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The Relationship between Zhedanov's Algebra $AW(3)$ and the Double Affine Hecke Algebra in the Rank One Case[⋆]

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Abstract. Zhedanov's algebra $AW(3)$ is considered with explicit structure constants such that, in the basic representation, the first generator becomes the second order q-difference operator for the Askey–Wilson polynomials. It is proved that this representation is faithful for a certain quotient of $AW(3)$ such that the Casimir operator is equal to a special constant. Some explicit aspects of the double affine Hecke algebra (DAHA) related to symmetric and non-symmetric Askey–Wilson polynomials are presented and proved without requiring knowledge of general DAHA theory. Finally a central extension of this quotient of $AW(3)$ is introduced which can be embedded in the DAHA by means of the faithful basic representations of both algebras.

Key words: Zhedanov's algebra $AW(3)$; double affine Hecke algebra in rank one; Askey–Wilson polynomials; non-symmetric Askey–Wilson polynomials

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Dedicated to the memory of Vadim Kuznetsov

Briefly after I had moved from CWI, Amsterdam to a professorship at the University of Amsterdam in 1992, Vadim Kuznetsov contacted me about the possibility to come to Amsterdam as a postdoc. We successfully applied for a grant. He arrived with his wife Olga and his son Simon in Amsterdam for a two-years stay during 1993–1995. I vividly remember picking them up at the airport and going in the taxi with all their stuff to their first apartment in Amsterdam, at the edge of the red light quarter. These were two interesting years, where we learnt a lot from each other. We wrote one joint paper, but Vadim wrote many further papers alone or with other coauthors during this period. We should have written more together, but our temperaments were too different for that. Vadim was always speeding up, while I wanted to ponder and to look for further extensions and relations with other work.

After his Amsterdam years Vadim had a marvelous career which led to prestigious UK grants, tenure in Leeds, and a lot of organizing of conferences and proceedings. We met several times afterwards. I visited for instance Leeds for one week, and Vadim was an invited speaker at the conference in Amsterdam in 2003 on the occasion of my sixtieth birthday.

1 Introduction

Zhedanov [16] introduced in 1991 an algebra $AW(3)$ with three generators K_0, K_1, K_2 and three relations in the form of q-commutators, which describes deeper symmetries of the Askey–

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Wilson polynomials. In fact, for suitable choices of the structure constants of the algebra, the Askey–Wilson polynomial $p_n(x)$ is the kernel of an intertwining operator between a representation of $AW(3)$ by q-difference operators on the space of polynomials in x and a representation by tridiagonal operators on the space of of infinite sequences $(c_n)_{n=1,2,\ldots}$. In the first representation K_1 is multiplication by x and K_0 is the second order q-difference operator for which the Askey–Wilson polynomials are eigenfunctions with explicit eigenvalues λ_n. In the second representation K_0 is the diagonal operator with diagonal elements λ_n and K_1 is the tridiagonal operator corresponding to the three-term recurrence relation for the Askey–Wilson polynomials. The formula for $p_n(x)$ expressing the intertwining property with respect to K_2 is the so-called *q-structure relation* for the Askey–Wilson polynomials (see [6]) and the relation for $AW(3)$ involving the q-commutator of K_1 and K_2 is the so-called *q-string equation* (see [4]). Terwilliger & Vidunas [15] showed that every Leonard pair satisfies the $AW(3)$ relations for a suitable choice of the structure constants.

In 1992, one year after Zhedanov's paper [16], Cherednik [2] introduced double affine Hecke algebras associated with root systems (DAHA's). This was the first of an important series of papers by the same author, where a representation of the DAHA was given in terms of q-difference-reflection operators (q-analogues of Dunkl operators), joint eigenfunctions of such operators were identified as non-symmetric Macdonald polynomials, and Macdonald's conjectures for ordinary (symmetric) Macdonald polynomials associated with root systems could be proved. For a nice exposition of this theory see Macdonald's recent book [7]. In particular, the DAHA approach to Macdonald–Koornwinder polynomials, due to several authors (see Sahi [11, 12], Stokman [14] and references given there) is also presented in [7]. The last chapter of [7] discusses the rank one specialization of these general results. For the DAHA of type A_1 (one parameter) this yields non-symmetric q-ultraspherical polynomials. For the DAHA of type $(C_1^\vee, C_1)$ (four parameters) the non-symmetric Askey–Wilson polynomials are obtained. These were earlier treated by Sahi [12] and by Noumi & Stokman [9]. See also Sahi's recent paper [13].

