset \mathcal{L}G_0 \cdot \mathcal{L}G_1$ acts on U via the Kac–Moody action (13)*. The subgroup $\mathcal{N}_0$ is the isotropy subgroup of $\langle \Lambda \rangle$ in $\mathcal{L}G_0$.*

As before, after writing $\mathcal{K}$ as a quotient, one gets a reduction theorem.

Theorem 8 ([31]). *The Poisson bracket* (10) *can be reduced to $\mathcal{K}$ and there exists a well-defined Poisson bracket $\{\ ,\ \}_1^R$ defined on a generating set of independent differential invariants.*

Example 6. As we saw before, the (left) Serret–Frenet equations for the $\mathbb{RP}^1$ case are given by

$$K = \begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix}$$

where $k = -\frac{1}{2}S(u)$. The splitting of the Lie algebra $\mathfrak{sl}(2)$ is given by

$$\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} \in \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1.$$

In this case $\Lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_{-1}$. The isotropic subgroup of $\langle \Lambda \rangle$ in G_0 is G_0 itself, and so $\mathcal{N} = \mathcal{L}G_0 \cdot \mathcal{L}G_1$ or subspace of lower triangular matrices. To reduce the bracket (10) we need to have a functional $h : \mathcal{K} \to \mathbb{R}$ and to find an extension $\mathcal{H} : \mathcal{M} \to \mathbb{R}$ such that

$$\left(\frac{\delta \mathcal{H}}{\delta L}(K)\right)_x + \left[K, \frac{\delta \mathcal{H}}{\delta L}(K)\right] \in \mathfrak{n}^0 \tag{17}$$

for any $K \in \mathcal{K}$, where $\mathfrak{n} = \mathcal{L}\mathfrak{g}_0 \oplus \mathcal{L}\mathfrak{g}_1$ and $\mathfrak{n}^0$ is its annihilator (which we can identify with $\mathcal{L}\mathfrak{g}_1 = \mathcal{L}\mathfrak{g}_{-1}^*$). Also, if $\mathcal{H}$ is an extension of h, its variational derivative at $\mathcal{K}$ will be given by

$$\frac{\delta \mathcal{H}}{\delta L}(K) = \begin{pmatrix} a & \frac{\delta h}{\delta k}(k) \\ b & -a \end{pmatrix},$$

where a and b are to be determined. On the other hand condition (17) translates into

$$\begin{pmatrix} a_x + b - k\frac{\delta h}{\delta k}(k) & \left(\frac{\delta h}{\delta k}(k)\right)_x - 2a \\ b_x + 2ka & -a_x - b + k\frac{\delta h}{\delta k}(k) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}.$$

From here $a = \frac{1}{2}\left(\frac{\delta h}{\delta k}(k)\right)_x$ and $b = k\frac{\delta h}{\delta k}(k) - \frac{1}{2}\left(\frac{\delta h}{\delta k}(k)\right)_{xx}$. We are now ready to find the reduced bracket. If $f, h : \mathcal{K} \to \mathbb{R}$ are two functionals and $\mathcal{F}$ and $\mathcal{H}$ are extensions vanishing on the $\mathcal{N}$-leaves, the reduced bracket is given by

$$\begin{aligned}
\{f, h\}_1^R(K) &= \int_{S^1} \operatorname{tr}\left(\frac{\delta \mathcal{F}}{\delta L}(K)\left\{\left(\frac{\delta \mathcal{H}}{\delta L}(K)\right)_x + \left[K, \frac{\delta \mathcal{H}}{\delta L}(K)\right]\right\}\right) dx \\
&= \int_{S^1} \operatorname{tr}\left(\begin{pmatrix} * & \frac{\delta f}{\delta k}(k) \\ * & * \end{pmatrix}\begin{pmatrix} 0 & 0 \\ k\left(\frac{\delta h}{\delta k}(k)\right)_x + (k\frac{\delta h}{\delta k}(k))_x - \frac{1}{2}\left(\frac{\delta h}{\delta k}(k)\right)_{xxx} & 0 \end{pmatrix}\right) dx \\
&= \int_{S^1} \frac{\delta f}{\delta k}(k)\left(-\frac{1}{2}D^3 + kD + Dk\right)\frac{\delta h}{\delta k}(k) dx.
\end{aligned}$$

The difference in coefficients as compared to the introductory example is due to the fact that $k = -\frac{1}{2}S(u)$ and not $S(u)$.

