# Global Stability of Dynamic Systems of High Order

*Mohammed BENALILI and Azzedine LANSARI*

*Department of Mathematics, B.P. 119, Faculty of Sciences,*
*University Abou-bekr BelKaïd, Tlemcen, Algeria*
E-mail: *m_benalili@mail.univ-tlemcen.dz, a_lansari@mail.univ-tlemcen.dz*

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**Abstract.** This paper deals with global asymptotic stability of prolongations of flows induced by specific vector fields and their prolongations. The method used is based on various estimates of the flows.

*Key words:* global stability; vector fields; prolongations of flows

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## 1 Introduction

Global stability of dynamic systems is a vast domain in ordinary differential equations and it is one of its main topics. Many works have been done in this context, we list some of them: [3, 4, 5, 6, 7, 8]. However, little is known in the stability of high order (see [10] and [2]). In this paper, we are concerned with the global asymptotic stability of prolongations of flows generated by some specific vector fields and their perturbations. The method used is based on various estimates of the flows and their prolongations. To justify the study of the dynamic of prolongations of flows, we consider the Lie algebra $\chi(\mathbb{R}^n)$ of vector fields on $\mathbb{R}^n$ endowed with the weak topology, which is the topology of the uniform convergence of vector fields and all their derivatives on a compact sets. The Lie bracket is a fundamental operation not only in differential geometry but in many fields of mathematics, such as dynamic and control theory. The invertibility of this latter is of many uses i.e. given any vector fields $X$, $Z$ find a vector field $Y$ such that $[X, Y] = Z$. In the case of vector fields $X$ defined in a neighborhood of a point $a$ with $X(a) \neq 0$ we have a positive answer: since in this case the vector field $X$ is locally of the form $\frac{\partial}{\partial x_1}$ and the solution is given by

$$Y(x_1, \dots, x_n) = \int_{-r}^{x_1} Z(t, x_2, \dots, x_n)dt,$$

where $\|x\| = \max\limits_{1\le i\le n} |x_i| < r$. In the case of singular vector fields, i.e. $X(a) = 0$ little is known. Consider a singular vector field $X$ defined in a neighborhood $U$ of the origin 0 with $X(0) = 0$ and let $\phi_t$ be the flow generated by $X$. Suppose that $X$ is complete and consider a vector field $Y$ defined on an open set $V \supset \phi_t(U)$ for all $t \in \mathbb{R}$. The transportation of a vector field $Y$ along the flow $\phi_t$ is defined as

$$(\phi_t)_*Y(x) = (D\phi_t \cdot Y) \circ \phi_{-t}(x)$$

and the derivative with respect to $t$ is given as follows

$$\frac{d}{dt}(\phi_t)_*Y = [(\phi_t)_*X, (\phi_t)_*Y].$$

Put $Y_t = -\int_0^t (\phi_s)_* Z ds$, then

$$[X,Y_t] = -\frac{d}{dt}\Big|_{t=0}(\phi_t)_* \int_0^t (\phi_s)_* Z ds = -\int_0^t \frac{d}{ds}(\phi_s)_* Z ds = Z - (\phi_t)_* Z.$$

So if $(\phi_t)_* Z$ converges to 0 and the integral $Y = -\int_0^{+\infty} (\phi_s)_* Z ds$ is convergent in the weak topology, then $Y$ is a solution of our equation.

As applications of the right invertibility of the bracket operation on germs of vector fields at a singular point we refer the reader to the papers by the authors [1, 2] (see also [10]).

## 2 Generalities

First we recall some definitions on global asymptotic stability as introduced in [9]. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$, $K \subset \mathbb{R}^n$ is a compact set and $f$ any smooth function on $\mathbb{R}^n$, we put

$$\|f\|_r^K = \sup_{x\in K} \max_{|\alpha|\le r} \|D^\alpha f(x)\| . \tag{1}$$

**Definition 1.** A point $a \in \mathbb{R}^n$ is said globally asymptotically stable (in brief $G.A.S.$) of the flow $\phi_t$ if

i) $a$ is an asymptotically stable (in brief $A.S.$) equilibrium of the flow $\phi_t$;

ii) for any compact set $K \subset \mathbb{R}^n$ and any $\varepsilon > 0$ there exists $T_K > 0$ such that for any $t \ge T_K$ we have $\|\phi_t(x) - a\| \le \epsilon$ for all $x \in K$.

**Definition 2.** The point $a \in \mathbb{R}^n$ is said globally asymptotically stable of order $r$ $(1 \le r \le \infty)$ for the flow $\phi_t$ if

i) $a$ is a $G.A.S.$ point for the flow $\phi_t$;

ii) for any compact set $K \subset \mathbb{R}^n$ and

$$\forall\, \epsilon > 0, \quad \exists\, T_K > 0 \;\text{ such that }\; \forall\, t \ge T_K \;\Rightarrow\; \|\phi_t - aI\|_r^K \le \epsilon,$$

where $I$ denotes the identity map.

A vector field $X$ will be called semi-complete if the $X$-flow $\phi_t = \exp(tX)$ is defined for all $t \ge 0$.

First we quote the following proposition which characterizes the uniform asymptotic stability, for a proof see the book of W. Hahn [5].

Let $(\phi)_t$ denote a flow defined on $\mathbb{R}^n$.

**Proposition 1.** *The origin 0 in $\mathbb{R}^n$ is G.A.S. point for the flow $\phi_t$ if for any ball $B(0,\rho)$, centered at 0 and of radius $\rho > 0$, there exist $t_0 \ge 0$ and functions $a$, $b$ such that*

$$\|\phi_t(x)\| \le a(\|x\|)b(t) \tag{2}$$

*with $a$ a continuous function on $B(0,\rho)$ monotonously increasing such that $a(0) = 0$ and $b$ is a continuous function defined for any $t \ge t_0$ monotonously decreasing such that $\lim\limits_{t\to+\infty} b(t) = 0$.*

## 3 Estimates of prolongations of flows

We start with some perturbations of linear vector fields.

### 3.1 Perturbation of linear vector fields

Consider the following linear vector field

$$X_1 = \sum_{i=1}^{n} \alpha_i x_i \frac{\partial}{\partial x_i},$$

where the coefficients $\alpha_i \in [a, b] \subset \mathbb{R}$ and are not all 0.

The $X_1$-flow, $\psi_t^1 = \exp(tX_1)$ is then

$$\psi_t^1(x) = xe^{\alpha t} = \left(x_1 e^{\alpha_1 t}, \dots, x_n e^{\alpha_n t}\right) \quad \forall\, t \in \mathbb{R} \tag{3}$$

and its estimates are given by

$$\|x\|\, e^{at} \leq \|\psi_t^1(x)\| \leq \|x\|\, e^{bt}. \tag{4}$$

Consider now a perturbation of the vector field $X_1$ of the form $Y_1 = X_1 + Z_1$, where $Z_1$ is a smooth vector field globally Lipschitzian on $\mathbb{R}^n$. The explicit form of the $Y_1$-flow is then

$$\psi_t^1(x) = xe^{At} + \int_0^t Z_1\left(\psi_s^1(x)\right) ds, \tag{5}$$

where $A = \begin{pmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \end{pmatrix}$.

**Lemma 1.** *If the perturbation $Z_1$ fulfills*

$$\|Z_1(x)\| \leq c_0 \quad \forall\, x \in \mathbb{R}^n \tag{6}$$

*then the vector field $Y_1$ is complete and the $Y_1$-flow satisfies the estimates*

$$\left(\|x\| - \frac{c_0}{a}\right) e^{bt} + \frac{c_0}{a} \leq \|\psi_t^1(x)\| \leq \left(\|x\| + \frac{c_0}{b}\right) e^{bt} - \frac{c_0}{b}.$$

**Proof.** Clearly the $Y_1$-flow $\psi_t^1$ is bounded for any $t \in [0, T]$ with $T < +\infty$ and any $x \in \mathbb{R}^n$. The same is true if we replace $t$ by $-t$. Then $\psi_t^1$ is complete.

Consider now the equation

$$\frac{1}{2}\frac{d}{dt}\left\|\psi_t^1(x)\right\|^2 = \left\langle \psi_t^1(x), \alpha\psi_t^1(x) + Z_1\left(\psi_t^1(x)\right)\right\rangle. \tag{7}$$

Letting $y = \|\psi_t^1(x)\|$, we deduce

$$ay^2 - c_0 y \leq \frac{1}{2}\frac{d}{dt}y^2 \leq by^2 + c_0 y, \qquad y(0) = \|x\|$$

and by integrating we obtain

$$\left(\|x\| - \frac{c_0}{a}\right) e^{bt} + \frac{c_0}{a} \leq y \leq \left(\|x\| + \frac{c_0}{b}\right) e^{bt} - \frac{c_0}{b}. \qquad \blacksquare$$

Let $B(0,1)$ be the open unit ball centered at the origin 0.

