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Bäcklund Transformation for the BC-Type Toda Lattice*

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Abstract. We study an integrable case of n-particle Toda lattice: open chain with boundary terms containing 4 parameters. For this model we construct a Bäcklund transformation and prove its basic properties: canonicity, commutativity and spectrality. The Bäcklund transformation can be also viewed as a discretized time dynamics. Two Lax matrices are used: of order 2 and of order $2n + 2$, which are mutually dual, sharing the same spectral curve.

Key words: Bäcklund transformation; Toda lattice; integrability; boundary conditions; classical Lie algebras

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1 Introduction

In the present paper we study the Hamiltonian system of n one-dimensional particles with coordinates x_j and canonical momenta X_j, $j = 1, \dots, n$:

$$\{X_j, X_k\} = \{x_j, x_k\} = 0, \qquad \{X_j, x_k\} = \delta_{jk}, \tag{1.1}$$

characterized by the Hamiltonian

$$H = \sum_{j=1}^{n} \frac{1}{2} X_j^2 + \sum_{j=1}^{n-1} \mathrm{e}^{x_{j+1}-x_j} + \alpha_1 \mathrm{e}^{x_1} + \frac{1}{2}\beta_1 \mathrm{e}^{2x_1} + \alpha_n \mathrm{e}^{-x_n} + \frac{1}{2}\beta_n \mathrm{e}^{-2x_n} \tag{1.2}$$

containing 4 arbitrary parameters: α_1, β_1, α_n, β_n.

The model was missing from the early lists of integrable cases of the Toda lattice [1, 2] based on Dynkin diagrams for simple affine Lie algebras. Its integrability was proved first in [3, 4, 5]. As for the more recent classifications, in [6] the model is enlisted as the case (i). In [7, 8] particular cases of the Hamiltonian (1.2) are assigned to the $C_n^{(1)}$ case with 'Morse terms'. For brevity, we refer to the model as 'BC-Toda lattice' emphasising the fact that each boundary term is a linear combination of the term $\sim \alpha$ corresponding to the root system B and of the term $\sim \beta$ corresponding to the root system C, see [1, 2, 7, 8].

In section 2 we review briefly the known facts about the integrability of the model using the approach developed in [3, 4] and based on the Lax matrix $L(u)$ of order 2 and the corresponding quadratic r-matrix algebra. In particular, we construct explicitly a generating function of the complete set of commuting Hamiltonians H_j ($j = 1, \dots, n$) which includes the physical Hamiltonian H (1.2).

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In Section 3 we describe the main result of our paper: construction of a Bäcklund transformation (BT) for our model as a one-parametric family of maps $\mathcal{B}_\lambda : (Xx) \mapsto (Yy)$ from the variables (Xx) to the variables (Yy). We construct the BT choosing an appropriate gauge (or Darboux) transformation of the local Lax matrices. In Section 4, adopting the Hamiltonian point of view developed in [9, 10], we prove the basic properties of the BT:

1. Preservation of the commuting Hamiltonians $\mathcal{B}_\lambda : H_j(X,x) \mapsto H_j(Y,y)$.
2. Canonicity: preservation of the Poisson bracket (1.1).
3. Commutativity: $\mathcal{B}_{\lambda_1} \circ \mathcal{B}_{\lambda_2} = \mathcal{B}_{\lambda_2} \circ \mathcal{B}_{\lambda_1}$.
4. Spectrality: the fact that the graph of the BT is a Lagrangian manifold on which the 2-form

$$\Omega \equiv \sum_{j=1}^{n} \big(\mathrm{d}X_j \wedge \mathrm{d}x_j - \mathrm{d}Y_j \wedge \mathrm{d}y_j\big) - \mathrm{d}\ln\Lambda \wedge \mathrm{d}\lambda \tag{1.3}$$

vanishes. Here Λ is an eigenvalue of the matrix $L(\lambda)$. In other words, the parameter λ of the BT and its exponentiated canonical conjugate Λ lie on the spectral curve of $L(u)$:

$$\det\big(\Lambda - L(\lambda)\big) = 0. \tag{1.4}$$

We also prove the following expansion of $\mathcal{B}_\lambda$ in λ^{-1}

$$\mathcal{B}_\lambda : f \mapsto f - 2\lambda^{-1}\{H, f\} + O(\lambda^{-2}), \qquad \lambda \to \infty. \tag{1.5}$$

which allows to interpret the BT as a discrete time dynamics approximating the continuous-time dynamics generated by the Hamiltonian (1.2).

In Section 5 we construct for our system an alternative Lax matrix $\mathcal{L}(v)$. The new Lax matrix of order $2n+2$ is dual to the matrix $L(u)$ of order 2 in the sense that they share the same spectral curve with the parameters u and v having been swapped:

$$\det\big(v - L(u)\big) = (-1)^{n+1} v \det\big(u - \mathcal{L}(v)\big). \tag{1.6}$$

In the same section we provide an interpretation of the BT in terms of the 'big' Lax matrix $\mathcal{L}(v)$ and establish a remarkable factorization formula for $\lambda^2 - \mathcal{L}^2(v)$.

The concluding Section 6 contains a summary and a discussion. All the technical proofs and tedious calculations are removed to the Appendices.

2 Integrability of the model

In demonstrating the integrability of the model we follow the approach to the integrable chains with boundary conditions developed in [3, 4] and use the notation of [9, 10].

The Lax matrix $L(u)$ for the BC-Toda lattice is constructed as the product

$$L(u) = K_-(u) T^t(-u) K_+(u) T(u) \tag{2.1}$$

of the following matrices (T^t stands for the matrix transposition).

The monodromy matrix $T(u)$ is itself the product

$$T(u) = \ell_n(u) \cdots \ell_1(u) \tag{2.2}$$

of the local Lax matrices

$$\ell_j(u) \equiv \ell(u; X_j, x_j) = \begin{pmatrix} u + X_j & -\mathrm{e}^{x_j} \\ \mathrm{e}^{-x_j} & 0 \end{pmatrix}, \tag{2.3}$$

each containing only the variables X_j, x_j describing a single particle. Note that $\operatorname{tr} T(u)$ is the generating function for the Hamiltonians of the periodic Toda lattice.

The matrices $K_\pm(u)$ containing the information about the boundary interactions are defined as [3, 4]

$$K_-(u) = \begin{pmatrix} u & -\alpha_1 \\ \alpha_1 & \beta_1 u \end{pmatrix}, \qquad K_+(u) = \begin{pmatrix} u & -\alpha_n \\ \alpha_n & \beta_n u \end{pmatrix}. \tag{2.4}$$

The significance of the Lax matrix $L(u)$ is that its spectrum is invariant under the dynamics generated by the Hamiltonian (1.2), the corresponding equations of motion $\mathrm{d}G/\mathrm{d}t \equiv \dot{G} = \{H, G\}$ for an observable G being

$$\dot{x}_j = X_j, \qquad j = 1, \ldots, n \tag{2.5}$$

and

$$\dot{X}_j = \mathrm{e}^{x_{j+1}-x_j} - \mathrm{e}^{x_j - x_{j-1}}, \qquad j = 2, \ldots, n-1, \tag{2.6a}$$

$$\dot{X}_1 = \mathrm{e}^{x_2 - x_1} - \alpha_1 \mathrm{e}^{x_1} - \beta_1 \mathrm{e}^{2x_1}, \tag{2.6b}$$

$$\dot{X}_n = -\mathrm{e}^{x_n - x_{n-1}} + \alpha_n \mathrm{e}^{-x_n} + \beta_n \mathrm{e}^{-2x_n}. \tag{2.6c}$$