As it happens, the companion bracket also reduces for $L_0 = \Lambda^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Indeed, it is given by

$$\begin{aligned}
\{f, h\}_2^R(K) &= \int_{S^1} \operatorname{tr}\left\{\frac{\delta \mathcal{F}}{\delta L}(K)\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \frac{\delta \mathcal{H}}{\delta L}(K)\right]\right\} \\
&= \int_{S^1} \operatorname{tr}\left\{\begin{pmatrix} \frac{1}{2}\left(\frac{\delta f}{\delta k}(k)\right)_x & \frac{\delta f}{\delta k}(k) \\ * & -\frac{1}{2}\left(\frac{\delta f}{\delta k}(k)\right)_x \end{pmatrix}\begin{pmatrix} -\frac{\delta h}{\delta k}(k) & 0 \\ \left(\frac{\delta h}{\delta k}(k)\right)_x & \frac{\delta h}{\delta k}(k) \end{pmatrix}\right\} dx \\
&= 2\int_{S^1} \frac{\delta f}{\delta k}(k) D \frac{\delta h}{\delta k}(k) dx.
\end{aligned}$$

It is not true in general that (11) is also reducible to $\mathcal{K}$. In fact, one finds that for $M = \mathbb{RP}^n$ and $G = \mathrm{PSL}(n+1)$ the second bracket (11) is also reducible to $\mathcal{K}$ when $L_0 = \Lambda^* \in \mathfrak{g}^*$. The resulting two brackets are the first and second Hamiltonian structure for Adler–Gel'fand–Dikii flows. But if M is the so-called Lagrangian Grassmannian, $G = \mathrm{Sp}(4)$, the second bracket is never reducible to $\mathcal{K}$ [34].

One interesting comment on the connection to AKNS representations: as before, the reduction of the bracket (10) is directly linked to geometric realizations. But the reduction of (11) indicates the existence of an AKNS representation and an integrable system. In the KdV example we described how the Serret–Frenet equations (6) were gauge equivalent to $\lambda = 0$ using the gauge (7). If we gauge the x-evolution of the KdV AKNS representation in Example 4 by that same element we get that the matrix A changes into

$$A_\lambda = \begin{pmatrix} 0 & 1 \\ q - \lambda^2 & 0 \end{pmatrix} = A - \lambda^2 L_0.$$

Therefore, up to a constant gauge, L_0 indicates the position of the spectral parameter in the KdV AKNS representation in Example 4. In fact, it goes further. One can prove that the coefficient h (as in (3)) of the λ-dependent realizations for KdV determined by this AKNS representation (that is, $h = q - 2\lambda^2$) is given by the variational derivative of the Hamiltonian functional used to write KdV as Hamiltonian system in the pencil $\{\ ,\ \}_1^R - \lambda^2\{\ ,\ \}_2^R$. In the same fashion, the NLS, when written in terms of κ and τ as in (2), has a second Hamiltonian structure obtained when reducing (11) with the choice $L_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. One can see that this system has an AKNS representation with x evolution given by

$$\rho_x = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau - \lambda \\ 0 & -\tau + \lambda & 0 \end{pmatrix} \rho$$

and where the t component is determined by the evolution induced on the *right* moving frame ρ by a u evolution whose invariant coefficients h and g as jn (1) are given by the variational derivative of the Hamiltonian used to write this Euclidean representation of NLS as Hamiltonian with respect to the pencil $\{\ ,\ \}_1^R - \lambda\{\ ,\ \}_2^R$. This must be a known fact on integrable systems, but we could not find it in the literature. For a complete description of this relation see [7].

Back to symmetric spaces. Recall that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, so that we can identify T_xM with $\mathfrak{g}_{-1}$. Recall also that $d\Phi_\rho(o) = (T_1 \dots T_n)$ is a classical moving frame.

Theorem 9 ([31]). *Assume ρ is as in the statement of Theorem 7. Assume $u(t,x)$ is a solution of the evolution*

$$u_t = d\Phi_\rho(o)\mathbf{r} = r_1T_1 + \dots + r_nT_n, \tag{18}$$

where $\mathbf{r} = (r_i)$ is a vector of differential invariants. Then, if there exists a Hamiltonian functional $h : \mathcal{K} \to \mathbb{R}$ and an extension of h, $\mathcal{H} : U \to \mathbb{R}$, constant of the leaves of $\mathcal{N}$ and such that $\left[\frac{\delta\mathcal{H}}{\delta L}(L)\right]_{-1} = \mathbf{r}$ (the subindex -1 indicates the component in $\mathfrak{g}_{-1}$), then the evolution induced on the generating system of differential invariants defined by $\mathcal{K}$ is Hamiltonian with associated Hamiltonian h.