**Lemma 2.** *If the perturbation $Z_1$ fulfills the estimates*

$$\begin{aligned} &\|Z_1(x)\| \le c_0' \|x\|^{1+m} \quad \forall\, x \in B(0,1) \text{ and any integer } m \ge 1, \\ &\|Z_1(x)\| \le c_0'' \|x\| \quad \text{for every } x \in \mathbb{R}^n \setminus B(0,1), \end{aligned} \tag{8}$$

*then $Y_1$ is complete and the $Y_1$-flow fulfills the following estimates for ant $t \ge 0$*

$$\begin{aligned} &\|x\| e^{a_0 t} \le \left\|\psi_t^1(x)\right\| \le \|x\| e^{b_0 t}, \\ &\|x\| e^{-b_0 t} \le \left\|\psi_{-t}^1(x)\right\| \le \|x\| e^{-a_0 t} \end{aligned} \tag{9}$$

*with $c_0 = \max\{c_0', c_0''\}$, $a_0 = a - c_0$ and $b_0 = b + c_0$.*

**Proof.** Taking account of the explicit form of the flow (5) and the estimates (8), we deduce that $Y_1$ is complete. If $x \in B(0,1)$ then $\|Z_1(x)\| \le c_0' \|x\|^{1+m} \le c_0' \|x\|$, letting $c_0 = \max\{c_0', c_0''\}$ then $\|Z_1(x)\| \le c_0 \|x\|$ for any $x \in \mathbb{R}^n$. If we put $y = \left\|\psi_t^1(x)\right\|$ the equation (7) leads to

$$(a - c_0)y \le \frac{d}{dt}y \le (b + c_0)y, \qquad y(0) = \|x\|$$

and putting $b_0 = b + c_0$, $a_0 = a - c_0$, we deduce the following estimates

$$\|x\| e^{a_0 t} \le y \le \|x\| e^{b_0 t} \quad \text{for any } t \ge 0.$$

The same is also true in the on $\mathbb{R}^n \setminus B(0,1)$. ∎

**Lemma 3.** *Suppose that all the coefficients $\alpha_i$ are negative, $a \le \alpha_i \le b < 0$.*

*If the perturbation $Z_1$ fulfills the estimates*

$$\|Z_1(x)\| \le c_0 \|x\|^{1+m} \qquad \text{for any } x \in \mathbb{R}^n \text{ and any integer } m \ge 1, \tag{10}$$

*then the vector field $Y_1$ is semi-complete and the $Y_1$-flow satisfies the estimates for any $t \ge 0$*

$$\begin{aligned} &\|x\| e^{at} \left(1 - \frac{c_0}{a} \|x\|^m (1 - e^{amt})\right)^{-\frac{1}{m}} \\ &\quad \le \left\|\psi_t^1(x)\right\| \le \|x\| e^{bt} \left(1 - \frac{c_0}{b} \|x\|^m (1 - e^{bmt})\right)^{-\frac{1}{m}}. \end{aligned} \tag{11}$$

**Proof.** By the relation (5) and the estimates (10), we deduce that the vector field $Y_1$ is semi-complete. Letting $y = \left\|\psi_t^1(x)\right\|$ and taking into account the equation (7) and the estimates (10) we deduce that

$$ay - c_0 y^{1+m} \le \frac{d}{dt}y \le by + c_0 y^{1+m}, \qquad y(0) = \|x\|$$

and by integration we have

$$\|x\| e^{at} \left(1 - \frac{c_0}{a} \|x\|^m (1 - e^{amt})\right)^{-\frac{1}{m}} \le y \le \|x\| e^{bt} \left(1 - \frac{c_0}{b} \|x\|^m (1 - e^{bmt})\right)^{-\frac{1}{m}}.$$ ∎

**Example 1.** Let the vector field

$$X_3 = \sum_{i=1}^{n} \left(\alpha_i x_i + \beta_i x_i^{1+m_i}\right) \frac{\partial}{\partial x_i}$$

such that all the coefficients fulfilling

$$a \le \alpha_i \le b < 0, \qquad a' \le \beta_i \le b' \le 0$$

and all the exponents $m_i$ are even positive integers with $0 < m_0' \leq m_i \leq m_0$. The associated flow $\phi_t^3 = \exp(tX_3)$ is the solution of the dynamic system

$$\frac{d}{dt}\phi_t(x) = X_3 \circ \phi_t(x), \qquad \phi_0(x) = x$$

or in coordinates

$$\frac{d}{dt}\left(\phi_t(x)\right)_i = \alpha_i \left(\phi_t(x)\right)_i + \beta_i \left(\phi_t(x)\right)_i^{1+m_i}, \qquad \phi_0(x) = x.$$

This latter is a Bernoulli type equation and its solution is given by

$$\left(\phi_t^3(x)\right)_i = x_i e^{\alpha_i t}\left(1 + \frac{\beta_i}{\alpha_i} x_i^{m_i}\left(1 - e^{\alpha_i m_i t}\right)\right)^{\frac{-1}{m_i}}. \tag{12}$$

The $X_3$-flow $\phi_t^3 = \exp(tX_3)$ then has the explicit form

$$\phi_t^3(x) = x e^{\alpha t}\left(1 + \frac{\beta}{\alpha} x^m \left(1 - e^{\alpha m t}\right)\right)^{\frac{-1}{m}}$$

and the following estimates are true, $\forall\, t \geq 0$

$$\|x\|\, e^{at} \leq \left\|\phi_t^3(x)\right\| \leq \|x\|\, e^{bt}. \tag{13}$$

### 3.2 Estimation of the $k^{\text{th}}$ prolongation of the $Y_1$-flow

Denote by $\eta_1^1(t,x,\nu) = D\psi_t^1(x)\nu$, where $\nu \in \mathbb{R}^n$, the first derivative with respect to $x$ of the $Y_1$-flow, solution of the dynamic system

$$\frac{d}{dt}\eta_1^1(t,x,\nu) = \left(D_y X_1 + D_y Z_1\right)\eta_1^1(t,x,\nu), \qquad \eta_1^1(0,x,\nu) = \nu$$

with $y = \psi_t^1(x)$.

**Lemma 4.** *If the perturbation $Z_1$ fulfills the estimate*

$$\|DZ_1(x)\| \leq c_1 \quad \textit{for any } x \in \mathbb{R}^n, \tag{14}$$

*then the derivative of the $Y_1$-flow is complete and has the following estimates, for any $t \geq 0$*

$$e^{a_1 t} \leq \left\|D\psi_t^1(x)\right\| \leq e^{b_1 t}, \qquad e^{-b_1 t} \leq \left\|D\psi_{-t}^1(x)\right\| \leq e^{-a_1 t} \tag{15}$$

*with $a_1 = a - c_1$ and $b_1 = b + c_1$ .*

**Proof.** Consider as in previous lemmas the following equation

$$\frac{1}{2}\frac{d}{dt}\left\|\eta_1^1(t,x,\nu)\right\|^2 = \left\langle \eta_1^1(t,x,\nu), \left(\alpha + DZ_1\right)\eta_1^1(t,x,\nu)\right\rangle \tag{16}$$

and put $z = \left\|\eta_1^1(t,x,\nu)\right\|$, so

$$(a - c_1)z^2 \leq \frac{1}{2}\frac{d}{dt}z^2 \leq (b + c_1)z^2, \qquad z(0) = \|\nu\| \tag{17}$$

and then

$$\|\nu\|\, e^{a_1 t} \leq z \leq \|\nu\|\, e^{b_1 t} \quad \text{for any} \quad t \geq 0 \quad \text{and} \quad \nu \in \mathbb{R}^n. \qquad \blacksquare$$

**Lemma 5.** *If the perturbation $Z_1$ fulfils the estimates*

$$\left\|D^l Z_1(x)\right\| \le c_l' \left\|x\right\|^{1-l+m} \text{ for any } x \in B(0,1) \text{ and all integers } m \ge 1,$$
$$\left\|D^l Z_1(x)\right\| \le c_l'' \left\|x\right\|^{1-l} \quad \forall\, x \in \mathbb{R}^n \setminus B(0,1)$$

*with $l = 0, 1$, then the first derivative of the $Y_1$-flow is complete and is estimated by, for any $t \ge 0$*

$$e^{a_1 t} \le \|D\psi_t^1(x)\| \le e^{b_1 t}, \qquad e^{-b_1 t} \le \|D\psi_{-t}^1(x)\| \le e^{-a_1 t} \tag{18}$$

*with $c_l = \max\{c_l', c_l''\}$, $a_l = a - c_l$ and $b_l = b + c_l$, $l = 0, 1$.*

**Proof.** For any $x \in B(0,1)$ we have $\|D^l Z_1(x)\| \le c_l' \|x\|^{1-l+m} \le c_l' \|x\|^{1-l}$ and letting $c_l = \max\{c_l', c_l''\}$, we get for any $x \in \mathbb{R}^n$ $\|D^l Z_1(x)\| \le c_l \|x\|^{1-l}$. By the same arguments as in previous lemmas we get the estimates (18). ■

**Lemma 6.** *Suppose that all the coefficients $\alpha_i$ are negative, $a \le \alpha_i \le b < 0$.*
*If the perturbation $Z_1$ fulfills the estimates*

$$\|Z_1(x)\| \le c_0 \|x\|^{1+m}, \quad \|DZ_1(x)\| \le c_1 \|x\|^m \quad \text{for all } x \in \mathbb{R}^n \text{ and any integers } m \ge 1.$$

