Symmetry Transformation in Extended Phase Space: the Harmonic Oscillator in the Husimi Representation*

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Abstract. In a previous work the concept of quantum potential is generalized into extended phase space (EPS) for a particle in linear and harmonic potentials. It was shown there that in contrast to the Schrödinger quantum mechanics by an appropriate extended canonical transformation one can obtain the Wigner representation of phase space quantum mechanics in which the quantum potential is removed from dynamical equation. In other words, one still has the form invariance of the ordinary Hamilton–Jacobi equation in this representation. The situation, mathematically, is similar to the disappearance of the centrifugal potential in going from the spherical to the Cartesian coordinates. Here we show that the Husimi representation is another possible representation where the quantum potential for the harmonic potential disappears and the modified Hamilton–Jacobi equation reduces to the familiar classical form. This happens when the parameter in the Husimi transformation assumes a specific value corresponding to Q-function.

Key words: Hamilton–Jacobi equation; quantum potential; Husimi function; extended phase space

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1 Introduction

According to the Bohm approach to quantum mechanics the quantum potential in the modified Hamilton–Jacobi equation may be equally looked at from the point of view of the Newton second law as a quantum force term [1]. Thus, in the causal interpretation, in addition to the external force, the quantum force derived from quantum potential, guides the trajectory of the quantum particle [2]. Takabayashi [3] and Muga [4] introduced the concept of quantum internal energy (or stress) as a consequence of the projection from the phase space representation to the configuration space representation. They argued that unlike the classical systems which have the kinetic and potential energies, quantum systems also have intrinsic internal energies associated with spatial localization and momentum dispersion emerging from their inherent extended natures suggesting a link to the Heisenberg position-momentum uncertainty principle [5]. Holland [6] investigated the de Broglie–Bohm law of motion using a variational formulation. He considered a quantum system and suggested a total Lagrangian for the interaction of a point like particle with the Schrödinger field. The interaction of the particle and field is attributed to a scalar potential that turns out to be the quantum potential. The connection of the quantum

*This paper is a contribution to the Proceedings of the Seventh International Conference “Symmetry in Nonlinear Mathematical Physics” (June 24–30, 2007, Kyiv, Ukraine). The full collection is available at http://www.emis.de/journals/SIGMA/symmetry2007.html potential with quantum fluctuations and quantum geometry in terms of Weyl's curvature has been studied by F. Shojai and A. Shojai [7] and Carroll [8]. Nevertheless, unlike the external potential, the quantum potential is not a pre-assigned function of the system coordinates and can only be derived from the wave function of the system [5] or from the corresponding quantum distribution functions used to calculate the average values of the observables [9]. This representation dependent property of the quantum potential allows one to find appropriate representations where the quantum potential could be removed from the modified Hamilton–Jacobi equation. Carroll [10] has shown that there are generalized quantum theories for which the quantum potential depends on the wave function. Using the extended phase space formulation of quantum mechanics [11, 12, 13], Nasiri [9] has shown that in the Wigner representation of phase space quantum mechanics [14] the quantum potential is removed from the dynamical equation of a particle in linear and harmonic potentials keeping the Hamilton–Jacobi equation form invariant. It seems that the Husimi representation could be another candidate to release the quantum potential. In Section 2, the EPS formalism is reviewed. In Section 3, the extended canonical transformations are introduced and are used to obtain the Husimi equation and the corresponding solution. In Section 4, an expression for the quantum potential is obtained in the EPS. In Section 5, an appropriate phase space representation is found in which the quantum potential is removed for the harmonic potential. Section 6 is devoted to the concluding remarks.

2 Review of EPS formalism

Assuming the phase space coordinates p and q to be independent variables on virtual trajectories allows one to define momenta π_p and π_q, conjugate to p and q, respectively. One may define an extended Lagrangian in the phase space as follows [11, 12]

$$\mathcal{L}(p,q,\dot{p},\dot{q}) = -\dot{p}q - \dot{q}p + \mathcal{L}^p(p,\dot{p}) + \mathcal{L}^q(q,\dot{q}), \tag{1}$$

where $\mathcal{L}^q$ and $\mathcal{L}^p$ are q and p space Lagrangians, satisfying the following Legendre transformation, respectively,

$$H\left(\frac{\partial \mathcal{L}^q}{\partial \dot{q}}, q\right) = \dot{q}\frac{\partial \mathcal{L}^q}{\partial \dot{q}} - \mathcal{L}^q(q,\dot{q}), \qquad H\left(p, \frac{\partial \mathcal{L}^p}{\partial \dot{p}}\right) = -\dot{p}\frac{\partial \mathcal{L}^p}{\partial \dot{p}} + \mathcal{L}^p(p,\dot{p}).$$