To prove the invariance of the spectrum of $L(u)$ we introduce the matrices $A_j(u)$

$$A_j(u) = \begin{pmatrix} -u & \mathrm{e}^{x_j} \\ -\mathrm{e}^{-x_{j-1}} & 0 \end{pmatrix}, \qquad j = 2, \ldots, n-1, \tag{2.7}$$

$$A_1(u) = \begin{pmatrix} -u & \mathrm{e}^{x_1} \\ -\alpha_1 - \beta_1 \mathrm{e}^{x_1} & 0 \end{pmatrix}, \qquad A_{n+1}(u) = \begin{pmatrix} -u & \alpha_n + \beta_n \mathrm{e}^{-x_n} \\ -\mathrm{e}^{-x_n} & 0 \end{pmatrix}, \tag{2.8}$$

which satisfy the easily verified identities

$$\dot{\ell}_j = A_{j+1}\ell_j - \ell_j A_j, \qquad j = 1, \ldots, n, \tag{2.9}$$

$$-\dot{K}_+ = 0 = K_+ A_{n+1}(u) + A^t_{n+1}(-u) K_+, \tag{2.10a}$$

$$\dot{K}_- = 0 = A_1(u) K_- + K_- A^t_1(-u). \tag{2.10b}$$

From (2.2) and (2.9) it follows immediately that

$$\dot{T}(u) = A_{n+1}(u) T(u) - T(u) A_1(u). \tag{2.11}$$

Then, using (2.1) and (2.10), we obtain the equality

$$\dot{L}(u) = \big[A_1(u), L(u)\big] \tag{2.12}$$

implying that the spectrum of $L(u)$ is preserved by the dynamics.

There are only two spectral invariants of a 2×2 matrix: the trace and the determinant. From (2.3) it follows that $\det \ell(u) = 1$ and, respectively, $\det T(u) = 1$, so, by (2.1), the determinant of $L(u)$

$$d(u) \equiv \det L(u) = \det K_-(u) \det K_+(u) = (\alpha_1^2 + \beta_1 u^2)(\alpha_n^2 + \beta_n u^2) \tag{2.13}$$

contains no dynamical variables Xx. The trace

$$t(u) \equiv \operatorname{tr} L(u) = \operatorname{tr} K_-(u) T^t(-u) K_+(u) T(u), \tag{2.14}$$

however, does contain dynamical variables and therefore can be used as a generating function of the integrals of motion, which can be chosen as the coefficients of the polynomial $t(u)$ of degree $2n+2$ in u. Note that $t(-u) = t(u)$ due to the symmetry

$$K_\pm^t(-u) = -K_\pm(u). \tag{2.15}$$

The leading coefficient of $t(u)$ at u^{2n+2} is a constant $(-1)^n$. Same is true for its free term

$$t(0) = \operatorname{tr} K_+(0)K_-(0) = -2\alpha_n\alpha_1 \tag{2.16}$$

due to the identity

$$MK_\pm(0)M^t = \det M \cdot K_\pm(0), \tag{2.17}$$

which holds for any matrix M.

We are left then with n nontrivial coefficients H_j

$$t(u) = (-1)^n u^{2n+2} - 2\alpha_n\alpha_1 + \sum_{j=1}^{n} H_j u^{2j} \tag{2.18}$$

which are integrals of motion $\dot{H}_j = 0$ since $\dot{t}(u) = 0$ due to (2.12).

The conserved quantities H_j are obviously polynomial in X, $\mathrm{e}^{\pm x}$. Their independence can easily be established by setting $\mathrm{e}^{\pm x} = 0$ in (2.3) and analysing the resulting polynomials in X. It is also easy to verify that the physical Hamiltonian (1.2) is expressed as

$$H = \frac{(-1)^{n+1}}{2} H_n. \tag{2.19}$$

The quantities H_j are also in involution

$$\{H_j, H_k\} = 0 \tag{2.20}$$

with respect to the Poisson bracket (1.1). Together with the independence of H_j, it constitutes the Liouville integrability of our system.

The commutativity (2.20) of H_j or, equivalently, of $t(u)$

$$\{t(u_1), t(u_2)\} = 0 \tag{2.21}$$

is proved in the standard way using the r-matrix technique [3, 4].

Let $\mathbf{1}$ be the unit matrix of order 2 and for any matrix L define

$$\overset{1}{L} \equiv L \otimes \mathbf{1}, \qquad \overset{2}{L} \equiv \mathbf{1} \otimes L. \tag{2.22}$$

We have then the quadratic Poisson brackets [10, 11]

$$\{\overset{1}{\ell}(u_1), \overset{2}{\ell}(u_2)\} = [r(u_1 - u_2), \overset{1}{\ell}(u_1)\overset{2}{\ell}(u_2)], \tag{2.23}$$

and, as a consequence,

$$\{\overset{1}{T}(u_1), \overset{2}{T}(u_2)\} = [r(u_1 - u_2), \overset{1}{T}(u_1)\overset{2}{T}(u_2)], \tag{2.24}$$

with the r-matrix

$$r(u) = \frac{\mathcal{P}}{u}, \tag{2.25}$$

where $\mathcal{P}$ is the permutation matrix $\mathcal{P} a \otimes b = b \otimes a$.

Let

$$\widetilde{r}(u) = r^{t_1}(u) = r^{t_2}(u), \tag{2.26}$$

t_1 and t_2 being, respectively, transposition with respect to the first and second component of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Then for both $\mathcal{T}(u) = T(u)K_-(u)T^t(-u)$ and $\mathcal{T}(u) = T^t(-u)K_+(u)T(u)$ we obtain the same Poisson algebra [3, 4]

$$\begin{aligned}\{\overset{1}{\mathcal{T}}(u_1), \overset{2}{\mathcal{T}}(u_2)\} = r(u_1 - u_2)\overset{1}{\mathcal{T}}(u_1)\overset{2}{\mathcal{T}}(u_2) - \overset{1}{\mathcal{T}}(u_1)\overset{2}{\mathcal{T}}(u_2)r(u_1 - u_2)\\ - \overset{1}{\mathcal{T}}(u_1)\widetilde{r}(u_1 + u_2)\overset{2}{\mathcal{T}}(u_2) + \overset{2}{\mathcal{T}}(u_2)\widetilde{r}(u_1 + u_2)\overset{1}{\mathcal{T}}(u_1),\end{aligned} \tag{2.27}$$

which ensures the commutativity (2.21) of $t(u)$.

3 Describing Bäcklund transformation

In this section we shall construct a Bäcklund transformation (BT) for our model. We shall stay in the framework of the Hamiltonian approach proposed in [9] and follow closely our previous treatment of the periodic Toda lattice [9, 10], with the necessary modifications taking into account the boundary conditions.

We are looking thus for a one-parametric family of maps $\mathcal{B}_\lambda : (Xx) \mapsto (Yy)$ from the variables (Xx) to the variables (Yy) characterised by the properties enlisted in the Introduction: *Invariance of Hamiltonians, Canonicity, Commutativity and Spectrality.*

The invariance of the commuting Hamiltonians H_j, or of their generating polynomial $t(u) = \operatorname{tr} L(u)$ will be ensured if we find an invertible matrix $M_1(u,\lambda)$ intertwining the matrices $L(u)$ depending on the variables Xx and Yy:

$$M_1(u,\lambda)L(u;Y,y) = L(u;X,x)M_1(u,\lambda). \tag{3.1}$$

To find $M_1(u,\lambda)$ let us look for a gauge transformation

$$M_{j+1}(u,\lambda)\ell(u;Y_j,y_j) = \ell(u;X_j,x_j)M_j(u,\lambda), \qquad j = 1,\ldots,n, \tag{3.2}$$

implying that $\det M_j$ does not depend on j. From (3.2) and (2.2) we obtain

$$M_{n+1}(u,\lambda)T(u;Y,y) = T(u;X,x)M_1(u,\lambda). \tag{3.3}$$

Let J be the the standard skew-symmetric matrix of order 2

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J^t = -J, \qquad J^2 = -\mathbf{1}, \tag{3.4}$$

and define the antipode M^a as

$$M^a \equiv -JMJ \tag{3.5}$$

for any matrix M of order 2. It is easy to see that

$$M^t M^a = M^a M^t = \det M. \tag{3.6}$$

Transposing (3.3) and using (3.6) together with the the fact that $\det M_j$ is independent of j we obtain the relation

$$T^t(-u;X,x)M^a_{n+1}(-u,\lambda) = M^a_1(-u,\lambda)T^t(-u;Y,y). \tag{3.7}$$

We shall be able to obtain (3.1) if we impose two additional relations

$$K_-(u)M_1^a(-u,\lambda) = M_1(u,\lambda)K_-(u), \tag{3.8a}$$
$$K_+(u)M_{n+1}(u,\lambda) = M_{n+1}^a(-u,\lambda)K_+(u). \tag{3.8b}$$