Comparison of Zhedanov's $AW(3)$ with the DAHA of type of type $(C_1^\vee, C_1)$, denoted by $\tilde{\mathfrak{H}}$, suggests some relationship. Both algebras are presented by generators and relations, the first has a representation by q-difference operators on the space of symmetric Laurent polynomials in z and the second has a representation by q-difference-reflection operators on the space of general Laurent polynomials in z. Since this representation of the DAHA is called the *basic representation* of $\tilde{\mathfrak{H}}$, I will call the just mentioned representation of $AW(3)$ also the *basic representation*. In the basic representation of $AW(3)$ the operator K_0 is equal to some operator D occurring in the basic representation of $\tilde{\mathfrak{H}}$ and involving reflections, provided D is restricted in its action to symmetric Laurent polynomials. This suggests that the basic representation of $AW(3)$ may remain valid if we represent K_0 by D, so that it involves reflection terms. It will turn out in this paper that this conjecture is correct in the A_1 case, i.e., when the Askey–Wilson parameters are restricted to the continuous q-ultraspherical case. In the general case the conjecture is true for a rather harmless central extension of $AW(3)$ involving a generator T_1, which will be identified with the familiar T_1 in $\tilde{\mathfrak{H}}$ which has in the basic representation of $\tilde{\mathfrak{H}}$ the symmetric Laurent polynomials as one of its two eigenspaces.

This paper does not suppose any knowledge about the general theory of double affine Hecke algebras and about Macdonald and related polynomials in higher rank. The contents of the paper are as follows. Section 2 presents $AW(3)$ and its relationship with Askey–Wilson polynomials. We add to $AW(3)$ one more relation expressing that the Casimir operator Q is equal to a special constant Q_0 (of course precisely the constant occurring for Q in the basic representation), and we denote the resulting quotient algebra by $AW(3, Q_0)$. Then it is shown that the basic representation of $AW(3, Q_0)$ is faithful. Section 3 discusses $\tilde{\mathfrak{H}}$ (the DAHA of type $(C_1^\vee, C_1)$), its basic representation, and the basis vectors for the 2-dimensional eigenspaces of the operator D in terms of Askey–Wilson polynomials. Section 4 gives an explicit expression for the non-symmetric Askey–Wilson polynomials which is in somewhat different terms than the explicit expression in [7, § 6.6]. Two presentations of $\tilde{\mathfrak{H}}$ by generators and relations of PBW-type are given in Section 5. The faithfulness of the basic representation is proved (a result which of course is also a special case of the known result in the case of general rank, see Sahi [11]). The main result of the present paper, the embedding of a central extension of $AW(3, Q_0)$ in $\tilde{\mathfrak{H}}$, is stated and proved in Section 6.

For the computations in this paper I made heavy use of computer algebra performed in *Mathematica*®. For reductions of expressions in non-commuting variables subject to relations I used the package *NCAlgebra* [8] within *Mathematica*®. *Mathematica* notebooks containing these computations will be available for downloading in `http://www.science.uva.nl/~thk/art/`.

Conventions

Throughout assume that q and a, b, c, d are complex constants such that

$$q \neq 0, \qquad q^m \neq 1 \ (m = 1, 2, \ldots), \qquad a, b, c, d \neq 0, \qquad abcd \neq q^{-m} \ (m = 0, 1, 2, \ldots). \tag{1.1}$$

Let e_1, e_2, e_3, e_4 be the elementary symmetric polynomials in a, b, c, d:

$$\begin{aligned} &e_1 := a + b + c + d, \qquad e_2 := ab + ac + bc + ad + bd + cd, \\ &e_3 := abc + abd + acd + bcd, \qquad e_4 := abcd. \end{aligned} \tag{1.2}$$

For (q-)Pochhammer symbols and (q-)hypergeometric series use the notation of [3]. In particular,

$$(a; q)_k := \prod_{j=0}^{k-1} (1 - aq^j), \qquad (a_1, \ldots, a_r; q)_k := (a_1; q)_k \cdots (a_r; q)_k,$$

$${}_r\phi_{r-1}\left(\begin{matrix} q^{-n}, a_2, \ldots, a_r \\ b_1, \ldots, b_{r-1} \end{matrix}; q, z\right) := \sum_{k=0}^{n} \frac{(q^{-n}, a_2, \ldots, a_r; q)_k}{(b_1, \ldots, b_{r-1}, q; q)_k} z^k.$$

For Laurent polynomials f in z the z-dependence will be written as $f[z]$. Symmetric Laurent polynomials $f[z] = \sum_{k=-n}^{n} c_k z^k$ (where $c_k = c_{-k}$) are related to ordinary polynomials $f(x)$ in $x = \frac{1}{2}(z + z^{-1})$ by $f(\frac{1}{2}(z + z^{-1})) = f[z]$.