*Then the estimates of the first derivation of the $Y_1$-flow are as follows, for any $t \ge 0$*

$$e^{at}\left(1 - \frac{c_0}{a}\|x\|^m\left(1 - e^{amt}\right)\right)^{-\frac{c_1}{mc_0}} \le \left\|D\psi_t^1(x)\right\| \le e^{bt}\left(1 - \frac{c_0}{b}\|x\|^m\left(1 - e^{bmt}\right)\right)^{-\frac{c_1}{mc_0}}.$$

**Proof.** Letting $y = \left\|\psi_t^1(x)\right\|$ and $z = \left\|\eta_1^1(t,x,\nu)\right\|$ in equation (16), we get

$$(a - c_1 y^m) z^2 \le \frac{1}{2}\frac{d}{dt} z^2 \le (b + c_1 y^m) z^2, \qquad z(0) = \|\nu\|$$

and taking into account the estimates given by the relation (11), we obtain

$$\|x\|^m e^{mat}\left(1 - \frac{c_0}{a}\|x\|^m\left(1 - e^{amt}\right)\right)^{-1} \le y^m \le \|x\|^m e^{mbt}\left(1 - \frac{c_0}{b}\|x\|^m\left(1 - e^{bmt}\right)\right)^{-1}$$

consequently

$$\|\nu\| \exp\left(at - c_1 \int_0^t \frac{\|x\|^m e^{mas} ds}{1 - \frac{c_0}{a}\|x\|^m\left(1 - e^{ams}\right)}\right)$$
$$\le z \le \|\nu\| \exp\left(bt + c_1 \int_0^t \frac{\|x\|^m e^{mbs} ds}{1 - \frac{c_0}{b}\|x\|^m\left(1 - e^{bms}\right)}\right)$$

which has the solution

$$\|\nu\| e^{at}\left(1 - \frac{c_0}{a}\|x\|^m\left(1 - e^{amt}\right)\right)^{-\frac{c_1}{mc_0}}$$
$$\le z \le \|\nu\| e^{bt}\left(1 - \frac{c_0}{b}\|x\|^m\left(1 - e^{bmt}\right)\right)^{-\frac{c_1}{mc_0}} \quad \text{for } \nu \in \mathbb{R}^n.$$ ■

**Example 2.** We consider the same vector field as in Example 1. Denote by $\xi_3^1(t,x,\nu) = D\phi_t^3(x)\nu$, $\forall\, \nu \in \mathbb{R}^n$, the first derivation of the $X_3$-flow. In coordinates, we have for any $i, j = 1, \ldots, n$,

$$\left(\phi_t^3(x)\right)_i = x_i e^{\alpha_i t}\left(1 + \frac{\beta_i}{\alpha_i} x_i^{m_i}\left(1 - e^{\alpha_i m_i t}\right)\right)^{\frac{-1}{m_i}}$$

so we deduce that

$$\frac{\partial}{\partial x_j}\left(\phi_t^3(x)\right)_i = e^{\alpha_i t}\left(1+\frac{\beta_i}{\alpha_i}x_i^{m_i}\left(1-e^{\alpha_i m_i t}\right)\right)^{-1-\frac{1}{m_i}}\delta_j^i$$

and by the estimates (13) we get

$$e^{at} \leq \left\|D\phi_t^3(x)\right\| \leq e^{bt}.$$

The second derivative is

$$\frac{\partial^2}{\partial x_i^2}\left(\phi_t^3(x)\right)_i = -(1+m_i)\frac{\beta_i}{\alpha_i}x_i^{-1+m_i}e^{\alpha_i t}\left(1-e^{\alpha_i m_i t}\right)\left(1+\frac{\beta_i}{\alpha_i}x_i^{m_i}\left(1-e^{\alpha_i m_i t}\right)\right)^{-2-\frac{1}{m_i}}.$$

Consequently, for $l=1,2$ and any $x \in B(0,\rho)$ with $\rho > 0$ arbitrary fixed, there are constants $M_l > 0$ such that

$$\left\|D^l\phi_t^3(x)\right\| \leq M_l e^{bt}.$$

### 3.3 Perturbation of a nonlinear vector field

Consider the nonlinear vector field

$$X_2 = \sum_{i=1}^{n}\beta_i x_i^{1+m_i}\frac{\partial}{\partial x_i} \quad \text{with all } \ m_i > 0 \ \text{ and all } \ \beta_i \leq 0.$$

The explicit form of the $X_2$-flow is then given by

$$\phi_t^2(x) = x(1-m\beta t x^m)^{\frac{-1}{m}} \tag{19}$$

for any $t \geq 0$ in the sense

$$\left(\phi_t^2(x)\right)_i = x_i(1-m_i\beta_i t x_i^{m_i})^{\frac{-1}{m_i}}, \qquad 1 \leq i \leq n.$$

**Lemma 7.** *If the following assumptions are true*
*i) all the coefficients $\beta_i$ are non positive, $-a' \leq \beta_i \leq -b' \leq 0$*
*ii) all the exponents $m_i$ are even positive integers; $0 < m_0 \leq m_i \leq m_0'$.*
*Then the vector field $X_2$ is semi-complete and the $X_2$-flow satisfies the estimates*

$$\|x\|\left(1+b'm_0 t\|x\|^{m_0}\right)^{\frac{-1}{m_0}} \leq \left\|\phi_t^2(x)\right\| \leq \|x\|\left(1+a'm_0't\|x\|^{m_0'}\right)^{\frac{-1}{m_0'}} \quad \textit{for any} \ \ t \geq 0. \tag{20}$$

**Proof.** Clearly the flow $\phi_t^2 = \exp(tX_2)$ given by (19) is semi-complete i.e. defined for all $t \geq 0$.
Consider the equation

$$\frac{1}{2}\frac{d}{dt}\|\phi_t^2(x)\|^2 = \left\langle\phi_t^2(x), \beta\left(\phi_t^2(x)\right)^{1+m}\right\rangle$$

and put $y = \phi_t^2(x)$, then

$$b'y^{2+m_0} \leq \frac{1}{2}\frac{d}{dt}y^2 \leq a'y^{2+m_0'}, \qquad y(0) = \|x\|$$

and we get the estimates given in (20). ∎

### 3.4 Estimation of the $k^{\text{th}}$ order derivation of the $X_2$-flow

Let $\xi_2^1(t,x,\nu) = D\phi_t^2(x)\nu$, $\forall\, \nu \in \mathbb{R}^n$ be the first derivation of the $X_2$-flow.

By formula (19), we get in coordinates

$$\frac{\partial}{\partial x_j}\left(\phi_t^2(x)\right)_i = \left(1 - m_i\beta_i t x_i^{m_i}\right)^{-1-\frac{1}{m_i}}\delta_i^j \quad \text{with} \quad \delta_i^j = \begin{cases} 1 & \text{if } \ i = j, \\ 0 & \text{if } \ i \neq j, \end{cases}$$

where $i, j = 1, \ldots, n$.

Consequently

$$\left(1 + b' m t \left\|x\right\|^{m_0}\right)^{-1-\frac{1}{m_0}} \leq \left\|D\phi_t^2(x)\right\| \leq \left(1 + a' m_0' t \left\|x\right\|^{m_0'}\right)^{-1-\frac{1}{m_0'}}. \tag{21}$$

To get the estimates of the second derivative, we put

$$w_i = 1 - m_i\beta_i t x_i^{m_i},$$

so

$$\frac{d}{dx_i} w_i = m_i(w_i - 1)x_i^{-1} \qquad \text{and} \qquad \frac{\partial}{\partial x_i}\left(\phi_t^2(x)\right)_i = {w_i}^{-1-\frac{1}{m_i}}.$$

Consequently

$$\frac{\partial^2}{\partial x_i^2}\left(\phi_t^2(x)\right)_i = (1+m_i)x_i^{-1}{w_i}^{-\frac{1}{m_i}}\left(w_i^{-2} - w_i^{-1}\right) = x_i^{-1}{w_i}^{-\frac{1}{m_i}}\left(\frac{a_1^2}{w_i} + \frac{a_2^2}{w_i^2}\right),$$

where $a_1^2$ and $a_2^2$ are real constants. Let $\rho > 0$ be any arbitrary and fixed real number, then for any $x \in B(0,\rho)$ and any $t \geq t_0 > 0$ and $l = 1, 2$ there is $M_l > 0$ such that

$$\left\|D^l\phi_t^2(x)\right\| \leq M_l t^{-1-\frac{1}{m_0'}}.$$

Suppose that for $l = 1, \ldots, k-1$, with fixed $k$, there exist constants $a_j^l$ and $M_l > 0$ such that

$$\frac{\partial^l}{\partial x_i^l}\left(\phi_t^2(x)\right)_i = x_i^{1-l}{w_i}^{-\frac{1}{m_i}}\sum_{j=1}^{l}\frac{a_j^l}{w_i^j},$$

where $a_j^l$ are real constants and

$$\left\|D^l\phi_t^2(x)\right\| \leq M_l t^{-1-\frac{1}{m_0'}} \quad \forall\, t > 0.$$

For the estimates of the $k^{\text{th}}$ derivative, we compute

$$\frac{\partial^k}{\partial x_i^k}\left(\phi_t^2(x)\right)_i = x_i^{1-k}{w_i}^{-\frac{1}{m_i}}\sum_{j=1}^{k}\frac{a_j^k}{w_i^j},$$

$$\frac{\partial^k}{\partial x_i^k}\left(\phi_t^2(x)\right)_i = \frac{d}{dx_i}x_i^{2-k}{w_i}^{-\frac{1}{m_i}}\sum_{j=1}^{k-1}\frac{a_j^{k-1}}{w_i^j}$$

$$= x_i^{1-k}{w_i}^{-\frac{1}{m_i}}\sum_{j=1}^{k-1}\left(\frac{a_j^{k-1}}{w_i^j}(1-k-jm_i) + \frac{a_j^{k-1}}{w_i^{j+1}}(1+jm_i)\right) = x_i^{1-k}{w_i}^{-\frac{1}{m_i}}\sum_{j=1}^{k}\frac{a_j^k}{w_i^j},$$

where $a_j^k$ are real constants.