Then, starting with the right-hand side $L(u;X,x)M_1(u,\lambda)$ of (3.1) and using (2.1) and (3.3) we obtain

$$\begin{aligned} L(u;X,x)M_1(u,\lambda) &= K_-(u)T^t(-u;X,x)K_+(u)T(u;X,x)M_1(u,\lambda) \\ &= K_-(u)T^t(-u;X,x)K_+(u)M_{n+1}(u,\lambda)T(u;Y,y) \end{aligned} \tag{3.9}$$

Using then (3.8b) to move $M_{n+1}(u,\lambda)$ through $K_+(u)$, then using (3.7) and finally (3.8a) we get, step by step,

$$\begin{aligned} L(u;X,x)M_1(u,\lambda) &= K_-(u)T^t(-u;X,x)M_{n+1}^a(-u,\lambda)K_+(u)T(u;Y,y) \\ &= K_-(u)M_1^a(-u,\lambda)T^t(-u;Y,y)K_+(u)T(u;Y,y) \\ &= M_1(u,\lambda)K_-(u)T^t(-u;Y,y)K_+(u)T(u;Y,y) \\ &= M_1(u,\lambda)L(u;Y,y) \end{aligned} \tag{3.10}$$

arriving finally at (3.1).

We have thus to find a set of matrices $M_j(u,\lambda)$, $j=1,\dots,n+1$ compatible with the conditions (3.2) and (3.8). A quick calculation shows that the so called DST-ansatz for M_j used in [9, 10] for the periodic Toda lattice contradicts the conditions (3.8).

The philosophy advocated in [10] requires that the ansatz for the gauge matrix $M_j(u)$ be chosen in the form of a Lax matrix satisfying the r-matrix Poisson bracket (2.23) with the same r-matrix (2.25) as the Lax operator $\ell(u)$. It was shown in [10] that the so-called DST-ansatz

$$M_j^{\mathrm{DST}}(u,\lambda) = \begin{pmatrix} u-\lambda+s_jS_j & -s_j \\ S_j & -1 \end{pmatrix} \tag{3.11}$$

serves well for the periodic Toda case. The above ansatz is however not compatible with the boundary conditions (3.8) and we have to use a more complicated ansatz for M_j in the form of the Lax matrix for the isotropic Heisenberg magnet (XXX-model):

$$M_j(u,\lambda) = \begin{pmatrix} u-\lambda+s_jS_j & s_j^2S_j - 2\lambda s_j \\ S_j & -u-\lambda+s_jS_j \end{pmatrix}, \qquad \det M_j(u,\lambda) = \lambda^2 - u^2. \tag{3.12}$$

The same gauge transformation was used in [12] for constructing a Q-operator for the quantum XXX-magnet.

Substituting (3.12) into (3.2) we obtain the relations

$$X_j = -\lambda + s_j^{-1}\mathrm{e}^{x_j} + s_{j+1}\mathrm{e}^{-x_j}, \tag{3.13a}$$
$$Y_j = \lambda - s_j^{-1}\mathrm{e}^{y_j} - s_{j+1}\mathrm{e}^{-y_j}, \tag{3.13b}$$
$$S_j = 2\lambda s_j^{-1} - s_j^{-2}\mathrm{e}^{x_j} - s_j^{-2}\mathrm{e}^{y_j}, \tag{3.13c}$$
$$S_{j+1} = \mathrm{e}^{-x_j} + \mathrm{e}^{-y_j}, \tag{3.13d}$$

for $j=1,\dots,n$, and from (3.8), respectively,

$$S_1 = \frac{2(\alpha_1+\beta_1\lambda s_1)}{1+\beta_1 s_1^2}, \qquad S_{n+1} = \frac{2(\lambda s_{n+1}-\alpha_n)}{\beta_n + s_{n+1}^2}. \tag{3.14}$$

Eliminating the variables S_j, we arrive to the equations defining the BT ($j = 1, \ldots, n$):

$$X_j = -\lambda + s_j^{-1} e^{x_j} + s_{j+1} e^{-x_j}, \tag{3.15a}$$

$$Y_j = \lambda - s_j^{-1} e^{y_j} - s_{j+1} e^{-y_j}. \tag{3.15b}$$

The variables s_j, $j = 1, \ldots, n+1$ in (3.15) are implicitly defined as functions of x, y and λ from the quadratic equations

$$\left(e^{-x_{j-1}} + e^{-y_{j-1}}\right) s_j^2 - 2\lambda s_j + \left(e^{x_j} + e^{y_j}\right) = 0, \qquad j = 2, \ldots, n \tag{3.16a}$$

$$\left(2\alpha_1 + \beta_1 e^{x_1} + \beta_1 e^{y_1}\right) s_1^2 - 2\lambda s_1 + \left(e^{x_1} + e^{y_1}\right) = 0, \tag{3.16b}$$

$$\left(e^{-x_n} + e^{-y_n}\right) s_{n+1}^2 - 2\lambda s_{n+1} + \left(2\alpha_n + \beta_n e^{-x_n} + \beta_n e^{-y_n}\right) = 0. \tag{3.16c}$$

Like in the periodic case [9, 10], the BT map $\mathcal{B}_\lambda : (Xx) \mapsto (Yy)$ is described implicitly by the equations (3.15). Unlike the periodic case, we have extra variables s_j. It is more convenient not to express s_j from equations (3.16) and to substitute them into (3.15) but rather define the BT by the whole set of equations (3.15) and (3.16).

Equations (3.15) and (3.16) are algebraic equations and therefore define (Yy) as multivalued functions of (Xx), which is a common situation with integrable maps [13].

In this paper, to avoid the complications of the real algebraic geometry we allow all our variables to be complex.

4 Properties of the Bäcklund transformation

Having defined the map $\mathcal{B}_\lambda : (Xx) \mapsto (Yy)$ in the previous section, we proceed to establish its properties from the list given in the Introduction.

4.1 Preservation of Hamiltonians

The equality $H_j(X, x) = H_j(Y, y)\ \forall\ \lambda$, or, equivalently, $t(u; X, x) = t(u; Y, y)$ holds by construction, being a direct consequence of (3.1).

4.2 Canonicity

The canonicity of the BT means that the variables $Y(X, x; \lambda)$ and $y(X, x; \lambda)$ have the same canonical Poisson brackets (1.1) as (Xx). An equivalent formulation can be given in terms of symplectic spaces and Lagrangian manifolds. Consider the $4n$-dimensional symplectic space V_{4n} with coordinates $XxYy$ and symplectic 2-form

$$\Omega_{4n} \equiv \sum_{j=1}^{n} \left(dX_j \wedge dx_j - dY_j \wedge dy_j\right). \tag{4.1}$$

Equations (3.15) and (3.16) define a $2n$-dimensional submanifold $\Gamma_{2n} \subset V_{4n}$ which can be considered as the graph $Y = Y(X, x; \lambda)$, $y = y(X, x; \lambda)$ of the BT (the parameter λ is assumed here to be a constant). The canonicity of the BT is then equivalent to the fact that the manifold Γ_{2n} is *Lagrangian*, meaning that: (a) it is *isotropic*, that is nullifies the form Ω_{4n}

$$\Omega_{4n}|_{\Gamma_{2n}} = 0, \tag{4.2}$$

and (b) it has maximal possible dimension for an isotropic manifold: $\dim \Gamma_{2n} = \frac{1}{2} \dim V_{4n}$.

