On Parametrization of the Linear $GL(4, C)$ and Unitary $SU(4)$ Groups in Terms of Dirac Matrices*

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Abstract. Parametrization of 4×4-matrices G of the complex linear group $GL(4, C)$ in terms of four complex 4-vector parameters (k, m, n, l) is investigated. Additional restrictions separating some subgroups of $GL(4, C)$ are given explicitly. In the given parametrization, the problem of inverting any 4×4 matrix G is solved. Expression for determinant of any matrix G is found: $\det G = F(k, m, n, l)$. Unitarity conditions $G^{+} = G^{-1}$ have been formulated in the form of non-linear cubic algebraic equations including complex conjugation. Several simplest solutions of these unitarity equations have been found: three 2-parametric subgroups G_1, G_2, G_3 – each of subgroups consists of two commuting Abelian unitary groups; 4-parametric unitary subgroup consisting of a product of a 3-parametric group isomorphic $SU(2)$ and 1-parametric Abelian group. The Dirac basis of generators Λ_k, being of Gell-Mann type, substantially differs from the basis λ_i used in the literature on $SU(4)$ group, formulas relating them are found – they permit to separate $SU(3)$ subgroup in $SU(4)$. Special way to list 15 Dirac generators of $GL(4, C)$ can be used $\{\Lambda_k\} = \{\alpha_i \oplus \beta_j \oplus (\alpha_i V \beta_j = \boldsymbol{K} \oplus \boldsymbol{L} \oplus \boldsymbol{M})\}$, which permit to factorize $SU(4)$ transformations according to $S = e^{i\vec{a}\vec{\alpha}} e^{i\vec{b}\vec{\beta}} e^{i\boldsymbol{k}\boldsymbol{K}} e^{i\boldsymbol{l}\boldsymbol{L}} e^{i\boldsymbol{m}\boldsymbol{M}}$, where two first factors commute with each other and are isomorphic to $SU(2)$ group, the three last ones are 3-parametric groups, each of them consisting of three Abelian commuting unitary subgroups. Besides, the structure of fifteen Dirac matrices Λ_k permits to separate twenty 3-parametric subgroups in $SU(4)$ isomorphic to $SU(2)$; those subgroups might be used as bigger elementary blocks in constructing of a general transformation $SU(4)$. It is shown how one can specify the present approach for the pseudounitary group $SU(2, 2)$ and $SU(3, 1)$.

Key words: Dirac matrices; linear group; unitary group; Gell-Mann basis; parametrization

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1 Introduction

The unitary groups play an important role in numerous research areas: quantum theory of massless particles, cosmology models, quantum systems with dynamical symmetry, nano-scale physics, numerical calculations concerning entanglement and other quantum information parameters, high-energy particle theory – let us just specify these several points:

- $SU(2, 2)$ and conformal symmetry, massless particles [7, 19, 20, 39, 72];
- classical Yang–Mills equations and gauge fields [64];
- quantum computation and control, density matrices for entangled states [2, 31, 65];
- geometric phases and invariants for multi-level quantum systems [55];

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- high-temperature superconductivity and antiferromagnets [36, 52];
- composite structure of quarks and leptons [67, 68, 51];
- $SU(4)$ gauge models [73, 29];
- classification of hadrons and their interactions [30, 34, 28].

Because of so many applications in physics, various parametrizations for the group elements of unitary group $SU(4)$ and related to it deserve special attention. Our efforts will be given to extending some classical technical approaches proving their effectiveness in simple cases of the linear and unitary groups $SL(2,C)$ and $SU(2)$, so that we will work with objects known by every physicist, such as Pauli and Dirac matrices. This paper, written for physicists, is self-contained in that it does not require any previous knowledge of the subject nor any advanced mathematics.

Let us start with the known example of spinor covering for complex Lorentz group: consider the 8-parametric 4×4 matrices in the quasi diagonal form [18, 32, 45]

$$G = \left| \begin{array}{cc} k_0 + \boldsymbol{k}\vec{\sigma} & 0 \\ 0 & m_0 - \boldsymbol{m}\vec{\sigma} \end{array} \right| .$$

The composition rules for parameters $k = (k_0, \boldsymbol{k})$ and $m = (m_0, \boldsymbol{m})$ are

$$k''_0 = k'_0 k_0 + \boldsymbol{k}'\boldsymbol{k}, \qquad \boldsymbol{k}'' = k'_0 \boldsymbol{k} + \boldsymbol{k}' k_0 + i\boldsymbol{k}' \times \boldsymbol{k},$$
$$m''_0 = m'_0 m_0 + \boldsymbol{m}'\boldsymbol{m}, \qquad \boldsymbol{m}'' = m'_0 \boldsymbol{m} + \boldsymbol{m}' m_0 - i\boldsymbol{m}' \times \boldsymbol{m}.$$

With two additional constraints on 8 quantities $k_0^2 - \boldsymbol{k}^2 = +1$, $m_0^2 - \boldsymbol{m}^2 = +1$, we will arrive at a definite way to parameterize a double (spinor) covering for complex Lorentz group $SO(4,C)$. At this, the problem of inverting of the G matrices with unit determinant $\det G$ is solved straightforwardly: $G = G(k_0, \boldsymbol{k}, m_0, \boldsymbol{m})$, $G^{-1} = G(k_0, -\boldsymbol{k}, m_0, -\boldsymbol{m})$. Transition from covering 4-spinor transformations to 4-vector ones is performed through the known relationship $G\gamma^a G^{-1} = \gamma^c L_c{}^a$ which determine $2 \Longrightarrow 1$ map from $\pm G$ to L.

There exists a direct connection between the above 4-dimensional vector parametrization of the spinor group $G(k_a, m_a)$ and the Fedorov's parametrization [32] of the group of complex orthogonal Lorentz transformations in terms of 3-dimensional vectors $\boldsymbol{Q} = \boldsymbol{k}/k_0$, $\boldsymbol{M} = \boldsymbol{m}/m_0$, with the simple composition rules for vector parameters

$$\boldsymbol{Q}'' = \frac{\boldsymbol{Q} + \boldsymbol{Q}' + i\boldsymbol{Q}' \times \boldsymbol{Q}}{1 + \boldsymbol{Q}'\boldsymbol{Q}}, \qquad \boldsymbol{M}'' = \frac{\boldsymbol{M} + \boldsymbol{M}' - i\boldsymbol{M}' \times \boldsymbol{M}}{1 + \boldsymbol{M}'\boldsymbol{M}}.$$

Evidently, the pair $(\boldsymbol{Q}, \boldsymbol{M})$ provides us with possibility to parameterize correctly orthogonal matrices only. Instead, the (k_a, m_a) represent correct parameters for the spinor covering group. When we are interested only in local properties of the spinor representations, no substantial differences between orthogonal groups and their spinor coverings exist. However, in opposite cases global difference between orthogonal and spinor groups may be very substantial as well as correct parametrization of them.

Restrictions specifying the spinor coverings for orthogonal subgroups are well known [32]. In particular, restriction to real Lorentz group $O(3,1)$ is achieved by imposing one condition (including complex conjugation) $(k, m) \Longrightarrow (k, k^*)$. The case of real orthogonal group $O(4)$ is achieved by a formal change (transition to real parameters) $(k_0, \boldsymbol{k}) \Longrightarrow (k_0, i\boldsymbol{k})$, $(m_0, \boldsymbol{m}) \Longrightarrow (m_0, i\boldsymbol{m})$, and the real orthogonal group $O(2,2)$ corresponds to transition to real parameters according to $(k_0, k_1, k_2, k_3) \Longrightarrow (k_0, k_1, k_2, ik_3)$, $(m_0, m_1, m_2, m_3) \Longrightarrow (m_0, m_1, m_2, im_3)$.

To parameterize 4-spinor and 4-vector transformations of the complex Lorentz group one may use curvilinear coordinates. The simplest and widely used ones are Euler's complex angles (see [32] and references in [18]). In general, on the basis of the analysis given by Olevskiy [58] about coordinates in the real Lobachevski space, one can propose 34 different complex coordinate systems appropriate to parameterize the complex Lorentz group and its double covering.

A particular, Euler angle parametrization is closely connected with cylindrical coordinates on the complex 3-sphere, one of 34 possible coordinates. Such complex cylindrical coordinates can be introduced by the following relations [18]:

$$k_0 = \cos\rho\cos z, \qquad k_3 = i\cos\rho\sin z, \qquad k_1 = i\sin\rho\cos\phi, \qquad k_2 = i\sin\rho\sin\phi,$$
$$m_0 = \cos R\cos Z, \qquad m_3 = i\cos R\sin Z, \qquad m_1 = i\sin R\Phi, \qquad m_2 = i\sin R\sin\Phi.$$

Here 6 complex variables are independent, (ρ, z, ϕ), (R, Z, Φ), additional restrictions are satisfied identically by definition. Instead of cylindrical coordinates in (ρ, z, ϕ) and (R, Z, Φ) one can introduce Euler's complex variables (α, β, γ) and (A, B, Γ) through the simple linear formulas:

$$\alpha = \phi + z, \qquad \beta = 2\rho, \qquad \gamma = \phi - z, \qquad A = \Phi + Z, \qquad B = 2R, \qquad \Gamma = \Phi - Z.$$

Euler's angles (α, β, γ) and (A, B, Γ) are referred to k_a, m_a-parameters by the formulas (see in [32])

$$\cos\beta = k_0^2 - k_3^2 + k_1^2 + k_2^2, \qquad \sin\beta = 2\sqrt{k_0^2 - k_3^2}\sqrt{-k_1^2 - k_2^2},$$
$$\cos\alpha = \frac{-ik_0k_1 + k_2k_3}{\sqrt{k_0^2 - k_3^2}\sqrt{-k_1^2 - k_2^2}}, \qquad \sin\alpha = \frac{-ik_0k_2 - k_1k_3}{\sqrt{k_0^2 - k_3^2}\sqrt{-k_1^2 - k_2^2}},$$
$$\cos\gamma = \frac{-ik_0k_1 - k_2k_3}{\sqrt{k_0^2 - k_3^2}\sqrt{-k_1^2 - k_2^2}}, \qquad \sin\gamma = \frac{-ik_0k_2 + k_1k_3}{\sqrt{k_0^2 - k_3^2}\sqrt{-k_1^2 - k_2^2}},$$
$$\cos B = m_0^2 - m_3^2 + m_1^2 + m_2^2, \qquad \sin B = 2\sqrt{m_0^2 - m_3^2}\sqrt{-m_1^2 - m_2^2},$$
$$\cos A = \frac{+im_0m_1 + m_2m_3}{\sqrt{m_0^2 - m_3^2}\sqrt{-m_1^2 - m_2^2}}, \qquad \sin A = \frac{+im_0m_2 - m_1m_3}{\sqrt{m_0^2 - m_3^2}\sqrt{-m_1^2 - m_2^2}},$$
$$\cos\Gamma = \frac{+im_0m_1 - m_2m_3}{\sqrt{m_0^2 - m_3^2}\sqrt{-m_1^2 - m_2^2}}, \qquad \sin\Gamma = \frac{+im_0m_2 + m_1m_3}{\sqrt{m_0^2 - m_3^2}\sqrt{-m_1^2 - m_2^2}}.$$

Complex Euler's angles as parameters for complex Lorentz group $SO(4,C)$ have a distinguished feature: 2-spinor constituents are factorized into three elementary Euler's transforms (σ^i stands for the known Pauli matrices):

$$B(k) = e^{-i\sigma^3\alpha/2}e^{i\sigma^1\beta/2}e^{+i\sigma^3\gamma/2} \in SL(2,C),$$
$$B(\bar{m}) = e^{-i\sigma^3\Gamma/2}e^{i\sigma^1 B/2}e^{+i\sigma^3 A/2} \in SL(2,C)'.$$

The main question is how to extend possible parameterizations of small orthogonal group $SO(4,C)$ and its double covering to bigger orthogonal and unitary groups[1]. To be concrete we are going to focus attention mainly on the group $SU(4)$ and its counterparts $SU(2,2)$, $SU(3,1)$.

There exist many publications on the subject, a great deal of facts are known – in the following we will be turning to them. A good classification of different approaches in parameterizing finite transformations of $SU(4)$ was done in the recent paper by A. Gsponer [35]. Recalling it, we will try to cite publications in appropriate places though many of them should be placed in several different subclasses – it is natural because all approaches are closely connected to each other.

[1] In this subject, especially concerned with generalized Euler angles, we have found out much from Murnaghan's book [56].

- **Canonical form** [53, 54, 56, 57, 60, 61, 75]. They use explicitly the full set of the Lie generators[2] so that the group element is expressed as the exponential of the linear combination

$$G = \exp_n[i(a_1\lambda_1 + \cdots + a_n\lambda_N)]$$

the infinite series of terms implied be exp – symbol is usually very difficult to be summed in closed form – though there exists many interesting examples of those:

- **Non-canonical forms** [3, 4, 8, 9, 11, 35, 38, 40, 42, 57, 59, 61, 63, 74]. As a consequence of the Baker–Campbell–Hausdorff theorem [1, 26, 37] it is possible to break-down the canonical form into a product

$$G = \exp_{n^{(1)}} \times \cdots \times \exp_{n^{(k)}}, \qquad n^{(1)} + \cdots + n^{(k)} = N.$$

with the hope that $\exp_{n^{(i)}}$ could be summed in closed form and also that these factors have simple properties. This possibility for the groups $SU(4)$ and $SU(2,2)$ will be discussed in more detail in sections below.

- **Product form** [12, 13, 14, 16, 27, 38, 56, 57]. An extreme non-canonical form is to factorize the general exponential into a product of n simplest 1-parametric exponentials

$$G = \exp[ia_1\lambda_1] \times \cdots \times \exp[ia_N\lambda_N].$$

- **Basic elements** (the main approach in the present treatment) [8, 9, 35, 40, 44, 45, 46, 47, 74]. This way is to expand the elements of the group (matrices or quaternions) into a sum over basis elements and to work with a linear decomposition of the matrices over basic ones:

$$G' = x'_n\lambda_m, \qquad G = x_n\lambda_m, \qquad \lambda_0 = I, \qquad k \in \{0, 1, \ldots, N\},$$
$$G'' = G'G, \qquad x''_k\lambda_k = x'_m\lambda_m x_n\lambda_n = x'_m x_n \lambda_m\lambda_n, \tag{1.1}$$

as by definition the relationships $\lambda_m\lambda_n = e_{mnk}\lambda_k$ must hold, the group multiplication rule for parameters x_k looks

$$x''_k = e_{mnk}x'_m x_n. \tag{1.2}$$

The main claim is that the all properties of any matrix group are straightforwardly determined by the bilinear function, the latter is described by structure constants e_{mnk} entering the multiplication rule $\lambda_m\lambda_n = e_{mnk}\lambda_k$.

- **Hamilton–Cayley form** [8, 9, 11, 12, 13, 14, 15, 16, 17, 33, 63]. It is possible to expand the elements of the group into a power series of linear combination of generators:

$$\lambda(a) = i(a_1\lambda_1 + \cdots + a_N\lambda_N),$$

because of Hamilton–Cayley theorem this series has three terms for $SU(3)$ and four terms for $SU(4)$:

$$SU(3), \quad G(a) = e_0(a)I + e_1(a)\lambda(a) + e_2(a)\lambda^2(a),$$
$$SU(4), \quad G(a) = e_0(a)I + e_1(a)\lambda(a) + e_2(a)\lambda^2(a) + e_3(a)\lambda^3(a).$$

[2]In the paper we will designate generators in Dirac basis by Λ_i whereas another set of generators mainly used in the literature will be referred as λ_i.

- **Euler-angles representations** [10, 21, 22, 23, 24, 25, 35, 42, 56, 57, 69]. In Euler-angles representations only a sub-set $\{\lambda_n\} \subset \{\lambda_N\}$ of the Lie generators are sufficient to produce the whole set (for $SU(N)$ we need only $2(N-1)$ generators). In that sense all other way to obtain the whole set of elements are not minimal.

In our opinion, we should search the most simplicity in mathematical sense while working with basic elements λ_k and the structure constants determining the group multiplication rule (1.1), (1.2).

The material of this paper is arranged as follows.

In Section 2 an arbitrary 4×4 matrix $G \in GL(4,C)$ is decomposed into sixteen Dirac matrices[3]

$$G = AI + iB\gamma^5 + iA_l\gamma^l + B_l\gamma^l\gamma^5 + F_{mn}\sigma_{mn} = \left| \begin{array}{cc} k_0 + \boldsymbol{k}\vec{\sigma} & n_0 - \boldsymbol{n}\vec{\sigma} \\ -l_0 - \boldsymbol{l}\vec{\sigma} & m_0 - \boldsymbol{m}\vec{\sigma} \end{array} \right|, \tag{1.3}$$

for definiteness we will use the Weyl spinor basis; four 4-dimensional vectors (k,m,l,n) are definite linear combinations of A, B, A_l, B_l, F_{mn} – see (2.4). In such parameters (2.3), the group multiplication law $G'' = G'G$ is found in explicit form.

Then we turn to the following problem: at given $G = G(k,m,n,l)$ one should find parameters of the inverse matrix: $G^{-1} = G(k',m',n',l')$ – expressions for (k',m',n',l') have been found explicitly (for details of calculation see [62]). Also, several equivalent expressions for determinant $\det G$ have been obtained, which is essential when going to special groups $SL(4,C)$ and its subgroups.

In Section 3, with the help of the expression for the inverse matrix $G^{-1}(k',m',l',n')$ we begin to consider the unitary group $SU(4)$. To this end, one should specify the requirement of unitarity $G^+ = G^{-1}$ to the above vector parametrization – so that unitarity conditions are given as non-linear cubic algebraic equations for parameters (k,m,l,n) including complex conjugation.

In Section 4 we have constructed three 2-parametric solutions of the produced equations of unitarity[4], these subgroups G_1, G_2, G_3 consist of two commuting Abelian unitary subgroups.

In Section 5 we have constructed a 4-parametric solution[5] – it may be factorized into two commuting unitary factors: $G = G_0 \otimes SU(2)$ – see (5.15).

The task of complete solving of the unitarity conditions seems to be rather complicated. In remaining part of the present paper we describe some relations of the above treatment to other considerations of the problem in the literature. We hope that the full general solution of the unitary equations obtained can be constructed on the way of combining different techniques used in the theory of the unitary group $SU(4)$ and it will be considered elsewhere.

We turn again to the explicit form of the Dirac basis and note that all 15 matrices are of Gell-Mann type: they have a zero-trace, they are Hermitian, besides their squares are unite:

$$\mathrm{Sp}\Lambda = 0, \qquad (\Lambda)^2 = I, \qquad (\Lambda)^+ = \Lambda, \qquad \Lambda \in \{\Lambda_k,\ k = 1,\dots,15\}.$$

Exponential function of any of them equals to

$$U_j = e^{ia_j\Lambda_j} = \cos a_j + i\sin a_j\Lambda_j, \qquad \det e^{ia_j\Lambda_j} = +1, \qquad U_j^+ = U_j^{-1}, \qquad a_i \in R.$$

Evidently, multiplying such 15 elementary unitary matrices (at real parameters x_i) gives again an unitary matrix

$$U = e^{ia_1\Lambda_1}e^{ia_2\Lambda_2}\cdots e^{ia_{14}\Lambda_{14}}e^{ia_{l5}\Lambda_{l5}}, \qquad U^+ = e^{-ia_l\Lambda_l}e^{-ia_k\Lambda_k}\cdots e^{-ia_j\Lambda_j}e^{-ia_i\Lambda_i}.$$

[3]That Dirac matrices-based approach was widely used in physical context (see [5, 8, 9, 6, 45, 48, 49, 50, 66] and especially [40]).

[4]At this, the unitarity equations may be considered as special eigenvalue problems in 2-dimensional space.

[5]The problem again is reduced to solving of a special eigenvalue problem in 2-dimensional space.

At this there arises one special possibility to determine extended Euler angles $a_1, \ldots, a_{15}$. For the group $SU(4)$ the Euler parametrization of that type was found in [69]. A method to solve the problem in [69] was based on the use yet known Euler parametrization for $SU(3)$ – the latter problem was solved in [25]. Extension to $SU(N)$ group was done in [70, 71]. Evident advantage of the Euler angles approach is its simplicity, and evident defect consists in the following: we do not know any simple group multiplication rule for these angles – even the known solution for $SU(2)$ is too complicated and cannot be used effectively in calculation.

In Section 6 the main question is how in Dirac parametrization one can distinguish $SU(3)$, the subgroup in $SU(4)$. In this connection, it should be noted that the basis λ_i used in [25] substantially differs from the above Dirac basis Λ_i – this peculiarity is closely connected with distinguishing the $SU(3)$ in $SU(4)$. In order to have possibility to compare two approaches we need exact connection between λ_i and Λ_i – we have found required formulas[6]. The separation of $SL(3,C)$ in $SL(4,C)$ is given explicitly, at this 3×3 matrix group is described with the help of 4×4 matrices[7]. The group law for parameters of $SL(3,C)$ is specified.

In Section 7 one different way to list 15 generators of $GL(4,C)$ is examined[8]

$$\alpha_1 = \gamma^0\gamma^2, \qquad \alpha_2 = i\gamma^0\gamma^5, \qquad \alpha_3 = \gamma^5\gamma^2, \qquad \beta_1 = i\gamma^3\gamma^1, \qquad \beta_2 = i\gamma^3, \qquad \beta_3 = i\gamma^1,$$

these two set commute with each others $\alpha_j\beta_k = \beta_k\alpha_j$, and their multiplications provides us with 9 remaining basis elements of fifteen:

$$\begin{array}{lll} A_1 = \alpha_1\beta_1, & B_1 = \alpha_1\beta_2, & C_1 = \alpha_1\beta_3, \\ A_2 = \alpha_2\beta_1, & B_2 = \alpha_2\beta_2, & C_2 = \alpha_2\beta_3, \\ A_3 = \alpha_3\beta_1, & B_3 = \alpha_3\beta_2, & C_3 = \alpha_3\beta_3. \end{array}$$

We turn to the rule of multiplying 15 generators α_i, β_i, A_i, B_i, C_i and derive its explicit form (see (7.3)).

Section 8 adds some facts to a factorized structure of $SU(4)$. To this end, between 9 generators we distinguish three sets of commuting ones

$$\boldsymbol{K} = \{A_1, B_2, C_3\}, \qquad \boldsymbol{L} = \{C_1, A_2, B_3\}, \qquad \boldsymbol{M} = \{B_1, C_2, A_3\},$$

an arbitrary element from $GL(4,C)$ can be factorized as follows[9]

$$S = e^{i\vec{a}\vec{\alpha}} e^{i\vec{b}\vec{\beta}} e^{i\boldsymbol{kK}} e^{i\boldsymbol{lL}} e^{i\boldsymbol{mM}}, \tag{1.4}$$

where $\boldsymbol{K}$, $\boldsymbol{L}$, $\boldsymbol{M}$ are 3-parametric groups, each of them consists of three Abelian commuting unitary subgroups[10]. On the basis of 15 matrices one can easily see 20 ways to separate $SU(2)$ subgroups, which might be used as bigger elementary blocks in constructing a general transformation[11].

In Sections 9 and 10 we specify our approach for pseudounitary groups $SU(2,2)$ and $SU(3,1)$ respectively. All generators Λ'_k of these groups can readily be constructed on the basis of the known Dirac generators of $SU(4)$ (see (9.1)).

[6] This problem evidently is related to the task of distinguishing $GL(3,C)$ in $GL(4,C)$ as well.

[7] Interesting arguments related to this point but in the quaternion approach are given in [35].

[8] Such a possibility is well-known – see [40]; our approach looks simpler and more symmetrical because we use the Weyl basis for Dirac matrices instead of the standard one as in [40].

[9] These facts were described in main parts in [40].

[10] Note that existence of three Abelian commuting unitary subgroups was shown in [40] as well.

[11] This possibility was studied partly in [13, 14] on the basis of the Hamilton–Cayley approach.