The first two terms in equation (1) constitute a total time derivative. The equations of motion are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{p}} - \frac{\partial \mathcal{L}}{\partial p} = \frac{d}{dt}\frac{\partial \mathcal{L}^p}{\partial \dot{p}} - \frac{\partial \mathcal{L}^p}{\partial p} = 0, \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}^q}{\partial \dot{q}} - \frac{\partial \mathcal{L}^q}{\partial q} = 0. \tag{2}$$

The p and q in equations (1) and (2) are not, in general, canonical pairs. They are so only on actual trajectories and through a proper choice of the initial values. This gives the freedom of introducing a second set of canonical momenta for both p and q. One does this through the extended Lagrangian. Thus

$$\pi_p = \frac{\partial \mathcal{L}}{\partial \dot{p}} = \frac{\partial \mathcal{L}^p}{\partial \dot{p}} - q, \qquad \pi_q = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}^q}{\partial \dot{q}} - p.$$

Evidently, π_p and π_q vanish on actual trajectories and remain non zero on virtual ones. From these extended momenta, one defines an extended Hamiltonian,

$$\mathcal{H}(\pi_p,\pi_q,p,q) = \dot{p}\pi_p + \dot{q}\pi_q - \mathcal{L} = H(p+\pi_q,q) - H(p,q+\pi_p). \tag{3}$$

Using the canonical quantization rule, the following postulates are outlined:

- Let p, q, π_p and π_q be operators in a Hilbert space, X, of all complex functions, satisfying the following commutation relations

$$\begin{aligned} &[\pi_q, q] = -i\hbar, \qquad \pi_q = -i\hbar\frac{\partial}{\partial q}, \qquad [\pi_p, p] = -i\hbar, \qquad \pi_p = -i\hbar\frac{\partial}{\partial p}, \\ &[p, q] = [\pi_p, \pi_q] = [p, \pi_q] = [q, \pi_p] = 0. \end{aligned} \tag{4}$$

By virtue of equations (4), the extended Hamiltonian, $\mathcal{H}$ will also be an operator in X.

- A state function $\chi(p, q, t) \in X$ is assumed to satisfy the following dynamical equation

$$\begin{aligned} i\hbar\frac{\partial\chi}{\partial t} = \mathcal{H}\chi &= \left[H\left(p - i\hbar\frac{\partial}{\partial q}, q\right) - H\left(p, q - i\hbar\frac{\partial}{\partial p}\right)\right]\chi \\ &= \sum\frac{(-i\hbar)^n}{n!}\left\{\frac{\partial^n H}{\partial p^n}\frac{\partial^n}{\partial q^n} - \frac{\partial^n H}{\partial q^n}\frac{\partial^n}{\partial p^n}\right\}\chi. \end{aligned} \tag{5}$$

- The averaging rule for an observable $O(p, q)$, a c-number operator in this formalism, is given as

$$\langle O(p, q)\rangle = \int O(p, q)\chi^*(p, q, t)dpdq.$$

To find the solutions for equation (5) one may assume

$$\chi(p, q, t) = F(p, q, t)e^{-\frac{ipq}{\hbar}}. \tag{6}$$

The phase factor comes out due to the total derivatives in the Lagrangian of equation (1), $-\frac{d(pq)}{dt}$. Substituting equation (6) in equation (5) and eliminating the exponential factor gives

$$H\left(-i\hbar\frac{\partial}{\partial q}, q\right) - H\left(p, -i\hbar\frac{\partial}{\partial p}\right) = i\hbar\frac{\partial F}{\partial t}. \tag{7}$$

Equation (7) has separable solutions of the form

$$F(p, q, t) = \psi(q, t)\phi^*(p, t),$$

where $\psi(q, t)$ and $\phi(p, t)$ are the solutions of the Schrödinger equation in the q and p representations, respectively.