One way of proving the canonicity is to present explicitly the generating function $\Phi_\lambda(y;x)$ of the canonical transformation, such that

$$X_j = \frac{\partial \Phi_\lambda}{\partial x_j}, \qquad Y_j = -\frac{\partial \Phi_\lambda}{\partial y_j}. \tag{4.3}$$

The required function is given by the expression

$$\Phi_\lambda(y;x) = \sum_{j=1}^{n} f_\lambda(y_j, s_{j+1}; x_j, s_j) + \varphi_\lambda^{(0)}(s_1) + \varphi_\lambda^{(n+1)}(s_{n+1}), \tag{4.4}$$

where

$$f_\lambda(y_j, s_{j+1}; x_j, s_j) = \lambda(2\ln s_j - x_j - y_j) + s_j^{-1}(\mathrm{e}^{x_j} + \mathrm{e}^{y_j}) - s_{j+1}(\mathrm{e}^{-x_j} + \mathrm{e}^{-y_j}), \tag{4.5a}$$

$$\varphi_\lambda^{(0)}(s_1) = -\lambda \ln\bigl(1 + \beta_1 s_1^2\bigr) - \frac{2\alpha_1}{\sqrt{\beta_1}} \arctan\bigl(\sqrt{\beta_1} s_1\bigr), \tag{4.5b}$$

$$\varphi_\lambda^{(n+1)}(s_{n+1}) = \lambda \ln\bigl(\beta_n + s_{n+1}^2\bigr) - \frac{2\alpha_n}{\sqrt{\beta_n}} \arctan\left(\frac{s_{n+1}}{\sqrt{\beta_n}}\right), \tag{4.5c}$$

and $s_j(x, y; \lambda)$ are defined implicitly through (3.16).

Equalities (4.3) can be verified by a direct, though tedious, computation. Another, more elegant, way is to use the argument from [10] based on imposing a set of constraints in an extended phase space, see Appendix A.

4.3 Commutativity

The commutativity $\mathcal{B}_{\lambda_1} \circ \mathcal{B}_{\lambda_2} = \mathcal{B}_{\lambda_2} \circ \mathcal{B}_{\lambda_1}$ of the BT follows from the preservation of the complete set of Hamiltonians and the canonicity by the standard argument [9, 10] based on Veselov's theorem [13] about the action-angle representation of integrable maps.

4.4 Spectrality

The spectrality property formulated first in [9] generalises the canonicity by allowing the parameter λ of the BT to be a dynamical variable like x and y.

Let us extend the symplectic space V_{4n} from section 4.2 to a $(4n + 2)$-dimensional space V_{4n+2} by adding two more coordinates λ, μ and defining the extension Ω_{4n+2} of symplectic form Ω_{4n} (4.1) as

$$\Omega_{4n+2} \equiv \Omega_{4n} - \mathrm{d}\mu \wedge \mathrm{d}\lambda = \sum_{j=1}^{n} \bigl(\mathrm{d}X_j \wedge \mathrm{d}x_j - \mathrm{d}Y_j \wedge \mathrm{d}y_j\bigr) - \mathrm{d}\mu \wedge \mathrm{d}\lambda. \tag{4.6}$$

Define the extended graph Γ_{2n+1} of the BT by equations (3.15) and a new equation

$$\mu = -\frac{\partial}{\partial \lambda} \Phi_\lambda(y;x). \tag{4.7}$$

The 2-form Ω_{4n+2} obviously vanishes on Γ_{2n+1}, and the manifold Γ_{2n+1} is lagrangian.

An amazing fact is that e^μ is proportional to an eigenvalue of the matrix $L(\lambda)$, see (1.4). In fact, the two eigenvalues of $L(\lambda)$ can be found explicitly to be

$$\Lambda = (\alpha_n^2 + \beta_n \lambda^2) \frac{1 + \beta_1 s_1^2}{\beta_n + s_{n+1}^2} \prod_{j=1}^{n} \bigl(-s_j^{-2} e^{x_j + y_j}\bigr), \tag{4.8a}$$

$$\widetilde{\Lambda} = (\alpha_1^2 + \beta_1\lambda^2)\frac{\beta_n + s_{n+1}^2}{1+\beta_1 s_1^2}\prod_{j=1}^{n}\bigl(-s_j^2 e^{-x_j-y_j}\bigr), \tag{4.8b}$$

see Appendix B for the proof.

Having the explicit formulae (4.8a) for Λ and (4.4) for $\Phi_\lambda(y;x)$ one can easily verify that

$$\Lambda = (-1)^n(\alpha_n^2 + \beta_n\lambda^2)\,\mathrm{e}^{\mu}. \tag{4.9}$$

4.5 Bäcklund transformation as discrete time dynamics

One of applications of a BT is that it might provide a discrete-time approximation of a continuous-time integrable system [14, 15]. Indeed, iterations of the canonical map $\mathcal{B}_\lambda$ generate a discrete time dynamics. Furthermore, if we find a point $\lambda = \lambda_0$ that (a) the map $\mathcal{B}_{\lambda_0}$ becomes the identity map, and (b) in a neighbourhood of λ_0 the infinitesimal map $\mathcal{B}_{\lambda_0+\varepsilon} \sim \varepsilon\{H,\cdot\}$ reproduces the Hamiltonian flow with the Hamiltonian (1.2), we can claim that $\mathcal{B}_\lambda$ is a discrete time approximation of the BC-Toda lattice. An attractive feature of this approximation is that, unlike some others [14], the discrete-time system and the continuous-time one share the same integrals of motion.

In our case $\lambda_0 = \infty$. Letting $\varepsilon = \lambda^{-1}$ and assuming the ansatz

$$y_j = x_j + O(\varepsilon), \qquad j = 1,\dots,n \tag{4.10}$$

we obtain from (3.16a) and (3.16b) the expansion

$$s_j = \varepsilon\mathrm{e}^{x_j} + O(\varepsilon^2), \qquad j = 1,\dots,n \tag{4.11a}$$

and from (3.16c) the expansion

$$s_{n+1} = \varepsilon(\alpha_n + \beta_n\mathrm{e}^{-x_n}) + O(\varepsilon^2). \tag{4.11b}$$

Substituting then expansions (4.10) into equation (3.13d) we obtain

$$S_j = 2\mathrm{e}^{-x_{j-1}} + O(\varepsilon), \qquad j = 2,\dots,n+1 \tag{4.12a}$$

and substituting expansion (4.11) for s_1 into formula (3.14) for S_1 we obtain

$$S_1 = (2\alpha_1 + \beta_1\mathrm{e}^{x_1}) + O(\varepsilon). \tag{4.12b}$$

Then from (3.12) we have

$$-\varepsilon M_j = \mathbf{1} + \varepsilon\bigl(u\mathbf{1} + 2A_j\bigr) + O(\varepsilon^2), \qquad j = 1,\dots,n+1, \tag{4.13}$$

where A_j coincides with the matrix (given by (2.7) and (2.8)) which describes the continuous-time dynamics of the Lax matrix. From (3.2) we obtain then

$$\ell(u;Y_j,y_j) = \ell(u;X_j,x_j) - 2\varepsilon\bigl(A_{j+1}\ell(u;X_j,x_j) - \ell(u;X_j,x_j)A_j\bigr) + O(\varepsilon^2), \tag{4.14}$$

for $j = 1,\dots,n+1$. Comparing the result to (2.9) we get the expansion (1.5).