Let us call the generating differential invariants $\mathbf{k}$. Given a Hamiltonian system $\mathbf{k_t} = \xi_h(\mathbf{k}) = \mathcal{P}(\mathbf{k})\frac{\delta h}{\delta \mathbf{k}}$, its associated geometric evolution (18) will be its geometric realization in M. Therefore they always exist, the previous theorem guarantees their existence. Indeed, one only needs to extend the functional h preserving the leaves. As explained in [31] and described in our running examples, one can find $\frac{\delta\mathcal{H}}{\delta L}(K)$ along $\mathcal{K}$ explicitly using a simple algebraic process. Then, the coefficients r_i of the realization are given by $\left(\frac{\delta\mathcal{H}}{\delta L}(K)\right)_{-1}$, as identified with the tangent to the manifold using our section of G/H. Some examples are given in [31, 33, 34, 35].

As we have previously pointed out, the second bracket does never reduce to the space of invariants when $G = \mathrm{Sp}(4)$, the Lagrangian Grassmannian. Still, one can select a submanifold of invariants and study reductions of the brackets on the submanifold where the other invariants vanish. This is equivalent to studying Hamiltonian evolutions of special types of flows on $\mathrm{Sp}(4)/H$. In particular, the author defined differential invariants of projective type in [33]. She then showed how both brackets (10) and (11) (for some choice of L_0) could be reduced on flows of curves with vanishing non-projective differential invariants. The reductions produced Hamiltonian structures and geometric realizations for integrable systems of KdV-type. This, and its implication for the geometry of G/H is described next.

4 Completely integrable systems of KdV type associated to differential invariants of projective type

There are some differential invariants of curves in symmetric spaces that one might call *of projective type*. They are generated by the action of the group on second frames (hence they cannot be found using a classical moving frame) and most of them closely resemble the Schwarzian derivative. In terms of the gradation $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, if we choose an appropriate moving frame, these invariants will appear in K_1, where $K = \rho^{-1}\rho_x = K_{-1} + K_0 + K_1$ is the graded splitting of the Serret–Frenet equation associated to the moving frame ρ.

Of course, the simplest example of differential invariants of projective-type are projective differential invariants. As it was shown in [31], a moving frame can be chosen so that all differential invariants appear in the $\mathfrak{g}_1$ component of the Serret–Frenet equations, while all entries outside $\mathfrak{g}_1$ are constant. More examples of differential invariants of projective type appear in [33, 34, 35] and [37].

In this series of papers the author showed how in many symmetric spaces one can find geometric realizations inducing evolutions of KdV type on the differential invariants of projective type. Indeed, if G/H is a symmetric space as above, it is known [27] that $\mathfrak{g}$ is the direct sum of the following simple Lie algebras:

1. $\mathfrak{g} = \mathfrak{sl}(p+q)$ with $p, q \in \mathbb{Z}^+$. If $q = 1$ then $G/H \equiv \mathbb{R}\mathrm{P}^n$. In general G/H is the *Grassmannian.*
2. $\mathfrak{g} = \mathfrak{sp}(2n)$, the manifold G/H is called the *Lagrangian Grassmannian* and it can be identified with the manifold of Lagrangian planes in $\mathbb{R}^{2n}$.
3. $\mathfrak{g} = \mathfrak{o}(n,n)$, the manifold G/H is called the manifold of *reduced pure spinors.*
4. $\mathfrak{g} = \mathfrak{o}(p+1, q+1)$ with $p, q \in \mathbb{Z}^+$. If $q = 0$ the manifold G/H is isomorphic to the *Möbius sphere*, the local model for flat conformal manifolds.
5. Two exceptional cases, $\mathfrak{g} = E_6$ and $\mathfrak{g} = E_7$.

We will next describe the situation for each one of the 1–4 cases above. The Grassmannian case (1) (other than $q = 1$) and the exceptional cases (5) have not yet been studied.

4.1 Projective case

Let $G = \mathrm{PSL}(n+1)$. If $g \in G$ then, locally

$$g = g_{-1}g_0g_1 = \begin{pmatrix} I & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Theta & 0 \\ 0 & (\det\Theta)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ v^T & 1 \end{pmatrix}$$

with $u, v \in \mathbb{R}^n$ and $\Theta \in \mathrm{GL}(n)$. If we define H by the choice $u = 0$, then $G/H \cong \mathbb{RP}^n$, the n-projective space. This factorization corresponds to the splitting given by the gradation (16). A section for G/H can be taken to be the g_{-1} factor. As with any other homogeneous space, the action of G on this section is completely determined by the relation

$$g \begin{pmatrix} I & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & g \cdot u \\ 0 & 1 \end{pmatrix} h$$

for some $h \in H$.