2 On parameters of inverse transformations G^{-1}

Arbitrary 4×4 matrix $G \in GL(4,C)$ can be decomposed in terms of 16 Dirac matrices (such an approach to the group $L(4,C)$ was discussed and partly developed in [5, 8, 9, 6, 45, 48, 49, 50, 66] and especially in [40]):

$$G = AI + iB\gamma^5 + iA_l\gamma^l + B_l\gamma^l\gamma^5 + F_{mn}\sigma_{mn}, \tag{2.1}$$

where

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2g^{ab}, \qquad \gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3,$$
$$\sigma^{ab} = \frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a), \qquad g^{ab} = \text{diag}(+1,-1,-1,-1).$$

Taking 16 coefficients A, B, A_l, B_l, F_{mn} as parameters in the group $G = G(A, B, A_l, B_l, F_{mn})$ one can establish the corresponding multiplication law for these parameters:

$$G' = A'I + iB'\gamma^5 + iA'_l\gamma^l + B'_l\gamma^l\gamma^5 + F'_{mn}\sigma_{mn},$$
$$G = AI + iB\gamma^5 + iA_l\gamma^l + B_l\gamma^l\gamma^5 + F_{mn}\sigma_{mn},$$
$$G'' = G'G = A''I + iB''\gamma^5 + iA''_l\gamma^l + B''_l\gamma^l\gamma^5 + F''_{mn}\sigma_{mn},$$

where

$$
\begin{aligned}
A'' &= A'A - B'B - A'_lA^l - B'_lB^l - \tfrac{1}{2}F'_{kl}F^{kl},\\
B'' &= A'B + B'A + A'_lB^l - B'_lA^l + \tfrac{1}{4}F'_{mn}F_{cd}\epsilon^{mncd},\\
A''_l &= A'A_l - B'B_l + A'_lA + B'_lB + A'^kF_{kl}\\
&\qquad + F'_{lk}A^k + \tfrac{1}{2}B'_kF_{mn}\epsilon_l{}^{kmn} + \tfrac{1}{2}F'_{mn}B_k\epsilon_l{}^{mnk},\\
B''_l &= A'B_l + B'A_l - A'_lB + B'_lA + B'^kF_{kl}\\
&\qquad + F'_{lk}B^k + \tfrac{1}{2}A'_kF_{mn}\epsilon^{kmn}{}_l + \tfrac{1}{2}F'_{mn}A_k\epsilon^{mnk}{}_l,\\
F''_{mn} &= A'F_{mn} + F'_{mn}A - (A'_mA_n - A'_nA_m) - (B'_mB_n - B'_nB_m)\\
&\qquad + A'_lB_k\epsilon^{lkmn} - B'_lA_k\epsilon^{lkmn} + \tfrac{1}{2}B'F_{kl}\epsilon^{kl}{}_{mn} + \tfrac{1}{2}F'_{kl}B\epsilon^{kl}{}_{mn} + (F'_{mk}F^k{}_n - F'_{nk}F^k{}_m).
\end{aligned} \tag{2.2}
$$

The latter formulas are correct in any basis for Dirac matrices. Below we will use mainly Weyl spinor basis:

$$\gamma^a = \begin{vmatrix} 0 & \bar\sigma^a \\ \sigma^a & 0 \end{vmatrix}, \qquad \sigma^a = (I, \sigma^j), \qquad \bar\sigma^a = (I, -\sigma^j), \qquad \gamma^5 = \begin{vmatrix} -I & 0 \\ 0 & +I \end{vmatrix}.$$

With this choice, let us make $3+1$-splitting in all the formulas:

$$G \in GL(4,C), \qquad G = \begin{vmatrix} k_0 + \boldsymbol{k}\vec\sigma & n_0 - \boldsymbol{n}\vec\sigma \\ -l_0 - \boldsymbol{l}\vec\sigma & m_0 - \boldsymbol{m}\vec\sigma \end{vmatrix}, \tag{2.3}$$

where complex 4-vector parameters (k, l, m, n) are defined by [18]:

$$
\begin{aligned}
&k_0 = A - iB, \qquad k_j = a_j - ib_j, \qquad m_0 = A + iB, \qquad m_j = a_j + ib_j,\\
&l_0 = B_0 - iA_0, \qquad l_j = B_j - iA_j, \qquad n_0 = B_0 + iA_0, \qquad n_j = B_j + iA_j.
\end{aligned} \tag{2.4}
$$

For such parameters (2.3), the composition rule (2.2) will look as follows:

$$(k'', m''; n'', l'') = (k', m'; n', l')(k, m; n, l),$$

$$
\begin{aligned}
&k_0'' = k_0'k_0 + \boldsymbol{k}'\boldsymbol{k} - n_0'l_0 + \boldsymbol{n}'\boldsymbol{l},\\
&\boldsymbol{k}'' = k_0'\boldsymbol{k} + \boldsymbol{k}'k_0 + i\boldsymbol{k}' \times \boldsymbol{k} - n_0'\boldsymbol{l} + \boldsymbol{n}'l_0 + i\boldsymbol{n}' \times \boldsymbol{l},\\
&m_0'' = m_0'm_0 + \boldsymbol{m}'\boldsymbol{m} - l_0'n_0 + \boldsymbol{l}'\boldsymbol{n},\\
&\boldsymbol{m}'' = m_0'\boldsymbol{m} + \boldsymbol{m}'m_0 - i\boldsymbol{m}' \times \boldsymbol{m} - l_0'\boldsymbol{n} + \boldsymbol{l}'n_0 - i\boldsymbol{l}' \times \boldsymbol{n},\\
&n_0'' = k_0'n_0 - \boldsymbol{k}'\boldsymbol{n} + n_0'm_0 + \boldsymbol{n}'\boldsymbol{m},\\
&\boldsymbol{n}'' = k_0'\boldsymbol{n} - \boldsymbol{k}'n_0 + i\boldsymbol{k}' \times \boldsymbol{n} + n_0'\boldsymbol{m} + \boldsymbol{n}'m_0 - i\boldsymbol{n}' \times \boldsymbol{m},\\
&l_0'' = l_0'k_0 + \boldsymbol{l}'\boldsymbol{k} + m_0'l_0 - \boldsymbol{m}'\boldsymbol{l},\\
&\boldsymbol{l}'' = l_0'\boldsymbol{k} + \boldsymbol{l}'k_0 + i\boldsymbol{l}' \times \boldsymbol{k} + m_0'\boldsymbol{l} - \boldsymbol{m}'l_0 - i\boldsymbol{m}' \times \boldsymbol{l}.
\end{aligned}
\tag{2.5}
$$

Now let us turn to the following problem: with given $G = G(k, m, n, l)$ one should find parameters of the inverse matrix: $G^{-1} = G(k', m', n', l')$. In other words, starting from

$$
G(k,m,n,l) = \left| \begin{array}{cccc}
+(k_0 + k_3) & +(k_1 - ik_2) & +(n_0 - n_3) & -(n_1 - in_2) \\
+(k_1 + ik_2) & +(k_0 - k_3) & -(n_1 + in_2) & +(n_0 + n_3) \\
-(l_0 + l_3) & -(l_1 - il_2) & +(m_0 - m_3) & -(m_1 - im_2) \\
-(l_1 + il_2) & -(l_0 - l_3) & -(m_1 + im_2) & +(m_0 + m_3)
\end{array} \right|,
\tag{2.6}
$$

one should calculate parameters of the inverse matrix G^{-1}. The problem turns to be rather complicated[12], the final result is ($D = \det G$, $(mn) \equiv m_0n_0 - \boldsymbol{mn}$, and so on)

$$
\begin{aligned}
&k_0' = D^{-1}[k_0(mm) + m_0(ln) + l_0(nm) - n_0(lm) + i\boldsymbol{l}(\boldsymbol{m} \times \boldsymbol{n})],\\
&\boldsymbol{k}' = D^{-1}[-\boldsymbol{k}(mm) - \boldsymbol{m}(ln) - \boldsymbol{l}(nm) + \boldsymbol{n}(lm) + 2\boldsymbol{l} \times (\boldsymbol{n} \times \boldsymbol{m})\\
&\qquad + im_0(\boldsymbol{n} \times \boldsymbol{l}) + il_0(\boldsymbol{n} \times \boldsymbol{m}) + in_0(\boldsymbol{l} \times \boldsymbol{m})],\\
&m_0' = D^{-1}[k_0(ln) + m_0(kk) - l_0(kn) + n_0lk) + i\boldsymbol{n}(\boldsymbol{l} \times \boldsymbol{k})],\\
&\boldsymbol{m}' = D^{-1}[-\boldsymbol{k}(ln) - \boldsymbol{m}(kk) + \boldsymbol{l}(kn) - \boldsymbol{n}(kl) + 2\boldsymbol{n} \times (\boldsymbol{l} \times \boldsymbol{k})\\
&\qquad + in_0(\boldsymbol{k} \times \boldsymbol{l}) + il_0(\boldsymbol{k} \times \boldsymbol{n}) + ik_0(\boldsymbol{n} \times \boldsymbol{l})],\\
&l_0' = D^{-1}[+k_0(ml) - m_0(kl) - l_0(km) - n_0(ll) + i\boldsymbol{m}(\boldsymbol{l} \times \boldsymbol{k})],\\
&\boldsymbol{l}' = D^{-1}[+\boldsymbol{k}(ml) - \boldsymbol{m}(kl) - \boldsymbol{l}(km) - \boldsymbol{n}(ll) + 2\boldsymbol{m} \times (\boldsymbol{k} \times \boldsymbol{l})\\
&\qquad + im_0(\boldsymbol{l} \times \boldsymbol{k}) + ik_0(\boldsymbol{l} \times \boldsymbol{m}) + il_0(\boldsymbol{m} \times \boldsymbol{k})],\\
&n_0' = D^{-1}[-k_0(nm) + m_0(kn) - l_0(nn) - n_0(km) + i\boldsymbol{k}(\boldsymbol{m} \times \boldsymbol{n})],\\
&\boldsymbol{n}' = D^{-1}[-\boldsymbol{k}(nm) + \boldsymbol{m}(kn) - \boldsymbol{l}(nn) - \boldsymbol{n}(km) + 2\boldsymbol{k} \times (\boldsymbol{m} \times \boldsymbol{n})\\
&\qquad + ik_0(\boldsymbol{m} \times \boldsymbol{n}) + im_0(\boldsymbol{k} \times \boldsymbol{n}) + in_0(\boldsymbol{m} \times \boldsymbol{k})].
\end{aligned}
\tag{2.7}
$$

Substituting equations (2.7) into equation $G^{-1}G = I$ one arrives at

$$
\begin{aligned}
&D = k_0'' = k_0'k_0 + \boldsymbol{k}'\boldsymbol{k} - n_0'l_0 + \boldsymbol{n}'\boldsymbol{l},\\
&0 = \boldsymbol{k}'' = k_0'\boldsymbol{k} + \boldsymbol{k}'k_0 + i\boldsymbol{k}' \times \boldsymbol{k} - n_0'\boldsymbol{l} + \boldsymbol{n}'l_0 + i\boldsymbol{n}' \times \boldsymbol{l},\\
&D = m_0'' = m_0'm_0 + \boldsymbol{m}'\boldsymbol{m} - l_0'n_0 + \boldsymbol{l}'\boldsymbol{n},\\
&0 = \boldsymbol{m}'' = m_0'\boldsymbol{m} + \boldsymbol{m}'m_0 - i\boldsymbol{m}' \times \boldsymbol{m} - l_0'\boldsymbol{n} + \boldsymbol{l}'n_0 - i\boldsymbol{l}' \times \boldsymbol{n},\\
&0 = n_0'' = k_0'n_0 - \boldsymbol{k}'\boldsymbol{n} + n_0'm_0 + \boldsymbol{n}'\boldsymbol{m},\\
&0 = \boldsymbol{n}'' = k_0'\boldsymbol{n} - \boldsymbol{k}'n_0 + i\boldsymbol{k}' \times \boldsymbol{n} + n_0'\boldsymbol{m} + \boldsymbol{n}'m_0 - i\boldsymbol{n}' \times \boldsymbol{m},\\
&0 = l_0'' = l_0'k_0 + \boldsymbol{l}'\boldsymbol{k} + m_0'l_0 - \boldsymbol{m}'\boldsymbol{l},\\
&0 = \boldsymbol{l}'' = l_0'\boldsymbol{k} + \boldsymbol{l}'k_0 + i\boldsymbol{l}' \times \boldsymbol{k} + m_0'\boldsymbol{l} - \boldsymbol{m}'l_0 - i\boldsymbol{m}' \times \boldsymbol{l}.
\end{aligned}
$$

[12]For more details see [62]; also see a preceding paper [41].

After calculation, one can prove these identities and find the determinant:

$$
\begin{aligned}
D = \det G(k,m,n,l) &= (kk)(mm) + (ll)(nn) + 2(mk)(ln) + 2(lk)(nm) - 2(nk)(lm) \\
&+ 2i[k_0 \boldsymbol{l}(\boldsymbol{m} \times \boldsymbol{n}) + m_0 \boldsymbol{k}(\boldsymbol{n} \times \boldsymbol{l}) + l_0 \boldsymbol{k}(\boldsymbol{n} \times \boldsymbol{m}) + n_0 \boldsymbol{l}(\boldsymbol{m} \times \boldsymbol{k})] \\
&+ 4(\boldsymbol{kn})(\boldsymbol{ml}) - 4(\boldsymbol{km})(\boldsymbol{nl}). \qquad (2.8)
\end{aligned}
$$

Let us specify several more simple subgroups.

Case A

Let 0-components k_0, m_0, l_0, n_0 be real-valued, and 3-vectors $\boldsymbol{k}$, $\boldsymbol{m}$, $\boldsymbol{l}$, $\boldsymbol{n}$ be imaginary. Performing in (2.5) the formal change (new vectors are real-valued)

$$
\boldsymbol{k} \Longrightarrow i\boldsymbol{k}, \qquad \boldsymbol{m} \Longrightarrow i\boldsymbol{m}, \qquad \boldsymbol{l} \Longrightarrow i\boldsymbol{l}, \qquad \boldsymbol{n} \Longrightarrow i\boldsymbol{n},
$$

$$
G = \left| \begin{array}{cc} k_0 + i\boldsymbol{k}\vec{\sigma} & n_0 - i\boldsymbol{n}\vec{\sigma} \\ -l_0 - i\boldsymbol{l}\vec{\sigma} & m_0 - i\boldsymbol{m}\vec{\sigma} \end{array} \right|, \qquad (2.9)
$$

then the multiplication rules (2.5) for sixteen real variables look as follows

$$
\begin{aligned}
k_0'' &= k_0' k_0 - \boldsymbol{k}'\boldsymbol{k} - n_0' l_0 - \boldsymbol{n}'\boldsymbol{l}, \\
\boldsymbol{k}'' &= k_0'\boldsymbol{k} + \boldsymbol{k}' k_0 - \boldsymbol{k}' \times \boldsymbol{k} - n_0'\boldsymbol{l} + \boldsymbol{n}' l_0 - \boldsymbol{n}' \times \boldsymbol{l}, \\
m_0'' &= m_0' m_0 - \boldsymbol{m}'\boldsymbol{m} - l_0' n_0 - \boldsymbol{l}'\boldsymbol{n}, \\
\boldsymbol{m}'' &= m_0'\boldsymbol{m} + \boldsymbol{m}' m_0 + \boldsymbol{m}' \times \boldsymbol{m} - l_0'\boldsymbol{n} + \boldsymbol{l}' n_0 + \boldsymbol{l}' \times \boldsymbol{n}, \\
n_0'' &= k_0' n_0 + \boldsymbol{k}'\boldsymbol{n} + n_0' m_0 - \boldsymbol{n}'\boldsymbol{m}, \\
\boldsymbol{n}'' &= k_0'\boldsymbol{n} - \boldsymbol{k}' n_0 - \boldsymbol{k}' \times \boldsymbol{n} + n_0'\boldsymbol{m} + \boldsymbol{n}' m_0 + \boldsymbol{n}' \times \boldsymbol{m}, \\
l_0'' &= l_0' k_0 - \boldsymbol{l}'\boldsymbol{k} + m_0' l_0 + \boldsymbol{m}'\boldsymbol{l}, \\
\boldsymbol{l}'' &= l_0'\boldsymbol{k} + \boldsymbol{l}' k_0 - \boldsymbol{l}' \times \boldsymbol{k} + m_0'\boldsymbol{l} - \boldsymbol{m}' l_0 + \boldsymbol{m}' \times \boldsymbol{l}.
\end{aligned}
$$

Correspondingly, expression for determinant (2.8) becomes

$$
\begin{aligned}
D &= [kk][mm] + [ll][nn] + 2[mk][ln] + 2[lk][nm] - 2[nk][lm] \\
&+ 2[k_0 \boldsymbol{l}(\boldsymbol{m} \times \boldsymbol{n}) + m_0 \boldsymbol{k}(\boldsymbol{n} \times \boldsymbol{l}) + l_0 \boldsymbol{k}(\boldsymbol{n} \times \boldsymbol{m}) + n_0 \boldsymbol{l}(\boldsymbol{m} \times \boldsymbol{k})] \\
&+ 4(\boldsymbol{kn})(\boldsymbol{ml}) - 4(\boldsymbol{km})(\boldsymbol{nl}),
\end{aligned}
$$

where the notation is used: $[ab] = a_0 b_0 + \boldsymbol{ab}$.

Case B

Equations (2.5) permit the following restrictions:

$$
m_a = k_a^*, \qquad l_a = n_a^*,
$$

and become

$$
\begin{aligned}
k_0'' &= k_0' k_0 + \boldsymbol{k}'\boldsymbol{k} - n_0' n_0^* + \boldsymbol{n}'\boldsymbol{n}^*, \\
\boldsymbol{k}'' &= k_0'\boldsymbol{k} + \boldsymbol{k}' k_0 + i\boldsymbol{k}' \times \boldsymbol{k} - n_0'\boldsymbol{n}^* + \boldsymbol{n}' n_0^* + i\boldsymbol{n}' \times \boldsymbol{n}^*, \\
n_0'' &= k_0' n_0 - \boldsymbol{k}'\boldsymbol{n} + n_0' k_0^* + \boldsymbol{n}'\boldsymbol{k}^*, \\
\boldsymbol{n}'' &= k_0'\boldsymbol{n} - \boldsymbol{k}' n_0 + i\boldsymbol{k}' \times \boldsymbol{n} + n_0'\boldsymbol{k}^* + \boldsymbol{n}' k_0^* - i\boldsymbol{n}' \times \boldsymbol{k}^*.
\end{aligned}
$$

Determinant D is given by

$$
\begin{aligned}
D &= (kk)(kk)^* + (nn)^*(nn) + 2(k^*k)(n^*n) + 2(n^*k)(nk^*) - 2(nk)(nk)^* \\
&+ 2i[k_0 \boldsymbol{k}^*(\boldsymbol{n} \times \boldsymbol{n}^*) - k_0^* \boldsymbol{k}(\boldsymbol{n}^* \times \boldsymbol{n}) + n_0^* \boldsymbol{n}(\boldsymbol{k} \times \boldsymbol{k}^*) - n_0 \boldsymbol{n}^*(\boldsymbol{k}^* \times \boldsymbol{k})] \\
&+ 4(\boldsymbol{kn})(\boldsymbol{k}^*\boldsymbol{n}^*) - 4(\boldsymbol{kk}^*)(\boldsymbol{nn}^*).
\end{aligned}
$$

Case C

In (2.9) one can impose additional restrictions

$$m_0 = k_0, \qquad l_0 = n_0, \qquad \boldsymbol{m} = -\boldsymbol{k}, \qquad \boldsymbol{l} = -\boldsymbol{n}; \tag{2.10}$$

at this $G(k_0, \boldsymbol{k}, n_0, \boldsymbol{n})$ looks

$$G = \begin{vmatrix} (k_0 + i\boldsymbol{k}\vec{\sigma}) & (n_0 - i\boldsymbol{n}\vec{\sigma}) \\ -(n_0 - i\boldsymbol{n}\vec{\sigma}) & (k_0 + i\boldsymbol{k}\vec{\sigma}) \end{vmatrix};$$

and the composition rule is

$$\begin{aligned} k_0'' &= k_0' k_0 - \boldsymbol{k}'\boldsymbol{k} - n_0' n_0 + \boldsymbol{n}'\boldsymbol{n}, \\ \boldsymbol{k}'' &= k_0'\boldsymbol{k} + \boldsymbol{k}' k_0 - \boldsymbol{k}' \times \boldsymbol{k} + n_0'\boldsymbol{n} + \boldsymbol{n}' n_0 + \boldsymbol{n}' \times \boldsymbol{n}, \\ n_0'' &= k_0' n_0 + \boldsymbol{k}'\boldsymbol{n} + n_0' k_0 + \boldsymbol{n}'\boldsymbol{k}, \\ \boldsymbol{n}'' &= k_0'\boldsymbol{n} - \boldsymbol{k}' n_0 - \boldsymbol{k}' \times \boldsymbol{n} - n_0'\boldsymbol{k} + \boldsymbol{n}' k_0 - \boldsymbol{n}' \times \boldsymbol{k}. \end{aligned}$$

Determinant equals to

$$\begin{aligned} \det G = {} & [kk][kk] + [nn][nn] + 2(kk)(nn) + 2(nk)(nk) - 2[nk][nk]) \\ & + 4(\boldsymbol{kn})(\boldsymbol{kn}) - 4(\boldsymbol{kk})(\boldsymbol{nn}). \end{aligned}$$

Case D

There exists one other subgroup defined by

$$n_a = 0, \qquad l_a = 0, \qquad G = \begin{vmatrix} (k_0 + \boldsymbol{k}\vec{\sigma}) & 0 \\ 0 & (m_0 - \boldsymbol{m}\vec{\sigma}) \end{vmatrix},$$

the composition law (2.5) becomes simpler

$$\begin{aligned} & k_0'' = k_0' k_0 + \boldsymbol{k}'\boldsymbol{k}, \qquad \boldsymbol{k}'' = k_0'\boldsymbol{k} + \boldsymbol{k}' k_0 + i\boldsymbol{k}' \times \boldsymbol{k}, \\ & m_0'' = m_0' m_0 + \boldsymbol{m}'\boldsymbol{m}, \boldsymbol{m}'' = m_0'\boldsymbol{m} + \boldsymbol{m}' m_0 - i\boldsymbol{m}' \times \boldsymbol{m}, \end{aligned}$$

as well as the determinant D

$$\det G = (kk)(mm).$$

If one additionally imposes two requirements $(kk) = +1$, $(mm) = +1$, the Case D describes spinor covering for special complex rotation group $SO(4, C)$; this most simple case was considered in detail in [18].

It should be noted that the above general expression (2.8) for determinant can be transformed to a shorter form

$$\begin{aligned} \det G = {} & (kk)(mm) + (nn)(ll) + 2[kn][ml] \\ & - 2(k_0\boldsymbol{n} + n_0\boldsymbol{k} - i\boldsymbol{k} \times \boldsymbol{n})(m_0\boldsymbol{l} + l_0\boldsymbol{m} + i\boldsymbol{m} \times \boldsymbol{l}), \end{aligned}$$

which for the three Cases A, B, C becomes yet simpler:

$$\begin{aligned} \text{(A)}: \quad & \det G = [kk][mm] + [nn][ll] + 2(kn)(ml) \\ & \qquad + 2(k_0\boldsymbol{n} + n_0\boldsymbol{k} + \boldsymbol{k} \times \boldsymbol{n})(m_0\boldsymbol{l} + l_0\boldsymbol{m} - \boldsymbol{m} \times \boldsymbol{l}), \\ \text{(B)}: \quad & \det G = (kk)(k^*k^*) + (nn)(n^*n^*) + 2[kn][k^*n^*] \\ & \qquad - 2(k_0\boldsymbol{n} + n_0\boldsymbol{k} - i\boldsymbol{k} \times \boldsymbol{n})(k_0^*\boldsymbol{n}^* + n_0^*\boldsymbol{k}^* + i\boldsymbol{k}^* \times \boldsymbol{n}^*), \\ \text{(C)}: \quad & \det G = [kk]^2 + [nn]^2 + 2(kn)^2 - 2(k_0\boldsymbol{n} + n_0\boldsymbol{k} + \boldsymbol{k} \times \boldsymbol{n})^2. \end{aligned}$$

3 Unitarity condition

Now let us turn to consideration of the unitary group $SU(4)$. One should specify the requirement of unitarity $G^+ = G^{-1}$ to the above vector parametrization. Taking into account the formulas

$$G^+ = \left| \begin{array}{cc} k_0^* + \boldsymbol{k}^*\vec{\sigma} & -l_0^* - \boldsymbol{l}^*\vec{\sigma} \\ n_0^* - \boldsymbol{n}^*\vec{\sigma} & m_0^* - \boldsymbol{m}^*\vec{\sigma} \end{array} \right| , \qquad G^{-1} = \left| \begin{array}{cc} k_0' + \boldsymbol{k}'\vec{\sigma} & n_0' - \boldsymbol{n}'\vec{\sigma} \\ -l_0' - \boldsymbol{l}'\vec{\sigma} & m_0' - \boldsymbol{m}'\vec{\sigma} \end{array} \right| , \tag{3.1}$$

which can be represented differently

$$G^+ = G(k_0^*, \boldsymbol{k}^*; m_0^*, \boldsymbol{m}^*; -l_0^*, \boldsymbol{l}^*, -n_0^*, \boldsymbol{n}^*), \qquad G^{-1} = G(k_0', \boldsymbol{k}'; m_0', \boldsymbol{m}'; n_0', \boldsymbol{n}', l_0', \boldsymbol{l}'),$$

we arrive at

$$\begin{array}{llll} k_0^* = k_0', & \boldsymbol{k}^* = \boldsymbol{k}', & m_0^* = m_0', & \boldsymbol{m}^* = \boldsymbol{m}', \\ -l_0^* = n_0', & \boldsymbol{l}^* = \boldsymbol{n}', & -n_0^* = l_0', & \boldsymbol{n}^* = \boldsymbol{l}'. \end{array} \tag{3.2}$$

With the use of expressions for parameters of the inverse matrix with additional restriction $\det G = +1$ equations (3.2) can be rewritten as

$$\begin{aligned} k_0^* &= +k_0(mm) + m_0(ln) + l_0(nm) - n_0(lm) + i\boldsymbol{l}(\boldsymbol{m} \times \boldsymbol{n}), \\ m_0^* &= +m_0(kk) + k_0(nl) + n_0(lk) - l_0(nk) - i\boldsymbol{n}(\boldsymbol{k} \times \boldsymbol{l}), \\ \boldsymbol{k}^* &= -\boldsymbol{k}(mm) - \boldsymbol{m}(ln) - \boldsymbol{l}(nm) + \boldsymbol{n}(lm) + 2\boldsymbol{l} \times (\boldsymbol{n} \times \boldsymbol{m}) \\ &\quad + im_0(\boldsymbol{n} \times \boldsymbol{l}) + il_0(\boldsymbol{n} \times \boldsymbol{m}) + in_0(\boldsymbol{l} \times \boldsymbol{m}), \\ \boldsymbol{m}^* &= -\boldsymbol{m}(kk) - \boldsymbol{k}(nl) - \boldsymbol{n}(lk) + \boldsymbol{l}(nk) + 2\boldsymbol{n} \times (\boldsymbol{l} \times \boldsymbol{k}) \\ &\quad - ik_0(\boldsymbol{l} \times \boldsymbol{n}) - in_0(\boldsymbol{l} \times \boldsymbol{k}) - il_0(\boldsymbol{n} \times \boldsymbol{k}), \\ l_0^* &= +k_0(nm) - m_0(kn) + l_0(nn) + n_0(km) + i\boldsymbol{k}(\boldsymbol{n} \times \boldsymbol{m}), \\ n_0^* &= +m_0(lk) - k_0(ml) + n_0(ll) + l_0(mk) - i\boldsymbol{m}(\boldsymbol{l} \times \boldsymbol{k}), \\ \boldsymbol{l}^* &= -\boldsymbol{k}(nm) + \boldsymbol{m}(kn) - \boldsymbol{l}(nn) - \boldsymbol{n}(km) + 2\boldsymbol{k} \times (\boldsymbol{m} \times \boldsymbol{n}) \\ &\quad + ik_0(\boldsymbol{m} \times \boldsymbol{n}) + im_0(\boldsymbol{k} \times \boldsymbol{n}) + in_0(\boldsymbol{m} \times \boldsymbol{k}), \\ \boldsymbol{n}^* &= -\boldsymbol{m}(kl) + \boldsymbol{k}(ml) - \boldsymbol{n}(ll) - \boldsymbol{l}(mk) + 2\boldsymbol{m} \times (\boldsymbol{k} \times \boldsymbol{l}) - im_0(\boldsymbol{k} \times \boldsymbol{l}) \\ &\quad - ik_0(\boldsymbol{m} \times \boldsymbol{l}) - il_0(\boldsymbol{k} \times \boldsymbol{m}). \end{aligned} \tag{3.3}$$

Thus, the known form for parameters of the inverse matrix G^{-1} makes possible to write easily relations (3.3) representing the unitarity condition for group $SU(4)$. Here there are 16 equations for 16 variables; evidently, not all of them are independent.