5 Dual Lax matrix

Many integrable systems possess a pair of Lax matrices sharing the same spectral curve with the parameters u and v swapped like in (1.6), see [16] for a list of examples and a discussion. In particular, the periodic n-particle Toda lattice has two Lax matrices: the 'small' one, of order 2 [11], and the 'big' one, of order n [17]. For various degenerate cases of the BC-Toda lattice 'big' Lax matrices are also known [2, 7, 8, 17].

In this section we present a new Lax matrix of order $2n+2$ for the most general, 4-parametric BC-Toda lattice. Here we describe the result, removing the detailed derivation to Appendix C.

Let E_{jk} be the square matrix of order $2n+2$ with the only nonzero entry $(E_{jk})_{jk}=1$. The Lax matrix $\mathcal{L}(v)$ is then described for the generic case $n\geq 3$ as

$$\begin{aligned}
\mathcal{L}(v) &= \sum_{j,k=1}^{n} \mathcal{L}_{jk}E_{jk} \\
&= \sum_{j=2}^{n} \mathrm{e}^{x_j-x_{j-1}} E_{j,j-1} + \sum_{j=1}^{n}\big(-X_jE_{jj}+E_{j,j+1}\big) \\
&\quad - \sum_{j=1}^{n-1} \mathrm{e}^{x_{j+1}-x_j} E_{2n+2-j,2n+1-j} + \sum_{j=1}^{n}\big(X_jE_{2n+2-j,2n+2-j}-E_{2n+2-j,2n+3-j}\big) \\
&\quad + \left(\alpha_n\mathrm{e}^{-x_n}+\frac{\beta_n}{2}\mathrm{e}^{-2x_n}\right)\big(E_{n+1,n}-E_{n+2,n+1}\big) \\
&\quad + \frac{\beta_n}{2}\mathrm{e}^{-x_n-x_{n-1}}\big(E_{n+3,n}-E_{n+2,n-1}\big) - E_{n+1,n+2} \\
&\quad - \left(\alpha_1\mathrm{e}^{x_1}+\frac{\beta_1}{2}\mathrm{e}^{2x_1}\right)\big(E_{2n+2,2n+1}+v^{-1}E_{1,2n+2}\big) \\
&\quad + \frac{\beta_1}{2v}\mathrm{e}^{x_1+x_2}\big(E_{2,2n+1}-E_{1,2n}\big) - vE_{2n+2,1}
\end{aligned} \tag{5.1}$$

and consists of a bulk 'Jacobian' strip (the main diagonal and two adjacent diagonals) which reproduces the Lax matrix for the open Toda lattice together with boundary blocks containing parameters $\alpha_1\beta_1\alpha_n\beta_n$. We do not consider here the special case of small dimensions $n=1,2$ when the two boundary blocks interfere with each other and the structure of the Lax matrices becomes more complicated

To help visualise the matrix $\mathcal{L}(v)$ we present an illustration for the case $n=3$, using the shorthand notation $\xi_j\equiv\mathrm{e}^{x_j}$, $\eta_j\equiv\mathrm{e}^{y_j}$:

$$\mathcal{L}(v) = \begin{pmatrix}
-X_1 & 1 & 0 & 0 & 0 & -\frac{\beta_1}{2v}\xi_1\xi_2 & 0 & \frac{\alpha_1}{v}\xi_1+\frac{\beta_1}{2v}\xi_1^2 \\
\frac{\xi_2}{\xi_1} & -X_2 & 1 & 0 & 0 & 0 & \frac{\beta_1}{2v}\xi_1\xi_2 & 0 \\
0 & \frac{\xi_3}{\xi_2} & -X_3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\alpha_3}{\xi_3}+\frac{\beta_3}{2\xi_3^2} & 0 & -1 & 0 & 0 & 0 \\
0 & -\frac{\beta_3}{2\xi_2\xi_3} & 0 & -\frac{\alpha_3}{\xi_3}-\frac{\beta_3}{2\xi_3^2} & X_3 & -1 & 0 & 0 \\
0 & 0 & \frac{\beta_3}{2\xi_2\xi_3} & 0 & -\frac{\xi_3}{\xi_2} & X_2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\xi_2}{\xi_1} & X_1 & -1 \\
-v & 0 & 0 & 0 & 0 & 0 & -\alpha_1\xi_1-\frac{\beta_1}{2}\xi_1^2 & 0
\end{pmatrix}. \tag{5.2}$$

The matrix $\mathcal{L}(v)$ possesses the symmetry

$$\mathcal{L}(v) = -C_v\mathcal{L}^t(v)C_v^{-1}, \tag{5.3}$$

where

$$C_v = -vE_{2n+2,2n+2} + \sum_{j=1}^{2n+1} E_{j,2n+2-j} = \left(\begin{array}{ccccc|c} 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & -v \end{array}\right) \tag{5.4}$$

(note that $C_v^{-1} = C_{v^{-1}}$).

The matrix $\mathcal{L}(v)$ shares the same spectral curve with the 'small' Lax operator $L(u)$ satisfying the determinantal identity (1.6) and thus generates the same commuting Hamiltonians H_j.

The Lax matrix $\mathcal{L}(v)$ of order $2n+2$ seems to be new. When one or more of the constants $\alpha_1\beta_1\alpha_n\beta_n$ vanish it degenerates (with a drop of dimension) into known Lax matrices for simple affine Lie algebras [2, 7, 8, 17]. For the general 4-parametric case a Lie-algebraic interpretation of $\mathcal{L}(v)$ is still unknown. In particular, it is an interesting question whether $\mathcal{L}(v)$ satisfies a kind of r-matrices Poisson algebra.

Inozemtsev [5] presented a different Lax matrix for the BC-Toda lattice, of order $2n$ instead of $2n+2$ and with a more complicated dependence on the spectral parameter. The relation of these two Lax matrices is yet to be investigated.

For the dynamics (2.5), (2.6) we have an analog of the Lax equation (2.12):

$$\dot{\mathcal{L}(v)} = [\mathcal{A}(v), \mathcal{L}(v)] \tag{5.5}$$

with $\mathcal{A}(v)$ defined as

$$\begin{aligned} \mathcal{A}(v) = {} & \sum_{j=1}^{n} \big(X_j E_{jj} - E_{j,j+1} - X_j E_{2n+2-j,2n+2-j} + E_{2n+2-j,2n+3-j}\big) + E_{n+1,n+1} \\ & + vE_{2n+2,1} - \frac{\beta_1}{2} \mathrm{e}^{2x_1} \big(E_{2n+2,2n+1} + v^{-1} E_{1,2n+2}\big) + \frac{\beta_n}{2} \mathrm{e}^{-2x_n} \big(E_{n+1,n} - E_{n+2,n}\big) \end{aligned} \tag{5.6}$$

and satisfying

$$\mathcal{A}(v) C_v + C_v \mathcal{A}^t(v) = 0. \tag{5.7}$$

The analog of the formula (3.1) for the Bäcklund transformation is

$$\mathcal{M}(v,\lambda)\mathcal{L}(v;Y,y) = \mathcal{L}(v;X,x)\mathcal{M}(v,\lambda), \tag{5.8a}$$

$$\widetilde{\mathcal{M}}(v,\lambda)\mathcal{L}(v;X,x) = \mathcal{L}(v;Y,y)\widetilde{\mathcal{M}}(v,\lambda), \tag{5.8b}$$

where $\mathcal{M}(v)$ is given by

$$\begin{aligned} \mathcal{M}(v) = {} & \sum_{j,k=1}^{n} \mathcal{M}_{jk} E_{jk} = -\sum_{j=2}^{n} \frac{\xi_j}{\eta_{j-1}} E_{j,j-1} + \sum_{j=1}^{n} \left(\frac{s_{j+1}}{\eta_j} - \frac{\xi_j}{s_j}\right) E_{jj} + E_{j,j+1} \\ & + \sum_{j=1}^{n-1} \frac{\eta_{j+1}}{\xi_j} E_{2n+2-j,2n+1-j} + \sum_{j=1}^{n} \left(\frac{s_{j+1}}{\xi_j} - \frac{\eta_j}{s_j}\right) E_{2n+2-j,2n+2-j} - E_{2n+2-j,2n+3-j} \\ & + \left(\frac{\alpha_n}{\xi_n} + \frac{\beta_n}{2\xi_n^2}\right) \big(E_{n+1,n} - E_{n+2,n+1}\big) + \frac{\beta_n}{2\xi_n\xi_{n-1}} \big(E_{n+3,n} - E_{n+2,n-1}\big) - E_{n+1,n+2} \\ & - \left(\alpha_1\xi_1 + \frac{\beta_1\xi_1^2}{2}\right) \big(E_{2n+2,2n+1} + v^{-1} E_{1,2n+2}\big) + \frac{\beta_1\xi_1\xi_2}{2v} \big(E_{2,2n+1} - E_{1,2n}\big) - vE_{2n+2,1}, \end{aligned} \tag{5.9}$$