So we resume

**Proposition 2.** *Suppose that*

*i) all the coefficients satisfy* $\beta_i \le 0$, $-a' \le \beta_i \le -b'$,

*ii) the exponents* $m_i$ *are even natural numbers such that* $0 < m_0 \le m_i \le m_0'$.

*Let* $\rho > 0$ *be any arbitrary fixed real number. For any* $x \in B(0,\rho)$, *for any* $t \ge t_0 > 0$ *and* $\forall\, k \ge 1$ *there exist a constant* $M_k > 0$ *such that*

$$\left\| D^k \phi_t^2(x) \right\| \le M_k t^{-1-\frac{1}{m_0'}}. \tag{22}$$

### 3.5 Estimates of the $Y_2$-flow

Let

$$Y_2 = \sum_{i=1}^{n} \left( \beta_i x_i^{1+m_i} + Z_{2i}(x) \right) \frac{\partial}{\partial x_i}$$

the perturbation of the nonlinear vector field $X_2$ and denote by $\psi_t^2 = \exp(tY_2)$ the solution of the dynamic system

$$\frac{d}{dt}\psi_t^2(x) = Y_2 \circ \psi_t^2(x), \qquad \psi_0^2(x) = x.$$

In coordinates we have, $i = 1, \dots, n$,

$$\frac{\partial}{\partial t}\psi_{2,i}(t,x) = \beta_i \psi_{2,i}^{1+m_i}(t,x) + Z_{2i}\left(\psi_t^2(x)\right), \qquad \psi_{2,i}(0,x) = x_i.$$

Putting

$$y_i(t) = \psi_{2,i}^{-m_i}(t,x)$$

and

$$\psi_t^2(x) = y^{\frac{-1}{m}}(t) = \left( y_1^{\frac{-1}{m_1}}(t), \dots, y_n^{\frac{-1}{m_n}}(t) \right)$$

we get

$$y_i'(t) = -m_i \psi_{2,i}^{-1-m_i}(t,x) \frac{\partial}{\partial t}\psi_{2,i}(t,x).$$

The Cauchy problem reads as

$$y_i'(t) = -m_i\beta_i - m_i \left(y_i(t)\right)^{1+\frac{1}{m_i}} Z_{2i}(y^{\frac{-1}{m}}(t)), \qquad y_i(0) = x_i^{-m_i}$$

and has the following solution

$$y_i(t) = x_i^{-m_i} - m_i\beta_i t - m_i \int_0^t y_i(s)^{1+\frac{1}{m_i}} Z_{2i}(y^{\frac{-1}{m}}(s)ds,$$

i.e.

$$\psi_{2,i}(t,x) = x_i \left( 1 - m_i\beta_i t x_i^{m_i} - m_i x_i^{m_i} \int_0^t \psi_i(s,x)^{-1-m_i} Z_{2i}(\psi_s^2(x))ds \right)^{-\frac{1}{m_i}},$$

so we have the explicit form of the $Y_2$-flow

$$\psi_t^2(x) = x \left( 1 - m\beta t x^m - m x^m \int_0^t \psi_s^2(x)^{-1-m} Z_2(\psi_s^2(x))ds \right)^{-\frac{1}{m}}. \tag{23}$$

Now we will estimate the $Y_2$-flow.

**Lemma 8.** *Suppose that*

*i) all the coefficients satisfy* $\beta_i \leq 0$, $-a' \leq \beta_i \leq -b'$*;*

*ii) the exponents* $m_i$ *are even natural numbers with* $0 < m_0 \leq m_i \leq m_0'$*;*

*iii)*

$$\|Z_{2i}(x)\| \leq c_0' |x_i|^{2+m_i} \quad \text{if } x \in B(0,1),$$
$$\|Z_{2i}(x)\| \leq c_0'' |x_i|^{1+m_i} \quad \text{if } x \in \mathbb{R}^n \setminus B(0,1)$$

*with* $c_0 = \max\{c_0', c_0''\}$, $b_0 = b' - c_0 > 0$, $a_0 = a' + c_0$.

*Then*

1) *the vector field* $Y_2$ *is semi-complete;*

2) *the* $Y_2$*-flow has the estimates*

$$\|x\| (1 + a_0 m_0 t \|x\|^{m_0})^{\frac{-1}{m_0}} \leq \|\psi_t^2(x)\| \leq \|x\| \left(1 + b_0 m_0' t \|x\|^{m_0'}\right)^{\frac{-1}{m_0'}}; \tag{24}$$

3) *let* $\rho > 0$ *and* $t_0 > 0$ *be fixed, then for any* $x \in B(0,\rho)$ *and any* $t \geq t_0 > 0$ *there is a constant* $M_0 > 0$ *such that*

$$\|\psi_t^2(x)\| \leq M_0 \|x\| t^{-\frac{1}{m_0'}}. \tag{25}$$

**Proof.** Let $x \in B(0,1)$, by assumption we have $\|Z_{2i}(x)\| \leq c_0' |x_i|^{2+m_i} \leq c_0' |x_i|^{1+m_i}$, put $c_0 = \max\{c_0', c_0''\}$ then for any $x \in \mathbb{R}^n$ we deduce $\|Z_{2i}(x)\| \leq c_0 |x_i|^{1+m_i}$. Now taking account of the relation (23) we deduce that for any $t \in [0,T]$

$$\left\|\psi_t^2(x)\right\| \leq \|x\| \left(1 + mt \|x\|^m (b' - c_0)\right)^{-\frac{1}{m}} \leq \|x\|$$

hence the vector $Y_2$ is semi-complete, i.e. defined for all $t \geq 0$.

Consider the equation

$$\frac{1}{2}\frac{d}{dt}\| \left(\psi_t^2(x)\right)_i \|^2 = \left\langle \left(\psi_t^2(x)\right)_i, \beta_i \left(\psi_t^2(x)\right)_i^{1+m_i} + Z_{2i}\left(\psi_t^2(x)\right) \right\rangle$$

we get $y_i = \| \left(\psi_t^2(x)\right)_i \|$ and $y_i(0) = |x_i|$, so we deduce

$$\frac{1}{2}\frac{d}{dt} y_i^2 \leq (\beta_i + c_0) y_i^{2+m_i} \leq -(b' - c_0) y_i^{2+m_i}$$

and

$$\frac{1}{2}\frac{d}{dt} y_i^2 \geq (\beta_i - c_0) y_i^{2+m_i} \geq -(a' + c_0) y_i^{2+m_i}.$$

We put $b_0 = b' - c_0$ and $a_0 = a' + c_0$, the solutions are estimated as

$$(|x_i|^{-m_i} + a_0 m_i t)^{-\frac{1}{m_i}} \leq \| \left(\psi_t^2(x)\right)_i \| \leq (|x_i|^{-m_i} + b_0 m_i t)^{-\frac{1}{m_i}}. \tag{26}$$

Hence, we have the estimate (25). ■

Now, we estimate the first derivation of the $Y_2$-flow. Let $\eta_2^1(t,x,\nu) = D\psi_t^2(x)\nu$, $\forall\, \nu \in \mathbb{R}^n$ the solution of the dynamic system

$$\frac{d}{dt}\eta_2^1(t,x,\nu) = (D_y X_2 + D_y Z_2)\, \eta_2^1(t,x,\nu), \qquad \eta_2^1(0,x,\nu) = \nu$$

with $y = \psi_t^2(x)$.

**Lemma 9.** *Suppose that*

*i) the coefficients are such that* $\beta_i \le 0$, $-a' \le \beta_i \le -b'$*;*

*ii) the coefficients* $m_i$ *are even natural numbers,* $0 < m_0 \le m_i \le m_0'$*;*

*iii)*

$$\|D^l Z_{2i}(x)\| \le c_l' \, |x_i|^{2-l+m_i} \quad \text{if } x \in B(0,1),$$
$$\|D^l Z_{2i}(x)\| \le c_l'' \, |x_i|^{1-l+m_i} \quad \text{if } x \in \mathbb{R}^n \setminus B(0,1)$$

*with* $l = 0, 1$*;*

*iv)*

$$a_0 = a' + c_0, \qquad b_0 = b' - c_0 > 0$$

*and*

$$a_1 = a'(1+m_0) + c_1, \qquad b_1 = b'(1+m_0) - c_1 > 0$$

*with* $c_l = \max\{c_l', c_l''\}$.