Let us write down several simpler cases.

Case A

With formal change[13]

$$\boldsymbol{k} \Longrightarrow i\boldsymbol{k}, \qquad \boldsymbol{m} \Longrightarrow i\boldsymbol{m}, \qquad \boldsymbol{l} \Longrightarrow i\boldsymbol{l}, \qquad \boldsymbol{n} \Longrightarrow i\boldsymbol{n}, \tag{3.4}$$

equations (3.3) give

$$\begin{aligned} k_0 &= +k_0[mm] + m_0[ln] + l_0[nm] - n_0[lm] + \boldsymbol{l}(\boldsymbol{m} \times \boldsymbol{n}), \\ m_0 &= +m_0[kk] + k_0[nl] + n_0[lk] - l_0[nk] - \boldsymbol{n}(\boldsymbol{k} \times \boldsymbol{l}), \end{aligned}$$

[13] Let 0-components k_0, m_0, l_0, n_0 be real-valued, and 3-vectors $\boldsymbol{k}$, $\boldsymbol{m}$, $\boldsymbol{l}$, $\boldsymbol{n}$ be imaginary.

$$\begin{aligned}
&\boldsymbol{k} = \boldsymbol{k}[mm] + \boldsymbol{m}[ln] + \boldsymbol{l}[nm] - \boldsymbol{n}[lm] + 2\boldsymbol{l} \times (\boldsymbol{n} \times \boldsymbol{m}) \\
&\qquad + m_0(\boldsymbol{n} \times \boldsymbol{l}) + l_0(\boldsymbol{n} \times \boldsymbol{m}) + n_0(\boldsymbol{l} \times \boldsymbol{m}), \\
&\boldsymbol{m} = +\boldsymbol{m}[kk] + \boldsymbol{k}[nl] + \boldsymbol{n}[lk] - \boldsymbol{l}[nk] + 2\boldsymbol{n} \times (\boldsymbol{l} \times \boldsymbol{k}) \\
&\qquad - k_0(\boldsymbol{l} \times \boldsymbol{n}) - n_0(\boldsymbol{l} \times \boldsymbol{k}) - l_0(\boldsymbol{n} \times \boldsymbol{k}), \\
&l_0 = +k_0[nm] - m_0[kn] + l_0[nn] + n_0[km] + \boldsymbol{k}(\boldsymbol{n} \times \boldsymbol{m}), \\
&n_0 = +m_0[lk] - k_0[ml] + n_0[ll] + l_0[mk] - \boldsymbol{m}(\boldsymbol{l} \times \boldsymbol{k}), \\
&\boldsymbol{l} = +\boldsymbol{k}[nm] - \boldsymbol{m}[kn] + \boldsymbol{l}[nn] + \boldsymbol{n}[km] + 2\boldsymbol{k} \times (\boldsymbol{m} \times \boldsymbol{n}) \\
&\qquad + k_0(\boldsymbol{m} \times \boldsymbol{n}) + m_0(\boldsymbol{k} \times \boldsymbol{n}) + n_0(\boldsymbol{m} \times \boldsymbol{k}), \\
&\boldsymbol{n} = +\boldsymbol{m}[kl] - \boldsymbol{k}[ml] + \boldsymbol{n}[ll] + \boldsymbol{l}[mk] + 2\boldsymbol{m} \times (\boldsymbol{k} \times \boldsymbol{l}) \\
&\qquad - m_0(\boldsymbol{k} \times \boldsymbol{l}) - k_0(\boldsymbol{m} \times \boldsymbol{l}) - l_0(\boldsymbol{k} \times \boldsymbol{m}).
\end{aligned}$$

Here there are 16 equations for 16 real-valued variables.

Case B

Let

$$\begin{aligned}
&m_0 = k_0^*, \qquad \boldsymbol{m} = \boldsymbol{k}^*, \qquad l_0 = n_0^*, \qquad \boldsymbol{l} = \boldsymbol{n}^*, \\
&k_0 = m_0^*, \qquad \boldsymbol{k} = \boldsymbol{m}^*, \qquad n_0 = l_0^*, \qquad \boldsymbol{n} = \boldsymbol{l}^*,
\end{aligned}$$

or symbolically $m = k^*$, $l = n^*$. The unitarity relations become

$$\begin{aligned}
&k_0^* = +k_0(k^*k^*) + k_0^*(n^*n) + n_0^*(nk^*) - n_0(n^*k^*) + i\boldsymbol{n}^*(\boldsymbol{k}^* \times \boldsymbol{n}), \\
&\boldsymbol{k}^* = -\boldsymbol{k}(k^*k^*) - \boldsymbol{k}^*(n^*n) - \boldsymbol{n}^*(nk^*) + \boldsymbol{n}(n^*k^*) \\
&\qquad + 2\boldsymbol{n}^* \times (\boldsymbol{n} \times \boldsymbol{k}^*) + ik_0^*(\boldsymbol{n} \times \boldsymbol{n}^*) + in_0^*(\boldsymbol{n} \times \boldsymbol{k}^*) + in_0(\boldsymbol{l} \times \boldsymbol{m}), \\
&n_0^* = +k_0^*(n^*k) - k_0(k^*n^*) + n_0(n^*n^*) + n_0^*(k^*k) - i\boldsymbol{k}^*(\boldsymbol{n}^* \times \boldsymbol{k}), \\
&\boldsymbol{n}^* = -\boldsymbol{k}^*(kn^*) + \boldsymbol{k}(k^*n^*) - \boldsymbol{n}(n^*n^*) - \boldsymbol{n}^*(k^*k) \\
&\qquad + 2\boldsymbol{k}^* \times (\boldsymbol{k} \times \boldsymbol{n}^*) - ik_0^*(\boldsymbol{k} \times \boldsymbol{n}^*) - ik_0(\boldsymbol{k}^* \times \boldsymbol{n}^*) - in_0^*(\boldsymbol{k} \times \boldsymbol{k}^*),
\end{aligned}$$

and 8 conjugated ones

$$\begin{aligned}
&k_0 = +k_0^*(kk) + k_0(nn^*) + n_0(n^*k) - n_0^*(nk) - i\boldsymbol{n}(\boldsymbol{k} \times \boldsymbol{n}^*), \\
&\boldsymbol{k} = -\boldsymbol{k}^*(kk) - \boldsymbol{k}(nn^*) - \boldsymbol{n}(n^*k) + \boldsymbol{n}^*(nk) \\
&\qquad + 2\boldsymbol{n} \times (\boldsymbol{n}^* \times \boldsymbol{k}) - ik_0(\boldsymbol{n}^* \times \boldsymbol{n}) - in_0(\boldsymbol{n}^* \times \boldsymbol{k}) - in_0^*(\boldsymbol{n} \times \boldsymbol{k}), \\
&n_0 = +k_0(nk^*) - k_0^*(kn) + n_0^*(nn) + n_0(kk^*) + i\boldsymbol{k}(\boldsymbol{n} \times \boldsymbol{k}^*), \\
&\boldsymbol{n} = -\boldsymbol{k}(nk^*) + \boldsymbol{k}^*(kn) - \boldsymbol{n}^*(nn) - \boldsymbol{n}(kk^*) \\
&\qquad + 2\boldsymbol{k} \times (\boldsymbol{k}^* \times \boldsymbol{n}) + ik_0(\boldsymbol{k}^* \times \boldsymbol{n}) + ik_0^*(\boldsymbol{k} \times \boldsymbol{n}) + in_0(\boldsymbol{k}^* \times \boldsymbol{k}).
\end{aligned}$$

It may be noted that latter relations are greatly simplified when $n = 0$, or when $k = 0$. Firstly, let us consider the case $n = 0$:

$$k_0^* = +k_0(k^*k^*), \qquad \boldsymbol{k}^* = -\boldsymbol{k}(k^*k^*).$$

Taking in mind the identity

$$\det G = (kk)(kk)^* = +1 \qquad \Longrightarrow \qquad (kk) = +1, \qquad (kk)^* = +1,$$

we arrive at $k_0^* = +k_0$, $\boldsymbol{k}^* = -\boldsymbol{k}$. It has sense to introduce the real-valued vector c_a:

$$k_0^* = +k_0 = c_0, \qquad \boldsymbol{k}^* = -\boldsymbol{k} : \qquad \boldsymbol{k} = i\boldsymbol{c},$$

then matrix G is

$$G(k, m = k^*, 0, 0) = \begin{vmatrix} c_0 + i\boldsymbol{c}\vec{\sigma} & 0 \\ 0 & c_0 - i\boldsymbol{c}\vec{\sigma} \end{vmatrix} \sim SU(2).$$

Another possibility is realized when $k = 0$:

$$n_0^* = +n_0(nn)^*, \qquad \boldsymbol{n}^* = -\boldsymbol{n}(nn)^*.$$

With the use of identity

$$\det G = (nn)(nn)^* = +1 \qquad \Longrightarrow \qquad (nn) = +1, \qquad (nn)^* = +1,$$

we get

$$n_0^* = +n_0 = c_0, \qquad \boldsymbol{n}^* = -\boldsymbol{n}, \qquad \boldsymbol{n} \equiv i\boldsymbol{c},$$

Corresponding matrices $G(0, 0, n, l = n^*)$ make up a special set of unitary matrices

$$G = \begin{vmatrix} 0 & c_0 - i\boldsymbol{c}\vec{\sigma} \\ -(c_0 + i\boldsymbol{c}\vec{\sigma}) & 0 \end{vmatrix}, \qquad G^+ = \begin{vmatrix} 0 & -(c_0 - i\boldsymbol{c}\vec{\sigma}) \\ (c_0 + i\boldsymbol{c}\vec{\sigma}) & 0 \end{vmatrix}. \tag{3.5}$$

However, it must be noted that these matrices (3.5) do not provide us with any subgroup because $G^2 = -I$.

Case C

Now in equations (3.4) one should take

$$m_0 = k_0, \qquad l_0 = n_0, \qquad \boldsymbol{m} = -\boldsymbol{k}, \qquad \boldsymbol{l} = -\boldsymbol{n},$$

then

$$\begin{aligned}
&k_0 = +k_0[kk] + k_0(nn) + n_0(nk) - n_0[nk], \\
&\boldsymbol{k} = \boldsymbol{k}[kk] - \boldsymbol{k}(nn) - \boldsymbol{n}(nk) - \boldsymbol{n}[nk] + 2\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{k}), \\
&n_0 = +k_0(nk) - k_0[kn] + n_0[nn] + n_0(kk), \\
&\boldsymbol{n} = -\boldsymbol{k}(kn) - \boldsymbol{k}[kn] + \boldsymbol{n}[nn] - \boldsymbol{n}(kk) + 2\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{n}).
\end{aligned} \tag{3.6}$$

4 2-parametric subgroups in $SU(4)$

To be certain in correctness of the produced equations of unitarity, one should try to solve them at least in several most simple particular cases. For instance, let us turn to the Case C and specify equations (3.6) for a subgroup arising when $k = (k_0, k_1, 0, 0)$ and $n = (n_0, n_1, 0, 0)$:

$$\begin{aligned}
&k_0 = +k_0[kk] + k_0(nn) + n_0(nk) - n_0[nk], \\
&k_1 = +k_1[kk] - k_1(nn) - n_1(nk) - n_1[nk], \\
&n_0 = +k_0(nk) - k_0[kn] + n_0[nn] + n_0(kk), \\
&n_1 = -k_1(kn) - k_1[kn] + n_1[nn] - n_1(kk),
\end{aligned} \tag{4.1}$$

they are four non-linear equations for four real variables. It may be noted that equations (4.1) can be regarded as two eigenvalue problems in two dimensional space (with eigenvalue +1):

$$\begin{vmatrix} (k_0^2 + n_0^2) - 1 + (k_1^2 - n_1^2) & -2n_1k_1 \\ -2n_1k_1 & (k_0^2 + n_0^2) - 1 - (k_1^2 - n_1^2) \end{vmatrix} \begin{vmatrix} k_0 \\ n_0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix},$$

$$\left| \begin{array}{cc} (k_1^2+n_1^2)-1+(k_0^2-n_0^2) & -2n_0k_0 \\ -2n_0k_0 & (k_1^2+n_1^2)-1-(k_0^2-n_0^2) \end{array} \right| \left| \begin{array}{c} k_1 \\ n_1 \end{array} \right| = \left| \begin{array}{c} 0 \\ 0 \end{array} \right| .$$

The determinants in both problems must be equated to zero:

$$[(k_0^2+n_0^2)-1]^2-(k_1^2-n_1^2)^2-4n_1^2k_1^2=0,$$
$$[(k_1^2+n_1^2)-1]^2-(k_0^2-n_0^2)^2-4n_0^2k_0^2=0,$$

or

$$[(k_0^2+n_0^2)-1]^2-(k_1^2+n_1^2)^2=0, \qquad [(k_1^2+n_1^2)-1]^2-(k_0^2+n_0^2)^2=0.$$

The latter equations may be rewritten in factorized form:

$$[(k_0^2+n_0^2)-1-(k_1^2+n_1^2)][(k_0^2+n_0^2)-1+(k_1^2+n_1^2)]=0,$$
$$[(k_1^2+n_1^2)-1-(k_0^2+n_0^2)][(k_1^2+n_1^2)-1+(k_0^2+n_0^2)]=0.$$

They have the structure: $AC=0$, $BC=0$. Four different cases arise.

(1) Let $C=0$, then

$$k_0^2+n_0^2+k_1^2+n_1^2=+1. \tag{4.2}$$

(2) Now, let $A=0$, $B=0$, but a contradiction arises: $A+B=0$, $A+B=-2$.

(3)–(4) There are two simples cases:

$$A=0,C=0 \qquad k_0^2+n_0^2=1, \qquad k_1=0,n_1=0, \tag{4.3}$$
$$B=0,C=0 \qquad k_1^2+n_1^2=1, \qquad k_0=0,n_0=0. \tag{4.4}$$

Evidently, (4.3) and (4.4) can be regarded as particular cases of the above variant (4.2). Now, one should take into account additional relation

$$\det G=[kk][kk]+[nn][nn]+2(kk)(nn)$$
$$+2(nk)(nk)-2[nk][nk])+4(\boldsymbol{kn})(\boldsymbol{kn})-4(\boldsymbol{kk})(\boldsymbol{nn})=1,$$

which can be transformed to

$$\det G=(k_0^2+k_1^2+n_0^2+n_1^2)^2-4(k_1n_0+k_0n_1)^2=+1. \tag{4.5}$$

Both equations (4.2) and (4.5) are to be satisfied

$$(k_0^2+n_0^2+k_1^2+n_1^2)=1, \qquad (k_0^2+k_1^2+n_0^2+n_1^2)^2-1-4(k_1n_0+k_0n_1)^2=0,$$

from where it follows

$$k_1n_0+k_0n_1=0, \qquad k_0^2+n_0^2+k_1^2+n_1^2=+1.$$

They specify a 2-parametric unitary subgroup in $SU(4)$

$$G_1^+=G_1^{-1}, \qquad \det G_1=+1,$$
$$k_1n_0+k_0n_1=0, \qquad k_0^2+n_0^2+k_1^2+n_1^2=+1,$$

$$G_1=\left| \begin{array}{cc} k_0+ik_1\sigma^1 & n_0-in_1\sigma^1 \\ -n_0+in_1\sigma^1 & k_0+ik_1\sigma^1 \end{array} \right| = \left| \begin{array}{cccc} k_0 & ik_1 & n_0 & -in_1 \\ ik_1 & k_0 & -in_1 & n_0 \\ -n_0 & in_1 & k_0 & ik_1 \\ in_1 & -n_0 & ik_1 & k_0 \end{array} \right| . \tag{4.6}$$

Two analogous subgroups are possible:

$$G_2^+ = G_2^{-1}, \qquad \det G_2 = +1,$$
$$k_2 n_0 + k_0 n_2 = 0, \qquad k_0^2 + n_0^2 + k_2^2 + n_2^2 = +1,$$
$$G_2 = \begin{vmatrix} k_0 + ik_2\sigma^2 & n_0 - in_2\sigma^2 \\ -n_0 + in_2\sigma^2 & k_0 + ik_2\sigma^2 \end{vmatrix} = \begin{vmatrix} k_0 & k_2 & n_0 & -n_2 \\ -k_2 & k_0 & n_2 & n_0 \\ -n_0 & n_2 & k_0 & k_2 \\ -n_2 & -n_0 & -k_2 & k_0 \end{vmatrix}; \tag{4.7}$$

$$G_3^+ = G_3^{-1}, \qquad \det G_3 = +1,$$
$$k_3 n_0 + k_0 n_3 = 0, \qquad k_0^2 + n_0^2 + k_3^2 + n_3^2 = +1,$$
$$G_3 = \begin{vmatrix} (k_0 + ik_3) & 0 & (n_0 - in_3) & 0 \\ 0 & (k_0 - ik_3) & 0 & (n_0 + in_3) \\ -(n_0 - in_3) & 0 & (k_0 + ik_3) & 0 \\ 0 & -(n_0 + in_3) & 0 & (k_0 - ik_3) \end{vmatrix}. \tag{4.8}$$

Let us consider the latter subgroup (4.8) in some detail. The multiplication law for parameters is

$$k_0'' = k_0'k_0 - k_3'k_3 - n_0'n_0 + n_3'n_3, \qquad k_3'' = k_0'k_3 + k_3'k_0 + n_0'n_3 + n_3'n_0,$$
$$n_0'' = k_0'n_0 + k_3'n_3 + n_0'k_0 + n_3'k_3, \qquad n_3'' = k_0'n_3 - k_3'n_0 - n_0'k_3 + n_3'k_0.$$

For two particular cases (see (4.3) and (4.4)), these formulas take the form:

$$\begin{array}{lll} \{G_3^{\prime 0}\}: & k_3^2 + n_3^2 = 1, & k_0 = 0, \qquad n_0 = 0, \\ & k_0'' = -k_3'k_3 + n_3'n_3, & k_3'' = 0, \\ & n_0'' = +k_3'n_3 + n_3'k_3, & n_3'' = 0, \\ \{G^0\}: & k_0^2 + n_0^2 = 1, & k_3 = 0, \qquad n_3 = 0, \\ & k_0'' = k_0'k_0 - n_0'n_0, & \\ & n_0'' = k_0'n_0 + n_0'k_0. & \end{array} \tag{4.9}$$

Therefore, multiplying of any two elements from $G_3^{\prime 0}$ does not lead us to any element from $G_3^{\prime 0}$, instead belonging to G^0: $G_3^{0\prime} G_3^0 \in G^0$. Similar result would be achieved for G_1 and G_2: $G_1^{0\prime} G_1^0 \in G^0$, $G_2^{0\prime} G_2^0 \in G^0$. In the subgroup given by (4.9) one can easily see the structure of the 1-parametric Abelian subgroup:

$$k_0 = \cos\alpha, \qquad n_0 = \sin\alpha,$$
$$G^0(\alpha) = \begin{vmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha \end{vmatrix}, \qquad \alpha \in [0, 2\pi]. \tag{4.10}$$

In the same manner, similar curvilinear parametrization can be readily produced for 2-parametric groups (4.6)–(4.8). For definiteness, for the subgroup G_3 such coordinates are given by

$$k_0 = \cos\alpha\cos\rho, \qquad k_3 = \cos\alpha\sin\rho,$$
$$n_0 = \sin\alpha\cos\rho, \qquad -n_3 = \sin\alpha\sin\rho, \alpha \in [0, 2\pi],$$

and matrix G_3 is

$$G_3(\rho,\alpha) = \left| \begin{array}{cccc} \cos\alpha e^{i\rho} & 0 & \sin\alpha e^{i\rho} & 0 \\ 0 & \cos\alpha e^{-i\rho} & 0 & \sin\alpha e^{-i\rho} \\ -\sin\alpha e^{i\rho} & 0 & \cos\alpha e^{i\rho} & 0 \\ 0 & -\sin\alpha e^{-i\rho} & 0 & \cos\alpha e^{-i\rho} \end{array} \right| . \tag{4.11}$$

One may note that equation (4.11) at $\rho = 0$ will coincide with $G^0(\alpha)$ in (4.10): $G_3(\rho = 0, \alpha) = G^0(\alpha)$. Similar curvilinear parametrization may be introduced for two other subgroups, G_1 and G_2.

One could try to obtain more general result just changing real valued curvilinear coordinates on complex. However it is easily verified that it is not the case: through that change though there arise subgroups but they are not unitary. Indeed, let the matrix (4.10) be complex: then unitarity condition gives

$$\cos\alpha\cos\alpha^* + \sin\alpha\sin\alpha^* = 1, \qquad -\cos\alpha\sin\alpha^* + \sin\alpha\cos\alpha^* = 0.$$

These two equations can be satisfied only by a real valued α. In the same manner, the the formal change $\{G_1, G_2, G_3\} \Longrightarrow \{G_1^C, G_2^C, G_3^C\}$ again provides us with non-unitary subgroups.