The corresponding splitting of the Lie algebra is given by

$$V = V_{-1} + V_0 + V_1 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & -\mathrm{tr}A \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b^T & 0 \end{pmatrix} \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

We can identify the first term above with the tangent to the manifold.

The following theorem describes the type of moving frame and invariant manifold we will choose. It is essential to choose a simple enough representation for $\mathcal{K}$ so that one can readily recognize the result of the reduction. Naturally, any choice of moving frame will produce a choice for $\mathcal{K}$ and a Hamiltonian structure. But showing the equivalence of Poisson structures is a non-trivial problem, and hence its recognition is an important part of the problem. As we said before, all differential invariants are of projective-type.

Theorem 10 (reformulation of Wilczynski [50]). *There exists a left moving frame ρ along nondegenerate curves in $\mathbb{RP}^n$ such that its Serret–Frenet equations are defined by matrices of the form*

$$K = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ k_1 & k_2 & \dots & k_n & 0 \end{pmatrix},$$

where k_i, $i = 1, \dots, n$ are, in general, a generating combination of the Wilczynski projective invariants *and their derivatives.*

For the precise relation between these invariants and Wylczynski's invariants, see [7]. The following result was originally obtained by Drinfel'd and Sokolov in [12]. Their description of the quotient is not the same as ours, but [31] showed that our reduction and theirs are equivalent.

Theorem 11 ([12]). *Assume $\mathcal{K}$ is represented by matrices of the form above. Then, the reduction of* (10) *to $\mathcal{K}$ is given by the Adler–Gel'fand–Dikii (AGD) bracket or second Hamiltonian structure for generalized KdV. The bracket* (11) *reduces for the choice $L_0 = e_n^*$ and its reduction is the first Hamiltonian structure for generalized KdV equations.*

Finally, the following theorem identifies geometric realizations for any flow Hamiltonian with respect to the reduction of (10); in particular, it provides geometric realizations for generalized KdV equations or AGD flows. The case $n = 2$ was originally proved in [18].

Theorem 12 ([36]). *Assume $u : J \subset \mathbb{R}^2 \to \mathrm{PSL}(n+1)/H$ is a solution of*

$$u_t = h_1 T_1 + h_2 T_2 + \dots + h_n T_n,$$

where T_i form a projective classical moving frame.

Then, $\mathbf{k} = (k_i)$ *satisfies an equation of the form*

$$\mathbf{k}_t = P\mathbf{h},$$

where $\mathbf{k} = (k_1, \ldots, k_n)^T$, $\mathbf{h} = (h_1, \ldots, h_n)^T$ *and where* P *is the Poisson tensor defining the* ***Adler–Gel'fand–Dikii Hamiltonian structure****. In particular, we obtain a projective geometric realization for a* ***generalized KdV system*** *of equations.*

In our next section we look at some cases for which not all differential invariants of curves are of projective type.

4.2 The Lagrangian Grassmannian and the manifold of reduced pure Spinors

These two examples are different, but their differential invariants of projective type behave similarly and so we will present them in a joint section.

Lagrangian Grassmannian. Let $G = \mathrm{Sp}(2n)$. If $g \in G$ then, locally

$$g = g_{-1}g_0g_1 = \begin{pmatrix} I & u \\ 0 & I \end{pmatrix} \begin{pmatrix} \Theta & 0 \\ 0 & \Theta^{-T} \end{pmatrix} \begin{pmatrix} I & 0 \\ S & I \end{pmatrix}$$

with u and S symmetric $n \times n$ matrices and $\Theta \in \mathrm{GL}(n)$. Again, this factorization corresponds to the splitting given by the gradation (16). The subgroup H is locally defined by the choice $u = 0$ and a local section of the quotient can be represented by g_{-1}. As usual, the action of the group is determined by the relation

$$g \begin{pmatrix} I & u \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & g \cdot u \\ 0 & I \end{pmatrix} h$$

for some $h \in H$. The corresponding splitting of the algebra is given by

$$V = V_{-1} + V_0 + V_1 = \begin{pmatrix} 0 & S_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix},$$

where S_1 and S_2 are symmetric matrices and $A \in \mathrm{gl}(n)$. The manifold G/H is usually called the *Lagrangian Grassmanian* in $\mathbb{R}^{2n}$ and it is identified with the manifold of Lagrangian planes in $\mathbb{R}^{2n}$.

The following theorem describes a representation of the manifold $\mathcal{K}$ for curves of Lagrangian planes in $\mathbb{R}^{2n}$ under the above action of $\mathrm{Sp}(2n)$.