2 Zhedanov's algebra $AW(3)$

Zhedanov [16] introduced an algebra $AW(3)$ with three generators K_0, K_1, K_2 and with three relations

$$\begin{aligned} &[K_0, K_1]_q = K_2, \\ &[K_1, K_2]_q = B\, K_1 + C_0\, K_0 + D_0, \\ &[K_2, K_0]_q = B\, K_0 + C_1\, K_1 + D_1, \end{aligned}$$

where

$$[X, Y]_q := q^{\frac{1}{2}} XY - q^{-\frac{1}{2}} YX$$

is the q-commutator and where the structure constants B, C_0, C_1, D_0, D_1 are fixed complex constants. He also gave a *Casimir operator*

$$Q := \left(q^{-\frac{1}{2}} - q^{\frac{3}{2}}\right) K_0 K_1 K_2 + q K_2^2 + B(K_0 K_1 + K_1 K_0) + q C_0 K_0^2 + q^{-1} C_1 K_1^2$$

$$+ (1+q)D_0K_0 + (1+q^{-1})D_1K_1,$$

which commutes with the generators.

Clearly, $AW(3)$ can equivalently be described as an algebra with two generators K_0, K_1 and with two relations

$$(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 = B\,K_1 + C_0\,K_0 + D_0, \tag{2.1}$$
$$(q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 = B\,K_0 + C_1\,K_1 + D_1. \tag{2.2}$$

Then the Casimir operator Q can be written as

$$\begin{aligned} Q = (K_1K_0)^2 &- (q^2+1+q^{-2})K_0(K_1K_0)K_1 + (q+q^{-1})K_0^2K_1^2 + (q+q^{-1})(C_0K_0^2 + C_1K_1^2) \\ &+ B\big((q+1+q^{-1})K_0K_1 + K_1K_0\big) + (q+1+q^{-1})(D_0K_0 + D_1K_1). \end{aligned} \tag{2.3}$$

Let the structure constants be expressed in terms of a, b, c, d by means of e_1, e_2, e_3, e_4 (see (1.2)) as follows:

$$\begin{aligned} B &:= (1-q^{-1})^2(e_3+qe_1), \\ C_0 &:= (q-q^{-1})^2, \\ C_1 &:= q^{-1}(q-q^{-1})^2e_4, \\ D_0 &:= -q^{-3}(1-q)^2(1+q)(e_4+qe_2+q^2), \\ D_1 &:= -q^{-3}(1-q)^2(1+q)(e_1e_4+qe_3). \end{aligned} \tag{2.4}$$

Then there is a representation (the *basic representation*) of the algebra $AW(3)$ with structure constants (2.4) on the space $\mathcal{A}_{\rm sym}$ of symmetric Laurent polynomials $f[z] = f[z^{-1}]$ as follows:

$$(K_0f)[z] = (D_{\rm sym}f)[z], \qquad (K_1f)[z] = ((Z+Z^{-1})f)[z] := (z+z^{-1})f[z], \tag{2.5}$$

where $D_{\rm sym}$, given by

$$\begin{aligned} (D_{\rm sym}f)[z] := &\frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}\big(f[qz]-f[z]\big) \\ &+ \frac{(a-z)(b-z)(c-z)(d-z)}{(1-z^2)(q-z^2)}\big(f[q^{-1}z]-f[z]\big) + (1+q^{-1}abcd)f[z], \end{aligned} \tag{2.6}$$

is the second order operator having the *Askey–Wilson polynomials* (see [1], [3, § 7.5], [5, § 3.1]) as eigenfunctions. It can indeed be verified that the operators K_0, K_1 given by (2.5) satisfy relations (2.1), (2.2) with structure constants (2.4), and that the Casimir operator Q becomes the following constant in this representation:

$$(Qf)(z) = Q_0\,f(z), \tag{2.7}$$

where

$$\begin{aligned} Q_0 := q^{-4}(1-q)^2\Big(&q^4(e_4-e_2) + q^3(e_1^2-e_1e_3-2e_2) \\ &- q^2(e_2e_4+2e_4+e_2) + q(e_3^2-2e_2e_4-e_1e_3) + e_4(1-e_2)\Big). \end{aligned} \tag{2.8}$$

Let $AW(3,Q_0)$ be the algebra generated by K_0, K_1 with relations (2.1), (2.2) and

$$Q = Q_0, \tag{2.9}$$

assuming the structure constants (2.4). Then the basic representation of $AW(3)$ is also a representation of $AW(3,Q_0)$.