*Then the first derivation of the* $Y_2$*-flow has the following estimates, for any* $t > 0$

$$\left(1 + b_0 m_0 t \, \|x\|^{m_0}\right)^{-\frac{a_1}{b_0 m_0}} \le \|D\psi_t^2(x)\| \le \left(1 + a_0 m_0' t \, \|x\|^{m_0'}\right)^{-\frac{b_1}{a_0 m_0'}}. \tag{27}$$

*Let* $\rho > 0$ *be arbitrary and fixed for any* $x \in B(0,\rho)$*, and any* $t \ge t_0 > 0$ *there is a constant* $M_1 > 0$ *such that*

$$\|D\psi_t^2(x)\| \le M_1 t^{-\frac{b_1}{a_0 m_0'}}. \tag{28}$$

**Proof.** Let $x \in B(0,1)$, for $l = 0, 1$ we have

$$\|D^l Z_{2i}(x)\| \le c_l' \, |x_i|^{2-l+m_i} \le c_l' \, |x_i|^{1-l+m_i}.$$

Let $c_l = \max\{c_l', c_l''\}$ then for $x \in \mathbb{R}^n$ one has

$$\|D^l Z_{2i}(x)\| \le c_l \, |x_i|^{1-l+m_i}.$$

Consider the equation

$$\frac{1}{2}\frac{d}{dt}\|\eta_2^1(t,x,\nu)\|^2 = \left\langle \eta_2^1(t,x,\nu), \left(D_y X_2 + D_y Z_2\right) \eta_2^1(t,x,\nu)\right\rangle$$

and put $z(t) = \|\eta_2^1(t,x,\nu)\|$ with $z(0) = \|\nu\|$, then

$$\frac{1}{2}\frac{d}{dt}z^2 \le \sup_{i=1,\dots,n} \left(((1+m_i)\beta_i + c_1)\| \left(\psi_t^2(x)\right)_i \|^{m_i}\right) z^2 \le z^2 \sup_{i=1,\dots,n} \left(-b_1 \| \left(\psi_t^2(x)\right)_i \|^{m_i}\right)$$

and

$$\frac{1}{2}\frac{d}{dt}z^2 \ge \inf_{i=1,\dots,n} \left(((1+m_i)\beta_i - c_1)\| \left(\psi_t^2(x)\right)_i \|^{m_i}\right) z^2 \ge z^2 \inf_{i=1,\dots,n} \left(-a_1 \| \left(\psi_t^2(x)\right)_i \|^{m_i}\right).$$

The solutions fulfill the following estimates

$$\|\nu\| \exp \inf_{i=1,\dots,n} \left(-a_1 \int_0^t \| \left(\psi_s^2(x)\right)_i \|^{m_i} ds\right)$$
$$\le z(t) \le \|\nu\| \exp \sup_{i=1,\dots,n} \left(-b_1 \int_0^t \| \left(\psi_s^2(x)\right)_i \|^{m_i} ds\right)$$

with, by (26)

$$\frac{|x_i|^{m_i}}{1+a_0m_it\,|x_i|^{m_i}} \le \|\left(\psi_t^2(x)\right)_i\|^{m_i} \le \frac{|x_i|^{m_i}}{1+b_0m_it\,|x_i|^{m_i}}.$$

So we deduce

$$\|\nu\| \exp \inf_{i=1,\dots,n}\left(-a_1\int_0^t \frac{|x_i|^{m_i}}{1+b_0m_is\,|x_i|^{m_i}}ds\right)$$
$$\le z(t) \le \|\nu\| \exp \sup_{i=1,\dots,n}\left(-b_1\int_0^t \frac{|x_i|^{m_i}}{1+a_0m_is\,|x_i|^{m_i}}ds\right).$$

Consequently the solutions satisfy

$$\|\nu\| \inf_{i=1,\dots,n}(1+b_0m_it\,|x_i|^{m_i})^{-\frac{a_1}{b_0m_i}} \le z(t) \le \|\nu\| \sup_{i=1,\dots,n}(1+a_0m_it\,|x_i|^{m_i})^{-\frac{b_1}{a_0m_i}}.$$

Then there are constants $m_0>0$ and $m_0'>0$ such that

$$\|\nu\|\,(1+b_0m_0t\,\|x\|^{m_0})^{-\frac{a_1}{b_0m_0}} \le \|D\psi_t^2(x)\nu\|$$
$$\le \|\nu\|\left(1+a_0m_0't\,\|x\|^{m_0'}\right)^{-\frac{b_1}{a_0m_0'}} \quad \forall\, \nu\in\mathbb{R}^n \text{ and for any } t>0.$$

Hence, we have the estimate (28). ■

### 3.6 Perturbation of binomial vector fields

Let

$$Y_3=\sum_{i=1}^n\left(\alpha_ix_i+\beta_ix_i^{1+m_i}+Z_{3i}(x)\right)\frac{\partial}{\partial x_i}$$

with $a\le\alpha_i\le b<0$, $a'\le\beta_i\le b'\le 0$ and $0<m_0\le m_i\le m_0'$, be the perturbation of the binomial vector field $X_3$ and let $\psi_t^3=\exp(tY_3)$ be the $Y_3$-flow which is the solution of the dynamic system

$$\frac{d}{dt}\psi_t(x)=Y_3\circ\psi_t(x), \qquad \psi_0(x)=x$$

and in coordinates, we get

$$\frac{\partial}{\partial t}\psi_{3,i}(t,x)=\alpha_i\psi_{3,i}(t,x)+\beta_i\psi_{3,i}^{1+m_i}(t,x)+Z_{3,i}\left(\psi_t^3(x)\right), \qquad \psi_i(0,x)=x_i$$

which is a Bernoulli type equation and by the same method as in the proof of previous lemmas and with putting

$$y_i(t)=\psi_{3,i}^{-m_i}(t,x)$$

and

$$\psi_t^3(x)=y^{\frac{-1}{m}}(t)=\left(y_1^{\frac{-1}{m_1}}(t),\dots,y_n^{\frac{-1}{m_n}}(t)\right)$$

we get the solution

$$\psi_{3,i}(t,x) = x_i e^{\alpha_i t}\left(1+\frac{\beta_i}{\alpha_i}x_i^{m_i}(1-e^{\alpha_i m_i t})\right.$$
$$\left.- m_i x_i^{m_i}\int_0^t \left[\psi_{3,i}(s,x)\right]^{-1-m_i} Z_{3,i}(\psi_s^3(x))e^{\alpha_i m_i s}ds\right)^{\frac{-1}{m_i}}$$

and the implicit form of the $Y_3$-flow reads as

$$\psi_t^3(x) = xe^{\alpha t}\left(1+\frac{\beta}{\alpha}x^m(1-e^{\alpha m t}) - mx^m\int_0^t \left[\psi_s^3(x)\right]^{-1-m} Z_3(\psi_s^3(x))e^{\alpha m s}ds\right)^{-\frac{1}{m}}. \tag{29}$$

## 3.7 Estimation of the $Y_3$-flow

By the same arguments as in the previous, we get the following estimates of the $Y_3$-flow.

**Lemma 10.** *If the following assumptions are true*
*i) all the coefficients* $\alpha_i$ *are negative,* $-a \le \alpha_i \le -b < 0$*;*
*ii) all the coefficients* $\beta_i$ *are non positive,* $-a' \le \beta_i \le -b'$*;*
*iii) the exponents* $m_i$ *are even natural numbers with* $0 < m_0 \le m_i \le m_0'$*;*
*iv)*

$$\|Z_{3i}(x)\| \le c_0' \left|x_i\right|^{2+m_i} \quad \text{if } x \in B(0,1),$$
$$\|Z_{3i}(x)\| \le c_0'' \left|x_i\right|^{1+m_i} \quad \text{if } x \in \mathbb{R}^n \setminus B(0,1)$$

*with* $c_0 = \max\{c_0', c_0''\}$, $b_0 = b' - c_0 > 0$, $a_0 = a' + c_0$.
*Then*
1) *there exist constants* $m > 0$ *and* $m' > 0$ *such that the* $Y_3-$*flow has the estimates,* $\forall\, t \ge 0$

$$\|x\| e^{-at}\left(1+\frac{a_0}{a}\|x\|^m(1-e^{-amt})\right)^{-\frac{1}{m}}$$
$$\le \|\psi_t^3(x)\| \le \|x\| e^{-bt}\left(1+\frac{b_0}{b}\|x\|^{m'}(1-e^{-bm't})\right)^{-\frac{1}{m'}};$$

2) *for any* $t > 0$ *there are positive constants* $c_1$ *and* $c_2$ *such that*

$$c_1\|x\| e^{-at} \le \|\psi_t^3(x)\| \le c_2\|x\| e^{-bt}; \tag{30}$$

3) *the vector field* $Y_3$ *is semi-complete.*

By similar calculations as in previous lemmas, we get the following estimates to the first derivative of the $Y_3$-flow.