It should be noted that each of three 2-parametric subgroup G_1, G_2, G_3, in addition to $G_0(\alpha)$, contains one additional Abelian unitary subgroup:

$$K_1 = \left| \begin{array}{cc} k_0 + ik_1\sigma^1 & 0 \\ 0 & k_0 + ik_1\sigma^1 \end{array} \right| , \qquad K_1 \subset G_1, \qquad k_0^2 + k_1^2 = 1,$$

$$K_2 = \left| \begin{array}{cc} k_0 + ik_2\sigma^2 & 0 \\ 0 & k_0 + ik_2\sigma^2 \end{array} \right| , \qquad K_2 \subset G_2, \qquad k_0^2 + k_2^2 = 1,$$

$$K_3 = \left| \begin{array}{cc} k_0 + ik_3\sigma^3 & 0 \\ 0 & k_0 + ik_3\sigma^3 \end{array} \right| , \qquad K_3 \subset G_3, \qquad k_0^2 + k_3^2 = 1.$$

It may be easily verified that

$$G_1 = G_0K_1 = K_1G_0, \qquad G_2 = G_0K_2 = K_2G_0, \qquad G_3 = G_0K_3 = K_3G_0.$$

Indeed

$$G_0(\alpha)K_1 = K_1G_0(\alpha) = \left| \begin{array}{cc} \cos\alpha k_0 + i\cos\alpha k_1\sigma^1 & \sin\alpha k_0 + i\sin\alpha k_1 \\ -\sin\alpha k_0 - i\sin\alpha k_1 & \cos\alpha k_0 + i\cos\alpha k_1\sigma^1 \end{array} \right| ,$$

and with notation

$$\cos\alpha k_0 = k_0', \qquad \cos\alpha k_1 = k_1', \qquad \sin\alpha k_0 = n_0', \qquad \sin\alpha k_1 = -n_1',$$
$$k_0'n_1' + k_1'n_0' = 0, \qquad k_0'^2 + k_3'^2 + n_0'^2 + n_3'^2 = 1$$

we arrive at

$$G_0K_1 = K_1G_0 = \left| \begin{array}{cc} k_0' + ik_1'\sigma^1 & n_0' - in_1' \\ -n_0' + in_1' & k_0' + ik_1'\sigma^1 \end{array} \right| \subset G_1.$$

5 4-parametric unitary subgroup

Let us turn again to the subgroup in $GL(4, C)$ given by Case C (see (2.10)):

$$G = \left| \begin{array}{cc} (k_0 + \boldsymbol{k}\vec{\sigma}) & (n_0 - \boldsymbol{n}\vec{\sigma}) \\ -(n_0 - \boldsymbol{n}\vec{\sigma}) & (k_0 + \boldsymbol{k}\vec{\sigma}) \end{array} \right| ,$$

when the unitarity equations look as follows:

$$
\begin{aligned}
k_0 &= +k_0[kk] + k_0(nn) + n_0(nk) - n_0[nk],\\
n_0 &= +k_0(nk) - k_0[kn] + n_0[nn] + n_0(kk),\\
\boldsymbol{k} &= \boldsymbol{k}[kk] - \boldsymbol{k}(nn) - \boldsymbol{n}(nk) - \boldsymbol{n}[nk] + 2\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{k}),\\
\boldsymbol{n} &= -\boldsymbol{k}(kn) - \boldsymbol{k}[kn] + \boldsymbol{n}[nn] - \boldsymbol{n}(kk) + 2\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{n}).
\end{aligned}
$$

They can be rewritten as four eigenvalue problems:

$$
\left|\begin{array}{cc} [kk] + (nn) & (nk) - [nk] \\ (nk) - [nk] & (kk) + [nn] \end{array}\right| \left|\begin{array}{c} k_0 \\ n_0 \end{array}\right| = (+1) \left|\begin{array}{c} k_0 \\ n_0 \end{array}\right|, \tag{5.1}
$$

$$
\begin{aligned}
&\left|\begin{array}{cc} +([kk] - [nn]) & -2(nk) \\ -2(nk) & -([kk] - [nn]) \end{array}\right| \left|\begin{array}{c} k_1 \\ n_1 \end{array}\right| = (+1) \left|\begin{array}{c} k_1 \\ n_1 \end{array}\right|,\\
&\left|\begin{array}{cc} +([kk] - [nn]) & -2(nk) \\ -2(nk) & -([kk] - [nn]) \end{array}\right| \left|\begin{array}{c} k_2 \\ n_2 \end{array}\right| = (+1) \left|\begin{array}{c} k_2 \\ n_2 \end{array}\right|,\\
&\left|\begin{array}{cc} +([kk] - [nn]) & -2(nk) \\ -2(nk) & -([kk] - [nn]) \end{array}\right| \left|\begin{array}{c} k_3 \\ n_3 \end{array}\right| = (+1) \left|\begin{array}{c} k_3 \\ n_3 \end{array}\right|.
\end{aligned} \tag{5.2}
$$

These equations have the same structure

$$
\left|\begin{array}{cc} A & C \\ C & B \end{array}\right| \left|\begin{array}{c} Z_1 \\ Z_2 \end{array}\right| = \lambda \left|\begin{array}{c} Z_1 \\ Z_2 \end{array}\right|,
$$

where $\lambda = +1$. Non-trivial solutions may exist only if

$$
\det \left|\begin{array}{cc} A - \lambda & C \\ C & B - \lambda \end{array}\right| = 0,
$$

which gives two different eigenvalues

$$
\lambda_1 = \frac{A + B + \sqrt{(A - B)^2 + 4C^2}}{2}, \qquad \lambda_2 = \frac{A + B + \sqrt{(A - B)^2 + 4C^2}}{2}.
$$

In explicit form, equations (5.1) looks as follows:

$$
\begin{aligned}
&\left|\begin{array}{cc} A & C \\ C & B \end{array}\right| \left|\begin{array}{c} k_0 \\ n_0 \end{array}\right| = \lambda \left|\begin{array}{c} k_0 \\ n_0 \end{array}\right|,\\
&A = (k_0^2 + n_0^2) + (\boldsymbol{k}^2 - \boldsymbol{n}^2), \qquad B = (k_0^2 + n_0^2) - (\boldsymbol{k}^2 - \boldsymbol{n}^2), \qquad C = -2\boldsymbol{k}\boldsymbol{n},\\
&\lambda_1 = (k_0^2 + n_0^2) + \sqrt{(\boldsymbol{k}^2 - \boldsymbol{n}^2)^2 + 4(\boldsymbol{k}\boldsymbol{n})^2},\\
&\lambda_2 = (k_0^2 + n_0^2) - \sqrt{(\boldsymbol{k}^2 - \boldsymbol{n}^2)^2 + 4(\boldsymbol{k}\boldsymbol{n})^2}.
\end{aligned} \tag{5.3}
$$

The eigenvalue $\lambda = +1$ might be constructed by two ways:

$$
\begin{aligned}
\lambda_1 = +1, &\qquad k_0^2 + n_0^2 = 1 - \sqrt{(\boldsymbol{k}^2 - \boldsymbol{n}^2)^2 + 4(\boldsymbol{k}\boldsymbol{n})^2},\\
\lambda_2 = +1, &\qquad k_0^2 + n_0^2 = 1 + \sqrt{(\boldsymbol{k}^2 - \boldsymbol{n}^2)^2 + 4(\boldsymbol{k}\boldsymbol{n})^2}.
\end{aligned} \tag{5.4}
$$

These two relations (5.4) are equivalent to the following one:

$$
(1 - k_0^2 - n_0^2)^2 = (\boldsymbol{k}^2 - \boldsymbol{n}^2)^2 + 4(\boldsymbol{k}\boldsymbol{n})^2.
$$

Thus, equations (5.3) have two different types:
Type I

$$
\begin{aligned}
&(A-1)k_0 + Cn_0 = 0, \qquad Ck_0 + (B-1)n_0 = 0,\\
&k_0^2 + n_0^2 = 1 - \sqrt{(\boldsymbol{k}^2 - \boldsymbol{n}^2)^2 + 4(\boldsymbol{kn})^2},\\
&k_0^2 + n_0^2 < +1, \qquad (\boldsymbol{k}^2 - \boldsymbol{n}^2)^2 + 4(\boldsymbol{kn})^2 < +1;
\end{aligned}
\tag{5.5}
$$

Type II

$$
\begin{aligned}
&(A-1)k_0 + Cn_0 = 0, \qquad Ck_0 + (B-1)n_0 = 0, \\
&k_0^2 + n_0^2 = 1 + \sqrt{(\boldsymbol{k}^2 - \boldsymbol{n}^2)^2 + 4(\boldsymbol{kn})^2}, \qquad k_0^2 + n_0^2 > +1.
\end{aligned}
\tag{5.6}
$$

Now let us turn to equations (5.2). They have the form

$$
\begin{aligned}
&\begin{vmatrix} A & C \\ C & -A \end{vmatrix} \begin{vmatrix} k_i \\ n_i \end{vmatrix} = \lambda \begin{vmatrix} k_i \\ n_i \end{vmatrix}, \qquad i = 1,2,3,\\
&A = k_0^2 + \boldsymbol{k}^2 - n_0^2 - \boldsymbol{n}^2, \qquad B = -A, \qquad C = -2(k_0 n_0 - \boldsymbol{kn}),\\
&\lambda_1 = +\sqrt{(k_0^2 + \boldsymbol{k}^2 - n_0^2 - \boldsymbol{n}^2)^2 + 4(k_0 n_0 - \boldsymbol{kn})^2},\\
&\lambda_2 = -\sqrt{(k_0^2 + \boldsymbol{k}^2 - n_0^2 - \boldsymbol{n}^2)^2 + 4(k_0 n_0 - \boldsymbol{kn})^2}.
\end{aligned}
$$

As we are interested only in positive eigenvalue $\lambda = +1$, we must use only one possibility $\lambda = +1 = \lambda_1$, so that

$$
\begin{aligned}
&(A-1)\boldsymbol{k} + C\boldsymbol{n} = 0, \qquad C\boldsymbol{k} - (A+1)\boldsymbol{n} = 0,\\
&1 = (k_0^2 + \boldsymbol{k}^2 - n_0^2 - \boldsymbol{n}^2)^2 + 4(k_0 n_0 - \boldsymbol{kn})^2.
\end{aligned}
\tag{5.7}
$$

Vector condition in (5.7) says that $\boldsymbol{k}$ and $\boldsymbol{n}$ are (anti)collinear:

$$
\boldsymbol{k} = K\boldsymbol{e}, \qquad \boldsymbol{n} = N\boldsymbol{e}, \qquad \boldsymbol{e}^2 = 1, \qquad \boldsymbol{e} \in S_2,
\tag{5.8}
$$

so that (5.7) give

$$
\begin{aligned}
&(A-1)K + CN = 0, \qquad CK - (A+1)N = 0,\\
&1 = (k_0^2 + K^2 - n_0^2 - N^2)^2 + 4(k_0 n_0 - KN)^2,\\
&A = k_0^2 + K^2 - n_0^2 - N^2, C = -2(k_0 n_0 - KN).
\end{aligned}
\tag{5.9}
$$

With notation (5.8), equations (5.5)–(5.6) take the form:
Type I

$$
(A-1)k_0 + Cn_0 = 0, \qquad Ck_0 + (B-1)n_0 = 0, \qquad k_0^2 + n_0^2 = 1 - (K^2 + N^2),
$$

Type II

$$
(A-1)k_0 + Cn_0 = 0, \qquad Ck_0 + (B-1)n_0 = 0, \qquad k_0^2 + n_0^2 = 1 + (K^2 + N^2),
\tag{5.10}
$$

where

$$
A = (k_0^2 + n_0^2) + (K^2 - N^2), \qquad B = (k_0^2 + n_0^2) - (K^2 - N^2), \qquad C = -2KN.
\tag{5.11}
$$

Therefore, we have 8 variables $\boldsymbol{e}$, k_0, n_0, K, N and the set of equations, (5.9)–(5.11) for them. Its solving turns to be rather involving, so let us formulate only the final result:

$$k_0, \qquad \boldsymbol{k} = K\boldsymbol{e}, \qquad n_0, \qquad \boldsymbol{n} = N\boldsymbol{e},$$
$$k_0^2 + K^2 + n_0^2 + N^2 = +1, \qquad k_0 N + n_0 K = 0,$$
$$G = \begin{vmatrix} (k_0 + iK\boldsymbol{e}\vec{\sigma}) & (n_0 - iN\boldsymbol{e}\vec{\sigma}) \\ -(n_0 - iN\boldsymbol{e}\vec{\sigma}) & (k_0 + iK\boldsymbol{e}\vec{\sigma}) \end{vmatrix}. \tag{5.12}$$

It should be noted that

$$\det G = (k_0^2 + K^2 + n_0^2 + N^2)^2 = +1.$$

The unitarity of the matrices (5.12) may be verified by direct calculation. Indeed,

$$G^+ = \begin{vmatrix} (k_0 - iK\boldsymbol{e}\vec{\sigma}) & -(n_0 + iN\boldsymbol{e}\vec{\sigma}) \\ (n_0 + iN\boldsymbol{e}\vec{\sigma}) & (k_0 - iK\boldsymbol{e}\vec{\sigma}) \end{vmatrix},$$

and further for $GG^+ = I$ we get (by 2×2 blocks)

$$(GG^+)_{11} = k_0^2 + K^2 + n_0^2 + N^2 = +1, \qquad (GG^+)_{12} = -2i(n_0 K + k_0 N)(\boldsymbol{e}\vec{\sigma}) = 0,$$
$$(GG^+)_{22} = k_0^2 + K^2 + n_0^2 + N^2 = +1, \qquad (GG^+)_{21} = +2i(n_0 K + k_0 N)(\boldsymbol{e}\vec{\sigma}) = 0.$$

One different way to parameterize (5.12) can be proposed. Indeed, relations (5.12) are

$$k_0, \qquad \boldsymbol{k} = K\boldsymbol{e}, \qquad n_0, \qquad \boldsymbol{n} = N\boldsymbol{e},$$
$$k_0^2(1 + \frac{K^2}{k_0^2}) + n_0^2(1 + \frac{N^2}{n_0^2}) = +1, \qquad \frac{K}{k_0} = -\frac{N}{n_0} \equiv W,$$

or

$$k_0, \qquad \boldsymbol{k} = k_0 W\boldsymbol{e}, \qquad n_0, \qquad \boldsymbol{n} = -n_0 W\boldsymbol{e},$$
$$(k_0^2 + n_0^2)(1 + W^2) = +1, \qquad K = k_0 W, \qquad N = -n_0 W, \qquad 0 \le k_0^2 + n_0^2 \le 1.$$

Therefore, matrix G can be presented as follows:

$$G = \begin{vmatrix} k_0(1 + iW\boldsymbol{e}\vec{\sigma}) & n_0(1 + iW\boldsymbol{e}\vec{\sigma}) \\ -n_0(1 + iW\boldsymbol{e}\vec{\sigma}) & k_0(1 + iW\boldsymbol{e}\vec{\sigma}) \end{vmatrix}, \tag{5.13}$$
$$(k_0^2 + n_0^2)(1 + W^2) = +1 \qquad \Longrightarrow \qquad W = \pm\sqrt{\frac{1}{k_0^2 + n_0^2} - 1}.$$

Evidently, it suffices to take positive values for W. The constructed subgroup (5.13) depends upon four parameters k_0, n_0, $\boldsymbol{e}$:

$$0 \le k_0^2 + n_0^2 \le 1, \qquad \boldsymbol{e}^2 = 1, \qquad (k_0^2 + n_0^2)(1 + W^2) = +1.$$

Let us establish the law of multiplication for four parameters k_0, n_0, $\boldsymbol{W} = W\boldsymbol{e}$:

$$G'' = \begin{vmatrix} k_0'(1 + i\boldsymbol{W}'\vec{\sigma}) & n_0'(1 + i\boldsymbol{W}'\vec{\sigma}) \\ -n_0'(1 + i\boldsymbol{W}'\vec{\sigma}) & k_0'(1 + i\boldsymbol{W}'\vec{\sigma}) \end{vmatrix} \begin{vmatrix} k_0(1 + i\boldsymbol{W}\vec{\sigma}) & n_0(1 + i\boldsymbol{W}\vec{\sigma}) \\ -n_0(1 + i\boldsymbol{W}\vec{\sigma}) & k_0(1 + i\boldsymbol{W}\vec{\sigma}) \end{vmatrix}$$

or by 2×2 blocks

$$(11) = (k_0' k_0 - n_0' n_0)(1 + i\boldsymbol{W}'\vec{\sigma})(1 + i\boldsymbol{W}\vec{\sigma}),$$

$$(12) = (k_0' n_0 + n_0' k_0)(1 + i\boldsymbol{W}'\vec{\sigma})(1 + i\boldsymbol{W}\vec{\sigma}),$$
$$(21) = -(k_0' n_0 + n_0' k_0)(1 + i\boldsymbol{W}'\vec{\sigma})(1 + i\boldsymbol{W}\vec{\sigma}),$$
$$(22) = (k_0' k_0 - n_0' n_0)(1 + i\boldsymbol{W}'\vec{\sigma})(1 + i\boldsymbol{W}\vec{\sigma}).$$

As $(11) = (22)$, $(12) = -(21)$; further one can consider only two blocks:

$$(11) = (k_0' k_0 - n_0' n_0)(1 - \boldsymbol{W}'\boldsymbol{W}) \left(1 + i \frac{\boldsymbol{W}' + \boldsymbol{W} - \boldsymbol{W}' \times \boldsymbol{W}}{1 - \boldsymbol{W}'\boldsymbol{W}} \vec{\sigma} \right),$$
$$(12) = (k_0' n_0 + n_0' k_0)(1 - \boldsymbol{W}'\boldsymbol{W}) \left(1 + i \frac{\boldsymbol{W}' + \boldsymbol{W} - \boldsymbol{W}' \times \boldsymbol{W}}{1 - \boldsymbol{W}'\boldsymbol{W}} \vec{\sigma} \right).$$

So the composition rules should be

$$k_0'' = (k_0' k_0 - n_0' n_0)(1 - \boldsymbol{W}'\boldsymbol{W}), \qquad n_0'' = (k_0' n_0 + n_0' k_0)(1 - \boldsymbol{W}'\boldsymbol{W}),$$
$$\boldsymbol{W}'' = \frac{\boldsymbol{W}' + \boldsymbol{W} - \boldsymbol{W}' \times \boldsymbol{W}}{1 - \boldsymbol{W}'\boldsymbol{W}}.$$

The later formula coincides with the Gibbs multiplication rule (see in [32]) for 3-dimensional rotation group $SO(3, R)$. It remains to prove the identity:

$$(k_0''^2 + n_0''^2)(1 + W''^2) = +1$$

which reduces to

$$(k_0' k_0 - n_0' n_0)^2 + (k_0' n_0 + n_0' k_0)^2 \left[(1 - \boldsymbol{W}'\boldsymbol{W})^2 + (\boldsymbol{W}' + \boldsymbol{W} - \boldsymbol{W}' \times \boldsymbol{W})^2 \right]. \tag{5.14}$$

First terms are

$$(k_0' k_0 - n_0' n_0)^2 + (k_0' n_0 + n_0' k_0)^2 = (k_0'^2 + n_0'^2)(k_0^2 + n_0^2).$$

Second term is

$$(1 - \boldsymbol{W}'\boldsymbol{W})^2 + (\boldsymbol{W}' + \boldsymbol{W} - \boldsymbol{W}' \times \boldsymbol{W})^2 = (1 + \boldsymbol{W}'^2)(1 + \boldsymbol{W}^2).$$

Therefore, (5.14) takes the form

$$(k_0'^2 + n_0'^2)(k_0^2 + n_0^2)(1 + \boldsymbol{W}'^2)(1 + \boldsymbol{W}^2) = 1$$

which is identity due to equalities

$$(k_0'^2 + n_0'^2)(1 + \boldsymbol{W}'^2) = 1, \qquad (k_0^2 + n_0^2)(1 + \boldsymbol{W}^2) = 1.$$

It is matter of simple calculation to introduce curvilinear parameters for such an unitary subgroup:

$$\boldsymbol{e} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$
$$k_0 = \cos\alpha\cos\rho, \qquad K = \cos\alpha\sin\rho, \qquad n_0 = \sin\alpha\cos\rho, \qquad N = -\sin\alpha\sin\rho,$$

and G looks as follows

$$G = \left| \begin{array}{cc} \Delta & \Sigma \\ -\Sigma & \Delta \end{array} \right|, \qquad \Delta = \left| \begin{array}{cc} \cos\alpha(\cos\rho + i\sin\rho\cos\theta) & i\cos\alpha\sin\rho\sin\theta e^{-i\phi} \\ i\cos\alpha\sin\rho\sin\theta e^{i\phi} & \cos\alpha(\cos\rho - i\sin\rho\cos\theta \end{array} \right|,$$
$$\Sigma = \left| \begin{array}{cc} \sin\alpha(\cos\rho + i\sin\rho\cos\theta) & +i\sin\alpha\sin\rho\sin\theta e^{-i\phi} \\ i\sin\alpha\sin\rho\sin\theta e^{i\phi} & \sin\alpha(\cos\rho - i\sin\rho\cos\theta) \end{array} \right|.$$

It should be noted that one one may factorize 4-parametric element into two unitary factors, 1-parametric and 3-parametric. Indeed, let us consider the product of commuting unitary groups, isomorphic to Abelian group G_0 and $SU(2)$:

$$G = G_0 \otimes SU(2) = SU(2) \otimes G_0 = \left| \begin{array}{cc} k'_0 & n'_0 \\ -n'_0 & k'_0 \end{array} \right| \left| \begin{array}{cc} a_0 + i\boldsymbol{a}\vec{\sigma} & 0 \\ 0 & a_0 + i\boldsymbol{a}\vec{\sigma} \end{array} \right|$$

$$= \left| \begin{array}{cc} k'_0 a_0 + ik'_0 \boldsymbol{a}\vec{\sigma} & n'_0 a_0 + in'_0 \boldsymbol{a}\vec{\sigma} \\ -n'_0 a_0 - in'_0 \boldsymbol{a}\vec{\sigma} & k'_0 a_0 + ik'_0 \boldsymbol{a}\vec{\sigma} \end{array} \right|, \qquad k_0'^2 + n_0'^2 = 1, \qquad a_0^2 + \boldsymbol{a}^2 = 1,$$

with the notation

$$k'_0 a_0 = k_0, \qquad k'_0 \boldsymbol{a} = k_0 \boldsymbol{W}, \qquad n'_0 a_0 = n_0, \qquad n'_0 \boldsymbol{a} = n_0 \boldsymbol{W},$$
$$(k_0^2 + n_0^2)(1 + W^2) = (k_0'^2 a_0^2 + n_0'^2 a_0^2)(1 + W^2) = (k_0'^2 + n_0'^2)(a_0^2 + \boldsymbol{a}^2) = 1,$$

takes the form

$$G_0 \otimes SU(2) = SU(2) \otimes G_0 = \left| \begin{array}{cc} k_0(1 + i\boldsymbol{W}\vec{\sigma}) & n_0(1 + i\boldsymbol{W}\vec{\sigma}) \\ -n_0(1 + i\boldsymbol{W}\vec{\sigma}) & k_0(1 + i\boldsymbol{W}\vec{\sigma}) \end{array} \right| = G. \tag{5.15}$$

Let us summarize the main results of the previous sections:

> Parametrization of 4×4 matrices G of the complex linear group $GL(4,C)$ in terms of four complex vector-parameters $G = G(k,m,n,l)$ is developed and the problem of inverting any 4×4 matrix G is solved. Expression for determinant of any matrix G is found: $\det G = F(k,m,n,l)$. Unitarity conditions have been formulated in the form of non-linear cubic algebraic equations including complex conjugation. Several simplest solutions of these unitarity equations have been found: three 2-parametric subgroups G_1, G_2, G_3 – each of subgroups consists of two commuting Abelian unitary groups; 4-parametric unitary subgroup consisting of a product of a 3-parametric group isomorphic $SU(2)$ and 1-parametric Abelian group.

The task of full solving of the unitarity conditions seems to be rather complicated and it will be considered elsewhere. In the remaining part of the present paper we describe some relations of the above treatment to other considerations of the problem in the literature. The relations described give grounds to hope that the full general solution of the unitary equations obtained can be constructed on the way of combining different techniques used in the theory of the unitary group $SU(4)$.

6 On subgroups $GL(3,C)$ and $SU(3)$, expressions for Gell-Mann matrices through the Dirac basis

In this section the main question is how in the Dirac parametrization one can distinguish $GL(3,C)$, subgroup in $GL(4,C)$. To this end, let us turn to the explicit form of the Dirac basis (the Weyl representation is used; at some elements the imaginary unit i is added)

$$\gamma^5 = \left| \begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right|, \qquad \gamma^0 = \left| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right|, \qquad i\gamma^5\gamma^0 = \left| \begin{array}{cccc} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{array} \right|,$$

$$i\gamma^1 = \left| \begin{array}{cccc} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{array} \right|, \qquad \gamma^5\gamma^1 = \left| \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right|, \qquad i\gamma^2 = \left| \begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right|, \tag{6.1}$$

$$\gamma^5\gamma^2 = \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \qquad i\gamma^3 = \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{vmatrix}, \qquad \gamma^5\gamma^3 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix},$$

$$2\sigma^{01} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, \qquad 2\sigma^{02} = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{vmatrix}, \qquad 2\sigma^{03} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

$$2i\sigma^{12} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \qquad 2i\sigma^{23} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \qquad 2i\sigma^{31} = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{vmatrix}.$$

All these 15 matrices Λ_i are of Gell-Mann type: they have a zero-trace, they are Hermitian, besides, their squares are unite:

$$\mathrm{Sp}\,\Lambda = 0, \qquad (\Lambda)^2 = I, \qquad (\Lambda)^+ = \Lambda, \qquad \Lambda \in \{\Lambda_k : k = 1, \ldots, 15\}.$$

Exponential function of any of them equals to

$$U = e^{ia\Lambda} = \cos a + i \sin a\Lambda, \qquad \det e^{ia\Lambda} = +1, \qquad U^+ = U^{-1}, \qquad a \in R.$$

Evidently, multiplying of such 15 elementary unitary matrices (at real parameters x_i) results in an unitary matrix

$$U = e^{ia_1\Lambda_1} e^{ia_2\Lambda_2} \cdots e^{ia_{14}\Lambda_{14}} e^{ia_{l5}\Lambda_{l5}}.$$

At this there arise 15 generalized angle-variables $a_1, \ldots, a_{15}$. Evident advantage of this approach is its simplicity, and evident defect consists in the following: we do not know any simple group multiplication rule for these angles.

It should be noted that the basis λ_i used in [69] substantially differs from the above Dirac basis Λ_i – this peculiarity is closely connected with distinguishing $SU(3)$ in $SU(4)$. This problem is evidently related to the task of distinguishing $GL(3, C)$ in $GL(4, C)$ as well.