The Askey–Wilson polynomials are given by

$$p_n\big(\tfrac12(z+z^{-1});a,b,c,d\mid q\big):=\frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,az^{-1}\\ ab,ac,ad\end{matrix};q,q\right). \tag{2.10}$$

These polynomials are symmetric in a, b, c, d (although this cannot be read off from (2.10)). We will work with the renormalized version which is *monic* as a Laurent polynomial in z (i.e., the coefficient of z^n equals 1):

$$\begin{aligned}P_n[z]=P_n[z;a,b,c,d\mid q]&:=\frac{1}{(abcdq^{n-1};q)_n}\,p_n\big(\tfrac12(z+z^{-1});a,b,c,d\mid q\big)\\ &=a^{-n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k\,(az,az^{-1};q)_k\,(abq^k,acq^k,adq^k;q)_{n-k}\,q^k}{(q;q)_k\,(abcdq^{n+k-1};q)_{n-k}}.\end{aligned} \tag{2.11}$$

Note that the monic Askey–Wilson polynomials $P_n[z]$ are well-defined for all n under condition (1.1).

The eigenvalue equation involving $D_{\rm sym}$ is

$$D_{\rm sym}P_n=\lambda_nP_n,\qquad \lambda_n:=q^{-n}+abcdq^{n-1}. \tag{2.12}$$

Under condition (1.1) all eigenvalues in (2.12) are distinct.

The three-term recurrence relation for the monic Askey–Wilson polynomials (see [5, (3.1.5)]) is as follows:

$$\begin{aligned}&(z+z^{-1})P_n[z]=P_{n+1}[z]+\beta_nP_n[z]+\gamma_nP_{n-1}[z]\qquad (n\ge1),\\ &(z+z^{-1})P_0[z]=P_1[z]+\beta_0P_0[z],\end{aligned} \tag{2.13}$$

$$\beta_n:=q^{n-1}\,\frac{(1-q^n-q^{n+1})e_3+qe_1+q^{2n-1}e_3e_4-q^{n-1}(1+q-q^{n+1})e_1e_4}{(1-q^{2n-2}e_4)(1-q^{2n}e_4)}, \tag{2.14}$$

$$\begin{aligned}\gamma_n:=\;&(1-q^{n-1}ab)(1-q^{n-1}ac)(1-q^{n-1}ad)(1-q^{n-1}bc)(1-q^{n-1}bd)(1-q^{n-1}cd)\\ &\times\frac{(1-q^n)(1-q^{n-2}e_4)}{(1-q^{2n-3}e_4)(1-q^{2n-2}e_4)^2(1-q^{2n-1}e_4)}.\end{aligned} \tag{2.15}$$

From this we see that $P_n[z]$ remains well-defined if the condition $a,b,c,d\ne0$ in (1.1) is omitted. It also follows from (2.12) and (2.13)–(2.15) that the representation (2.5) of $AW(3)$ is not necessarily irreducible, but that it has $1\in\mathcal{A}_{\rm sym}$ as a cyclic element. The representation will become irreducible if we moreover require that none of ab, ac, ad, bc, bd, cd equals q^{-m} for some $m=0,1,2,\ldots$.

We now show that $AW(3,Q_0)$ has the elements

$$K_0^n(K_1K_0)^lK_1^m\qquad (m,n=0,1,2,\ldots,\quad l=0,1) \tag{2.16}$$

as a basis and that the representation (2.5) of $AW(3,Q_0)$ is faithful.

Lemma 2.1. *Each element of $AW(3,Q_0)$ can be written as a linear combination of elements (2.16).*

Proof. $AW(3,Q_0)$ is spanned by elements $K_\alpha=K_{\alpha_1}\cdots K_{\alpha_k}$, where $\alpha=(\alpha_1,\ldots,\alpha_k)$, $\alpha_i=0$ or 1. Let $\rho(\alpha)$ the number of pairs (i,j) such that $i<j$, $\alpha_i=1$, $\alpha_j=0$. K_α has the form (2.16) iff $\rho(\alpha)=0$ or 1. We will show that each K_α with $\rho(\alpha)>1$ can be written as a linear combination of elements K_β with $\rho(\beta)<\rho(\alpha)$. Indeed, if $\rho(\alpha)>1$ then K_α must have a substring $K_1K_1K_0$ or $K_1K_0K_0$ or $K_1K_0K_1K_0$. By substitution of relations (2.1), (2.2) or (2.9) (with (2.3)), respectively, we see that each such string is a linear combination of elements K_β with $\rho(\beta)<\rho(\alpha)$. ∎

Theorem 2.2. *The elements* (2.16) *form a basis of* $AW(3,Q_0)$ *and the representation* (2.5) *of* $AW(3,Q_0)$ *is faithful.*