3 The extended canonical transformation

The canonical transformations that leaves the extended Hamilton equations form invariant are the extended canonical transformations [11]. Let us consider the following linear transformation

$$\begin{aligned} &p \to p' = p + \alpha\frac{\hbar^2}{f^2}\pi_p + \beta\pi_q, \qquad \pi_p \to \pi_{p'} = \pi_p, \\ &q \to q' = q + \alpha\pi_q + \beta\pi_p, \qquad \pi_q \to \pi_{q'} = \pi_q, \end{aligned} \tag{8}$$

where α and β are parameters to be determined and f is a positive parameter. The corresponding generator is

$$G(\pi_q, \pi_p) = G_1(\pi_q, \pi_p) + G_2(\pi_q, \pi_p) = \left(\frac{\hbar^2}{f^2}\frac{1}{2}\pi_p^2 + \frac{1}{2}\pi_q^2\right) + (\pi_p\pi_q),$$

and, the corresponding similarity transformation for the finite parameters $\alpha = -\frac{f}{2i\hbar}$ and $\beta = \frac{1}{2}$ becomes [15]

$$\hat{T} = e^{i\alpha\frac{\hat{G_1}}{\hbar}+i\beta\frac{\hat{G_2}}{\hbar}} = e^{\frac{\hbar^2}{4f}\frac{\partial^2}{\partial p^2}+\frac{f}{4}\frac{\partial^2}{\partial q^2}-\frac{i\hbar}{2}\frac{\partial^2}{\partial q\partial p}}. \tag{9}$$

It can be easily shown that the Wigner representation can be obtained by a canonical transformation or by the corresponding unitary transformation in the EPS [11]. The same technique could be applied to obtain the Wigner equation from the Schrödinger equation in the phase space [16] showing the relation between the EPS technique and the Later one. The above transformation is identical with a canonical transformation in the Wigner representation as follows

$$P \to p' = P + \alpha\frac{\hbar^2}{f^2}\pi_P, \qquad \pi_P \to \pi_{p'} = \pi_P, \qquad Q \to q' = Q + \alpha\pi_Q, \qquad \pi_Q \to \pi_{q'} = \pi_Q,$$

where the corresponding generator is

$$G'(\pi_Q, \pi_P) = \left(\frac{1}{2}\frac{\hbar^2}{f^2}\pi_P^2 + \frac{1}{2}\pi_Q^2\right).$$

The corresponding similarity transformation becomes

$$\hat{T}' = e^{i\alpha\frac{\hat{G}'}{\hbar}} = e^{\frac{\hbar^2}{4f}\frac{\partial^2}{\partial P^2}+\frac{f}{4}\frac{\partial^2}{\partial Q^2}}.$$

The set of p', q', $\pi_{p'}$, $\pi_{q'}$ now constitutes the Husimi representation [17]. The corresponding operation $\hat{T}'$ on the Wigner equation will give the evolution equation of the Husimi positive functions as follows

$$\hat{T}'\left(i\hbar\frac{\partial P_w}{\partial t}\right) = \hat{T}'(\mathcal{H}_w P_w),$$

where $\mathcal{H}_w$ and P_w are the Wigner operator and the Wigner function. Defining $\mathcal{H}_h = \hat{T}'\mathcal{H}_w\hat{T}'^{-1}$ as the Husimi operator and $P_h = \hat{T}'P_w$ as the Husimi function [18], one obtains the Husimi equation as follows

$$i\hbar\frac{\partial P_h}{\partial t} = \mathcal{H}_h P_h. \tag{10}$$

It can be easily shown that the Husimi and Wigner functions are related by the following integral transform

$$P_h(p', q', t) = \frac{1}{\pi\hbar}\int dQ \int dP e^{-\frac{(Q-q')^2}{f}-\frac{f(P-p')^2}{\hbar^2}} P_w(P, Q, t). \tag{11}$$

Equation (11), as the Husimi function, defines a class of non-negative functions. In the case of harmonic oscillator or radiation field, equation (11) for $f = \frac{\hbar}{m\omega}$ reduces to the well known Q-function [15]. We will come back to this point later in Section 5.

4 Quantum potential and generalization to the EPS

To generalize the concept of the quantum potential into EPS we have to obtain a p space counterpart of the quantum potential. However, unlike the q space, it does not have a simple form for a general potential $V(q)$ in p space (see details in [5] and [9]). As an example we consider the harmonic potential and obtain the modified Hamilton–Jacobi equation and quantum potential term.