Theorem 13 ([34]). *There exists a left moving frame* ρ *along a generic curve of Lagrangian planes such that its Serret–Frenet equations are given by*

$$K = \rho^{-1}\rho_x = \begin{pmatrix} K_0 & I \\ K_1 & K_0 \end{pmatrix},$$

where K_0 *is skew-symmetric and contains all differential invariants of order* 4*, and where* $K_1 = -\frac{1}{2}\mathcal{S}_d$*. The matrix* $\mathcal{S}_d$ *is diagonal and contains in its diagonal the eigenvalues of the* ***Lagrangian Schwarzian derivative*** *(Ovsienko* [43]*)*

$$\mathcal{S}(u) = u_1^{-1/2}\left(u_3 - \frac{3}{2}u_2u_1^{-1}u_2\right)(u_1^{-1/2})^T.$$

The entries of K_0 *and* K_1 *are functionally independent differential invariants for curves of Lagrangian planes in* $\mathbb{R}^{2n}$ *under the action of* $\mathrm{Sp}(2n)$*. They generate all other differential invariants. The* n *differential invariants that appear in* K_1 *are the invariants of projective type.*

Now we describe some of the geometric flows that preserve the value $K_0 = 0$. Therefore, geometric flows as below will affect only invariants of projective type, if proper initial conditions are chosen.

Theorem 14 ([34]). *Assume $u : J \subset \mathbb{R}^2 \to \mathrm{Sp}(2n)/H$ is a flow solution of*

$$u_t = \Theta^T u_1^{1/2} \mathbf{h} \, u_1^{1/2} \Theta, \tag{19}$$

where $\Theta(x,t) \in O(n)$ is the matrix diagonalizing $\mathcal{S}(u)$ (i.e., $\Theta\mathcal{S}(u)\Theta^T = \mathcal{S}_d$) and where $\mathbf{h}$ is a symmetric matrix of differential invariants. Assume $\mathbf{h}$ is diagonal. Then the flow preserves $K_0 = 0$.

Finally, our next theorem gives integrable PDEs with geometric realizations as geometric flows of Lagrangian planes.

Theorem 15 ([34]). *Let $\mathcal{K}_1$ be the submanifold of $\mathcal{K}$ given by $K_0 = 0$. Then, the reduced bracket on $\mathcal{K}$ restricts to $\mathcal{K}_1$ to induce a decoupled system of n second Hamiltonian structures for KdV. Bracket* (11) *also reduces to $\mathcal{K}_1$ (even though it does not in general reduce to $\mathcal{K}$ for any value of L_0). The reduction for the choice*

$$L_0 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}^* \in g_1$$

is a decoupled system of n first KdV Hamitonian structures.

Furthermore, assume $u(t,x)$ is a flow solution of (19) *with $\mathbf{h} = \mathcal{S}_d$. Then* (19) *becomes the Lagrangian Schwarzian KdV evolution*

$$u_t = u_3 - \frac{3}{2} u_2 u_1^{-1} u_2.$$

If we choose initial conditions for which $K_0 = 0$, then the differential invariants $\mathcal{S}_d$ of the flow satisfy the equation

$$(\mathcal{S}_d)_t = \left(\mathbf{D}^3 + \mathcal{S}_d\mathbf{D} + (\mathcal{S}_d)_x\right)\mathbf{h},$$

where $\mathbf{D}$ is the diagonal matrix with $\frac{d}{dx}$ down its diagonal.

Accordingly, if we choose $\mathbf{h} = \mathcal{S}_d$, then $\mathcal{S}_d$ is the solution of a *decoupled system of n KdV equations*

$$(\mathcal{S}_d)_t = (\mathcal{S}_d)_{xxx} + 3\mathcal{S}_d(\mathcal{S}_d)_x.$$

Reduced pure Spinors. A parallel description can be given for a different case, that of $G = O(n,n)$. In this case, if $g \in G$, locally

$$g = g_{-1}g_0g_1 = \begin{pmatrix} I-u & -u \\ u & I+u \end{pmatrix} \frac{1}{2} \begin{pmatrix} \Theta^{-1}+\Theta^T & \Theta^T-\Theta^{-1} \\ \Theta^T-\Theta^{-1} & \Theta^{-1}+\Theta^T \end{pmatrix} \begin{pmatrix} I-Z & Z \\ -Z & I+Z \end{pmatrix},$$

where u and S are now skew-symmetric matrices and where $\Theta \in \mathrm{GL}(n,\mathbb{R})$. The corresponding gradation of the algebra is given by $V = V_{-1} + V_0 + V_1 \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with

$$V_{-1} = V_{-1}(y) = \begin{pmatrix} -y & -y \\ y & y \end{pmatrix}, \qquad V_0 = V_0(C) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \qquad V_1 = V_1(z) = \begin{pmatrix} -z & z \\ -z & z \end{pmatrix},$$

y and z skew symmetric and $C = A + B$ given by the symmetric (B) and skew-symmertric (A) components of C.