Proof. Because of Lemma 2.1 it is sufficient to show that the operators

$$\begin{aligned}&(D_{\rm sym})^n\,(Z+Z^{-1})^m \qquad (m,n=0,1,2,\ldots),\\ &(D_{\rm sym})^{n-1}\,(Z+Z^{-1})\,D_{\rm sym}\,(Z+Z^{-1})^{m-1} \qquad (m,n=1,2,\ldots)\end{aligned} \tag{2.17}$$

acting on $\mathcal{A}_{\rm sym}$ are linearly independent. By (2.12) and (2.13) we have for all j:

$$\begin{aligned}&(D_{\rm sym})^n\,(Z+Z^{-1})^m\,P_j[z]=\lambda_{j+m}^n P_{j+m}[z]+\cdots,\\ &(D_{\rm sym})^{n-1}\,(Z+Z^{-1})\,D_{\rm sym}\,(Z+Z^{-1})^{m-1}\,P_j[z]=\lambda_{j+m}^{n-1}\lambda_{j+m-1}P_{j+m}[z]+\cdots,\end{aligned} \tag{2.18}$$

where the right-hand sides give expansions in terms of $P_k[z]$ with k running from $j+m$ downwards.

Suppose that the operators (2.17) are not linearly independent. Then

$$\begin{aligned}&\sum_{k=0}^m\sum_l a_{k,l}(D_{\rm sym})^l\,(Z+Z^{-1})^k\\ &\qquad+\sum_{k=1}^m\sum_l b_{k,l}(D_{\rm sym})^{l-1}\,(Z+Z^{-1})\,D_{\rm sym}\,(Z+Z^{-1})^{k-1}=0\end{aligned} \tag{2.19}$$

for certain coefficients $a_{k,l}$, $b_{k,l}$ such that for some l $a_{m,l}\neq 0$ or $b_{m,l}\neq 0$. Then it follows from (2.18) that for all j, when we let the left-hand side of (2.19) act on $P_j[z]$, the coefficient of $P_{j+m}[z]$ yields:

$$\sum_l(a_{m,l}\lambda_{j+m}^l+b_{m,l}\lambda_{j+m}^{l-1}\lambda_{j+m-1})=0. \tag{2.20}$$

By (2.12) we have, writing $x=q^{j+m}$ and $u=q^{-1}abcd$,

$$\lambda_{j+m}=x^{-1}+ux,\qquad \lambda_{j+m-1}=qx^{-1}+q^{-1}ux.$$

We can consider the identity (2.20) as an identity for Laurent polynomials in x. Since the left-hand side vanishes for infinitely many values of x, it must be identically zero. Let n be the maximal l for which $a_{m,l}\neq 0$ or $b_{m,l}\neq 0$. Then, in particular, the coefficients of x^{-n} and x^n in the left-hand side of (2.20) must be zero. This gives explicitly:

$$a_{m,n}+qb_{m,n}=0,\qquad u^na_{m,n}+q^{-1}u^nb_{m,n}=0.$$

This implies $a_{m,n}=b_{m,n}=0$, contradicting our assumption. ■

Remark 2.3. Note that we have 6 structure constants B, C_0, C_1, D_0, D_1, Q_0 depending on 4 parameters a, b, c, d. However, 2 degrees of freedom in the structure coefficients are caused by scale transformations. Indeed, the scale transformations $K_0\to c_0K_0$ and $K_1\to c_1K_1$ induce the following transformations on the structure coefficients:

$$B\to c_0c_1B,\quad C_0\to c_1^2C_0,\quad C_1\to c_0^2C_1,\quad D_0\to c_0c_1^2D_0,\quad D_1\to c_0^2c_1D_1,\quad Q_0\to c_0^2c_1^2Q_0.$$

But these scale transformations also affect the basic representation. This becomes $K_0=c_0D_{\rm sym}$, $K_1=c_1(Z+Z^{-1})$.

3 The double affine Hecke algebra of type $(C_1^\vee, C_1)$

Recall condition (1.1). The double affine Hecke algebra of type $(C_1^\vee, C_1)$, denoted by $\tilde{\mathfrak{H}}$ (see [7, § 6.4]), is generated by Z, Z^{-1}, T_1, T_0 with relations $ZZ^{-1} = 1 = Z^{-1}Z$ and

$$(T_1 + ab)(T_1 + 1) = 0, \tag{3.1}$$
$$(T_0 + q^{-1}cd)(T_0 + 1) = 0, \tag{3.2}$$
$$(T_1Z + a)(T_1Z + b) = 0, \tag{3.3}$$
$$(qT_0Z^{-1} + c)(qT_0Z^{-1} + d) = 0. \tag{3.4}$$

Here I have used the notation of [13], which is slightly different from the notation in [7, § 6.4]. Conditions on q, a, b, c, d in [7] are more strict than in (1.1). This will give no problem, as can be seen by checking all results hereafter from scratch.