**Lemma 11.** *Suppose that*
*i) all the coefficients* $\alpha_i$ *are negative,* $-a \le \alpha_i \le -b < 0$*;*
*ii) all the coefficients* $\beta_i$*are non positive,* $-a' \le \beta_i \le -b'$*;*
*iii) the exponents* $m_i$ *are even natural numbers such that* $0 < m_0 \le m_i \le m_0'$*;*
*iv)*

$$\|D^l Z_{3i}(x)\| \le c_l' \left|x_i\right|^{2-l+m_i} \quad \text{if } x \in B(0,1),$$
$$\|D^l Z_{3i}(x)\| \le c_l'' \left|x_i\right|^{1-l+m_i} \quad \text{if } x \in \mathbb{R}^n \setminus B(0,1)$$

*with* $l = 0, 1$*;*

*v)*

$$a_0 = a' + c_0, \qquad b_0 = b' - c_0 > 0$$

*and*

$$a_1 = a'(1 + m_0) + c_1, \qquad b_1 = b'(1 + m_0) - c_1 > 0$$

*with* $c_l = \max\{c_l', c_l''\}$.

*Then there exist constants* $m > 0$ *and* $m' > 0$ *such that for any* $t \geq 0$

$$e^{-at}\left(1 + \frac{b_0}{b}\|x\|^m\left(1 - e^{-bmt}\right)\right)^{-\frac{a_1}{b_0 m}} \leq \|D\psi_t^3(x)\| \leq e^{-bt}\left(1 + \frac{a_0}{a}\|x\|^{m'}\left(1 - e^{-am't}\right)\right)^{-\frac{b_1}{a_0 m'}}$$

*and for any* $t \geq 0$*, there is a constant* $M_1 > 0$ *such that*

$$\left\|D\psi_t^3(x)\right\| \leq M_1 e^{-bt}. \tag{31}$$

# 4 Global stability of prolongations of flows

With notations of the previous sections, we will give global stability of some flows.

## 4.1 Global stability of the $Y_1$-flow

**Lemma 12.** *Let the vector fields*

$$Y_1 = \sum_{i=1}^{n}\left(\alpha_i x_i + Z_{1i}(x)\right)\frac{\partial}{\partial x_i}$$

*with the following assumptions*

*i) all the coefficients are negative,* $-a \leq \alpha_i \leq -b < 0$*;*

*ii)*

$$\|Z_1(x)\| \leq c_0' \|x\|^{1+m} \quad \forall\, x \in B(0,1) \ \text{ and } \forall m \geq 1,$$
$$\|Z_1(x)\| \leq c_0'' \|x\| \quad \forall\, x \in \mathbb{R}^n \setminus B(0,1);$$

*iii)* $b_0 = b - c_0 > 0$*, where* $c_0 = \max\{c_0', c_0''\}$.

*Then the origin 0 is a globally asymptotically stable equilibrium to the* $Y_1$*-flow* $\psi_t^1$ *on* $\mathbb{R}^n$.

**Proof.** Let $\psi_t^1 = \exp(tY_1)$ be the $Y_1$-flow, then by the assumptions and the estimates given by Lemma 2 we get that

$$\left\|\psi_t^1(x)\right\| \leq \|x\| e^{-b_0 t} \quad \forall\, t \geq 0 \text{ and } \forall x \in \mathbb{R}^n$$

and by Proposition 1, the origin 0 is *G.A.S.* for $\psi_t^1$ on $\mathbb{R}^n$. ∎

**Example 3.** We consider the vector field

$$X_3 = \sum_{i=1}^{n}\left(\alpha_i x_i + \beta_i x_i^{1+m_i}\right)\frac{\partial}{\partial x_i}$$

of Example 1 with $a \leq \alpha_i \leq b < 0$, $a' \leq \beta_i \leq b' \leq 0$. The $X_3$-flow $\phi_t^3 = \exp(tX_3)$ is then given by

$$\phi_t^3(x) = xe^{\alpha t}\left(1 + \frac{\beta}{\alpha}x^m\left(1 - e^{\alpha m t}\right)\right)^{\frac{-1}{m}}.$$

Let $\rho > 0$ be arbitrary and fixed real number. By the estimates (13), we have for any $x \in B(0,\rho)$ and any $t \geq t_0 \geq 0$

$$\left\|\phi_t^3(x)\right\| \leq \|x\| e^{-bt}.$$

By Proposition1 the origin 0 is a $G.A.S.$ for the flow $\phi_t^3$ on $\mathbb{R}^n$.

### 4.2 Global stability of the first prolongation of the $Y_1$-flow

**Lemma 13.** *With the same assumptions as in Lemma* 12 *and the following conditions*

$$\|DZ_1(x)\| \leq c_1' \|x\|^m \quad \forall\, x \in B(0,1) \text{ and } \forall\, m \geq 1,$$
$$\|DZ_1(x)\| \leq c_1'' \quad \forall\, x \in \mathbb{R}^n \setminus B(0,1)$$

*with* $b_1 = b - c_1 > 0$ *and* $c_1 = \max\{c_1', c_1''\}$.

*Then the origin* 0 *is a globally asymptotically stable for the first prolongation of the* $Y_1$*-flow* $\psi_t^1$ *on* $\mathbb{R}^n$.

**Proof.** By the estimates (18) and the hypothesis we deduce that

$$\|D\psi_t^1(x)\nu\| \leq \|\nu\| e^{-b_1 t} \quad \forall\, t > 0, \quad \forall\, \nu \in \mathbb{R}^n$$

and by Proposition 1,we obtain that the origin 0 is a $G.A.S.$ equilibrium on $\mathbb{R}^n$ for $\eta_1^1(t,x,v) = D\psi_t^1(x)\nu$. ■

### 4.3 Global stability of the $k^{\text{th}}$ prolongation of the $Y_1$-flow

Suppose that

i) all the coefficients are negative, $-a \leq \alpha_i \leq -b < 0$;

ii) for any $l = 1, \ldots, k-1$

$$\|D^l Z_1(x)\| \leq c_l' \|x\|^{1-l+m} \quad \text{for any } x \in B(0,1) \text{ and for any integer } m \geq l-1,$$
$$\|D^l Z_1(x)\| \leq c_l'' \quad \forall\, x \in \mathbb{R}^n \setminus B(0,1),$$
$$a_0 = a + c_0, \qquad b_0 = b - c_0 > 0,$$
$$a_1 = a + c_1, \qquad b_1 = b - c_1 > 0$$

with $c_l = \max\{c_l', c_l''\}$, $b_l = c_l\ \forall\, l \geq 2$.

Put $\eta_1^l(t,x,\nu,\ldots,\nu) = D^k\psi_t^1(x)\nu^k$, where $\nu \in \mathbb{R}^n$. Since by Lemmas 12 and 13 the origin 0 is an $G.A.S.$ equilibrium for $\eta_1^l$, with $l = 0, 1$, on $\mathbb{R}^n$, we suppose that this property remains true for $l = 0, 1, \ldots, k-1$ with $k \geq 2$ i.e. for any $\rho > 0$ and any $x \in B(0,\rho)$ there exist constants $M_l > 0$ such that for any $t \geq t_0 > 0$

$$\|D^l\psi_t^1(x)\| \leq M_l e^{-b_1 t}.$$

We will show that the origin 0 is a $G.A.S.$ equilibrium for $\eta_1^k$ on $\mathbb{R}^n$. $\eta_1^k(t,x,\nu,\ldots,\nu) = D^k\psi_t^1(x)\nu^k$ is solution of the dynamic system

$$\frac{d}{dt}\eta_1^k = D_y Y_1 \cdot \eta_1^k + G_1^k(t,x,\nu), \qquad \eta_1^k(0,x,\nu,\ldots,\nu) = \nu$$

with $y = \psi_t^1(x)$ and

$$G_1^k(t,x,\nu) = \sum_{l=2}^{k} D_y^l Y_1(y) \sum_{\substack{i_1+\cdots+i_l=k \\ i_j>0}} \left( \prod_{j=1}^{l} D^{i_j}\psi_t^1(x)\nu^{i_j} \right)$$
$$= \sum_{l=2}^{k-1} D_y^l Z_1(y) \sum_{\substack{i_1+\cdots+i_l=k \\ i_j>0}} \left( \prod_{j=1}^{l} D^{i_j}\psi_t^1(x)\nu^{i_j} \right) + D_y^k Z_1(y) \left(D\psi_t^1(x)\nu\right)^k .$$

Consequently we get

$$\eta_1^k(t,x,\nu,\ldots,\nu) = D\psi_t^1(x)\nu + \int_0^t D\Psi_{t-s}^1(\psi_s^1(x))G_1^k(s,x,\nu)ds.$$

The integral is well defined at $s=0$, since

$$\lim_{s\to 0^+} D\psi_{t-s}^1(\psi_s^1(x)) = D\psi_t^1(x)$$

and there exist constants $A_l > 0$ such that

$$\lim_{s\to 0^+} G_1^k(s,x,\nu) = \sum_{l=2}^{k} A_l D_y^l Z_1(y)\nu^k.$$

We will show that it converges uniformly with respect to $x$ as $t + \infty$. Put

$$I_k = \int_0^t \|D\psi_{t-s}^1(\psi_s^1(x))\| \|G_1^k(s,x,\nu)\| ds.$$

Since $\|D^l Z_1(x)\| \le c_l \ \forall\ l \ge 1,\ \forall\ x \in \mathbb{R}^n$, there are constants $b_l > 0$ such that $\forall\ y \in \mathbb{R}^n$, $\|D_y^l Y_1(y)\| \le b_l$ and by the assumption of recurrence there exist constants $M_l > 0$ such that

$$\|D^l\psi_t^1(x)\| \le M_l e^{-b_1 t} \quad \forall\, t \ge 0.$$

We deduce that there is a constant $C_k > 0$ such that

$$I_k \le \sum_{l=2}^{k} b_l M_l \int_0^t e^{-b_1(t-s+sl)} ds \le C_k e^{-b_1 t}.$$