Assume now that $G = O(2m,2m)$. This case has been worked out in [33]. The homogeneous space is locally equivalent to the manifold of reduced pure spinors in the sense of [4]. The odd dimensional spinor case is worked out in [37]. Although somehow similar it is more cumbersome to describe, so we refer the reader to [37].

Theorem 16 ([33]). *Let u be a generic curve in $O(2m, 2m)/H$. There exists a left moving frame ρ such that the left Serret–Frenet equations associated to ρ are defined by*

$$K = V_{-1}(J) + V_0(R) + \frac{1}{8}V_1(\mathcal{D}),$$

where $J = \begin{pmatrix} 0 & I_m \\ -I_m & o \end{pmatrix}$ *and R is of the form*

$$R = \begin{pmatrix} R_1 & R_2 \\ R_3 & -R_1^T \end{pmatrix} \in \mathrm{Sp}(2m)$$

with R_2 and R_3 symmetric, $R_1 \in \mathfrak{gl}(m)$. The matrix R contains in the entries off the diagonals of R_i, $i = 1, 2, 3$, a generating set of independent fourth order differential invariants. The diagonals of R_i, $i = 1, 2, 3$ contain a set of $3m$ independent and generating differential invariants of order 5 for $m > 3$ and of order 5 and higher if $m \leq 3$. The matrix $\mathcal{D}$ is the skew-symmetric diagonalization of the Spinor Schwarzian derivative.

The Spinor Schwarzian derivative [33] is described as follows: if u is a generic curve represented by skew-symmetric matrices, u_1 is non degenerate and can be brought to a normal form using $\mu \in \mathfrak{gl}(2m)$. The matrix μ is determined up to an element of the symplectic group $\mathrm{Sp}(2m)$ (see [33]) and

$$\mu u_1 \mu^T = J.$$

We define the *Spinor Schwarzian derivative* to be

$$S(u) = \mu \left(u_3 - \frac{3}{2} u_2 u_1^{-1} u_2 \right) \mu^T,$$

again, unique up to the action of $\mathrm{Sp}(2m)$. One can then prove [33] that, for a generic curve, the Schwarzian derivative can be diagonalize using an element of the symplectic group. That is, there exists $\theta \in \mathrm{Sp}(2m)$ such that

$$\theta S(u) \theta^T = \mathcal{D} = \begin{pmatrix} 0 & \mathbf{d} \\ -\mathbf{d} & 0 \end{pmatrix}$$

with $\mathbf{d}$ diagonal. The matrix $\mathcal{D}$ is the one appearing in the Serret–Frenet equations and it contains in its entries m differential invariants *of projective type*.

The Spinor case seems to be different from others and the behavior of the Poisson brackets (10) and (11) is not completely understood yet. Still we do know how the invariants of projective type behave under the analogous of the KdV Schwarzian evolution. That is described in the following theorem. The *Spinor KdV Schwarzian evolution* is defined by the equation

$$u_t = u_3 - \frac{3}{2} u_2 u_1^{-1} u_2.$$

Theorem 17 ([33]). *Let ρ is a moving frame for which normalization equations of fourth order are defined by constants c_4. Assume that, as the fourth order invariants vanish, $[R, \widehat{R}] = \widehat{\widehat{R}} +$ block diagonals, where $\widehat{R}$ and $\widehat{\widehat{R}}$ are any matrices whose only non-zero entries are in the same position as the nonzero normalized entries in R. Assume also that $[R, [R, \widehat{R}]]^d = 0$ for $\widehat{R}$ as above, where ^{d} indicates the diagonals in the main four blocks.*

Then, if we choose initial conditions with vanishing fourth order invariants, these remain zero under the KdV Schwarzian flow, $\mathcal{D}_t$ and $(R^d)_t$ decouple, and $\mathcal{D}$ evolves as

$$\mathcal{D}_t J = \mathcal{D}_{xxx} J + 3\mathcal{D}\mathcal{D}_x,$$

following a decoupled system of KdV equations.

Although the hypothesis of this theorem seem to be very restrictive, they aren't. In fact, they are easily achieved when we construct moving frames in a dimension larger that 4, although they are more restrictive for the lower dimensions. The need for these conditions was explained in [33].