In order to have possibility to compare two approaches we need exact connection between λ_i and Λ_i. In [69] the following Gell-Mann basis for $SU(4)$ were used:

$$\lambda_1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad \lambda_2 = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad \lambda_3 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\lambda_4 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad \lambda_5 = \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad \lambda_6 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\lambda_7 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad \lambda_9 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix},$$

$$\lambda_{10} = \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \qquad \lambda_{11} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \lambda_{12} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix},$$

$$\lambda_{13} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \qquad \lambda_{14} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{vmatrix}, \qquad \lambda_{15} = \frac{1}{\sqrt{6}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix}. \tag{6.2}$$

All the λ excluding λ_8, λ_{15} possess the same property:

$$\lambda_i^3 = +\lambda_i, \qquad i \neq 8, 15.$$

The minimal polynomials for λ_8, λ_{15} can be easily found. Indeed,

$$(\lambda_8)^2 = \frac{1}{3} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad (\lambda_8)^3 = \frac{1}{3\sqrt{3}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

therefore

$$(\lambda_8)^3 = \frac{2}{3}\lambda_8 - \frac{1}{\sqrt{3}}(\lambda_8)^2.$$

Analogously, for λ_{15} we have

$$(\lambda_{15})^2 = \frac{1}{6} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{vmatrix}, \qquad (\lambda_{15})^3 = \frac{1}{6\sqrt{6}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -27 \end{vmatrix},$$

and

$$(\lambda_{15})^3 = \frac{1}{2}\lambda_{15} - \frac{2}{\sqrt{6}}(\lambda_{15})^2.$$

Comparing Λ_i and λ_i one can readily derive the linear combinations:

$$\begin{array}{lll} \gamma^0 + \gamma^5\gamma^3 = 2\lambda_4, & \gamma^0 - \gamma^5\gamma^3 = 2\lambda_{11}, & i\gamma^5\gamma^0 + i\gamma^3 = 2\lambda_5, \\ i\gamma^5\gamma^0 - i\gamma^3 = 2\lambda_{12}, & \gamma^5\gamma^1 + i\gamma^2 = 2\lambda_6, & \gamma^5\gamma^1 - i\gamma^2 = 2\lambda_9, \\ i\gamma^1 + \gamma^5\gamma^2 = 2\lambda_{10}, & i\gamma^1 - \gamma^5\gamma^2 = 2\lambda_7, & 2\sigma^{01} + 2i\sigma^{23} = 2\lambda_1, \\ 2\sigma^{01} - 2i\sigma^{23} = -2\lambda_{13}, & 2\sigma^{02} + 2i\sigma^{31} = 2\lambda_2, & 2\sigma^{02} - 2i\sigma^{31} = -2\lambda_{14}, \end{array}$$

and additional six combinations

$$2\sigma^{03} + 2i\sigma^{12} = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad 2\sigma^{03} - 2i\sigma^{12} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & +2 \end{vmatrix},$$

$$\gamma^5 + 2\sigma^{03} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}, \qquad \gamma^5 - 2\sigma^{03} = \begin{vmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\gamma^5 + 2i\sigma^{12} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad \gamma^5 - 2i\sigma^{12} = \begin{vmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix},$$

they should contain three linearly independent matrices. Those three linearly independent matrices might be chosen in different ways. Let us introduce the notation:

$$a = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right|, \qquad b = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right|, \qquad c = \left| \begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right|,$$

$$A = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{array} \right|, \qquad B = \left| \begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right|, \qquad C = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right|.$$

The matrices a, b, c have the 3×3 blocks different from zero, so they could be generators for $SU(3)$ transformations; whereas A, B, C may be generators only of the group $SU(4)$. All six matrices a, b, c, A, B, C have the same minimal polynomial:

$$\lambda^3 = \lambda.$$

Linear space to which these six matrices a, b, c, A, B, C belong is 3-dimensional. Indeed, one easily obtains

$$c = b - a, \qquad C - A = b, \qquad C - B = a, \qquad B - A = c,$$
$$\text{basis } \{a, b, C\} \qquad \Longrightarrow \qquad c = b - a, \qquad A = C - b, \qquad B = C - a.$$

These relations can be rewritten differently

$$a = b - c, \qquad C - A = b, \qquad C - B = b - c, \qquad B - A = c,$$
$$\text{basis } \{b, c, A\} \qquad \Longrightarrow \qquad a = b - c, \qquad C = A + b, \qquad B = A + c$$

or

$$b = a + c, \qquad C - A = b, \qquad C - B = b - c, \qquad B - A = c,$$
$$\text{basis } \{a, c, B\} \qquad \Longrightarrow \qquad b = a + c, \qquad C = B + a, \qquad A = B - c.$$

One should note that in the basis λ_i (6.2) the corresponding three linearly independent elements λ_3, λ_8, λ_{15} are taken as follows:

$$\lambda_3 = \tfrac{1}{2} 2\sigma^{03} + \tfrac{1}{2} 2i\sigma^{12}, \qquad \lambda_8 = -\tfrac{1}{\sqrt{3}}\gamma^5 + \tfrac{1}{2\sqrt{3}} 2\sigma^{03} - \tfrac{1}{2\sqrt{3}} 2i\sigma^{12},$$
$$\lambda_{15} = -\tfrac{1}{\sqrt{6}}\gamma^5 - \tfrac{1}{\sqrt{6}} 2\sigma^{03} + \tfrac{1}{\sqrt{6}} 2i\sigma^{12},$$

their minimal polynomials look

$$(\lambda_3)^3 = \lambda_3, \qquad (\lambda_8)^3 = \tfrac{2}{3}\lambda_8 - \tfrac{1}{\sqrt{3}}(\lambda_8)^2, \qquad (\lambda_{15})^3 = \tfrac{1}{2}\lambda_{15} - \tfrac{2}{\sqrt{6}}(\lambda_{15})^2.$$

It is the matter of simple calculation to find relationships between λ_3, λ_8, λ_{15} and the basis $\{a, b, C\}$:

$$\lambda_3 = a, \qquad \lambda_8 = \tfrac{1}{\sqrt{3}} a - \tfrac{2}{\sqrt{3}} b, \qquad \lambda_{15} = \tfrac{1}{\sqrt{6}} a + \tfrac{1}{\sqrt{6}} b - \tfrac{3}{\sqrt{6}} C. \tag{6.3}$$

In the following we will use the notation

$$a = \lambda_3, \qquad b = \lambda_8', \qquad C = \lambda_{15}',$$

so the previous formulas (6.3) will read $(\{\lambda_3, \lambda_8, \lambda_{15}\} \Longleftrightarrow \{\lambda_3, \lambda'_8, \lambda'_{15}\})$

$$\lambda_3 = \lambda_3, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\lambda_3 - \frac{2}{\sqrt{3}}\lambda'_8, \qquad \lambda_{15} = \frac{1}{\sqrt{6}}\lambda_3 + \frac{1}{\sqrt{6}}\lambda'_8 - \frac{3}{\sqrt{6}}\lambda'_{15}.$$

The inverse relations are

$$\lambda_3 = \lambda_3, \qquad \lambda'_8 = \frac{1}{2}\lambda_3 - \frac{\sqrt{3}}{2}\lambda_8, \qquad \lambda'_{15} = -\frac{\sqrt{6}}{3}\lambda_{15} + \frac{1}{2}\lambda_3 - \frac{1}{2\sqrt{3}}\lambda_8. \tag{6.4}$$

Now, starting with the linear decomposition of $G \in GL(4,C)$ the in Dirac basis (2.1):

$$\begin{aligned} G &= a_0 I + ib_0\gamma^5 + iA_0\gamma^0 + iA_k\gamma^k + B_0\gamma^0\gamma^5 + B_k\gamma^k\gamma^5 \\ &\quad + a_k 2\sigma_{0k} + b_1 2\sigma_{23} + b_2 2\sigma_{31} + b_3 2\sigma_{12} \\ &= a_0 I + ib_0\gamma^5 + iA_0\gamma^0 + A_k(i\gamma^k) + iB_0(i\gamma^5\gamma^0) - B_k(\gamma^5\gamma^k) \\ &\quad + a_k(2\sigma_{0k}) - ib_1(2i\sigma_{23}) - ib_2(2i\sigma_{31}) - ib_3(2i\sigma_{12}), \end{aligned}$$

with the help of the formulas

$$\begin{aligned} &\gamma^0 + \gamma^5\gamma^3 = 2\lambda_4, \qquad \gamma^0 - \gamma^5\gamma^3 = 2\lambda_{11}, \qquad i\gamma^5\gamma^0 + i\gamma^3 = 2\lambda_5, \\ &i\gamma^5\gamma^0 - i\gamma^3 = 2\lambda_{12}, \qquad \gamma^5\gamma^1 + i\gamma^2 = 2\lambda_6, \qquad \gamma^5\gamma^1 - i\gamma^2 = 2\lambda_9, \\ &i\gamma^1 + \gamma^5\gamma^2 = 2\lambda_{10}, \qquad i\gamma^1 - \gamma^5\gamma^2 = 2\lambda_7, \qquad 2\sigma^{01} + 2i\sigma^{23} = 2\lambda_1, \\ &2\sigma^{01} - 2i\sigma^{23} = -2\lambda_{13}, \qquad 2\sigma^{02} + 2i\sigma^{31} = 2\lambda_2, \qquad 2\sigma^{02} - 2i\sigma^{31} = -2\lambda_{14}, \\ &2\sigma^{03} + 2i\sigma^{12} = 2\lambda_3, \qquad 2\sigma^{03} - 2i\sigma^{12} = 2(\lambda'_{15} - \lambda'_8), \qquad \gamma^5 + 2\sigma^{03} = 2\lambda'_{15}, \\ &\gamma^5 - 2\sigma^{03} = 2(\lambda'_8 - \lambda_3), \qquad \gamma^5 + 2i\sigma^{12} = 2\lambda'_8, \qquad \gamma^5 - 2i\sigma^{12} = 2(\lambda'_{15} - \lambda_3) \end{aligned}$$

and inverse ones

$$\begin{aligned} &\gamma^0 = \lambda_4 + \lambda_{11}, \qquad \gamma^5\gamma^3 = \lambda_4 - \lambda_{11}, \qquad i\gamma^5\gamma^0 = \lambda_5 + \lambda_{12}, \qquad i\gamma^3 = \lambda_5 - \lambda_{12}, \\ &\gamma^5\gamma^1 = \lambda_6 + \lambda_9, \qquad i\gamma^2 = \lambda_6 - \lambda_9, \qquad i\gamma^1 = \lambda_{10} + \lambda_7, \qquad \gamma^5\gamma^2 = \lambda_{10} - \lambda_7, \\ &2i\sigma^{23} = \lambda_1 + \lambda_{13}, \qquad 2\sigma^{01} = \lambda_1 - \lambda_{13}, \qquad 2i\sigma^{31} = \lambda_2 + \lambda_{14}, \qquad 2\sigma^{02} = \lambda_2 - \lambda_{14}, \\ &2\sigma^{03} = \lambda_3 - (\lambda'_8 - \lambda'_{15}), \qquad 2i\sigma^{12} = \lambda_3 + (\lambda'_8 - \lambda'_{15}), \qquad \gamma^5 = -\lambda_3 + (\lambda'_8 + \lambda'_{15}), \end{aligned}$$

we will arrive at

$$\begin{aligned} G &= a_0 I + (a_1 - ib_1)\lambda_1 + (a_2 - ib_2)\lambda_2 + (iA_0 - B_3)\lambda_4 + (A_3 + iB_0)\lambda_5 + (A_2 - B_1)\lambda_6 \\ &\quad + (A_1 + B_2)\lambda_7 + (a_3 - ib_3 - ib_0)\lambda_3 + (-ib_3 + ib_0 - a_3)\lambda'_8 + (ib_0 + a_3 + ib_3)\lambda'_{15} \\ &\quad + (-B_1 - A_2)\lambda_9 + (A_1 - B_2)\lambda_{10} + (iA_0 + B_3)\lambda_{11} + (-A_3 + iB_0)\lambda_{12} \\ &\quad + (-a_1 - ib_1)\lambda_{13} + (-a_2 - ib_2)\lambda_{14}. \end{aligned}$$

In variables (k, m, l, n) (see (2.3), (2.4))

$$\begin{aligned} &B_0 - iA_0 = l_0, \qquad B_j - iA_j = l_j, \qquad B_0 + iA_0 = n_0, \qquad B_j + iA_j = n_j, \\ &a_0 - ib_0 = k_0, \qquad a_j - ib_j = k_j, \qquad a_0 + ib_0 = m_0, \qquad a_j + ib_j = m_j \end{aligned}$$

the previous expansion looks

$$\begin{aligned} G &= \frac{1}{2}(k_0 + m_0)I + k_1\lambda_1 + k_2\lambda_2 + \frac{1}{2}[(n_0 - n_3) - (l_0 + l_3)]\lambda_4 + \frac{1}{2i}[-(n_0 - n_3) - (l_0 + l_3)]\lambda_5 \\ &\quad + \frac{1}{2}[-(n_1 + in_2) - (l_1 - il_2)]\lambda_6 + \frac{1}{2i}[(n_1 + in_2) - (l_1 - il_2)]\lambda_7 + \left[k_3 + \frac{1}{2}(k_0 - m_0)\right]\lambda_3 \\ &\quad + \left[-m_3 + \frac{1}{2}(m_0 - k_0)\right]\lambda'_8 + \left[m_3 + \frac{1}{2}(m_0 - k_0)\right]\lambda'_{15} + \frac{1}{2}[-(n_1 - in_2) - (l_1 + il_2)]\lambda_9 \\ &\quad + \frac{1}{2i}[(n_1 - in_2) - (l_1 + il_2)]\lambda_{10} + \frac{1}{2}[(n_0 + n_3) - (l_0 - l_3)]\lambda_{11} \end{aligned}$$

$$+ \tfrac{1}{2i}[-(n_0+n_3)-(l_0-l_3)]\lambda_{12} - m_1\lambda_{13} - m_2\lambda_{14}.$$

Let coefficients at λ_9, λ_{10}, λ_{11}, λ_{12}, λ_{13}, λ_{14} be equal to zero:

$$(n_1 - in_2) + (l_1 + il_2) = 0, \qquad (n_1 - in_2) - (l_1 + il_2) = 0, \qquad (n_0 + n_3) - (l_0 - l_3) = 0,$$
$$(n_0 + n_3) + (l_0 - l_3) = 0, \qquad m_1 = 0, \qquad m_2 = 0. \tag{6.5}$$

Note that we do not require vanishing of the coefficient at λ'_{15}:

$$\lambda'_{15}\left(m_3 + \frac{m_0 - k_0}{2}\right) \neq 0.$$

As a result we have a subgroup of 4×4 matrices defined by 10 complex parameters. At this four elements are diagonal matrices:

$$I, \qquad \lambda_3, \qquad \lambda'_8, \qquad \lambda'_{15},$$

all other matrices have on the diagonal only zeros. Equations (6.5) give

$$in_2 = n_1, \qquad n_3 = -n_0, \qquad il_2 = -l_1, \qquad l_3 = l_0, \qquad m_1 = 0, \qquad m_2 = 0, \tag{6.6}$$

so that any matrix $G(k_a, n_0, n_1, l_0, l_1, m_0, m_3)$ is decomposed according to

$$G = k_1\lambda_1 + k_2\lambda_2 + (n_0 - l_0)\lambda_4 + i(n_0 + l_0)\lambda_5 + (-n_1 - l_1)\lambda_6 + i(-n_1 + l_1)\lambda_7 + \tfrac{1}{2}(k_0 + m_0)I$$
$$+ \left[k_3 + \tfrac{1}{2}(k_0 - m_0)\right]\lambda_3 + \left[-m_3 + \tfrac{1}{2}(m_0 - k_0)\right]\lambda'_8 + \left[m_3 + \tfrac{1}{2}(m_0 - k_0)\right]\lambda'_{15}. \tag{6.7}$$

Explicit form of the matrices parameterized by (6.7) can be obtained from representation for arbitrary element of $GL(4, C)$ (2.6)

$$G(k,m,n,l) = \left|\begin{array}{cccc} +(k_0+k_3) & +(k_1-ik_2) & +(n_0-n_3) & -(n_1-in_2) \\ +(k_1+ik_2) & +(k_0-k_3) & -(n_1+in_2) & +(n_0+n_3) \\ -(l_0+l_3) & -(l_1-il_2) & +(m_0-m_3) & -(m_1-im_2) \\ -(l_1+il_2) & -(l_0-l_3) & -(m_1+im_2) & +(m_0+m_3) \end{array}\right|$$

with additional restrictions (6.6):

$$G = \left|\begin{array}{cccc} k_0+k_3 & k_1-ik_2 & +2n_0 & 0 \\ k_1+ik_2 & k_0-k_3 & -2n_1 & 0 \\ -2l_0 & -2l_1 & m_0-m_3 & 0 \\ 0 & 0 & 0 & m_0+m_3 \end{array}\right|. \tag{6.8}$$

If additionally one requires $m_0 + m_3 = 1$, then

$$G = \left|\begin{array}{cccc} k_0+k_3 & k_1-ik_2 & +2n_0 & 0 \\ k_1+ik_2 & k_0-k_3 & -2n_1 & 0 \\ -2l_0 & -2l_1 & 1-2m_3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right|$$

with decomposition rule

$$G = k_1\lambda_1 + k_2\lambda_2 + (n_0 - l_0)\lambda_4 + i(n_0 + l_0)\lambda_5 + (-n_1 - l_1)\lambda_6 + i(-n_1 + l_1)\lambda_7$$
$$+ \tfrac{1}{2}(1 + k_0 - m_3)I + \left[k_3 + \tfrac{1}{2}(k_0 + m_3 - 1)\right]\lambda_3$$
$$+ \left[-m_3 + \tfrac{1}{2}(1 - m_3 - k_0)\right]\lambda'_8 + \tfrac{1}{2}(1 - k_0 + m_3)\lambda'_{15}. \tag{6.9}$$

In the diagonal part of (6.9), there are four independent matrices because equation (6.9) represents 4×4 matrix with the structure

$$G \sim \begin{vmatrix} GL(3,C) & 0 \\ 0 & 1 \end{vmatrix}.$$

To deal with the matrices from $GL(3,C)$, in the diagonal part of (6.9) one should separate only a 3×3 block:

$$\begin{aligned}
\mathrm{Diag} &= \tfrac{1}{2}(1+k_0-m_3)I^{(3)} + \left[k_3 + \tfrac{1}{2}(k_0+m_3-1)\right]\lambda_3^{(3)} \\
&\quad + \left[-m_3 + \tfrac{1}{2}(1-m_3-k_0)\right]\lambda_8'^{(3)} + \tfrac{1}{2}(1-k_0+m_3)\lambda_{15}'^{(3)} = \\
&= \tfrac{1}{2}(1+k_0-m_3)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \left[k_3+\tfrac{1}{2}(k_0+m_3-1)\right]\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \\
&\quad + \left[-m_3+\tfrac{1}{2}(1-m_3-k_0)\right]\begin{vmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \tfrac{1}{2}(1-k_0+m_3)\begin{vmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}.
\end{aligned}$$

Resolving $\lambda_{15}'^{(3)}$ in terms of $I^{(3)}$, $\lambda_3^{(3)}$, $\lambda_8'^{(3)}$:

$$\lambda_{15}'^{(3)} = -\tfrac{1}{3}I^{(3)} + \tfrac{1}{3}\lambda_3^{(3)} + \tfrac{1}{3}\lambda_8'^{(3)},$$

we arrive at a 3-term relation:

$$\mathrm{Diag} = \tfrac{1}{3}(1+2k_0-2m_3)I^{(3)} + \left[k_3+\tfrac{1}{3}(k_0+2m_3-1)\right]\lambda_3^{(3)} + \tfrac{1}{3}(-4m_3-2k_0+2)\lambda_8'^{(3)}.$$

The group law for parameters of $SL(3,C)$ has the form (the notation $M = 1-2m_3$ is used)

$$\begin{aligned}
&k_0'' = k_0'k_0 + \boldsymbol{k}'\boldsymbol{k} + 2(-n_0'l_0 + n_1'l_1), \\
&(\boldsymbol{k}'')_1 = (k_0'\boldsymbol{k} + \boldsymbol{k}'k_0 + i\boldsymbol{k}'\times\boldsymbol{k})_1 + 2(-n_0'l_1 + n_1'l_0), \\
&(\boldsymbol{k}'')_2 = (k_0'\boldsymbol{k} + \boldsymbol{k}'k_0 + i\boldsymbol{k}'\times\boldsymbol{k})_2 + 2(-in_0'l_1 - in_1'l_0), \\
&(\boldsymbol{k}'')_3 = (k_0'\boldsymbol{k} + \boldsymbol{k}'k_0 + i\boldsymbol{k}'\times\boldsymbol{k})_3 + 2(-n_0'l_0 - n_1'l_1), \\
&n_0'' = (k_0'+k_3')n_0 - (k_1'-ik_2')n_1 + n_0'M, \\
&n_1'' = (k_0'-k_3')n_1 - (k_1'+ik_2')n_0 + n_1'M, \\
&l_0'' = l_0'(k_0+k_3) + l_1'(k_1+ik_2) + M'l_0, \\
&l_1'' = l_0'(k_1-ik_2) + l_1'(k_0-k_3) + M'l_1, \\
&M'' = M'M - 4(l_0'n_0 - l_1'n_1).
\end{aligned}$$

These rules determine multiplication of the matrices

$$G = \begin{vmatrix} k_0+k_3 & k_1-ik_2 & +2n_0 & 0 \\ k_1+ik_2 & k_0-k_3 & -2n_1 & 0 \\ -2l_0 & -2l_1 & M & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

If additionally, in equation (6.8) one requires

$$n_0 = 0, \qquad n_1 = 0, \qquad l_0 = 0, \qquad l_1 = 0, \qquad m_3 = 0, \qquad m_0 = 1,$$

then

$$G = \left| \begin{array}{cccc} k_0 + k_3 & k_1 - ik_2 & 0 & 0 \\ k_1 + ik_2 & +k_0 - k_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| ,$$

with the decomposition rule

$$G = k_1\lambda_1 + k_2\lambda_2 + \tfrac{1}{2}(1 + k_0)I + \left[k_3 - \tfrac{1}{2}(1 - k_0)\right] \lambda_3 + \tfrac{1}{2}(1 - k_0)\lambda'_8 + \tfrac{1}{2}(1 - k_0)\lambda'_{15}.$$

One can readily verify that the 2×2 block is given by

$$G^{(2)}(k_a) = k_0 I^{(2)} + k_1\lambda_1^{(2)} + k_2\lambda_2^{(2)} + k_3\lambda_3^{(2)}.$$

7 On the multiplication law for $GL(4, C)$ in Dirac basis

In the Gell-Mann basis λ_i, an element of $GL(4, C)$ is

$$\begin{aligned} G = {} & a_0 I + (a_1 - ib_1)\lambda_1 + (a_2 - ib_2)\lambda_2 + (iA_0 - B_3)\lambda_4 + (A_3 + iB_0)\lambda_5 + (A_2 - B_1)\lambda_6 \\ & + (A_1 + B_2)\lambda_7 + (a_3 - ib_3 - ib_0)\lambda_3 + (-ib_3 + ib_0 - a_3)\lambda'_8 + (-B_1 - A_2)\lambda_9 \\ & + (A_1 - B_2)\lambda_{10} + (iA_0 + B_3)\lambda_{11} + (-A_3 + iB_0)\lambda_{12} + (-a_1 - ib_1)\lambda_{13} \\ & + (-a_2 - ib_2)\lambda_{14} + (ib_0 + a_3 + ib_3)\lambda'_{15}, \end{aligned}$$

or in variables (k, m, l, n):

$$\begin{aligned} G = {} & \tfrac{1}{2}(k_0 + m_0)I + k_1\lambda_1 + k_2\lambda_2 + \tfrac{1}{2}[(n_0 - n_3) - (l_0 + l_3)]\lambda_4 + \tfrac{1}{2i}[-(n_0 - n_3) - (l_0 + l_3)]\lambda_5 \\ & + \tfrac{1}{2}[-(n_1 + in_2) - (l_1 - il_2)]\lambda_6 + \tfrac{1}{2i}[(n_1 + in_2) - (l_1 - il_2)]\lambda_7 + \left[k_3 + \tfrac{1}{2}(k_0 - m_0)\right] \lambda_3 \\ & + \left[-m_3 + \tfrac{1}{2}(m_0 - k_0)\right] \lambda'_8 + \tfrac{1}{2}[-(n_1 - in_2) - (l_1 + il_2)]\lambda_9 \\ & + \tfrac{1}{2i}[(n_1 - in_2) - (l_1 + il_2)]\lambda_{10} + \tfrac{1}{2}[(n_0 + n_3) - (l_0 - l_3)]\lambda_{11} \\ & + \tfrac{1}{2i}[-(n_0 + n_3) - (l_0 - l_3)]\lambda_{12} - m_1\lambda_{13} - m_2\lambda_{14} + \left[m_3 + \tfrac{1}{2}(m_0 - k_0)\right] \lambda'_{15}. \end{aligned}$$

The problem is to establish the multiplication rule $G'' = G'G$ in λ-basis:

$$x''_k\lambda_k = x'_m\lambda_m x_n\lambda_n = x'_m x_n\lambda_m\lambda_n.$$

As by definition the relationships $\lambda_m\lambda_n = e_{mnk}\lambda_k$ must hold, the multiplication rule is

$$x''_k = e_{mnk}x'_m x_n. \tag{7.1}$$

The main claim is that the all properties of the $GL(4, C)$ with all its subgroups are determined by the bilinear function (7.1), the latter is described by structure constants e_{mnk}. It is evident that these group constants should be simpler in the Dirac basis Λ_i than in the basis λ_i. Our next task is to establish the multiplication law $G'' = G'G$ in Λ-basis:

$$\Lambda_m\Lambda_n = E_{mnk}\Lambda_k, \qquad X''_k = E_{mnk}X'_m X_n.$$

Before searching for structural constants E_{mnk}, let us introduce a special way to list the Dirac basis Λ_i:

$$\begin{array}{lllll} \alpha_1 = \gamma^0\gamma^2, & \alpha_2 = i\gamma^0\gamma^5, & \alpha_3 = \gamma^5\gamma^2 & \alpha_i^2 = I, & \alpha_1\alpha_2 = i\alpha_3, \alpha_2\alpha_1 = -i\alpha_1, \\ \beta_1 = i\gamma^3\gamma^1, & \beta_2 = i\gamma^3, & \beta_3 = i\gamma^1, & \beta_i^2 = I, & \beta_1\beta_2 = i\beta_3, \beta_2\beta_1 = -i\beta_3, \end{array}$$

these two set commute with each others $\alpha_j\beta_k = \beta_k\alpha_j$, and their multiplications provides us with 9 remaining basis elements of $\{\Lambda_k\}$:

$$
\begin{aligned}
&A_1 = \alpha_1\beta_1 = -\gamma^5, && B_1 = \alpha_1\beta_2 = \gamma^5\gamma^1, && C_1 = \alpha_1\beta_3 = \gamma^3\gamma^5,\\
&A_2 = \alpha_2\beta_1 = -i\gamma^2, && B_2 = \alpha_2\beta_2 = -i\gamma^1\gamma^2, && C_2 = \alpha_2\beta_3 = -i\gamma^2\gamma^3,\\
&A_3 = \alpha_3\beta_1 = \gamma^0, && B_3 = \alpha_3\beta_2 = \gamma^0\gamma^1, && C_3 = \alpha_3\beta_3 = \gamma^0\gamma^3.
\end{aligned} \tag{7.2}
$$