(using again the notation $\xi_j \equiv \mathrm{e}^{x_j}$, $\eta_j \equiv \mathrm{e}^{y_j}$) and $\widetilde{\mathcal{M}}(v)$ is defined as

$$\widetilde{\mathcal{M}}(v) \equiv C_v \mathcal{M}^t(v) C_v^{-1}. \tag{5.10}$$

One of common ways to obtain a Bäcklund transformation is from factorising a Lax matrix in two different ways, see [18] for Toda lattices and [13] for other integrable models. For our model we also have a remarkable factorisation, only instead of $\mathcal{L}(v)$ we have to take its square:

$$\lambda^2 - \mathcal{L}^2(v; X, x) = \mathcal{M}(v,\lambda)\widetilde{\mathcal{M}}(v,\lambda), \tag{5.11a}$$

$$\lambda^2 - \mathcal{L}^2(v; Y, y) = \widetilde{\mathcal{M}}(v,\lambda)\mathcal{M}(v,\lambda). \tag{5.11b}$$

6 Discussion

The method for constructing a Bäcklund transformation presented in this paper seems to be quite general and applicable as well to other integrable $sl(2)$-type chains with the boundary conditions treatable within the framework developed in [3, 4].

There is little doubt that a similar BT can be constructed for the D-type Toda lattice and a more general Inozemtsev's Toda lattice [5] with the boundary terms like

$$\frac{a_1}{\sinh^2 \frac{x_1}{2}} + \frac{b_1}{\sinh^2 x_1} + \frac{a_n}{\sinh^2 \frac{x_n}{2}} + \frac{b_n}{\sinh^2 x_n}$$

since those, as shown in [20], can also be described in the formalism based on the boundary K matrices (2.1) and the Poisson algebra (2.27).

The 'big' Lax matrix $\mathcal{L}(v)$ still awaits a proper Lie-algebraic interpretation. Obtaining a BT from the factorisation of $\lambda^2 - \mathcal{L}^2$ like in (5.11) might prove to be useful for other integrable systems related to classical Lie algebras.

It is well known that the quantum analog of a BT is the so-called Q-operator [21], see also [9]. Examples of Q-operators for quantum integrable chains with a boundary have been constructed recently for the XXX magnet [12] and for the Toda lattices of B, C and D types [22]. Our results for the BC-Toda lattice agree with those of [22], the generating function of the BT being a classical limit of the kernel of the Q-operator. Hopefully, our results will help to construct the Q-operator for the general 4-parametric quantum BC-Toda lattice.

A Proof of canonicity

Here we adapt to the BC-Toda case the argument from [10] developed originally for the periodic case. The trick is to obtain the graph Γ_{2n} of the BT as a projection of another manifold in a bigger symplectic space, the mentioned manifold being Lagrangian for trivial reason.

Consider the 8-dimensional symplectic space W_8 with coordinates $XxYySsTt$ and the symplectic form

$$\omega_8 \equiv \mathrm{d}X \wedge \mathrm{d}x + \mathrm{d}S \wedge \mathrm{d}s - \mathrm{d}Y \wedge \mathrm{d}y - \mathrm{d}T \wedge \mathrm{d}t. \tag{A.1}$$

The matrix relation

$$M(u,\lambda; T, t)\ell(u; Y, y) = \ell(u; X, x)M(u,\lambda; S, s) \tag{A.2}$$

is equivalent to 4 relations

$$X = -\lambda + s^{-1}\mathrm{e}^{x} + t\mathrm{e}^{-x}, \tag{A.3a}$$

$$Y = \lambda - s^{-1}\mathrm{e}^{y} - t\mathrm{e}^{-y}, \tag{A.3b}$$

$$S = 2\lambda s^{-1} - s^{-2}\mathrm{e}^{x} - s^{-2}\mathrm{e}^{y}, \tag{A.3c}$$

$$T = \mathrm{e}^{-x} + \mathrm{e}^{-y}, \tag{A.3d}$$

defining a 4-dimensional submanifold $\mathcal{G}_4 \subset W_8$. The fact that $\mathcal{G}_4$ is Lagrangian, that is $\omega_8|_{\mathcal{G}_4} = 0$, is proved by presenting explicitly the generating function

$$f_\lambda(y,t;x,s) = \lambda(2\ln s - x - y) + s^{-1}(\mathrm{e}^{x} + \mathrm{e}^{y}) - t(\mathrm{e}^{-x} + \mathrm{e}^{-y}), \tag{A.4}$$

such that

$$X = \frac{\partial f_\lambda}{\partial x}, \qquad S = \frac{\partial f_\lambda}{\partial s}, \qquad Y = -\frac{\partial f_\lambda}{\partial y}, \qquad T = -\frac{\partial f_\lambda}{\partial t}. \tag{A.5}$$

An alternative proof [10] is based on the fact that $\ell(u)$ and $M(u,\lambda)$ are symplectic leaves of the same Poisson algebra (2.23).

Relation (A.2) defines thus a canonical transformation from $XxSs$ to $YyTt$.

Let us take n copies $W_8^{(j)}$ of W_8 decorating the variables $XxYySsTt$ with the indices $j = 1,\dots,n$ and impose on them n matrix relations obtained from (A.2) by adding subscript j to all variables. We obtain then a Lagrangian manifold $\mathcal{G}_{4n} = \otimes_{j=1}^{n}\mathcal{G}_4^{(j)}$ in the $8n$-dimensional symplectic space $W_{8n} = \oplus_{j=1}^{n} W_8^{(j)}$ with the symplectic form $\omega_{8n} = \sum_{j=1}^{n}\omega_8^{(j)}$ and the corresponding canonical transformation with the generating function $\sum_{j=1}^{n} f_\lambda(y_j,t_j;x_j,s_j)$.

Let us also introduce 4 additional variables T_0, t_0 and S_{n+1}, s_{n+1} serving as coordinates in the 4-dimensional symplectic space W_4 with the symplectic form $\omega_4 \equiv \mathrm{d}S_{n+1}\wedge \mathrm{d}s_{n+1} - \mathrm{d}T_0 \wedge \mathrm{d}t_0$. The relations

$$T_0 = \frac{2(\alpha_1 + \beta_1\lambda t_0)}{1+\beta_1 t_0^2}, \qquad S_{n+1} = \frac{2(\lambda s_{n+1} - \alpha_n)}{\beta_n + s_{n+1}^2} \tag{A.6}$$

define then a 2-dimensional Lagrangian submanifold $\mathcal{G}_2 \subset W_4$ characterised by the generating function $\varphi = \varphi_\lambda^{(0)}(t_0) + \varphi_\lambda^{(n+1)}(s_{n+1})$ with $\varphi_\lambda^{(0)}$ and $\varphi_\lambda^{(n+1)}$ defined by (4.5b) and (4.5c), respectively:

$$T_0 = -\frac{\partial\varphi_\lambda}{\partial t_0}, \qquad S_{n+1} = \frac{\partial\varphi_\lambda}{\partial s_{n+1}}. \tag{A.7}$$

We end up with the $(8n+4)$-dimensional symplectic space $W_{8n+4} = W_{8n} + W_4$, symplectic form $\omega_{8n+4} = \omega_{8n} + \omega_4$, and the $(4n+2)$-dimensional Lagrangian submanifold $\mathcal{G}_{4n+2} = \mathcal{G}_{4n}\times\mathcal{G}_2 \subset W_{8n+4}$ defined by the generating function

$$F_\lambda = \varphi_\lambda^{(0)}(t_0) + \varphi_\lambda^{(n+1)}(s_{n+1}) + \sum_{j=1}^{n} f_\lambda(y_j,t_j;x_j,s_j). \tag{A.8}$$

The final step is to impose $2n+2$ constraints

$$T_j = S_{j+1}, \quad t_j = s_{j+1}, \qquad j = 0,\dots,n, \tag{A.9}$$

which define a subspace $W_{6n+2} \subset W_{8n+4}$ of dimension $(8n+4)-(2n+2) = 6n+2$ and respective $2n$-dimensional submanifold $\mathcal{G}_{2n} = \mathcal{G}_{4n+2}\cap W_{6n+2}$.