This is the first case where the manifold of vanishing non-projective invariants is not preserved. Not only the vanishing of fifth order invariants is not preserved, but the system *blows up* when we approach the submanifold of vanishing fifth order invariants. This situation (and the analogous one for the odd dimensional case) is not well understood. It is possible that choosing normalization equations for which c_4 involve derivatives of the third order differential invariants will produce a better behaved moving frame. Or perhaps the fifth order differential invariants are also of projective type and the Hamiltonian behavior is more complicated that appears to be. The main problem understanding this case is the choice of a moving frame that simplifies the study of the reduced evolution. In the spinorial case such a choice is highly not trivial and we do not know whether there is no such choice or it is just very involved. It is also not known whether or not (10) and (11) reduce to $\mathcal{K}_1$. For more information see [33] and [37].

The Lagrangian and Spinorial examples describe a certain type of behavior (evolving as a decoupled system of KdVs) of invariants of projective type that is certainly different from our first example, that of $\mathbb{RP}^n$. Still, there is a third behavior that is different from these two. These evolutions of KdV type appear in conformal manifolds. In the conformal case we have only two differential invariants of projective type, and their Hamiltonian behavior is that of a complexly coupled system of KdVs. That is what we describe in the next subsection.

4.3 Conformal case

In this section we study the case of $G = O(p+1, q+1)$ acting on $\mathbb{R}^{p+q}$ as described in [42]. The case $q = 0$ was originally studied in [35]. Using the gradation appearing in [27] we can locally factor an element of the group as $g = g_1 g_0 g_{-1}$ (this is a factorization for a right moving frame, not a left one, so it is slightly different from the previous examples), with $g_i \in G_i$ where

$$g_{-1}(Y) = \begin{pmatrix} 1 - \frac{1}{2}\|Y\|^2 & -Y_1^T & -\frac{1}{2}\|Y\|^2 & Y_2^T \\ Y_1 & I_p & Y_1 & 0 \\ \frac{1}{2}\|Y\|^2 & Y_1^T & 1 + \frac{1}{2}\|Y\|^2 & -Y_2^T \\ Y_2 & 0 & Y_2 & I_q \end{pmatrix}, \qquad g_0(a, b, \Theta) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & \Theta_{11} & 0 & \Theta_{12} \\ b & 0 & a & 0 \\ 0 & \Theta_{21} & 0 & \Theta_{22} \end{pmatrix},$$

$$g_1(Z) = \begin{pmatrix} 1 - \frac{1}{2}\|Z\|^2 & Z_1^T & \frac{1}{2}\|Z\|^2 & Z_2^T \\ -Z_1 & I_p & Z_1 & 0 \\ -\frac{1}{2}\|Z\|^2 & Z_1^T & 1 + \frac{1}{2}\|Z\|^2 & Z_2^T \\ Z_2 & 0 & -Z_2 & I_q \end{pmatrix}.$$

The splitting $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ and $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is into p and q components, $\|X\|^2$ is given by the flat metric of signature (p, q) and also $\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \in O(p, q)$, $a^2 - b^2 = 1$. Without loosing generality we will assume that the flat metric is given by $\|X\|^2 = X^T J X$, where $J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. The corresponding splitting in the algebra is given by

$$V_{-1}(y) = \begin{pmatrix} 0 & -y_1^T & 0 & y_2^T \\ y_1 & 0 & y_1 & 0 \\ 0 & y_1^T & 0 & -y_2^T \\ y_2 & 0 & y_2 & 0 \end{pmatrix}, \qquad V_0(\alpha, A) = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & A_{11} & 0 & A_{12} \\ \alpha & 0 & 0 & 0 \\ 0 & A_{21} & 0 & A_{22} \end{pmatrix},$$

$$V_1(z) = \begin{pmatrix} 0 & z_1^T & 0 & z_2^T \\ -z_1 & 0 & z_1 & 0 \\ 0 & z_1^T & 0 & z_2^T \\ z_2 & 0 & -z_2 & 0 \end{pmatrix},$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ are the p and q components and where $A = (A_{ij}) \in \mathfrak{o}(p,q)$. The algebra structure can be described as

$$[V_0(\alpha, A), V_1(z)] = V_1(JAJz + \alpha z), \qquad [V_0(\alpha, A), V_{-1}(y)] = V_{-1}(Ay - \alpha y),$$
$$[V_1(z), V_{-1}(y)] = 2V_0\left(z^T y, Jzy^T J - yz^T\right), \qquad [V_0(\alpha, A), V_0(\beta, B)] = V_0(0, [A, B]).$$

With this factorization one chooses H to be defined by $Y = 0$ and uses G_{-1} as a local section of G/H (as such $Y = -u$, not u).

As before, the following theorem describes convenient choices of moving frames.