So for any $x \in \mathbb{R}^n$ one has

$$\lim_{t\to+\infty} I_k \le \sum_{l=2}^{k} M_l b_l \|\nu\|^l \int_0^{+\infty} e^{-b_1 s} ds = \frac{1}{b_1} \sum_{l=2}^{k} M_l b_l \|\nu\|^l$$

and the integral $I_k$ is uniformly convergent with respect to $x \in \mathbb{R}^n$ as $t \to +\infty$. Consequently

$$\lim_{t\to+\infty} \|\eta_1^k\| = \lim_{t\to+\infty} \|D\psi_t^1(x)\nu\| + \int_0^{+\infty} \lim_{t\to+\infty} \|D\psi_{t-s}^1(\psi_s^1(x))\| \|G_1^k(s,x,\nu)\| ds = 0$$

and there is a constant $M_k' > 0$ such that

$$\|\eta_1^k\| \le \|D\psi_t^1(x)\nu\| + \int_0^{+\infty} \|D\psi_{t-s}^1(\psi_s^1(x))\| \|G_1^k(s,x,\nu)\| ds \le M_k' \|\nu\|^k e^{-b_1 t}.$$

This show by Proposition 1 that the origin 0 is a $G.A.S.$ equilibrium to $\eta_1^k$ on $\mathbb{R}^n$. We formulate our proving as follows

**Proposition 3.** *Let $k \ge 0$ be any integer. The origin 0 is a G.A.S. equilibrium of order $k$ for the $Y_1$-flow and there is a constant $M_k > 0$ such that $\forall\ t > 0$*

$$\|D^k\psi_t^1(x)\| \le M_k e^{-b_1 t}, \qquad \|D^k\psi_{-t}^1(x)\| \le M_k e^{a_1 t}. \tag{32}$$

# 5 Global stability of a flow generated by nonlinear perturbed vector fields

First we will start with monomial vector fields.

## 5.1 Global stability of the $X_2$-flow

Let

$$X_2 = \sum_{i=1}^{n} \beta_i x_i^{1+m_i} \frac{\partial}{\partial x_i}$$

with

(i) all the coefficients $\beta_i \leq 0$ such that $-a' \leq \beta_i \leq -b'$;

(ii) all the exponents $m_i$ are even natural integers with $0 < m_0 \leq m_i \leq m_0'$.

Let $\phi_t^2 = \exp(tX_2)$ be the $X_2$-flow. By the estimations (19) we obtain

$$\left\|\phi_t^2(x)\right\| \leq \|x\| \left(1 + a' m_0' t \|x\|^{m_0'}\right)^{\frac{-1}{m_0'}}.$$

Let $\rho > 0$ be arbitrary fixed, for any $x \in B(0,\rho)$ and any $t \geq t_0 > 0$ there is a constant $M_0 > 0$ such that

$$\|\phi_t^2(x)\| \leq M_0 \|x\| t^{-\frac{1}{m_0'}}.$$

By Proposition 1, the origin is a globally asymptotically stable equilibrium to the flow $\phi_t^2$ on $\mathbb{R}^n$.

Let $l = 1, 2, \ldots$ any positive integer. By Proposition 2, we have: for any fixed $\rho > 0$, and all $x \in B(0,\rho)$ and $t \geq t_0 > 0$, there exist constants $M_l > 0$ and $M_l' > 0$ such that

$$\|D^l \phi_t^2(x)\| \leq M_l\, t^{-1-\frac{1}{m_0'}} \qquad \text{and} \qquad \|D^l \phi_0^2(x)\| \leq M_l'.$$

So the origin 0 is a $G.A.S.$ equilibrium for $D^l \phi_t^2(x)$ on $\mathbb{R}^n$.

Resuming our proving, we get

**Proposition 4.** *Let $k \geq 0$ be any integer. Under the above conditions (i) and (ii), the origin 0 is a G.A.S. of order $k$ for the $X_2$-flow on $\mathbb{R}^n$.*

## 5.2 Global stability of high order of the $Y_2$-flow

Let

$$Y_2 = \sum_{i=1}^{n} \left(\beta_i x_i^{1+m_i} + Z_{2,i}(x)\right) \frac{\partial}{\partial x_i}$$

be a smooth vector field on $\mathbb{R}^n$ such that

i) all the coefficients $\beta_i \leq 0$ are non negative with $-a' \leq \beta_i \leq -b'$;

ii) $m_i$ are even natural numbers with $0 < m_0 \leq m_i \leq m_0'$;

iii) for $k = 0, \ldots, 1 + m_i$

$$\|D^k Z_{2i}(x)\| \leq c_k' \,|x_i|^{2-k+m_i} \quad \text{if } x \in B(0,1)\,;$$
$$\|D^k Z_{2i}(x)\| \leq c_k'' \,|x_i|^{1-k+m_i} \quad \text{if } x \in \mathbb{R}^n \setminus B(0,1)\,;$$

iv) for any $k \geq 2 + m_i$

$$\|D^k Z_{2i}(x)\| \leq c_k;$$

v)

$$a_0 = a' + c_0, \qquad a_1 = a'(1+m_0) + c_1,$$
$$b_0 = b' - c_0 > 0, \qquad b_1 = b'(1+m_0) - c_1 > a_0 m_0'$$

with $c_k = \max\{c_k', c_k''\}$.

**Remark 1.** If $x \in B(0,1)$ then $\|D^k Z_{2i}(x)\| \leq c_k' |x_i|^{2-k+m_i} \leq c_k' |x_i|^{1-k+m_i}$. Putting $c_l = \max\{c_l', c_l''\}$, we deduce that for any $x \in \mathbb{R}^n$ have $\|D^k Z_{2i}(x)\| \leq c_k |x_i|^{1-k+m_i}$.

### 5.2.1 Global stability of the $Y_2$-flow on $\mathbb{R}^n$

Let $\psi_t^2 = \exp(tY_2)$ be the $Y_2$-flow and let $\rho > 0$ be arbitrary and fixed, so by the estimates (25) for all $x \in B(0,\rho)$ and all $t \geq t_0 > 0$ there is a constant $M_0 > 0$ such that

$$\|\psi_t^2(x)\| \leq M_0 \|x\| t^{-\frac{1}{m_0'}}.$$

So by Proposition 1, the origin 0 is a $G.A.S.$ equilibrium for the $Y_2$-flow $\psi_t^2$ on $\mathbb{R}^n$.

### 5.2.2 Global stability of prolongation of the $Y_2$-flow on $\mathbb{R}^n$

We proceed by recurrence. Since it is already true for $k = 0$, we suppose that for any $l = 1, \ldots, k-1$, with $k \geq 2$, the origin 0 is a $G.A.S.$ to $D^l\psi_t^2(x)$ on $\mathbb{R}^n$ that is to say for any fixed $\rho > 0$, all $x \in B(0,\rho)$ and all $t \geq t_0 > 0$ there are constants $M_l > 0$ such that

$$\|D^l\psi_t^2(x)\| \leq M_l t^{-\frac{b_1}{a_0 m_0'}} \qquad \text{and} \qquad \|D^l\psi_0^2(x)\| \leq M_l'.$$

We will show that 0 is a $G.A.S.$ for $D^k\psi_t^2(x)$ on $\mathbb{R}^n$.

Put $\eta_2^k(t,x,\nu,\ldots,\nu) = D^k\psi_t^2(x)\nu^k \ \forall\, \nu \in \mathbb{R}^n$ which is solution of the dynamic system

$$\frac{d}{dt}\eta_2^k = D_y Y_2 \cdot \eta_2^k + G_2^k(t,x,\nu), \qquad \eta_2^k(0,x,\nu,\ldots,\nu) = \nu$$

with $y = \psi_t^2(x)$ and

$$G_2^k(t,x,\nu) = \sum_{l=2}^{k} D_y^l Y_2(y) \sum_{\substack{i_1+\cdots+i_l=k \\ i_j>0}} \left( \prod_{j=1}^{l} D^{i_j}\psi_t^2(x)\nu^{i_j} \right).$$

By the method of the resolvent, we deduce

$$\eta_2^k(t,x,\nu,\ldots,\nu) = D\psi_t^2(x)\nu + \int_0^t D\psi_{t-s}^2(\psi_s^2(x)) G_2^k(s,x,\nu)ds.$$

Clearly the integral

$$I_k^1 = \int_0^1 \|D\psi_{t-s}^2(\psi_s^2(x))\| \|G_2^k(s,x,\nu)\| ds$$

is well defined at $s = 0$ and $s = t$, since

$$\lim_{s\to 0^+} D\psi^2_{t-s}(\psi^2_s(x)) = D\psi^2_t(x).$$

By the recurrent assumption $D^l\psi^2_0(x)$ are bounded and there exist constants $A_l > 0$ such that

$$\lim_{s\to 0^+} \|G^k_2(s, x, \nu)\| \leq \sum_{l=2}^{k} A_l \|D^l_x Y_2(x)\nu^l\|.$$

In the same way

$$\lim_{s\to t^-} D\psi^2_{t-s}(\psi^2_s(x)) = \text{identity}.$$

Now, we have to show that

$$I^2_k = \int_1^t \|D\psi^2_{t-s}(\psi^2_s(x))\| \|G^k_2(s, x, \nu)\| ds$$

converges uniformly on any compact set $K \subset \mathbb{R}^n$ as $t \to 0$.