Constraints (A.9) allow to eliminate the variables Tt. The space W_{6n+2} splits then into the direct sum $W_{6n+2} = V_{4n} + W_{2n+2}$ of the space W_{4n} with coordinates $X_jx_jY_jy_j$ $(j = 1,\dots,n)$

and W_{2n+2} with coordinates $S_j s_j$ ($j = 1, \dots, n+1$). Using (A.9) we obtain that $\mathrm{d}T_j \wedge \mathrm{d}t_j - \mathrm{d}S_{j+1} \wedge \mathrm{d}s_{j+1} = 0$ and therefore the symplectic form ω_{8n+4} restricted on W_{6n+2}

$$\omega_{8n+4}|_{W_{6n+2}} = \sum_{j=1}^{n} \big(\mathrm{d}X_j \wedge \mathrm{d}x_j - \mathrm{d}Y_j \wedge y_j\big), \tag{A.10}$$

degenerates: it vanishes on W_{2n+2} and remains nondegenerate on V_{4n}. In fact, on V_{4n} the form ω_{8n+4} coincides with the standard symplectic form (4.1).

$$\omega_{8n+4}|_{V_{4n}} = \Omega_{4n}. \tag{A.11}$$

After the elimination of the variables Tt from equations (A.3) and (A.6), the resulting set of equations defining the submanifold $\mathcal{G}_{2n} = \mathcal{G}_{4n+2} \cap W_{6n+2} \subset W_{6n+2}$ coincides with equations (3.13) and (3.14) defining the BT.

As we have seen in Section 3, the variables $S_j s_j$ can also be eliminated leaving a $2n$ dimensional submanifold $\Gamma_{2n} \subset V_{4n}$ coinciding with the graph of the BT discussed in Section 4.2. By construction, Γ_{2n} is the projection of $\mathcal{G}_{2n}$ from W_{6n+2} onto V_{4n} parallel to W_{2n+2}. Furthermore, Γ_{2n} is Lagrangian since ω_{8n+4} vanishes on $\mathcal{G}_{4n+2}$, therefore on $\mathcal{G}_{2n} = \mathcal{G}_{4n+2} \cap W_{6n+2}$, and therefore on Γ_{2n}. The canonicity of the BT is thus established geometrically, without tedious calculations.

The same argument as in [10] shows that the generating function Φ_λ of the Lagrangian submanifold Γ_{2n} is obtained by setting $t_j = s_{j+1}$ in (A.8), which produces formula (4.4).

B Proof of spectrality

Here we provide the proof of formulae (4.8) for the eigenvalues of $L(\lambda)$. For the proof we use an observation from [10] and show that the eigenvectors of $L(\lambda)$ are given by null-vectors of $M_1(\pm\lambda, \lambda)$.

After setting $u = -\lambda$ in (3.12) the matrix M_j becomes a projector

$$M_j(-\lambda, \lambda) = \begin{pmatrix} -2\lambda + s_j S_j & s_j^2 S_j - 2\lambda s_j \\ S_j & s_j S_j \end{pmatrix} = \begin{pmatrix} -2\lambda + s_j S_j \\ S_j \end{pmatrix} \begin{pmatrix} 1 & s_j \end{pmatrix} \tag{B.1}$$

with the null-vector

$$\sigma_j \equiv \begin{pmatrix} -s_j \\ 1 \end{pmatrix}, \qquad M_j(-\lambda, \lambda)\sigma_j = 0. \tag{B.2}$$

Let us set $u = -\lambda$ in the matrix equality (3.1) and apply it to the vector σ_1. By (B.2), the right-hand side gives 0. Therefore, $L(-\lambda)\sigma_1$ should be proportional to the same null-vector σ_1 of $M_j(-\lambda, \lambda)$, and σ_1 is an eigenvector of $L(-\lambda)$.

To find the corresponding eigenvalue Λ, use the factorised expression (2.1) of $L(-\lambda)$ and apply it to σ_1. Using (2.3) we obtain

$$\ell(-\lambda; Y_j, y_j)\sigma_j = -s_j \mathrm{e}^{-y_j} \sigma_{j+1}, \tag{B.3}$$

hence

$$T(u; Y, y)\sigma_1 = \sigma_{n+1} \prod_{j=1}^{n} \big(-s_j \mathrm{e}^{-y_j}\big). \tag{B.4}$$

From (3.5) and (3.12) we obtain

$$M_j^a(u, \lambda) = \begin{pmatrix} -u - \lambda + s_j S_j & -S_j \\ 2\lambda s_j - s_j^2 S_j & u - \lambda + s_j S_j \end{pmatrix}, \tag{B.5}$$

hence

$$M_j^a(\lambda,\lambda) = \begin{pmatrix} -2\lambda + s_jS_j & -S_j \\ (2\lambda s_j - s_j^2)S_j & s_jS_j \end{pmatrix} = \begin{pmatrix} -1 \\ s_j \end{pmatrix} \begin{pmatrix} 2\lambda - s_jS_j & S_j \end{pmatrix}, \tag{B.6}$$

the corresponding null-vector being

$$\widetilde{\sigma}_j \equiv \begin{pmatrix} S_j \\ s_jS_j - 2\lambda \end{pmatrix}, \qquad M_j^a\widetilde{\sigma}_j = 0. \tag{B.7}$$

A direct calculation using (2.3) and (3.13d) yields

$$\ell_j^t(\lambda; Y_j, y_j)\widetilde{\sigma}_{j+1} = s_j\mathrm{e}^{-x_j}\widetilde{\sigma}_j \tag{B.8}$$

and, consequently,

$$T^t(\lambda; Y, y)\widetilde{\sigma}_{n+1} = \widetilde{\sigma}_1 \prod_{j=1}^{n} \big(s_j\mathrm{e}^{-x_j}\big). \tag{B.9}$$

From (2.4) we get, respectively, the identities

$$K_+(-\lambda)\sigma_{n+1} = \frac{1}{2}(\beta_n + s_{n+1}^2)\widetilde{\sigma}_{n+1}, \qquad K_-(-\lambda)\widetilde{\sigma}_1 = 2\frac{\alpha_1^2 + \beta_1\lambda^2}{1 + \beta_1 s_1^2}\,\widetilde{\sigma}_1. \tag{B.10}$$

Using the above formulae we are able to move σ_1 through all the factors constituting $L(-\lambda)$ and obtain the equality

$$L(-\lambda; Y, y)\sigma_1 = \Lambda\sigma_1, \tag{B.11}$$

where Λ is given by (4.8a). Note that Λ is an eigenvalue of $L(\lambda)$ as well since $\Lambda(\lambda) = \Lambda(-\lambda)$. The second eigenvalue $\widetilde{\Lambda}$ (4.8b) of $L(\lambda)$ is obtained from

$$\Lambda\widetilde{\Lambda} = \det L(\lambda) \equiv d(\lambda) = (\alpha_n^2 + \beta_n\lambda^2)(\alpha_1^2 + \beta_1\lambda^2), \tag{B.12}$$

see (2.13).