The extended Hamiltonian of equation (3) for the harmonic potential, $V(q) = \frac{1}{2}kq^2$, becomes

$$\mathcal{H} = \frac{\pi_q^2}{2m} + \frac{p}{m}\pi_q - \frac{1}{2}k\pi_p^2 - kq\pi_p.$$

The state function χ in equation (5) is, in general, a complex function. Thus we assume

$$\chi(p,q,t) = \mathcal{R}(p,q,t)e^{\frac{i\mathcal{S}(p,q,t)}{\hbar}},$$

where $\mathcal{R}(p,q,t)$ is the amplitude and $\mathcal{S}(p,q,t)$ is the phase given by

$$\mathcal{S}(p,q,t) = \int^t \mathcal{L}(p,q,\dot{p},\dot{q},t')dt',$$

where $\mathcal{L}(p,q,\dot{p},\dot{q},t')$ is defined by equation (1). Using equation (1), one gets

$$\mathcal{S}(p,q,t) = \mathcal{S}^p + \mathcal{S}^q - pq,$$

where $\mathcal{S}^p$ and $\mathcal{S}^q$ are defined as follows, respectively [9]

$$\mathcal{S}^q(q,t) = \int^t \mathcal{L}^q(q,\dot{q},t')dt', \qquad \mathcal{S}^p(p,t) = \int^t \mathcal{L}^p(p,\dot{p},t')dt'.$$

Equation (5) now gives

$$\begin{aligned} i\hbar\left(\frac{\partial\mathcal{R}}{\partial t} + \frac{i\mathcal{R}}{\hbar}\frac{\partial\mathcal{S}}{\partial t}\right) &= -\frac{\hbar^2}{2m}\left[\frac{\partial^2\mathcal{R}}{\partial q^2} + \frac{2i}{\hbar}\frac{\partial\mathcal{R}}{\partial q}\frac{\partial\mathcal{S}}{\partial q} + \frac{i\mathcal{R}}{\hbar}\frac{\partial^2\mathcal{S}}{\partial q^2} - \frac{\mathcal{R}}{\hbar^2}\left(\frac{\partial\mathcal{S}}{\partial q}\right)^2\right] \\ &\quad - \frac{i\hbar p}{m}\left(\frac{\partial\mathcal{R}}{\partial q} + \frac{i\mathcal{R}}{\hbar}\frac{\partial\mathcal{S}}{\partial q}\right) + \frac{k\hbar^2}{2}\left[\frac{\partial^2\mathcal{R}}{\partial p^2} + \frac{2i}{\hbar}\frac{\partial\mathcal{R}}{\partial p}\frac{\partial\mathcal{S}}{\partial p} + \frac{i\mathcal{R}}{\hbar}\frac{\partial^2\mathcal{S}}{\partial p^2} - \frac{\mathcal{R}}{\hbar^2}\left(\frac{\partial\mathcal{S}}{\partial p}\right)^2\right] \\ &\quad + i\hbar kq\left(\frac{\partial\mathcal{R}}{\partial p} + \frac{i\mathcal{R}}{\hbar}\frac{\partial\mathcal{S}}{\partial p}\right). \end{aligned} \tag{12}$$

Assuming

$$\pi_p = \frac{\partial\mathcal{S}(p,q,t)}{\partial p}, \qquad \pi_q = \frac{\partial\mathcal{S}(p,q,t)}{\partial q}.$$

The real part of equation (12) gives

$$\frac{\partial\mathcal{S}}{\partial t} - \frac{\hbar^2}{2m}\frac{1}{\mathcal{R}}\frac{\partial^2\mathcal{R}}{\partial q^2} + \frac{\hbar^2 k}{2}\frac{1}{\mathcal{R}}\frac{\partial^2\mathcal{R}}{\partial p^2} + \mathcal{H} = 0, \tag{13}$$

equation (13) is the modified Hamilton–Jacobi equation for the harmonic potential in the EPS. The second and third terms in equation (13) together define the quantum potential in extended phase space. The second term is the EPS counterpart of quantum potential in q space and the third term is the same thing in p space.