Let $x \in K$, by the relations (26) and (28) we get for all $t \geq 0$

$$\|x\| \left(1 + a_0m_0t \|x\|^{m_0}\right)^{\frac{-1}{m_0}} \leq \|\psi^2_t(x)\| \leq \|x\| \left(1 + b_0m'_0t \|x\|^{m'_0}\right)^{\frac{-1}{m'_0}},$$
$$\left(1 + b_0m_0t \|x\|^{m_0}\right)^{-\frac{a_1}{b_0m_0}} \leq \|D\psi^2_t(x)\| \leq \left(1 + a_0m'_0t \|x\|^{m'_0}\right)^{-\frac{b_1}{a_0m'_0}}.$$

So $\|y\| = \|\psi^2_t(x)\| \leq \|x\|$ and $\left\|D\psi^2_{t-s}(\psi^2_s(x))\right\|$ is bounded. Since for any $x \in \mathbb{R}^n$ and any $l = 1, \ldots, 1+m_i$, $\|D^l Z_{2i}(x)\| \leq c_l |x_i|^{1-l+m_i}$ then $D^l_y Y_2(y)$ are bounded. Now by the assumption of recurrence there exist constants $M_l > 0$ such that for any $t > 0$

$$\|D^l\psi^2_t(x)\| \leq M_l t^{-\frac{b_1}{a_0m'_0}}$$

with $a_0m'_0 < b_1$ i.e. $\frac{b_1}{a_0m'_0} > 1$, and we deduce the existence of constants $C_l > 0$ such that

$$\lim_{t\to+\infty} I^2_k \leq \sum_{l=2}^{k} C_l \int_1^{+\infty} s^{-\frac{lb_1}{a_0m'_0}} ds \leq \sum_{l=2}^{k} C_l \left(\frac{lb_1}{a_0m'_0} - 1\right)^{-1}.$$

The integral $I^2_k$ converges uniformly on any compact $K \subset \mathbb{R}^n$ as $t \to +\infty$.

Now since the integral is well defined at $s = 0$, then

$$\lim_{t\to 0} \|\eta^k_2(t, x, \nu, \ldots, \nu)\| \leq \lim_{t\to 0} \|D\psi^2_t(x)\nu\| = \|\nu\|$$

hence there is a constant $M'_k > 0$ such that

$$\|D^k\psi^2_0(x)\| \leq M'_k.$$

In the same way as above the integral $\int_0^t \|D\psi^2_{t-s}(\psi^2_s(x))\| \|G^k_2(s, x, \nu)\| ds$ is well defined and putting $\tau = \frac{s}{t}$ we obtain

$$\eta^k_2(t, x, \nu, \ldots, \nu) = D\psi^2_t(x)\nu + t \int_0^1 D\psi^2_{t(1-\tau)}(\psi^2_{t\tau}(x)) G^k_2(t\tau, x, \nu) d\tau.$$

Since $b_1 = b'(1+m_0) - c_1 > a_0 m_0'$ , by the estimates (26) and (28), we deduce the existence of a constant $M_k > 0$ such that

$$\begin{aligned}\|\eta_2^k(t,x,\nu,\dots,\nu)\| &\leq \|D\psi_t^2(x)\nu\| + t\int_0^1 \|G_2^k(t\tau,x,\nu)\|d\tau \\ &\leq \|D\psi_t^2(x)\nu\| + t\sum_{l=2}^{k}\int_0^1 \frac{(t\tau)^{-\frac{lb_1}{a_0m_0'}}\|x\|^{1+m_0'-l}}{\left(1+b_0m_0't\tau\|x\|^{m_0'}\right)^{\frac{1+m_0'-l}{m_0'}}}d\tau \leq M_k t^{-\frac{b_1}{a_0m_0'}}.\end{aligned}$$

Which shows that the origin 0 is a $G.A.S.$ equilibrium for $\eta_2^k$ on $\mathbb{R}^n$. We formulate this fact as

**Proposition 5.** *Let $k \geq 0$ be any integer. Under the above conditions (i), (ii), (iii), (iv) and (v), the origin 0 is a G.A.S. of order $k$ on $\mathbb{R}^n$ for the $Y_2$-flow and there is a constant $M_k > 0$ such that for any $t \geq t_0 > 0$*

$$\|D^k\psi_t^2(x)\| \leq M_k t^{-\frac{b_1}{a_0m_0'}}. \tag{33}$$

### 5.3 Global stability of prolongations of the $Y_3$-flow

Let

$$Y_3 = \sum_{i=1}^{n}\left(\alpha_i x_i + \beta_i x_i^{1+m_i} + Z_{3i}(x)\right)\frac{\partial}{\partial x_i}$$

with

i) all the coefficient $\alpha_i$ are negative with $-a \leq \alpha_i \leq -b$;

ii) all the coefficients $\beta_i \leq 0$ and $-a' \leq \beta_i \leq -b'$;

iii) the exponents $m_i$ are even natural numbers with $0 < m_0 \leq m_i \leq m_0'$;

iv) For any $k = 0,\dots,1+m_i$

$$\begin{aligned}&\|D^kZ_{3i}(x)\| \leq c_k'\,|x_i|^{2-k+m_i} \quad \text{if } x \in B\,(0,1)\,,\\ &\|D^kZ_{3i}(x)\| \leq c_k''\,|x_i|^{1-k+m_i} \quad \text{if } x \in \mathbb{R}^n \setminus B\,(0,1)\,;\end{aligned}$$

v) for any $k \geq 2+m_i$

$$\|D^kZ_{3i}(x)\| \leq c_k;$$

vi)

$$\begin{aligned}&a_0 = a' + c_0, \qquad a_1 = a'(1+m_0) + c_1,\\ &b_0 = b' - c_0 > 0, \qquad b_1 = b'(1+m_0) - c_1 > 0\end{aligned}$$

with $c_k = \max\{c_k', c_k''\}$.

**Remark 2.** If $x \in B\,(0,1)$ then $\|D^kZ_{3i}(x)\| \leq c_k'\,|x_i|^{2-k+m_i} \leq c_k'\,|x_i|^{1-k+m_i}$.
Let $c_l = \max\{c_l', c_l''\}$, for any $x \in \mathbb{R}^n$ one has $\|D^kZ_{3i}(x)\| \leq c_k\,|x_i|^{1-k+m_i}$.

### 5.3.1 Global stability of the $Y_3$-flow $\mathbb{R}^n$

Denote by $\psi_t^3 = \exp(tY_3)$, by the estimates (30), we have

$$\|\psi_t^3(x)\| \leq C\|x\|e^{-bt} \quad \forall\, t > 0 \text{ and } \forall\, x \in \mathbb{R}^n,$$

where $C > 0$ is a constant. So by Proposition 1, 0 is a $G.A.S.$ on $\mathbb{R}^n$. We proceed by recurrence; since the property is true in case $k = 0$, we assume that the property remains true for any $l = 1, \ldots, k-1$, with $k$ fixed i.e. 0 is a global $G.A.S.$ of $\eta_3^l(t, x, \nu, \ldots \nu) = \|D^l\psi_t^3(x)\nu^k\|$ on $\mathbb{R}^n$ and there exist constants $M_l > 0$ such that for any $t > 0$

$$\|D^l\psi_t^3(x)\| \leq M_l e^{-bt}.$$

We will show that 0 is a $G.A.S.$ equilibrium to $\eta_3^k$ on $\mathbb{R}^n$.

$\eta_3^k(t, x, \nu, \ldots, \nu)$ is a solution to the dynamic system

$$\frac{d}{dt}\eta_3^k = D_y\eta_3^k + G_3^k(t, x, \nu)$$

with $y = \psi_t^3(x)$ and

$$G_3^k(t, x, \nu) = \sum_{l=2}^{k} D_y^l Y_3(y) \sum_{\substack{i_1+\cdots+i_l=k \\ i_j>0}} \left( \prod_{j=1}^{l} D^{i_j}\psi_t^3(x)\nu^{i_j} \right).$$

By the method of the resolvent, we get

$$\eta_3^k(t, x, \nu, \ldots, \nu) = D\psi_t^3(x)\nu + \int_0^t D\psi_{t-s}^3(\psi_s^3(x))G_3^k(s, x, \nu)ds$$

and by the same argument as for the $Y^1$-flow, we deduce that for any integer $k \geq 0$ there exist a constant $M_k$ such that $\forall\, t \geq 0$

$$\|D^k\psi_t^3(x)\| \leq M_k \|x\| e^{-bt}.$$

By Proposition 1, we have

**Proposition 6.** *Under the above conditions (i), (ii), (iii), (iv), (v) and (vi), the origin 0 is a G.A.S. equilibrium of order $k$ on $\mathbb{R}^n$ to the $Y_3$-flow .*

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