From (3.1) and (3.2) and the non-vanishing of a, b, c, d we see that T_1 and T_0 are invertible:

$$T_1^{-1} = -a^{-1}b^{-1}T_1 - (1 + a^{-1}b^{-1}), \tag{3.5}$$
$$T_0^{-1} = -qc^{-1}d^{-1}T_0 - (1 + qc^{-1}d^{-1}). \tag{3.6}$$

Put

$$Y := T_1T_0, \tag{3.7}$$
$$D := Y + q^{-1}abcdY^{-1} = T_1T_0 + q^{-1}abcdT_0^{-1}T_1^{-1}, \tag{3.8}$$
$$Z_{\rm sym} := Z + Z^{-1}. \tag{3.9}$$

By (3.1) and (3.2) D commutes with T_1 and T_0. By (3.1) and (3.3) $Z_{\rm sym}$ commutes with T_1.

The algebra $\tilde{\mathfrak{H}}$ has a faithful representation, the so-called *basic representation*, on the space $\mathcal{A}$ of Laurent polynomials $f[z]$ as follows:

$$(Zf)[z] := z\,f[z], \tag{3.10}$$
$$(T_1f)[z] := \frac{(a+b)z - (1+ab)}{1-z^2}\,f[z] + \frac{(1-az)(1-bz)}{1-z^2}\,f[z^{-1}], \tag{3.11}$$
$$(T_0f)[z] := \frac{q^{-1}z((cd+q)z - (c+d)q)}{q-z^2}\,f[z] - \frac{(c-z)(d-z)}{q-z^2}\,f[qz^{-1}]. \tag{3.12}$$

The representation property is from [7, § 6.4] or by straightforward computation. The faithfulness is from [7, (4.7.4)] or by an independent proof later in this paper.

Now we can compute:

$$\begin{aligned}(Yf)[z] &= \frac{z\big(1+ab-(a+b)z\big)\big((c+d)q-(cd+q)z\big)}{q(1-z^2)(q-z^2)}\,f[z]\\
&\quad+\frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}\,f[qz]\\
&\quad+\frac{(1-az)(1-bz)\big((c+d)qz-(cd+q)\big)}{q(1-z^2)(1-qz^2)}\,f[z^{-1}]\\
&\quad+\frac{(c-z)(d-z)\big(1+ab-(a+b)z\big)}{(1-z^2)(q-z^2)}\,f[qz^{-1}],\end{aligned} \tag{3.13}$$

$$\begin{aligned}(Df)[z] &= \frac{(1-q)z(1-az)(1-bz)\,\big((q+1)(cd+q)z-q(c+d)(1+z^2)\big)}{q(1-z^2)(q-z^2)(1-qz^2)}\,f[z^{-1}]\\
&\quad+\frac{(1-q)z(c-z)(d-z)\,\big((a+b)(q+z^2)-(ab+1)(q+1)z\big)}{(1-z^2)(q-z^2)(q^2-z^2)}\,f[qz^{-1}]\end{aligned}$$

$$
\begin{aligned}
&+\Big((a+b)(cd+q)(q+z^2)+q(ab+1)(c+d)(1+z^2)\\
&-\big((q+1)(cd+q)(ab+1)+2q(a+b)(c+d)\big)z\Big)\,\frac{z}{q(1-z^2)(q-z^2)}\,f(z) \qquad (3.14)\\
&+\frac{(c-z)(d-z)(aq-z)(bq-z)}{(q-z^2)(q^2-z^2)}f[q^{-1}z]+\frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}f[qz].
\end{aligned}
$$

If we compare (3.14) and (2.6) then we see that

$$(Df)[z]=(D_{\mathrm{sym}}f)[z] \qquad \text{if} \qquad f[z]=f[z^{-1}].$$

In particular, if we apply D to the Askey–Wilson polynomial $P_n[z]$ given by (2.11) then we obtain from (2.12) that

$$DP_n=\lambda_n P_n. \tag{3.15}$$

By (3.1) and (3.2) the operators T_1 and T_0, acting on $\mathcal{A}$ as given by (3.11), (3.12) have two eigenvalues. We can characterize the eigenspaces.

Proposition 3.1. *T_1 given by* (3.11) *has eigenvalues $-ab$ and -1. $T_1f=-ab\,f$ iff f is symmetric. If a, b are distinct from a^{-1}, b^{-1} then $T_1f=-f$ iff $f[z]=z^{-1}(1-az)(1-bz)g[z]$ for some symmetric Laurent polynomial g.*

Proof. We compute

$$(T_1f)[z]+ab\,f[z]=\frac{(1-az)(1-bz)}{1-z^2}\,(f[z^{-1}]-f[z]),$$

which settles the first assertion. We also compute

$$(T_1f)[z]+f[z]=\frac{(1-az)(1-bz)}{1-z^2}\,f[z^{-1}]-\frac{(a-z)(b-z)}{1-z^2}\,f[z].$$