The multiplication rules for basic elements

$$\alpha_1,\ \alpha_2,\ \alpha_3, \qquad \beta_1,\ \beta_2,\ \beta_3, \qquad A_1,\ A_2,\ A_3, \qquad B_1,\ B_2,\ B_3, \qquad C_1,\ C_2,\ C_3,$$

are

	α_1	α_2	α_3
α_1	I	$i\alpha_3$	$-i\alpha_2$
α_2	$-i\alpha_3$	I	$i\alpha_1$
α_3	$i\alpha_2$	$-i\alpha_1$	I

	β_1	β_2	β_3
α_1	A_1	B_1	C_1
α_2	A_2	B_2	C_2
α_3	A_3	B_3	C_3

	A_1	A_2	A_3
α_1	β_1	iA_3	$-iA_2$
α_2	$-iA_3$	β_1	iA_1
α_3	iA_2	$-iA_1$	β_1

	B_1	B_2	B_3
α_1	β_2	iB_3	$-iB_2$
α_2	$-iB_3$	β_2	iB_1
α_3	iB_2	$-iB_1$	β_2

	C_1	C_2	C_3
α_1	β_3	iC_3	$-iC_2$
α_2	$-iC_3$	β_3	iC_1
α_3	iC_2	$-iC_1$	β_3

	α_1	α_2	α_3
β_1	A_1	A_2	A_3
β_2	B_1	B_2	B_3
β_3	C_1	C_2	C_3

	β_1	β_2	β_3
β_1	I	$i\beta_3$	$-i\beta_2$
β_2	$-i\beta_3$	I	$i\beta_1$
β_3	$i\beta_2$	$-i\beta_1$	I

	A_1	A_2	A_3
β_1	α_1	α_2	α_3
β_2	$-iC_1$	$-iC_2$	$-iC_3$
β_3	iB_1	iB_2	iB_3

	B_1	B_2	B_3
β_1	iC_1	iC_2	iC_3
β_2	α_1	α_2	α_3
β_3	$-iA_1$	$-iA_2$	$-iA_3$

	C_1	C_2	C_3
β_1	$-iB_1$	$-iB_2$	$-iB_3$
β_2	iA_1	iA_2	iA_3
β_3	α_1	α_2	α_3

	α_1	α_2	α_3
A_1	β_1	iA_3	$-iA_2$
A_2	$-iA_3$	β_1	iA_1
A_3	iA_2	$-iA_1$	β_1

	β_1	β_2	β_3
A_1	α_1	iC_1	$-iB_1$
A_2	α_2	iC_2	$-iB_2$
A_3	α_3	iC_3	$-iB_3$

	A_1	A_2	A_3
A_1	I	$i\alpha_3$	$-i\alpha_2$
A_2	$-i\alpha_3$	I	$i\alpha_1$
A_3	$i\alpha_2$	$-i\alpha_1$	I

	B_1	B_2	B_3
A_1	$i\beta_3$	$-C_3$	C_2
A_2	C_3	$i\beta_3$	$-C_1$
A_3	$-C_2$	C_1	$i\beta_3$

	C_1	C_2	C_3
A_1	$-i\beta_2$	B_3	$-B_2$
A_2	$-B_3$	$-i\beta_2$	B_1
A_3	B_2	$-B_1$	$-i\beta_2$

	α_1	α_2	α_3
B_1	β_2	iB_3	$-iB_2$
B_2	$-iB_3$	β_2	iB_1
B_3	iB_2	$-iB_1$	β_2

	β_1	β_2	β_3
B_1	$-iC_1$	α_1	iA_1
B_2	$-iC_2$	α_2	iA_2
B_3	$-iC_3$	α_3	iA_3

	A_1	A_2	A_3
B_1	$-i\beta_3$	C_3	$-C_2$
B_2	$-C_3$	$-i\beta_3$	C_1
B_3	C_2	$-C_1$	$-i\beta_3$

	B_1	B_2	B_3
B_1	I	$i\alpha_3$	$-i\alpha_2$
B_2	$-i\alpha_3$	I	$i\alpha_1$
B_3	$i\alpha_2$	$-i\alpha_1$	I

	C_1	C_2	C_3
B_1	$i\beta_1$	$-A_3$	A_2
B_2	A_3	$i\beta_1$	$-A_1$
B_3	$-A_2$	A_1	$i\beta_1$

	α_1	α_2	α_3
C_1	β_3	iC_3	$-iC_2$
C_2	$-iC_3$	β_3	iC_1
C_3	iC_2	$-iC_1$	β_3

	β_1	β_2	β_3
C_1	iB_1	$-iA_1$	α_1
C_2	iB_2	$-iA_2$	α_2
C_3	iB_3	$-iA_3$	α_3

	A_1	A_2	A_3
C_1	$i\beta_2$	$-B_3$	B_2
C_2	B_3	$i\beta_2$	$-B_1$
C_3	$-B_2$	B_1	$i\beta_2$

	B_1	B_2	B_3
C_1	$-i\beta_1$	A_3	$-A_2\beta_1$
C_2	$-A_3$	$-i\beta_1$	A_1
C_3	A_2	$-A_1$	$-i\beta_1$

$$
\begin{array}{c|ccc}
 & C_1 & C_2 & C_3 \\
\hline
C_1 & I & i\alpha_3 & -i\alpha_2 \\
C_2 & -i\alpha_3 & I & i\alpha_1 \\
C_3 & i\alpha_2 & -i\alpha_1 & I
\end{array}
\qquad (7.3)
$$

These relations provide us with simple formulas for fifteen coordinates of the element of $GL(4,C)$

$$
\begin{aligned}
& G = \gamma I + a_j\alpha_j + b_j\beta_j + X_jA_j + Y_jB_j + Z_jC_j, \\
& \gamma = \tfrac{1}{4}\mathrm{Sp}\, G, \qquad a_j = \tfrac{1}{4}\mathrm{Sp}\,\alpha_j G, \qquad b_j = \tfrac{1}{4}\mathrm{Sp}\,\beta_j G, \\
& X_j = \tfrac{1}{4}\mathrm{Sp}\, A_j G, \qquad Y_j = \tfrac{1}{4}\mathrm{Sp} B_j G, \qquad Z_j = \tfrac{1}{4}\mathrm{Sp}\, C_j G.
\end{aligned}
$$

With the use of relations (7.3) an explicit form of the group law for $(15+1)$ parameters can be found:

$$
\begin{aligned}
& (\gamma' I + a'_i\alpha_i + b'_i\beta_i + X'_iA_i + Y'_iB_i + Z'_iC_i)(\gamma I + a_j\alpha_j + b_j\beta_j + X_jA_j + Y_jB_j + Z_jC_j) \\
& \quad = \gamma'\gamma I + (\gamma' a_j + a'_j\gamma)\alpha_j + (\gamma' b_j + b'_j\gamma)\beta_j + (\gamma' X_j + X'_j\gamma)A_j + (\gamma' Y_j + Y'_i\gamma)B_j \\
& \quad + (\gamma' Z_j + Z'_j\gamma)C_j + a'_1(a_1 + a_2 i\alpha_3 - a_3 i\alpha_2) + a'_2(-a_1 i\alpha_3 + a_2 + a_3 i\alpha_1) \\
& \quad + a'_3(a_1 i\alpha_2 - a_2 i\alpha_1 + a_3) + a'_1(b_1A_1 + b_2B_1 + b_3C_1) + a'_2(b_1A_2 + b_2B_2 + b_3C_2) \\
& \quad + a'_3(b_1A_3 + b_2B_3 + b_3C_3) + a'_1(X_1\beta_1 + X_2 iA_3 - X_3 iA_2) \\
& \quad + a'_2(-X_1 iA_3 + X_2\beta_1 + X_3 iA_1) + a'_3(X_1 iA_2 - X_2 iA_1 + X_3\beta_1) \\
& \quad + a'_1(Y_1\beta_2 + Y_2 iB_3 - Y_3 iB_2) + a'_2(-Y_1 iB_3 + Y_2\beta_2 + Y_3 iB_1) \\
& \quad + a'_3(Y_1 iB_2 - Y_2 iB_1 + Y_3\beta_2) + a'_1(Z_1\beta_3 + Z_2 iC_3 - Z_3 iC_2) \\
& \quad + a'_2(-Z_1 iC_3 + Z_2\beta_3 + Z_3 iC_1) + a'_3(Z_1 iC_2 - Z_2 iC_1 + Z_3\beta_3) \\
& \quad + b'_1(a_1A_1 + a_2A_2 + a_3A_3) + b'_2(a_1B_1 + a_2B_2 + a_3B_3) + b'_3(a_1C_1 + a_2C_2 + a_3C_3) \\
& \quad + b'_1(b_1 + b_2 i\beta_3 - b_3 i\beta_2) + b'_2(-b_1 i\beta_3 + b_2 + b_3 i\beta_1) + b'_3(b_1 i\beta_2 - b_2 i\beta_1 + b_3) \\
& \quad + b'_1(X_1\alpha_1 + X_2\alpha_2 + X_3\alpha_3) + b'_2(-X_1 iC_1 - X_2 iC_2 - X_3 iC_3) \\
& \quad + b'_3(X_1 iB_1 + X_2 iB_2 + X_3 iB_3) + b'_1(Y_1 iC_1 + Y_2 iC_2 + Y_3 iC_3) \\
& \quad + b'_2(Y_1\alpha_1 + Y_2\alpha_2 + Y_3\alpha_3) + b'_3(-Y_1 iA_1 - Y_2 iA_2 - Y_3 iA_3) \\
& \quad + b'_1(-Z_1 iB_1 - Z_2 iB_2 - Z_3 iB_3) + b'_2(Z_1 iA_1 + Z_2 iA_2 + Z_3 iA_3) \\
& \quad + b'_3(Z_1\alpha_1 + Z_2\alpha_2 + Z_3\alpha_3) + X'_1(a_1\beta_1 + a_2 iA_3 - a_3 iA_2) \\
& \quad + X'_2(-a_1 iA_3 + a_2\beta_1 + a_3 iA_1) + X'_3(a_1 iA_2 - a_2 iA_1 + a_3\beta_1) \\
& \quad + X'_1(b_1\alpha_1 + b_2 iC_1 - b_3 iB_1) + X'_2(b_1\alpha_2 + b_2 iC_2 - b_3 iB_2) + X'_3(b_1\alpha_3 + b_2 iC_3 - b_3 iB_3) \\
& \quad + X'_1(X_1 + X_2 i\alpha_3 - X_3 i\alpha_2) + X'_2(-X_1 i\alpha_3 + X_2 + X_3 i\alpha_1) + X'_3(X_1 i\alpha_2 - X_2 i\alpha_1 + X_3) \\
& \quad + X'_1(Y_1 i\beta_3 - Y_2C_3 + Y_3C_2) + X'_2(Y_1C_3 + Y_2 i\beta_3 - Y_3C_1) + X'_3(-Y_1C_2 + Y_2C_1 + Y_3 i\beta_3) \\
& \quad + X'_1(-Z_1 i\beta_2 + Z_2B_3 - Z_3B_2) + X'_2(-Z_1B_3 - Z_2 i\beta_2 + Z_3B_1) \\
& \quad + X'_3(Z_1B_2 - Z_2B_1 - Z_3 i\beta_2) + Y'_1(a_1\beta_2 + a_2 iB_3 - a_3 iB_2) \\
& \quad + Y'_2(-a_1 iB_3 + a_2\beta_2 + a_3 iB_1) + Y'_3(a_1 iB_2 - a_2 iB_1 + a_3\beta_2) \\
& \quad + Y'_1(-b_1 iC_1 + b_2\alpha_1 + b_3 iA_1) + Y'_2(-b_1 iC_2 + b_2\alpha_2 + b_3 iA_2) \\
& \quad + Y'_3(-b_1 iC_3 + b_2\alpha_3 + b_3 iA_3) + Y'_1(-X_1 i\beta_3 + X_2C_3 - X_3C_2) \\
& \quad + Y'_2(-X_1C_3 - X_2 i\beta_3 + X_3C_1) + Y'_3(X_1C_2 - X_2C_1 - X_3 i\beta_3) \\
& \quad + Y'_1(Y_1 + Y_2 i\alpha_3 - Y_3 i\alpha_2) + Y'_2(-Y_1 i\alpha_3 + Y_2 + Y_3 i\alpha_1) + Y'_3(Y_1 i\alpha_2 - Y_2 i\alpha_1 + Y_3) \\
& \quad + Y'_1(Z_1 i\beta_1 - Z_2A_3 + Z_3A_2) + Y'_2(Z_1A_3 + Z_2 i\beta_1 - Z_3A_1) \\
& \quad + Y'_3(-Z_1A_2 + Z_2A_1 + Z_3 i\beta_1) + Z'_1(a_1\beta_3 + a_2 iC_3 - a_3 iC_2) \\
& \quad + Z'_2(-a_1 iC_3 + a_2\beta_3 + a_3 iC_1) + Z'_3(a_1 iC_2 - a_2 iC_1 + a_3\beta_3)
\end{aligned}
$$

$$
\begin{aligned}
&+ Z_1'(b_1 iB_1 - b_2 iA_1 + b_3\alpha_1) + Z_2'(b_1 iB_2 - b_2 iA_2 + b_3\alpha_2) + Z_3'(b_1 iB_3 - b_2 iA_3 + b_3\alpha_3)\\
&+ Z_1'(X_1 i\beta_2 - X_2 B_3 + X_3 B_2) + Z_2'(X_1 B_3 + X_2 i\beta_2 - X_3 B_1)\\
&+ Z_3'(-X_1 B_2 + X_2 B_1 + X_3 i\beta_2) + Z_1'(-Y_1 i\beta_1 + Y_2 A_3 - Y_3 A_2)\\
&+ Z_2'(-Y_1 A_3 - Y_2 i\beta_1 + Y_3 A_1) + Z_3'(Y_1 A_2 - Y_2 A_1 - Y_3 i\beta_1)\\
&+ Z_1'(Z_1 + Z_2 i\alpha_3 - Z_3 i\alpha_2) + Z_2'(-Z_1 i\alpha_3 + Z_2 + Z_3 i\alpha_1) + Z_3'(Z_1 i\alpha_2 - Z_2 i\alpha_1 + Z_3).
\end{aligned}
$$

From these relations we arrive at the following composition rules:

$$
\begin{aligned}
\gamma'' &= \gamma'\gamma + (a_1'a_1 + a_2'a_2 + a_3'a_3) + (b_1'b_1 + b_2'b_2 + b_3'b_3)\\
&\quad + (X_1'X_1 + X_2'X_2 + X_3'X_3) + (Y_1'Y_1 + Y_2'Y_2 + Y_3'Y_3) + (Z_1'Z_1 + Z_2'Z_2 + Z_3'Z_3),\\
a_1'' &= (\gamma'a_1 + a_1'\gamma) + (b_1'X_1 + b_2'Y_1 + b_3'Z_1) + (X_1'b_1 + Y_1'b_2 + Z_1'b_3)\\
&\quad + i(a_2'a_3 - a_3'a_2) + i(X_2'X_3 - X_3'X_2) + i(Y_2'Y_3 - Y_3'Y_2) + i(Z_2'Z_3 - Z_3'Z_2),\\
a_2'' &= (\gamma'a_2 + a_2'\gamma) + (b_1'X_2 + b_2'Y_2 + b_3'Z_2) + (X_2'b_1 + Y_2'b_2 + Z_2'b_3)\\
&\quad + i(a_3'a_1 - a_1'a_3) + i(X_3'X_1 - X_1'X_3) + i(Y_3'Y_1 - Y_1'Y_3) + i(Z_3'Z_1 - Z_1'Z_3),\\
a_3'' &= (\gamma'a_3 + a_3'\gamma) + (b_1'X_3 + b_2'Y_3 + b_3'Z_3) + (X_3'b_1 + Y_3'b_2 + Z_3'b_3)\\
&\quad + i(a_1'a_2 - a_2'a_1) + i(X_1'X_2 - X_2'X_1) + i(Y_1'Y_2 - Y_2'Y_1) + i(Z_1'Z_2 - Z_2'Z_1),\\
b_1'' &= \gamma'b_1 + b_1'\gamma + i(b_2'b_3 - b_3'b_2) + (a_1'X_1 + a_2'X_2 + a_3'X_3) + (X_1'a_1 + X_2'a_2 + X_3'a_3)\\
&\quad + i(Y_1'Z_1 + Y_2'Z_2 + Y_3'Z_3) - i(Z_1'Y_1 + Z_2'Y_2 + Z_3'Y_3),\\
b_2'' &= \gamma'b_2 + b_2'\gamma + i(b_3'b_1 - b_1'b_3) + (a_1'Y_1 + a_2'Y_2 + a_3'Y_3) + (Y_1'a_1 + Y_2'a_2 + Y_3'a_3)\\
&\quad + i(Z_1'X_1 + Z_2'X_2 + Z_3'X_3) - i(X_1'Z_1 + X_2'Z_2 + X_3'Z_3),\\
b_3'' &= \gamma'b_3 + \gamma b_3' + i(b_1'b_2 - b_2'b_1) + (a_1'Z_1 + a_2'Z_2 + a_3'Z_3) + (Z_1'a_1 + Z_2'a_2 + Z_3'a_3)\\
&\quad + i(X_1'Y_1 + X_2'Y_2 + X_3'Y_3) - i(Y_1'X_1 + Y_2'X_2 + Y_3'X_3),\\
X_1'' &= (\gamma'X_1 + \gamma X_1') + (a_1'b_1 + a_1b_1') + i(Y_1'b_3 - Y_1b_3') + i(b_2'Z_1 - b_2Z_1')\\
&\quad + i(a_2'X_3 - a_3'X_2) - i(a_2X_3' - a_3X_2') + (Z_2Y_3' - Z_3Y_2') + (Z_2'Y_3 - Z_3'Y_2),\\
X_2'' &= (\gamma'X_2 + \gamma X_2') + (a_2'b_1 + a_2b_1') + i(Y_2'b_3 - Y_2b_3') + i(b_2'Z_2 - b_2Z_2')\\
&\quad + i(a_3'X_1 - a_1'X_2) - i(a_3X_1' - a_1X_3') + (Z_3Y_1' - Z_1Y_3') + (Z_3'Y_1 - Z_1'Y_3),\\
X_3'' &= (\gamma'X_3 + \gamma X_3') + (a_3'b_1 + a_3b_1') + i(Y_3'b_3 - Y_3b_3') + i(b_2'Z_3 - b_2Z_3')\\
&\quad + i(a_1'X_2 - a_2'X_1) - i(a_1X_2' - a_2X_1') + (Z_1Y_2' - Z_2Y_1') + (Z_1'Y_2 - Z_2'Y_1),\\
Y_1'' &= (\gamma'Y_1 + \gamma Y_1') + (a_1'b_2 + a_1b_2') + i(Z_1'b_1 - Z_1b_1') + i(b_3'X_1 - b_3X_1')\\
&\quad + i(a_2'Y_3 - a_3'Y_2) - i(a_2Y_3' - a_3Y_2') + (X_2Z_3' - X_3Z_2') + (X_2'Z_3 - X_3'Z_2),\\
Y_2'' &= (\gamma'Y_2 + \gamma Y_2') + (a_2'b_2 + a_2b_2') + i(Z_2'b_1 - Z_2b_1') + i(b_3'X_2 - b_3X_2')\\
&\quad + i(a_3'Y_1 - a_1'Y_3) - i(a_3Y_1' - a_1Y_3') + (X_3Z_1' - X_1Z_3') + (X_3'Z_1 - X_1'Z_3),\\
Y_3'' &= (\gamma'Y_3 + \gamma Y_3') + (a_3'b_2 + a_3b_2') + i(Z_3'b_1 - Z_3b_1') + i(b_3'X_3 - b_3X_3')\\
&\quad + i(a_1'Y_2 - a_2'Y_1) - i(a_1Y_2' - a_2Y_1') + (X_1Z_2' - X_2Z_1') + (X_1'Z_2 - X_2'Z_1),\\
Z_1'' &= (\gamma'Z_1 + \gamma Z_1') + (a_1'b_3 + a_1b_3') + i(Y_1b_1' - Y_1'b_1) + i(X_1'b_2 - X_1b_2')\\
&\quad + i(a_2'Z_3 - a_3'Z_2) - i(a_2Z_3' - a_3Z_2') + (Y_2X_3' - Y_3X_2') + (Y_2'X_3 - Y_3'X_2),\\
Z_2'' &= (\gamma'Z_2 + \gamma Z_2') + (a_2'b_3 + a_2b_3') + i(Y_2b_1' - Y_2'b_1) + i(X_2'b_2 - X_2b_2')\\
&\quad + i(a_3'Z_1 - a_1'Z_3) - i(a_3Z_1' - a_1Z_3') + (Y_3X_1' - Y_1X_3') + (Y_3'X_1 - Y_1'X_3),\\
Z_3'' &= (\gamma'Z_3 + \gamma Z_3') + (a_3'b_3 + a_3b_3') + i(Y_3b_1' - Y_3'b_1) + i(X_3'b_2 - X_3b_2')\\
&\quad + i(a_1'Z_2 - a_2'Z_1) - i(a_1Z_2' - a_2Z_1') + (Y_1X_2' - Y_2X_1') + (Y_1'X_2 - Y_2'X_1). \qquad (7.4)
\end{aligned}
$$

With the help of the index notation

$$
\boldsymbol{X} = \boldsymbol{C}^{(1)}, \qquad \boldsymbol{Y} = \boldsymbol{C}^{(2)}, \qquad \boldsymbol{Z} = \boldsymbol{C}^{(3)},
$$

it is easy to see a cyclic symmetry in the above relationships:

$$
\begin{aligned}
\gamma'' &= \gamma'\gamma + a'_k a_k + b'_k b_k + C_k^{(1)'} C_k^{(1)} + C_k^{(2)'} C_k^{(2)} + C_k^{(3)'} C_k^{(3)}, \\
a''_k &= \gamma' a_k + \gamma a'_k + (b'_1 C_k^{(1)} + b_1 C_k^{(1)'}) + (b'_2 C_k^{(2)} + b_2 C_k^{(2)'}) + (b'_3 C_k^{(3)}) + b_3 C_k^{(3)'}) \\
&\quad + i\epsilon_{kln} a'_l a_n + i\epsilon_{kln} C_l^{(1)'} C_n^{(1)} + i\epsilon_{kln} C_l^{(2)'} C_n^{(2)} + i\epsilon_{kln} C_l^{(3)'} C_n^{(3)}, \\
b''_k &= \gamma' b_k + \gamma b'_k + i\epsilon_{kln} b'_l b_n + (a'_1 C_1^{(k)} + a_1 C_1^{(k)'}) + (a'_2 C_2^{(k)} + a_2 C_2^{(k)'}) \\
&\quad + (a'_3 C_3^{(k)} + a_3 C_3^{(k)'}) + i\epsilon_{kln} C_m^{(l)'} C_m^{(n)}, \\
C_k^{(1)} &= \gamma' C_k^{(1)} + \gamma C_k^{(1)'} + (a'_k b_1 + a_k b'_1) + i\epsilon_{(1)ln} (C_k^{(l)'} b_n - C_k^{(l)} b'_n) \\
&\quad + i\epsilon_{kln} (a'_l C_n^{(1)} - a_l C_n^{(1)'}) + \epsilon_{kln} (C_l^{(2)'} C_n^{(3)} + C_l^{(2)} C_n^{(3)'}), \\
C_k^{(2)} &= \gamma' C_k^{(2)} + \gamma C_k^{(2)'} + (a'_k b_2 + a_k b'_2) + i\epsilon_{(2)ln} (C_k^{(l)'} b_n - C_k^{(l)} b'_n) \\
&\quad + i\epsilon_{kln} (a'_l C_n^{(2)} - a_l C_n^{(2)'}) + \epsilon_{kln} (C_l^{(3)'} C_n^{(1)} + C_l^{(3)} C_n^{(1)'}), \\
C_k^{(3)} &= \gamma' C_k^{(3)} + \gamma C_k^{(3)'} + (a'_k b_3 + a_k b'_3) + i\epsilon_{(3)ln} (C_k^{(l)'} b_n - C_k^{(l)} b'_n) \\
&\quad + i\epsilon_{kln} (a'_l C_n^{(3)} - a_l C_n^{(3)'}) + \epsilon_{kln} (C_l^{(1)'} C_n^{(2)} + C_l^{(1)} C_n^{(2)'}).
\end{aligned} \tag{7.5}
$$

It is readily seen that these group multiplication laws (7.4), (7.5) permit 15 two-parametric subgroups:

$$
\begin{aligned}
(\gamma, a) \in \{&(\gamma, a_1), (\gamma, a_2), (\gamma, a_3),\ (\gamma, b_1), (\gamma, b_2), \gamma, b_3),\ (\gamma, X_1), (\gamma, X_2), (\gamma, X_3), \\
&(\gamma, Y_1), (\gamma, Y_2), (\gamma, Y_3),\ (\gamma, Z_1), (\gamma, Z_2), (\gamma, Z_3)\}
\end{aligned}
$$

with the same composition law:

$$
\gamma'' = \gamma'\gamma + a' a, \qquad a'' = \gamma' a + \gamma a',
$$

which in variables $\gamma = W\cos\phi$, $a = iW\sin\phi$ takes the form

$$
W'' = W'W, \qquad \alpha'' = \alpha' + \alpha.
$$

The variable W is determined by $\det G(W, \alpha) = W^4$, the choice $W = 1$ guarantees $\det G = +1$.