Theorem 18 ([33]). *There exists a left moving frame ρ such that the Serret–Frenet equation for ρ is given by $\rho^{-1}\rho_x = K = K_1 + K_0 + K_{-1}$ where $K_{-1} = V_{-1}(e_1)$, $K_1 = V_1(k_1 e_1 + k_2 e_2)$ and $K_0 = V_0(0, \hat{K}_0)$ with*

$$\hat{K}_0 = \begin{pmatrix} A_0 & B_0 \\ B_0^T & 0 \end{pmatrix},$$

and

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -k_3 & -k_4 & \dots & -k_p \\ 0 & k_3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & k_p & 0 & \dots & 0 & 0 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ -k_{p+1} & -k_{p+2} & \dots & -k_{p+q} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

In this case there are only two generating differential invariants of projective type, namely k_1 and k_2. Again, the behavior of the Poisson brackets (10) and (11) with respect to this submanifold is spotless.

Theorem 19 ([33]). *Let $\mathcal{K}_1$ be the submanifold of $\mathcal{K}$ given by $K_0 = 0$. Then, the reduction of* (10) *to $\mathcal{K}$ restricts to $\mathcal{K}_1$ to induce the second Hamiltonian structure for a complexly coupled KdV system. Bracket* (11) *also reduced to $\mathcal{K}_1$ to produce the first Hamiltonian structure for this system.*

And, again, the geometric realization for complexly coupled KdV is found.

Theorem 20 ([33]). *Assume $u : J \subset \mathbb{R}^2 \to O(p+1, q+1)/H$ is a solution of*

$$u_t = h_1 T + h_2 N,$$

where T and N are conformal tangent and normal (see [33]*) and h_1, h_2 are any two functions of k_1, k_2 and their derivatives. Then the flow has a limit as $K_0 \to 0$. As $K_0 \to 0$, the evolution of k_1 and k_2 becomes*

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \begin{pmatrix} -\frac{1}{2}D^3 + k_1 D + Dk_1 & k_2 D + Dk_2 \\ k_2 D + Dk_2 & \frac{1}{2}D^3 - k_1 D - Dk_1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

If we choose $h_1 = k_1$ and $h_2 = k_2$, then the evolution is a complexly coupled system of KdV equations.

5 Discussion

The aim of this paper is to review some of the known evidence linking the character of differential invariant of curves in homogeneous spaces and the geometric realizations of integrable systems in those manifolds. In particular, we have described how projective geometry and geometric realizations of KdV-type evolutions seem to be very closely related. A similar case can perhaps be made for Schrödinger flows, mKdV and sine-Gordon flows as linked to Riemannian geometry. As we said before, several authors [1, 25, 26, 28, 29, 38, 46, 48, 49], have described geometric realizations of these evolutions on manifolds that have what amounts to be a classical *natural* moving frame, i.e., a frame whose derivatives of non tangential vectors have a tangential direction. This frame appears in Riemannian manifolds and is generated by the action of the group in first order frames. Thus, one could call the invariants they generate invariants of Riemannian-type.

In fact, the most interesting question is how the geometry of the manifold itself generates these geometric realizations. And, further, if a manifold hosts a geometric realization of an integrable system of a certain type, does that fact have any implications for the geometry of the manifold? In the case of projective geometry, the following conjecture due to M. Eastwood points us in this direction.

Conjecture. *In this type of symmetric spaces there exists a natural projective structure along curves that generates Hamiltonian structures of KdV type along some flows.*

In the conformal case $G = O(p + 1, q + 1)$ the two invariants of projective type are directly connected to invariant differential operators that appear in the work of Bailey and Eastwood (see [5, 6]). The authors defined conformal circles as solutions of a differential equation. The equation defines the curves together with a preferred parametrization. The parametrizations endow conformal circles with a projective structure (theirs is an explicit proof of Cartan's observation that a curve in a conformal manifold inherits a natural projective structure, see [10]). We now know that the vanishing of the differential equation in [5] implies the vanishing of both differential invariants of projective type found in [33]. Therefore, the complexly coupled system of KdV equations could be generated by the projective structure on conformal curves that Cartan originally described.

Natural projective structures on curves have only been described for the cases $O(p+1, q+1)$ [5] and $\mathrm{SL}(p+q)$ [6], but they do perhaps exist for $|r|$-graded parabolic manifolds. Thus, resolving this conjecture and its generalizations would help to understand the more general situation of parabolic manifolds. In [12] Drinfel'd and Sokolov described many evolutions of KdV-type linked to parabolic gradations of the Lie algebra $\mathfrak{g}$. It would be interesting to learn if parabolic manifolds (flag manifolds) can be used to generate geometric realizations for these systems.

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