5 Quantum potential-free representation

It was shown that in the Wigner representation the quantum potential was removed from the corresponding modified Hamilton–Jacobi equation for the linear and harmonic potentials [9]. Here, we look for another possible representation in which the quantum potential could be removed. With the use of the canonical transformation equation (8), its corresponding generator equation (9) and the Baker–Hausdorff relation, the new extended Hamiltonian for the Husimi function becomes

$$\begin{aligned}\mathcal{H}_h &= \exp\left[-\frac{f}{4\hbar^2}\pi_Q^2 - \frac{1}{4f}\pi_P^2\right]\mathcal{H}_w \exp\left[-\left(-\frac{f}{4\hbar^2}\pi_Q^2 - \frac{1}{4f}\pi_P^2\right)\right] \\ &= \mathcal{H}_w + i\left[-\frac{f}{4\hbar^2}\pi_Q^2 - \frac{1}{4f}\pi_P^2, \mathcal{H}_w\right] \\ &\quad + \frac{i^2}{2!}\left[-\frac{f}{4\hbar^2}\pi_Q^2 - \frac{1}{4f}\pi_P^2, \left[-\frac{f}{4\hbar^2}\pi_Q^2 - \frac{1}{4f}\pi_P^2, \mathcal{H}_w\right]\right] + \cdots .\end{aligned}$$

This Hamiltonian for the harmonic potential, $V(q) = \frac{1}{2}kq^2$ becomes

$$\mathcal{H}_h = \frac{P}{m}\pi_Q - kQ\pi_P + \left(\frac{i\hbar}{2mf} - \frac{ifk}{2\hbar}\right)\pi_Q\pi_P.$$

To have $\mathcal{H}_h$ in terms of the prime coordinates, one must follow the operator transformation rule. (See the paragraph before equation (10) and the Section 5.5 in [11].) The above Hamiltonian for $f = \frac{\hbar}{m\omega}$ changes to the Q-function Hamiltonian, $\mathcal{H}_Q$. It can be easily shown that the modified Hamilton–Jacobi equation for this Hamiltonian becomes

$$\mathcal{H}_Q + \frac{\partial \mathcal{S}_Q}{\partial t} = 0, \tag{14}$$

where $\mathcal{S}_Q$ is defined as Q-action. Equation (14) has the familiar form of the classical Hamilton–Jacobi equation. As noted before, equation (9) transforms equation (5) to the Husimi equation which for $f = \frac{\hbar}{m\omega}$ the Husimi equation transforms to the evolution equation for the Q-function [15]. Thus, we conclude that equation (14) in which the quantum potential disappears is in the Q representation.

6 Conclusions

In this paper the concept of the quantum potential in configuration space is generalized for the extended phase space. The Husimi functions are obtained by appropriate canonical transformations in the extended phase space. By means of the EPS formalism it is shown that the quantum potential is removed from the dynamical equation of the distribution function in the Husimi representation. Thus, the Hamilton–Jacobi equation takes its standard form in the extended phase space by excluding the quantum potential. Removing the quantum potential in the Husimi representation fixes the value of parameter involved and gives the well known Q-function. In addition to the present work, in a previous paper it was shown that the quantum potential was removed for the constant, the linear and the harmonic potentials in the Wigner representation [9]. Now we are planning to look for the representations where the quantum potential would be removed in the general case.

A Appendix

To obtain the differential form of equation (11), one can do as follows: we change the variables as

$$Q - q' = q, \qquad P - p' = p,$$

using the above variables, and one can rewrite equation (11) as follows:

$$P_h(p', q', t) = \frac{1}{\pi\hbar}\int dq \int dp e^{-\frac{q^2}{f} - \frac{fp^2}{\hbar^2}} P_w(p + p', q + q', t).$$

Now using the Taylor expansion, we obtain:

$$\int dq \int dp e^{-\frac{q^2}{f}-\frac{fp^2}{\hbar^2}} \left\{ P_w(p',q',t) + q\frac{\partial P_w(p,q,t)}{\partial q}\Big|_{q=q',p=p'} + p\frac{\partial P_w(p,q,t)}{\partial p}\Big|_{q=q',p=p'} \right.$$
$$\left. + \frac{q^2}{2}\frac{\partial^2 P_w(p,q,t)}{\partial q^2}\Big|_{q=q',p=p'} + \frac{p^2}{2}\frac{\partial^2 P_w(p,q,t)}{\partial p^2}\Big|_{q=q',p=p'} + \cdots \right\}$$
$$= P_w(p',q',t) + \frac{f}{4}\frac{\partial^2 P_w(p,q,t)}{\partial q^2}\Big|_{q=q',p=p'} + \frac{\hbar^2}{4f}\frac{\partial^2 P_w(p,q,t)}{\partial p^2}\Big|_{q=q',p=p'} + \cdots$$
$$= e^{\frac{f}{4}\frac{\partial^2}{\partial q^2}+\frac{\hbar^2}{4f}\frac{\partial^2}{\partial p^2}} P_w(p',q',t).$$

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