All 15 basis elements $\Lambda_{(\rho)} \in \{\alpha_k, \beta_k, A_k, B_k, C_k\}$ possess the same properties:

$$
\Lambda^+_{(\rho)} = \Lambda_{(\rho)}, \qquad \Lambda^2_{(\rho)} = I.
$$

Therefore, one can construct 15 different elementary unitary (at real valued parameters) matrices by one the same recipe:

$$
\begin{aligned}
U_{(\rho)} &= e^{i\phi_{(\rho)}\Lambda_{(\rho)}} = \cos\phi_{(\rho)} + i\sin\phi_{(\rho)}\Lambda_{(\rho)}, \\
U^+_{(\rho)} &= U^{-1}_{(\rho)} = e^{-i\phi_{(\rho)}\Lambda_{(\rho)}} = \cos\phi_{(\rho)} - i\sin\phi_{(\rho)}\Lambda_{(\rho)}.
\end{aligned}
$$

The whole set of unitary matrices $SU(4)$ may be constructed on the basis of a simple factorized formula:

$$
U = e^{i\phi_{(1)}\Lambda_{(1)}} \cdots e^{i\phi_{(15)}\Lambda_{(15)}}.
$$

The order of the factors is important. Every such order leads us to a definite parametrization for the group $SU(4)$ – all them seem to be equivalent.

In the end of the section let us write down the explicit form of these 15 elementary unitary transformations:

$$\alpha_1 = \begin{vmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{vmatrix}, \qquad \alpha_2 = \begin{vmatrix} 0 & i \\ -i & 0 \end{vmatrix}, \qquad \alpha_3 = \begin{vmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{vmatrix},$$
$$U_1^\alpha = \begin{vmatrix} \cos\phi + i\sin\phi\sigma_2 & 0 \\ 0 & \cos\phi - i\sin\phi\sigma_2 \end{vmatrix}, \qquad U_2^\alpha = \begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix},$$
$$U_3^\alpha = \begin{vmatrix} \cos\phi & i\sin\phi\sigma_2 \\ i\sin\phi\sigma_2 & \cos\phi \end{vmatrix}, \qquad \beta_1 = \begin{vmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{vmatrix}, \qquad \beta_2 = \begin{vmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{vmatrix},$$
$$\beta_3 = \begin{vmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{vmatrix}, \qquad U_1^\beta = \begin{vmatrix} \cos\phi + i\sin\phi\sigma_2 & 0 \\ 0 & \cos\phi + i\sin\phi\sigma_2 \end{vmatrix},$$
$$U_2^\beta = \begin{vmatrix} \cos\phi & \sin\phi\sigma_3 \\ -\sin\phi\sigma_3 & \cos\phi \end{vmatrix}, \qquad U_3^\beta = \begin{vmatrix} \cos\phi & \sin\phi\sigma_1 \\ -\sin\phi\sigma_1 & \cos\phi \end{vmatrix}, \qquad A_1 = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix},$$
$$A_2 = \begin{vmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{vmatrix}, \qquad A_3 = \begin{vmatrix} 0 & I \\ I & 0 \end{vmatrix}, \qquad U_1^A = \begin{vmatrix} \cos\phi + i\sin\phi & 0 \\ 0 & \cos\phi - i\sin\phi \end{vmatrix},$$
$$U_2^A = \begin{vmatrix} \cos\phi & -\sin\phi\sigma_2 \\ \sin\phi\sigma_2 & \cos\phi \end{vmatrix}, \qquad U_3^A = \begin{vmatrix} \cos\phi & i\sin\phi \\ i\sin\phi & \cos\phi \end{vmatrix}, \qquad B_1 = \begin{vmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{vmatrix},$$
$$B_2 = \begin{vmatrix} -\sigma_3 & 0 \\ 0 & -\sigma_3 \end{vmatrix}, \qquad B_3 = \begin{vmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{vmatrix}, \qquad U_1^B = \begin{vmatrix} \cos\phi & i\sin\phi\sigma_1 \\ i\sin\phi\sigma_1 & \cos\phi \end{vmatrix},$$
$$U_2^B = \begin{vmatrix} \cos\phi - i\sin\phi\sigma_3 & 0 \\ 0 & \cos\phi - i\sin\phi\sigma_3 \end{vmatrix},$$
$$U_3^B = \begin{vmatrix} \cos\phi - i\sin\phi\sigma_1 & 0 \\ 0 & \cos\phi + i\sin\phi\sigma_1 \end{vmatrix}, \qquad C_1 = \begin{vmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{vmatrix},$$
$$C_2 = \begin{vmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{vmatrix}, \qquad C_3 = \begin{vmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{vmatrix}, \qquad U_1^C = \begin{vmatrix} \cos\phi & -i\sin\phi\sigma_3 \\ -i\sin\phi\sigma_3 & \cos\phi \end{vmatrix},$$
$$U_2^C = \begin{vmatrix} \cos\phi - i\sin\phi\sigma_1 & 0 \\ 0 & \cos\phi - i\sin\phi\sigma_1 \end{vmatrix},$$
$$U_3^C = \begin{vmatrix} \cos\phi + i\sin\phi\sigma_3 & 0 \\ 0 & \cos\phi - i\sin\phi\sigma_3 \end{vmatrix}. \tag{7.6}$$

Certainly, these relations provide us with 15 elementary solutions of the unitarity equations (3.3). For instance, the generator α_2 gives rise to the above 1-parametric Abelian subgroup $G^0(\alpha)$; whereas the above 4-parametric subgroup $G_0 \times SU(2)$ (5.15) is generated by $(\alpha_2; \beta_1, B_2, C_2)$.

The question is how one could describe all combinations of the above 15 simple sub-solutions by a single unifying formula – the latter should evidently exist.

8 On factorization $SU(4)$ and the group fine-structure

On the basis of 9 matrices (7.2) one can construct six 3-dimensional sub-sets:

$$\boldsymbol{K} = \{A_1 = \alpha_1\beta_1,\ B_2 = \alpha_2\beta_2,\ C_3 = \alpha_3\beta_3\},$$
$$\boldsymbol{L} = \{C_1 = \alpha_1\beta_3,\ A_2 = \alpha_2\beta_1,\ B_3 = \alpha_3\beta_2\},$$
$$\boldsymbol{M} = \{B_1 = \alpha_1\beta_2,\ C_2 = \alpha_2\beta_3,\ A_3 = \alpha_3\beta_1\},$$
$$\boldsymbol{K}' = \{-C_1' = -\alpha_1\beta_3,\ -B_2' = -\alpha_2\beta_2,\ -C_3' = -\alpha_3\beta_3\},$$
$$\boldsymbol{L}' = \{-B_1' = -\alpha_1\beta_2,\ -A_2' = -\alpha_2\beta_1,\ -B_3' = -\alpha_3\beta_2\},$$
$$\boldsymbol{M}' = \{-A_1' = -\alpha_1\beta_1,\ -C_2' = -\alpha_2\beta_3,\ -B_3' = -\alpha_3\beta_2\},$$

(one may recall the rule to calculate the determinant of a 3×3 matrix) with the same commutation relations:

$$\Gamma_1\Gamma_2 = -\Gamma_3, \qquad \Gamma_2\Gamma_1 = -\Gamma_3, \qquad \Gamma_1\Gamma_2 - \Gamma_2\Gamma_1 = 0, \qquad \Gamma_1\Gamma_2 + \Gamma_2\Gamma_1 = -2\Gamma_3, \tag{8.1}$$

and analogous by cyclic symmetry. The whole set of the above 9 matrices coincides with

$$\vec{\alpha}, \quad \vec{\beta}, \quad \boldsymbol{K}, \quad \boldsymbol{L}, \quad \boldsymbol{M}, \tag{8.2}$$

or

$$\vec{\alpha}, \quad \vec{\beta}, \quad \boldsymbol{K}', \quad \boldsymbol{L}', \quad \boldsymbol{M}'.$$

It suffices to consider one variant, let it be (8.2). It seem reasonable to suppose that arbitrary element from $GL(4,C)$ can be factorized as follows

$$S = e^{i\vec{a}\vec{\alpha}} e^{i\vec{b}\vec{\beta}} e^{i\boldsymbol{k}\boldsymbol{K}} e^{i\boldsymbol{l}\boldsymbol{L}} e^{i\boldsymbol{m}\boldsymbol{M}}. \tag{8.3}$$

When all parameters are real-valued, the formula provides us with the rule to construct elements from $SU(4)$ group[14]. The order of factors might be different. Let us specify the group law for these 5 subsets. First are the two groups:

$$e^{i\vec{a}\vec{\alpha}} = \cos a + i \sin a(n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3), \qquad e^{i\vec{b}\vec{\beta}} = \cos b + i \sin b(n_1\beta_1 + n_2\beta_2 + n_3\beta_3).$$

They are isomorphic, so one can consider only the first one:

$$e^{i\vec{a}\vec{\alpha}} = \cos a + i \sin a(n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3) = x_0 - ix_1\alpha_1 - ix_2\alpha_2 - ix_3\alpha_3.$$

Multiplying two matrices we arrive at

$$\begin{aligned} &x_0'' = x_0'x_0 - x_1'x_1 - x_2'x_2 - x_3'x_3, \qquad x_1'' = x_0'x_1 + x_1'x_0 + (x_2'x_3 - x_2'x_3), \\ &x_2'' = x_0'x_2 + x_2'x_0 + (x_3'x_1 - x_1'x_3), \qquad x_3'' = x_0'x_3 + x_3'x_0 + (x_1'x_2 - x_2'x_1). \end{aligned} \tag{8.4}$$

Parameters (x_0, x_i) should obey

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \qquad \Longleftrightarrow \qquad \det e^{i\vec{a}\vec{\alpha}} = +1.$$

The inverse matrix looks

$$(x_0, \boldsymbol{x})^{-1} = (x_0, -\boldsymbol{x}).$$

With real (x_0, x_i) we have a group isomorphic to $SU(2)$, spinor covering for $SO(3,R)$:

$$\boldsymbol{c} = \frac{\boldsymbol{x}}{x_0}, \qquad \boldsymbol{c}'' = \frac{\boldsymbol{c}' + \boldsymbol{c} + \boldsymbol{c}' \times \boldsymbol{c}}{1 - \boldsymbol{c}'\boldsymbol{c}}.$$

At complex (x_0, x_i) we have a group isomorphic to $GL(2,C)$, spinor covering for $SO(3,C)$ or Lorentz group.

Now let us turn to finite transformations from remaining subsets. It is readily verified that these 1-parametric finite elements

$$e^{iy_1\Gamma_1} = \cos y_1 + i \sin y_1\Gamma_1, \qquad e^{iy_2\Gamma_1} = \cos y_2 + i \sin y_2\Gamma_2, \qquad e^{iy_3\Gamma_3} = \cos y_3 + i \sin y_3\Gamma_3,$$

[14] Just such a structure was described in [40].

commute with each other:

$$\begin{aligned}
e^{iy_1\Gamma_1}e^{iy_2\Gamma_2} &= (\cos y_1 + i\sin y_1\Gamma_1)(\cos y_2 + i\sin y_2\Gamma_2) = \\
&= \cos y_1\cos y_2 + i\cos y_1\sin y_2\Gamma_2 + i\cos y_2\sin y_1\Gamma_1 + \sin y_1\sin y_2\Gamma_3, \\
e^{iy_2\Gamma_2}e^{iy_1\Gamma_1} &= (\cos y_2 + i\sin y_2\Gamma_2)(\cos y_1 + i\sin y_1\Gamma_1) = \\
&= \cos y_2\cos y_1 + i\cos y_2\sin y_1\Gamma_1 + i\cos y_1\sin y_2\Gamma_2 + \sin y_2\sin y_1\Gamma_3,
\end{aligned}$$

that is $e^{iy_1\Gamma_1}e^{iy_2\Gamma_2} = e^{iy_2\Gamma_2}e^{iy_1\Gamma_1}$, and so on. Evidently, this property correlates with the commutative relations (8.1). Thus, each of tree subgroups can be constructed as multiplying of elementary 1-parametric commuting transformations. Their explicit forms are:

subgroup K

$$\begin{aligned}
&\boldsymbol{K} = \{A_1 = \alpha_1\beta_1,\ B_2 = \alpha_2\beta_2,\ C_3 = \alpha_3\beta_3\}, \\
&A_1 = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}, \qquad B_2 = \begin{vmatrix} -\sigma_3 & 0 \\ 0 & -\sigma_3 \end{vmatrix}, \qquad C_3 = \begin{vmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{vmatrix}, \\
&e^{ik_1K_1} = \cos k_1 + i\sin k_1A_1, \qquad e^{ik_2K_1} = \cos k_2 + i\sin k_2B_2, \\
&e^{ik_3K_3} = \cos k_3 + i\sin k_3C_3;
\end{aligned}$$

subgroup L

$$\begin{aligned}
&\boldsymbol{L} = \{C_1 = \alpha_1\beta_3,\ A_2 = \alpha_2\beta_1,\ B_3 = \alpha_3\beta_2\}, \\
&C_1 = \begin{vmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{vmatrix}, \qquad A_2 = \begin{vmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{vmatrix}, \qquad B_3 = \begin{vmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{vmatrix}, \\
&e^{il_1L_1} = \cos l_1 + i\sin l_1C_1, \qquad e^{il_2L_1} = \cos l_2 + i\sin l_2A_2, \qquad e^{il_3L_3} = \cos l_3 + i\sin l_3B_3;
\end{aligned}$$

subgroup M

$$\begin{aligned}
&\boldsymbol{M} = \{B_1 = \alpha_1\beta_2,\ C_2 = \alpha_2\beta_3,\ A_3 = \alpha_3\beta_1\}, \\
&B_1 = \begin{vmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{vmatrix}, \qquad C_2 = \begin{vmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{vmatrix}, \qquad A_3 = \begin{vmatrix} 0 & I \\ I & 0 \end{vmatrix}, \\
&e^{im_1M_1} = \cos m_1 + i\sin m_1B_1, \qquad e^{im_2M_1} = \cos m_2 + i\sin m_2C_2, \\
&e^{im_3M_3} = \cos m_3 + i\sin m_3A_3.
\end{aligned}$$

One additional note should be made. In the recent paper by A. Gsponer [35] on the quaternion approach to the problem of building the finite transformations from $SU(3)$ and $SU(4)$ an important point was to divide 15 basis 4×4 matrices into three sets:

set A of antisymmetrical matrices,

set S of symmetrical matrices,

set D of diagonal traceless ones.

It is easily seen that

set $A = \{\alpha_i \oplus \beta_i\}$;

set $S = \{A_2, A_3, B_1, B_3, C_1, C_2\} = \{\boldsymbol{L} \oplus \boldsymbol{M}\}$;

set $D = \{A_1, B_2, C_3 = \boldsymbol{K}\}$.

Turning again to relationship (8.3), let us rewrite it as follows

$$S = e^{i\vec{a}\vec{\alpha}} [e^{i k \boldsymbol{K}} e^{i l \boldsymbol{L}} e^{i m \boldsymbol{M}}] e^{i\vec{b}\vec{\beta}}.$$

which exactly corresponds to the structure used in [35] in connection with the Lanczos decomposition theorem [43].

Several last comments should be made. On the basis of 15 matrices

$$\begin{array}{llllll}
\alpha_1, & \alpha_2, & \alpha_3, & \beta_1, & \beta_2, & \beta_3, \\
A_1 = \alpha_1\beta_1, & & B_1 = \alpha_1\beta_2, & & C_1 = \alpha_1\beta_3, & \\
A_2 = \alpha_2\beta_1, & & B_2 = \alpha_2\beta_2, & & C_2 = \alpha_2\beta_3, & \\
A_3 = \alpha_3\beta_1, & & B_3 = \alpha_3\beta_2, & & C_3 = \alpha_3\beta_3 &
\end{array}$$

one can easily see the following 20 ways to separate $SU(2)$ subgroups (certainly, those arise at real-valued parameters; complex-valued parameters give rise to linear subgroups $GL(2,C)$):

$$\begin{array}{lll}
(\alpha_1,\alpha_2,\alpha_3), & (\beta_1,\beta_2,\beta_3), & \\
(\alpha_1,A_2,A_3), & (A_1,\alpha_2,A_3), & (A_1,A_2,\alpha_3), \\
(\alpha_1,B_2,B_3), & (B_1,\alpha_2,B_3), & (B_1,B_2,\alpha_3), \\
(\alpha_1,C_2,C_3), & (C_1,\alpha_2,C_3), & (C_1,C_2,\alpha_3), \\
(\beta_1,B_1,C_1), & (\beta_1,B_2,C_2), & (\beta_1,B_3,C_3), \\
(A_1,\beta_2,C_1), & (A_2,\beta_2,C_2), & (A_3,\beta_2,C_3), \\
(A_1,B_1,\beta_3), & (A_2,B_2,\beta_3), & (A_3,B_3,\beta_3).
\end{array} \qquad (8.5)$$

Certainly, they provide us with twenty different 3-parametric solutions of the unitarity equations (3.3). Such 3-subgroups might be used as bigger elementary blocks in constructing of a general transformation [25, 28].

For instance, for the variant from (8.5): $(\alpha_1, A_2, A_3) \Longrightarrow (a_1, X_2, X_3)$ the general multiplication law (7.4) gives

$$\gamma'' = \gamma'\gamma + a_1' a_1 + X_2' X_2 + X_3' X_3, \qquad a_1'' = \gamma' a_1 + a_1' \gamma + i(X_2' X_3 - X_3' X_2),$$
$$a_2'' = 0, \qquad a_3'' = 0, \qquad b_1'' = 0, \qquad b_2'' = 0, \qquad b_3'' = 0, \qquad X_1'' = 0,$$
$$X_2'' = \gamma' X_2 + \gamma X_2' + +i(-a_1' X_2) + a_1 X_3'), \qquad X_3'' = \gamma' X_3 + \gamma X_3' + i(a_1' X_2 - a_1 X_2'),$$
$$Y_1'' = 0, \qquad Y_2'' = 0, \qquad Y_3'' = 0, \qquad Z_1'' = 0, \qquad Z_2'' = 0, \qquad Z_3'' = 0,$$

that is

$$\gamma'' = \gamma'\gamma + a_1' a_1 + X_2' X_2 + X_3' X_3, \qquad a_1'' = \gamma' a_1 + a_1' \gamma + i(X_2' X_3 - X_3' X_2),$$
$$X_2'' = \gamma' X_2 + \gamma X_2' + i(X_3' a_1 - a_1' X_2), \qquad X_3'' = \gamma' X_3 + \gamma X_3' + i(a_1' X_2 - a_1 X_2'),$$

which coincides with equation (8.4). The same can be done for any other representative from (8.5).

9 On pseudo-unitary group $SU(2,2)$

As said, the Dirac basis was used previously [9, 40, 48, 50] in studying the exponentials for $SU(2,2)$ matrices. Let us show how the above formalism can apply to this pseudo-unitary group $SU(2,2)$. Transformations from $SU(2,2)$ should leave invariant the following form

$$(z_1^*, z_2^*, z_3^*, z_4^*) \left| \begin{array}{cccc} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right| \left| \begin{array}{c} z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \right|, \qquad z^+ \eta z = z'^+ \eta z',$$

which leads to

$$z' = Uz, \qquad \text{where} \qquad U^{+}\eta = \eta U^{-1}, \qquad \eta = \left| \begin{array}{cccc} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right| .$$

Any generator Λ' of those transformations must obey relation

$$U_k = e^{ia\Lambda'_k}, \qquad (\Lambda'_k)^{+}\eta = \eta\Lambda'_k, \qquad k \in \{1, \dots, 15\}$$

or allowing for identity $\eta = -\gamma^5$

$$(\Lambda'_k)^{+}\gamma^5 = \gamma^5\Lambda'_k, \qquad k \in \{1, \dots, 15\}.$$

All generators Λ'_k of the group $SU(2,2)$ can be readily constructed on the basis of the known generators Λ_k of $SU(4)$ (see (6.1))

$$\begin{array}{lll} \Lambda'_1 = \Lambda_1 = \gamma^5, & \Lambda'_1 = i\Lambda_1 = i\gamma^0, & \Lambda'_3 = i\Lambda_3 = -\gamma^5\gamma^0, \\ \Lambda'_4 = i\Lambda_4 = -\gamma^1, & \Lambda'_5 = i\Lambda_5 = i\gamma^5\gamma^1, & \Lambda'_6 = i\Lambda_6 = -\gamma^2, \\ \Lambda'_7 = i\Lambda_7 = i\gamma^5\gamma^2, & \Lambda'_8 = i\Lambda_8 = -\gamma^3, & \Lambda'_9 = i\Lambda_9 = i\gamma^5\gamma^3, \\ \Lambda'_{10} = \Lambda_{10} = 2\sigma^{01}, & \Lambda'_{11} = \Lambda_{11} = 2\sigma^{02}, & \Lambda'_{12} = \Lambda_{12} = 2\sigma^{03}, \\ \Lambda'_{13} = \Lambda_{13} = 2i\sigma^{12}, & \Lambda'_{14} = \Lambda_{14} = 2i\sigma^{23}, & \Lambda'_{15} = \Lambda_{15} = 2i\sigma^{31}. \end{array} \tag{9.1}$$

Basis elements may be listed as follows:

$$\begin{array}{lll} \alpha'_1 = \alpha_1 = \gamma^0\gamma^2, & \alpha'_2 = i\alpha_2 = -\gamma^0\gamma^5, & \alpha'_3 = i\alpha_3 = i\gamma^5\gamma^2 \\ (\alpha'_1)^2 = I, & (\alpha'_2)^2 = -I, & (\alpha'_3)^2 = -I, \\ \alpha'_2\alpha'_3 = -i\alpha_1, & \alpha'_3\alpha'_1 = i\alpha'_2, & \alpha'_1\alpha'_2 = i\alpha'_3; \\ \beta'_1 = \beta_1 = i\gamma^3\gamma^1, & \beta'_2 = i\beta_2 = -\gamma^3, & \beta'_3 = i\beta_3 = -\gamma^1, \\ (\beta'_1)^2 = I, & (\beta'_2)^2 = -I, & (\beta'_3)^2 = -I, \\ \beta'_2\beta'_3 = -i\beta_1, & \beta'_3\beta'_1 = i\beta'_2, & \beta'_1\beta'_2 = i\beta'_3. \end{array}$$

These two sets commute with each other: $\alpha_j\beta_k = \beta_k\alpha_j$; and their multiplications give remaining 9 elements:

$$\begin{array}{lll} A'_1 = A_1 = \alpha'_1\beta'_1 = -\gamma^5, & B'_1 = iB_1 = \alpha'_1\beta'_2 = i\gamma^5\gamma^1, & C'_1 = iC_1 = \alpha'_1\beta'_3 = i\gamma^3\gamma^5, \\ A'_2 = iA_2 = \alpha'_2\beta'_1 = \gamma^2, & B'_2 = -B_2 = \alpha'_2\beta'_2 = +i\gamma^1\gamma^2, & C'_2 = -C_2 = \alpha'_2\beta'_3 = +i\gamma^2\gamma^3, \\ A'_3 = iA_3 = \alpha'_3\beta'_1 = i\gamma^0, & B'_3 = -B_3 = \alpha'_3\beta'_2 = -\gamma^0\gamma^1, & C'_3 = -C_3 = \alpha'_3\beta'_3 = -\gamma^0\gamma^3. \end{array}$$