C Derivation of the dual Lax matrix

To construct the 'big' Lax operator $\mathcal{L}(v)$ from the 'small' one $L(u)$ we use the technique developed for the periodic the periodic Toda lattice [10, 19], with the necessary corrections to accommodate the boundary conditions.

Let θ_1 be an eigenvector of $L(u)$ with the eigenvalue v:

$$L(u)\theta_1 = v\theta_1, \qquad \theta_1 = \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix}. \tag{C.1}$$

Reading off the factors constituting the product $L(u)$, see (2.1), (2.2), define recursively the vectors θ_j

$$\theta_j = \begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix}, \qquad j = 1, \dots 2n+2, \tag{C.2}$$

by the relations

$$\theta_{j+1} = \ell(u; X_j, x_j)\theta_j, \qquad j = 1, \dots, n, \tag{C.3a}$$

$$\theta_{n+2} = K_+(u)\theta_{n+1}, \tag{C.3b}$$
$$\theta_{n+j+3} = \ell^t(-u; X_{n-j}, x_{n-j})\theta_{n+j+2}, \qquad j = 0,\ldots,n-1, \tag{C.3c}$$

and close the circuit with the equation

$$v\theta_1 = K_-(u)\theta_{2n+2}, \tag{C.3d}$$

which is equivalent to (C.1).

A recursive elimination of ψ_j results in the equations

$$u\varphi_1 = \varphi_2 - X_1\varphi_1 + \left(\frac{\alpha_1}{v}\mathrm{e}^{x_1} + \frac{\beta_1}{v}\mathrm{e}^{2x_1}\right)\varphi_{2n+2} - \frac{\beta_1}{v}\mathrm{e}^{2x_1}X_1\varphi_{2n+1} + \frac{\beta_1}{v}\mathrm{e}^{x_1+x_2}\varphi_{2n}, \tag{C.4a}$$
$$u\varphi_j = \varphi_{j+1} - X_j\varphi_j + \mathrm{e}^{x_j-x_{j-1}}\varphi_{n-1}, \qquad j = 2,\ldots,n \tag{C.4b}$$
$$u\varphi_{n+1} = \varphi_{n+2} + \alpha_n\mathrm{e}^{-x_n}\varphi_n, \tag{C.4c}$$
$$\begin{aligned}u\varphi_{n+2} = {} & -\varphi_{n+3} + X_n\varphi_{n+2} + (\alpha_n\mathrm{e}^{-x_n} + \beta_n\mathrm{e}^{-2x_n})\varphi_{n+1}\\ & - \beta_n\mathrm{e}^{-2x_n}X_n\varphi_n + \beta_n\mathrm{e}^{-x_n-x_{n-1}}\varphi_{n-1},\end{aligned} \tag{C.4d}$$
$$u\varphi_j = -\varphi_{j+1} + X_{2n+2-j}\varphi_j - \mathrm{e}^{x_{j-3}-x_{j-4}}\varphi_{j-1}, \qquad j = n+3,\ldots,2n+1, \tag{C.4e}$$
$$u\varphi_{2n+2} = v\varphi_1 - \alpha_1\mathrm{e}^{x_1}\varphi_{2n+1}. \tag{C.4f}$$

In order to simplify the 6-terms relations (C.4a) and (C.4d) and make the matrix $\mathcal{L}(v)$ more symmetric we perform an additional reversible change of variables

$$\varphi_1 = \widetilde{\varphi}_1 + \frac{\beta_1}{2v}\mathrm{e}^{2x_1}\widetilde{\varphi}_{2n+1}, \tag{C.5a}$$
$$\varphi_j = \widetilde{\varphi}_j, \qquad j = 2,\ldots,n+1, \tag{C.5b}$$
$$\varphi_{n+2} = \widetilde{\varphi}_{n+2} + \frac{\beta_n}{2}\mathrm{e}^{-2x_n}\widetilde{\varphi}_n, \tag{C.5c}$$
$$\varphi_j = -\widetilde{\varphi}_j, \qquad j = n+3,\ldots,2n+2. \tag{C.5d}$$

The resulting equations for $\widetilde{\varphi}_j$ read

$$u\widetilde{\varphi}_1 = \widetilde{\varphi}_2 - X_1\widetilde{\varphi}_1 - \left(\frac{\alpha_1}{v}\mathrm{e}^{x_1} + \frac{\beta_1}{2v}\mathrm{e}^{2x_1}\right)\widetilde{\varphi}_{2n+2} - \frac{\beta_1}{2v}\mathrm{e}^{x_1+x_2}\widetilde{\varphi}_{2n}, \tag{C.6a}$$
$$u\widetilde{\varphi}_2 = \widetilde{\varphi}_3 - X_2\widetilde{\varphi}_2 + \mathrm{e}^{x_2-x_1}\widetilde{\varphi}_1 + \frac{\beta_1}{2v}\mathrm{e}^{2x_1}\widetilde{\varphi}_{2n+1}, \tag{C.6b}$$
$$u\widetilde{\varphi}_j = \widetilde{\varphi}_{j+1} - X_j\widetilde{\varphi}_j + \mathrm{e}^{x_j-x_{j-1}}\widetilde{\varphi}_{n-1}, \qquad j = 3,\ldots,n \tag{C.6c}$$
$$u\widetilde{\varphi}_{n+1} = -\widetilde{\varphi}_{n+2} + \left(\alpha_n\mathrm{e}^{-x_n} + \frac{\beta_n}{2}\mathrm{e}^{-2x_n}\right)\widetilde{\varphi}_n, \tag{C.6d}$$
$$u\widetilde{\varphi}_{n+2} = -\widetilde{\varphi}_{n+3} + X_n\widetilde{\varphi}_{n+2} - \left(\alpha_n\mathrm{e}^{-x_n} + \frac{\beta_n}{2}\mathrm{e}^{-2x_n}\right)\widetilde{\varphi}_{n+1} - \frac{\beta_n}{2}\mathrm{e}^{-x_n-x_{n-1}}\widetilde{\varphi}_{n-1}, \tag{C.6e}$$
$$u\widetilde{\varphi}_{n+3} = -\widetilde{\varphi}_{n+4} + X_{n-1}\widetilde{\varphi}_{n+3} - \mathrm{e}^{x_n-x_{n-1}}\widetilde{\varphi}_{n+2} + \frac{\beta_n}{2}\mathrm{e}^{-2x_n}, \tag{C.6f}$$
$$u\widetilde{\varphi}_j = -\widetilde{\varphi}_{j+1} + X_{2n+2-j}\widetilde{\varphi}_j - \mathrm{e}^{x_{j-3}-x_{j-4}}\widetilde{\varphi}_{j-1}, \qquad j = n+4,\ldots,2n+1, \tag{C.6g}$$
$$u\widetilde{\varphi}_{2n+2} = -v\widetilde{\varphi}_1 - \left(\alpha_1\mathrm{e}^{x_1} + \frac{\beta_1}{2}\mathrm{e}^{2x_1}\right)\widetilde{\varphi}_{2n+1}. \tag{C.6h}$$

Introducing the vector Θ with $2n+2$ components $\widetilde{\varphi}_j$, $j = 1,\ldots,2n+2$ we can rewrite relations (C.6) in the matrix form

$$\mathcal{L}(v)\Theta = u\mathcal{L}(v)\Theta \tag{C.7}$$

with the matrix $\mathcal{L}(v)$ given by (5.1). It follows from (C.7) that u is an eigenvalue of $\mathcal{L}(v)$.

The rest of the formulae of Section 5 are obtained by a straitforward calculation not much different from the periodic case [10, 19].

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