This equals zero if $f[z]=z^{-1}(1-az)(1-bz)g[z]$ with g symmetric. On the other hand, if $(T_1f)[z]+f[z]=0$ and a, b are distinct from a^{-1}, b^{-1} then

$$(1-az)(1-bz)f[z^{-1}]=(a-z)(b-z)f[z]$$

and hence $f[z]=z^{-1}(1-az)(1-bz)g[z]$ for some Laurent polynomial g and we obtain $g[z]=g[z^{-1}]$. ■

Proposition 3.2. *T_0 given by* (3.12) *has eigenvalues $-q^{-1}cd$ and -1. $T_0f=-q^{-1}cd\,f$ iff $f[z]=f[qz^{-1}]$. If c, d are distinct from qc^{-1}, qd^{-1} then $T_0f=-f$ iff $f[z]=z^{-1}(c-z)(d-z)g[z]$ for some Laurent polynomial g satisfying $g[z]=g[qz^{-1}]$.*

Proof. We compute

$$(T_0f)[z]+q^{-1}cd\,f[z]=\frac{(c-z)(d-z)}{q-z^2}\,(f[z]-f[qz^{-1}]),$$

which settles the first assertion. We also compute

$$(T_0f)[z]+f[z]=\frac{(q-cz)(q-dz)}{q(q-z^2)}\,f[z]-\frac{q(c-z)(d-z)}{q(q-z^2)}\,f[qz^{-1}].$$

Then the second assertion is proved by similar arguments as in the proof of Proposition 3.1. ■

We now look for further explicit solutions of the eigenvalue equation

$$Df = \lambda_n f. \tag{3.16}$$

Clearly, the solution P_n (see (3.15)) also satisfies $T_1 P_n = -ab\, P_n$. In order to find further solutions of (3.16) we make an Ansatz for f as suggested by Propositions 3.1 and 3.2, namely $f[z] = z^{-1}(1-az)(1-bz)g[z]$ or $f[z] = g[q^{-\frac12}z]$ or $f[z] = z^{-1}(c-z)(d-z)g[q^{-\frac12}z]$, in each case with g symmetric. Then it turns out that (3.16) takes the form of the Askey–Wilson second order q-difference equation, but with parameters and sometimes also the degree changed. We thus obtain as further solutions f of (3.16) for $n \ge 1$:

$$Q_n[z] := a^{-1}b^{-1}z^{-1}(1-az)(1-bz)\, P_{n-1}[z; qa, qb, c, d \mid q], \tag{3.17}$$
$$P_n^\dagger[z] := q^{\frac12 n}\, P_n\big[q^{-\frac12}z; q^{\frac12}a, q^{\frac12}b, q^{-\frac12}c, q^{-\frac12}d \mid q\big], \tag{3.18}$$
$$Q_n^\dagger[z] := q^{\frac12(n-1)}z^{-1}(c-z)(d-z)\, P_{n-1}\big[q^{-\frac12}z; q^{\frac12}a, q^{\frac12}b, q^{\frac12}c, q^{\frac12}d \mid q\big]. \tag{3.19}$$

So we have for $n \ge 1$ four different eigenfunctions of D at eigenvalue $q^{-n} + abcdq^{n-1}$ which are also eigenfunction of T_1 or T_0:

$$T_1 P_n = -ab\, P_n, \qquad T_1 Q_n = -Q_n, \qquad T_0 P_n^\dagger = -q^{-1}cd\, P_n^\dagger, \qquad T_0 Q_n^\dagger = -Q_n^\dagger. \tag{3.20}$$

They all are Laurent polynomials of degree n with highest term z^n and lowest term const z^{-n}:

$$\begin{aligned} &P_n[z] = z^n + \cdots + z^{-n}, && Q_n[z] = z^n + \cdots + a^{-1}b^{-1}z^{-n},\\ &P_n^\dagger[z] = z^n + \cdots + q^n z^{-n}, && Q_n^\dagger[z] = z^n + \cdots + q^{n-1}cdz^{-n}. \end{aligned} \tag{3.21}$$

Since the eigenvalues λ_n are distinct for different n, it follows that D has a 1-dimensional eigenspace $\mathcal{A}_0$ at eigenvalue λ_0, consisting of the constant Laurent polynomials, and that it has a 2-dimensional eigenspace $\mathcal{A}_n$ at eigenvalue λ_n if $n \ge 1$, which has P_n and $P_n^\dagger$ as basis vectors, but which also has any other two out of P_n, Q_n, $P_n^\dagger$, $Q_n^\dagger$ as basis vectors, provided these two functions have the coefficients of z^{-n} distinct. Generically we can use any two out of these four as basis vectors. The basis consisting of P_n and $P_n^\dagger$ occurs in [7, § 6.6]. In the following sections we will work first with the basis consisting of P_n and $Q_n^\dagger$, but afterwards it will be more convenient to use P_n and Q_n.