Making in relations (7.6) a formal change in accordance with

$$\begin{array}{l} e^{ia\Lambda} = \cos a + i\sin a\Lambda, \qquad \Lambda' = i\Lambda, \\ \cos a + i\sin a\Lambda = e^{ia\Lambda} \quad \Longrightarrow \quad e^{ia\Lambda'} = \cos ia + i\sin ia\Lambda = \cosh a - \sinh a\Lambda, \end{array}$$

we arrive at explicit form of elementary pseudo-unitary $SU(2,2)$-transformations:

$$\alpha'_1 = \alpha_1 = \left| \begin{array}{cc} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{array} \right|, \qquad U_1^{\alpha} = \left| \begin{array}{cc} \cos\phi + i\sin\phi\sigma_2 & 0 \\ 0 & \cos\phi - i\sin\phi\sigma_2 \end{array} \right|,$$

$$\alpha'_2 = i\alpha_2 = \left| \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right|, \qquad U_2^{\alpha} = \left| \begin{array}{cc} \cosh\chi & -i\sinh\chi \\ i\sinh\chi & \cosh\chi \end{array} \right|,$$

$$
\alpha_3' = i\alpha_3 = \begin{vmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{vmatrix}, \qquad U_3^{\alpha} = \begin{vmatrix} \cosh\chi & -\sinh\chi\sigma_2 \\ \sinh\chi\sigma_2 & \cosh\chi \end{vmatrix},
$$
$$
\beta_1' = \beta_1 = \begin{vmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{vmatrix}, \qquad U_1^{\beta} = \begin{vmatrix} \cos\phi + i\sin\phi\sigma_2 & 0 \\ 0 & \cos\phi + i\sin\phi\sigma_2 \end{vmatrix},
$$
$$
\beta_2' = i\beta_2 = \begin{vmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{vmatrix}, \qquad U_2^{\beta} = \begin{vmatrix} \cosh\chi & i\sinh\chi\sigma_3 \\ -i\sinh\chi\sigma_3 & \cosh\chi \end{vmatrix},
$$
$$
\beta_3' = i\beta_3 = \begin{vmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{vmatrix}, \qquad U_3^{\beta} = \begin{vmatrix} \cosh\chi & i\sinh\chi\sigma_1 \\ -i\sinh\chi\sigma_1 & \cosh\chi \end{vmatrix},
$$
$$
A_1' = A_1 = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}, \qquad U_1^{A} = \begin{vmatrix} \cos\phi + i\sin\phi & 0 \\ 0 & \cos\phi - i\sin\phi \end{vmatrix},
$$
$$
A_2' = iA_2 = \begin{vmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{vmatrix}, \qquad U_2^{A} = \begin{vmatrix} \cosh\chi & -i\sinh\chi\sigma_2 \\ i\sinh\chi\sigma_2 & \cosh\chi \end{vmatrix},
$$
$$
A_3' = iA_3 = \begin{vmatrix} 0 & i \\ i & 0 \end{vmatrix}, \qquad U_3^{A} = \begin{vmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{vmatrix},
$$
$$
B_1' = iB_1 = \begin{vmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{vmatrix}, \qquad U_1^{B} = \begin{vmatrix} \cosh\chi & -\sinh\chi\sigma_1 \\ -\sinh\chi\sigma_1 & \cosh\chi \end{vmatrix},
$$
$$
B_2' = -B_2 = \begin{vmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{vmatrix}, \qquad U_2^{B} = \begin{vmatrix} \cos\phi + i\sin\phi\sigma_3 & 0 \\ 0 & \cos\phi + i\sin\phi\sigma_3 \end{vmatrix},
$$
$$
B_3' = -B_3 = \begin{vmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{vmatrix}, \qquad U_3^{B} = \begin{vmatrix} \cos\phi + i\sin\phi\sigma_1 & 0 \\ 0 & \cos\phi - i\sin\phi\sigma_1 \end{vmatrix},
$$
$$
C_1' = iC_1 = \begin{vmatrix} 0 & -i\sigma_3 \\ -i\sigma_3 & 0 \end{vmatrix}, \qquad U_1^{C} = \begin{vmatrix} \cosh\chi & \sinh\chi\sigma_3 \\ \sinh\chi\sigma_3 & \cosh\chi \end{vmatrix},
$$
$$
C_2' = -C_2 = \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{vmatrix}, \qquad U_2^{C} = \begin{vmatrix} \cos\phi i\sin\phi\sigma_1 & 0 \\ 0 & \cos\phi + i\sin\phi\sigma_1 \end{vmatrix},
$$
$$
C_3' = -C_3 = \begin{vmatrix} -\sigma_3 & 0 \\ 0 & +\sigma_3 \end{vmatrix}, \qquad U_3^{C} = \begin{vmatrix} \cos\phi - i\sin\phi\sigma_3 & 0 \\ 0 & \cos\phi + i\sin\phi\sigma_3 \end{vmatrix}. \tag{9.2}
$$

We can easily obtain unitarity equations for $SU(2,2)$ group, simple solutions to which are given by (9.2). Indeed, taking into account the formulas (see (3.1)

$$
G^{+}\eta = \begin{vmatrix} k_0^* + \boldsymbol{k}^*\vec{\sigma} & -l_0^* - \boldsymbol{l}^*\vec{\sigma} \\ n_0^* - \boldsymbol{n}^*\vec{\sigma} & m_0^* - \boldsymbol{m}^*\vec{\sigma} \end{vmatrix} \begin{vmatrix} -I & 0 \\ 0 & +I \end{vmatrix} = \begin{vmatrix} -k_0^* - \boldsymbol{k}^*\vec{\sigma} & -l_0^* - \boldsymbol{l}^*\vec{\sigma} \\ -n_0^* + \boldsymbol{n}^*\vec{\sigma} & m_0^* - \boldsymbol{m}^*\vec{\sigma} \end{vmatrix},
$$
$$
\eta G^{-1} = \begin{vmatrix} -I & 0 \\ 0 & I \end{vmatrix} \begin{vmatrix} k_0' + \boldsymbol{k}'\vec{\sigma} & n_0' - \boldsymbol{n}'\vec{\sigma} \\ -l_0' - \boldsymbol{l}'\vec{\sigma} & m_0' - \boldsymbol{m}'\vec{\sigma} \end{vmatrix} = \begin{vmatrix} -k_0' - \boldsymbol{k}'\vec{\sigma} & -n_0' + \boldsymbol{n}'\vec{\sigma} \\ -l_0' - \boldsymbol{l}'\vec{\sigma} & m_0' - \boldsymbol{m}'\vec{\sigma} \end{vmatrix},
$$

from $G^{+}\eta = \eta G^{-1}$ we produce

$$
\begin{aligned}
&k_0^* = k_0', \qquad \boldsymbol{k}^* = \boldsymbol{k}', \qquad m_0^* = m_0', \qquad \boldsymbol{m}^* = \boldsymbol{m}', \\
&l_0^* = n_0', \qquad \boldsymbol{l}^* = -\boldsymbol{n}', \qquad n_0^* = l_0', \qquad \boldsymbol{n}^* = -\boldsymbol{l}'.
\end{aligned}
$$

These relations differ from analogous ones (3.2) for $SU(4)$ group only in all signs of the second line. Therefore, the unitarity conditions for the group $SU(2,2)$ are (compare with (3.3))

$$
\begin{aligned}
&k_0^* = +k_0(mm) + m_0(ln) + l_0(nm) - n_0(lm) + i\boldsymbol{l}(\boldsymbol{m}\times\boldsymbol{n}), \\
&m_0^* = +m_0(kk) + k_0(nl) + n_0(lk) - l_0(nk) - i\boldsymbol{n}(\boldsymbol{k}\times\boldsymbol{l}), \\
&\boldsymbol{k}^* = -\boldsymbol{k}(mm) - \boldsymbol{m}(ln) - \boldsymbol{l}(nm) + \boldsymbol{n}(lm) + 2\boldsymbol{l}\times(\boldsymbol{n}\times\boldsymbol{m})
\end{aligned}
$$

$$
\begin{aligned}
&\quad + im_0(\boldsymbol{n}\times\boldsymbol{l}) + il_0(\boldsymbol{n}\times\boldsymbol{m}) + in_0(\boldsymbol{l}\times\boldsymbol{m}),\\
\boldsymbol{m}^* &= -\boldsymbol{m}(kk) - \boldsymbol{k}(nl) - \boldsymbol{n}(lk) + \boldsymbol{l}(nk) + 2\boldsymbol{n}\times(\boldsymbol{l}\times\boldsymbol{k})\\
&\quad - ik_0(\boldsymbol{l}\times\boldsymbol{n}) - in_0(\boldsymbol{l}\times\boldsymbol{k}) - il_0(\boldsymbol{n}\times\boldsymbol{k}),\\
-l_0^* &= +k_0(nm) - m_0(kn) + l_0(nn) + n_0(km) + i\boldsymbol{k}(\boldsymbol{n}\times\boldsymbol{m}),\\
-n_0^* &= +m_0(lk) - k_0(ml) + n_0(ll) + l_0(mk) - i\boldsymbol{m}(\boldsymbol{l}\times\boldsymbol{k}),\\
-\boldsymbol{l}^* &= -\boldsymbol{k}(nm) + \boldsymbol{m}(kn) - \boldsymbol{l}(nn) - \boldsymbol{n}(km) + 2\boldsymbol{k}\times(\boldsymbol{m}\times\boldsymbol{n})\\
&\quad + ik_0(\boldsymbol{m}\times\boldsymbol{n}) + im_0(\boldsymbol{k}\times\boldsymbol{n}) + in_0(\boldsymbol{m}\times\boldsymbol{k}),\\
-\boldsymbol{n}^* &= -\boldsymbol{m}(kl) + \boldsymbol{k}(ml) - \boldsymbol{n}(ll) - \boldsymbol{l}(mk) + 2\boldsymbol{m}\times(\boldsymbol{k}\times\boldsymbol{l}) - im_0(\boldsymbol{k}\times\boldsymbol{l})\\
&\quad - ik_0(\boldsymbol{m}\times\boldsymbol{l}) - il_0(\boldsymbol{k}\times\boldsymbol{m}).
\end{aligned}
$$

Several words on factorization $SU(2,2) = SU(1,1)\times[K\times L\times M]\times SU(1,1)$ and a further group fine-structure for $SU(2,2)$. On the basis of 9 matrices

$$
\begin{aligned}
&A_1' = A_1 = \alpha_1'\beta_1' = -\gamma^5, && B_1' = iB_1 = \alpha_1'\beta_2' = i\gamma^5\gamma^1, && C_1' = iC_1 = \alpha_1'\beta_3' = i\gamma^3\gamma^5,\\
&A_2' = iA_2 = \alpha_2'\beta_1' = \gamma^2, && B_2' = -B_2 = \alpha_2'\beta_2' = +i\gamma^1\gamma^2, && C_2' = -C_2 = \alpha_2'\beta_3' = i\gamma^2\gamma^3,\\
&A_3' = iA_3 = \alpha_3'\beta_1' = i\gamma^0, && B_3' = -B_3 = \alpha_3'\beta_2' = -\gamma^0\gamma^1, && C_3' = -C_3 = \alpha_3'\beta_3' = -\gamma^0\gamma^3
\end{aligned}
$$

one can construct three 3-dimensional subsets (omitting three others):

$$
\begin{aligned}
\boldsymbol{K}' &= \{A_1' = A_1 = \alpha_1'\beta_1',\ B_2' = -B_2 = \alpha_2'\beta_2',\ C_3' = -C_3 = \alpha_3'\beta_3'\},\\
\boldsymbol{L}' &= \{C_1' = iC_1 = \alpha_1'\beta_3',\ A_2' = iA_2 = \alpha_2'\beta_1',\ B_3 = -B_3 = \alpha_3'\beta_2'\},\\
\boldsymbol{M}' &= \{B_1' = iB_1 = \alpha_1'\beta_2',\ C_2' = -C_2 = \alpha_2'\beta_3',\ A_3' = iA_3 = \alpha_3'\beta_1'\},
\end{aligned}
$$

with the same commutation relations:

$$
\Gamma_1\Gamma_2 = +\Gamma_3, \qquad \Gamma_2\Gamma_1 = +\Gamma_3, \qquad \Gamma_1\Gamma_2 - \Gamma_2\Gamma_1 = 0, \qquad \Gamma_1\Gamma_2 + \Gamma_2\Gamma_1 = +2\Gamma_3,
$$

and analogous ones by cyclic symmetry. Arbitrary element from $SU(2,2)$ can be factorized as follows

$$
S = e^{i\vec{a}\vec{\alpha}'} e^{i\vec{b}\vec{\beta}'} e^{i\boldsymbol{k}\boldsymbol{K}'} e^{i\boldsymbol{l}\boldsymbol{L}'} e^{i\boldsymbol{m}\boldsymbol{M}'},
$$

all parameters are real-valued. Let us specify the group law for these 5 sub-sets. Two groups $e^{i\vec{a}\vec{\alpha}'}$, $e^{i\vec{b}\vec{\beta}'}$ are isomorphic so one can consider only the first one:

$$
\begin{aligned}
e^{i\vec{a}\vec{\alpha}'} &= I + i(a_1\alpha_1' + a_2\alpha_2' + a_3\alpha_3') - \tfrac{1}{2!}(a_1^2 - a_2^2 - a_3^2)\\
&\quad - i\tfrac{1}{3!}(a_1^2 - a_2^2 - a_3^2)(a_1\alpha_1' + a_2\alpha_2' + a_3\alpha_3') + \tfrac{1}{4!}(a_1^2 - a_2^2 - a_3^2)^2 + \cdots .
\end{aligned}
$$

In the variables

$$
a_i = a n_i, \qquad n_1^2 - n_2^2 - n_3^2 = 1
$$

we have

$$
\begin{aligned}
e^{i\vec{a}\vec{\alpha}'} &= I + i(n_1\alpha_1' + n_2\alpha_2' + n_3\alpha_3')a - \tfrac{1}{2!}a^2 - i\tfrac{1}{3!}(n_1\alpha_1' + n_2\alpha_2' + n_3\alpha_3')a^3\\
&\quad + \tfrac{1}{4!}a^4 + \tfrac{1}{5!}(n_1\alpha_1' + n_2\alpha_2' + in_3\alpha_3')a^5 - \tfrac{1}{6!}a^6 + \cdots ,
\end{aligned}
$$

that is

$$
e^{i\vec{a}\vec{\alpha}'} = \cos a + i\sin a(n_1\alpha_1' + n_2\alpha_2' + n_3\alpha_3') = x_0 - ix_1\alpha_1' - ix_2\alpha_2' - ix_3\alpha_3'.
$$

Multiplying two matrices we arrive at

$$x_0'' = x_0'x_0 - x_1'x_1 + x_2'x_2 + x_3'x_3, \qquad x_1'' = x_0'x_1 + x_1'x_0 - (x_2'x_3 - x_2'x_3),$$
$$x_2'' = x_0'x_2 + x_2'x_0 + (x_3'x_1 - x_1'x_3), \qquad x_3'' = x_0'x_3 + x_3'x_0 + (x_1'x_2 - x_2'x_1).$$

The inverse matrix looks

$$(x_0, \boldsymbol{x})^{-1} = (x_0, -\boldsymbol{x}).$$

Parameters (x_0, x_i) should obey the following condition

$$x_0^2 + x_1^2 - x_2^2 - x_3^2 = 1.$$

In the variables a, n_i it will look

$$\cos^2 a + \sin^2 a(n_1^2 - n_2^2 - n_3^2) = 1, \qquad n_1^2 - n_2^2 - n_3^2 = 1.$$

For three particular cases we will have:

$$\boldsymbol{n} = (1,0,0), \qquad \cos^2 a + \sin^2 a = 1, \qquad e^{ia\alpha_1'} = \cos a + i \sin a\alpha_1', \qquad a \in R;$$
$$\boldsymbol{n} = (0,i,0), \qquad \cos^2 a - \sin^2 a = 1,$$
$$a = ib, \qquad \cos ib = \cosh b, \qquad \sin ib = i \sinh b, \qquad b \in R,$$
$$e^{ia\alpha_2'} = \cos a + i \sin a n_2 \alpha_2' = \cosh b - i \sinh b\alpha_2';$$
$$\boldsymbol{n} = (0,0,i), \qquad \cos^2 a - \sin^2 a = 1,$$
$$a = ib, \qquad \cos ib = \cosh b, \qquad \sin ib = i \sinh b, \qquad b \in R,$$
$$e^{ia\alpha_3'} = \cos a + i \sin a n_3 \alpha_3' = \cosh b - i \sinh b\alpha_3'.$$

Now let us turn to finite transformations from remaining sub-sets $\boldsymbol{K}'$, $\boldsymbol{L}'$, $\boldsymbol{M}'$. Each of tree subgroups can be constructed as multiplying of elementary 1-parametric commuting transformations. Their explicit forms are:

subgroup K'

$$\boldsymbol{K}' = \{A_1' = A_1 = \alpha_1'\beta_1', \; B_2' = -B_2 = \alpha_2'\beta_2', \; C_3' = -C_3 = \alpha_3'\beta_3'\},$$
$$A_1' = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}, \qquad B_2' = \begin{vmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{vmatrix}, \qquad C_3' = \begin{vmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{vmatrix};$$

subgroup L'

$$\boldsymbol{L}' = \{C_1' = iC_1 = \alpha_1'\beta_3', \; A_2' = iA_2 = \alpha_2'\beta_1', \; B_3' = -B_3 = \alpha_3'\beta_2'\},$$
$$C_1' = \begin{vmatrix} 0 & -i\sigma_3 \\ -i\sigma_3 & 0 \end{vmatrix}, \qquad A_2' = \begin{vmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{vmatrix}, \qquad B_3' = \begin{vmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{vmatrix};$$

subgroup M'

$$\boldsymbol{M}' = \{B_1' = iB_1 = \alpha_1\beta_2, \; C_2' = -C_2 = \alpha_2\beta_3, \; A_3' = iA_3 = \alpha_3\beta_1\},$$
$$B_1' = \begin{vmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{vmatrix}, \qquad C_2' = \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{vmatrix}, \qquad A_3' = \begin{vmatrix} 0 & i \\ i & 0 \end{vmatrix}.$$

On the basis of 15 matrices

$$\alpha_1', \qquad \alpha_2', \qquad \alpha_3', \qquad \beta_1', \qquad \beta_2', \qquad \beta_3',$$

$$\begin{array}{lll} A_1' = \alpha_1'\beta_1', & B_1' = \alpha_1'\beta_2', & C_1' = \alpha_1'\beta_3', \\ A_2' = \alpha_2'\beta_1', & B_2' = \alpha_2'\beta_2', & C_2' = \alpha_2'\beta_3', \\ A_3' = \alpha_3'\beta_1', & B_3' = \alpha_3'\beta_2', & C_3' = \alpha_3'\beta_3' \end{array}$$

one can easily see the following 20 ways to separate $SU(1,1)$ subgroups:

$$\begin{array}{lll} (\alpha_1', \alpha_2', \alpha_3'), & (\beta_1', \beta_2', \beta_3'), & \\ (\alpha_1', A_2', A_3'), & (A_1', \alpha_2', A_3'), & (A_1', A_2', \alpha_3'), \\ (\alpha_1', B_2', B_3'), & (B_1', \alpha_2', B_3'), & (B_1', B_2', \alpha_3'), \\ (\alpha_1', C_2', C_3'), & (C_1', \alpha_2', C_3'), & (C_1', C_2', \alpha_3'), \\ (\beta_1', B_1', C_1'), & (\beta_1', B_2', C_2'), & (\beta_1', B_3', C_3'), \\ (A_1', \beta_2', C_1'), & (A_2', \beta_2', C_2'), & (A_3', \beta_2', C_3'), \\ (A_1, B_1, \beta_3), & (A_2, B_2, \beta_3), & (A_3, B_3, \beta_3). \end{array}$$

such 3-subgroups might be used as bigger elementary blocks in constructing a general transformation.

10 On pseudo-unitary group $SU(3, 1)$

Let us show how the above formalism can apply to the pseudo-unitary group $SU(3, 1)$. Transformations from $SU(3, 1)$ should leave invariant the following form

$$(z_1^*, z_2^*, z_3^*, z_4^*) \left| \begin{array}{cccc} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right| \left| \begin{array}{c} z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \right|, \qquad z^+ \eta z = z'^+ \eta z',$$

which leads to

$$z' = Uz, \qquad \text{where} \qquad U^+ \eta = \eta U^{-1}, \qquad \eta = \left| \begin{array}{cccc} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right|.$$

Any generator Λ' of those transformations must obey the relation

$$U_k = e^{ia\Lambda_k'}, \qquad (\Lambda_k')^+ \eta = \eta \Lambda_k'. \qquad k \in \{1, \ldots, 15\}.$$

The matrix η is a linear combination

$$\eta = \tfrac{1}{2}(2i\sigma^{12} - 2\sigma^{03} - \gamma^5 + I) = \tfrac{1}{2}(i\gamma^1\gamma^2 - \gamma^0\gamma^3 + i\gamma^0\gamma^1\gamma^2\gamma^3 + I).$$

All generators of the group $SU(3, 1)$ can readily be constructed on the basis of the known generators λ_k of $SU(4)$ (see (6.2))

$$\begin{array}{llllllllll} \lambda_1, & \lambda_2, & \lambda_3, & \lambda_4, & \lambda_5, & \lambda_6, & \lambda_7, & \lambda_8, & i\lambda_9, & i\lambda_{10}, \\ i\lambda_{11}, & i\lambda_{12}, & i\lambda_{13}, & i\lambda_{14}, & \lambda_{15}; & & & & & \end{array} \tag{10.1}$$

generator $\lambda_9, \ldots, \lambda_{14}$ are multiplied by imaginary unit i. Instead of λ_8, λ_{15} one can introduce other generator λ_8', λ_{15}' see (6.4) (diagonal generators are the same for group $SU(4)$, $SU(2, 2)$, and $SU(3, 1)$.

11 Discussion

Let us summarize the main point of the present treatment.

Parametrization of 4×4 matrices G of the complex linear group $GL(4, C)$ in terms of four complex 4-vector parameters (k, m, n, l) is investigated. Additional restrictions separating some subgroups of $GL(4, C)$ are given explicitly. In the given parametrization, the problem of inverting any 4×4 matrix G is solved. Expression for determinant of any matrix G is found: $\det G = F(k, m, n, l)$. Unitarity conditions $G^{+} = G^{-1}$ have been formulated in the form of non-linear cubic algebraic equations including complex conjugation. Several simplest solutions of these unitarity equations have been found: three 2-parametric subgroups G_1, G_2, G_3 – each of subgroups consists of two commuting Abelian unitary groups; 4-parametric unitary subgroup consisting of a product of a 3-parametric group isomorphic $SU(2)$ and 1-parametric Abelian group.

The Dirac basis of generators Λ_k, being of Gell-Mann type, substantially differs from the basis λ_i used in the literature on $SU(4)$ group, formulas relating them are found – they permit to separate $SU(3)$ subgroup in $SU(4)$. Special way to list 15 Dirac generators of $GL(4, C)$ can be used $\{\Lambda_k\} = \{\alpha_i \oplus \beta_j \oplus (\alpha_i\beta_j = \boldsymbol{K} \oplus \boldsymbol{L} \oplus \boldsymbol{M})\}$, which permit to factorize $SU(4)$ transformations according to $S = e^{i\vec{a}\vec{\alpha}} e^{i\vec{b}\vec{\beta}} e^{i\boldsymbol{k}\boldsymbol{K}} e^{i\boldsymbol{l}\boldsymbol{L}} e^{i\boldsymbol{m}\boldsymbol{M}}$, where two first factors commute with each other and are isomorphic to $SU(2)$ group, the three last are 3-parametric groups, each of them consists of three Abelian commuting unitary subgroups. Besides, the structure of fifteen Dirac matrices Λ_k permits to separate twenty 3-parametric subgroups in $SU(4)$ isomorphic to $SU(2)$; those subgroups might be used as bigger elementary blocks in constructing a general transformation $SU(4)$. It is shown how one can specify the present approach for the unitary group $SU(2,2)$ and $SU(3,1)$.

In principle, all different approaches used in the literature are closely related so that any result obtained within one technique may be easily translated to any other. There is no sense to persist in exploiting only one representation, thinking that it is much better than all others. Success should lie in combining different techniques. For instance, Euler angles-based approach provides us with the group elements in the separated variables form, which may be of a supreme importance at calculating matrix elements of the group. In turn, a factorized subgroup- based structure is of special interest in the particle physics and gauge theory of fundamental interaction. Geometrical properties of the groups, their global structure, differences between orthogonal groups and their double covering, and so on, seem to be most easily understood in terms of bilinear functions in space of linear parameters: $G = x_j \Lambda_j$, $x''j = e_{jkl} x'_k x_l$.

We have no ground to think that only exponential functions $e^{i\Lambda}$ are suitable for exploration into group structures. We may expect that in addition to Euler angles many other curvilinear coordinates might be of value for studying of the group structure. For instance, in the case of the group $SO(4, C)$ we have known 34 such coordinate systems owing to Olevskiy investigation [58] on 3-orthogonal coordinates in real Lobachevski space.

In conclusion, several words about possible application areas of the obtained results. The main argument in favor of constructing the theory of unitary groups $SU(4)$ (and related to it) in terms of Dirac matrices is the role of spinor methods being widely adopted in physics. Let us mention several problems most attractive for authors:

$SU(2,2)$ and conformal symmetry, massless particles;

classical Yang-Mills equations and gauge fields;

geometric phases for multi-level quantum systems;

composite structure of quarks and leptons;

$SU(4)$ gauge models.

In particular, description of the group $SU(2,2)$ in terms of matrices α^j, β^j should be of great benefit in investigation of *conformal symmetry in massless particles theory.* For instance, classical Maxwell equations in a medium can be presented in 4-dimensional complex matrix form with the use of two sets of matrices, exploited above:

$$\left(-i\frac{\partial}{\partial x^0}+\alpha^j\frac{\partial}{\partial x^j}\right)M+\left(-i\frac{\partial}{\partial x^0}+\beta^j\frac{\partial}{\partial x^j}\right)N=\frac{1}{\epsilon_0}\left|\begin{array}{c}\rho\\ \boldsymbol{j}/c\end{array}\right|,$$

where

$$M=\left|\begin{array}{c}0\\ \boldsymbol{M}\end{array}\right|,\qquad \boldsymbol{M}=\frac{1}{\epsilon_0}(\boldsymbol{D}+i\boldsymbol{H}/c)+(\boldsymbol{E}+ic\boldsymbol{B}),$$

$$N=\left|\begin{array}{c}0\\ \boldsymbol{N}\end{array}\right|,\qquad \boldsymbol{N}=\frac{1}{\epsilon_0}(\boldsymbol{D}-i\boldsymbol{H}/c)-(\boldsymbol{E}-ic\boldsymbol{B}).$$

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