4 Non-symmetric Askey–Wilson polynomials

Since T_1 and T_0 commute with D, the eigenspaces of D in $\mathcal{A}$ are invariant under $Y = T_1T_0$. We can find explicitly the eigenvectors of Y within these eigenspaces $\mathcal{A}_n$.

Theorem 4.1. *The non-symmetric Askey–Wilson polynomials*

$$E_{-n}[z] := \frac{1}{1-q^{n-1}cd}\,(P_n[z] - Q_n^\dagger[z]) \qquad (n = 1, 2, \ldots), \tag{4.1}$$
$$E_n[z] := \frac{q^n(1-q^{n-1}abcd)}{1-q^{2n-1}abcd}\, P_n[z] + \frac{1-q^n}{1-q^{2n-1}abcd}\, Q_n^\dagger[z] \qquad (n = 1, 2, \ldots), \tag{4.2}$$
$$E_0[z] := 1 \tag{4.3}$$

span the one-dimensional eigenspaces of Y within $\mathcal{A}_n$ with the following eigenvalues:

$$YE_{-n} = q^{-n}\, E_{-n} \qquad (n = 1, 2, \ldots), \tag{4.4}$$
$$YE_n = q^{n-1}abcd\, E_n \qquad (n = 0, 1, 2, \ldots). \tag{4.5}$$

The coefficients of highest and lowest terms in E_{-n} and E_n are:

$$E_{-n}[z] = z^{-n} + \cdots + \mathrm{const}\, z^{n-1} \qquad (n = 1, 2, \ldots), \tag{4.6}$$

$$E_n[z] = z^n + \cdots + \left(1 - \frac{(1-q^n)(1-q^{n-1}cd)}{1-q^{2n-1}abcd}\right) z^{-n} \qquad (n = 1, 2, \ldots). \tag{4.7}$$

Proof. Clearly, by their definition, E_{-n} and E_n are in $\mathcal{A}_n$, while (4.6), (4.7) follow from (3.21). Equation (4.5) for $n = 0$ follows from (3.7) and Propositions 3.1 and 3.2. For the proof of (4.4), (4.5) we use a q-difference equation for Askey–Wilson polynomials (see [3, (7.7.7)], [5, (3.1.8)]):

$$\frac{P_n[q^{-\frac12}z; a, b, c, d \mid q] - P_n[q^{\frac12}z; a, b, c, d \mid q]}{(q^{-\frac12 n} - q^{\frac12 n})(z - z^{-1})} = P_{n-1}\big[z; q^{\frac12}a, q^{\frac12}b, q^{\frac12}c, q^{\frac12}d \mid q\big]. \tag{4.8}$$

The expression $(YE_{-n})[z] - q^{-n}E_{-n}[z]$ $(n = 1, 2, \ldots)$ only involves terms $P_n[w; a, b, c, d \mid q]$ for $w = z, qz, q^{-1}z$ and terms $P_{n-1}[w; q^{\frac12}a, q^{\frac12}b, q^{\frac12}c, q^{\frac12}d \mid q]$ for $w = q^{-\frac12}z, q^{\frac12}z$, as can be seen from (4.1), (3.19) and (3.13). Now twice substitute in this expression (4.8) with z replaced by $q^{-\frac12}z$ and $q^{\frac12}z$, respectively. Then we arrive at an expression only involving terms $P_n[w; a, b, c, d \mid q]$ for $w = z, qz, q^{-1}z$. By (2.6) it can be recognized as $((D_{\rm sym}P_n)[z] - (q^{-n} + abcdq^{n-1})P_n[z])/(1-q^n)$, which equals zero by (2.12). This settles (4.4). The reduction of the expression $(YE_n)[z] - q^{n-1}abcdE_n[z]$ $(n = 1, 2, \ldots)$ can be done in a completely similar way. Here we arrive at the expression $((D_{\rm sym}P_n)[z] - (q^{-n} + abcdq^{n-1})P_n[z])/(1 - q^{1-2n}(abcd)^{-1})$, which equals zero. ■

Remark 4.2. By condition (1.1) all eigenvalues of Y on $\mathcal{A}$ (see (4.4), (4.5)) are distinct. So for all $n \in \mathbb{Z}$ $E_n[z]$ is the unique Laurent polynomial of degree $|n|$ which satisfies (4.4) or (4.5) and has coefficient of z^n equal to 1. Moreover, for $n \geq 1$, E_{-n} is the unique element of $\mathcal{A}_n$ of the form (4.6), and E_n is the unique element of $\mathcal{A}_n$ of the form (4.7)

Remark 4.3. The occurrence of the q-difference equation (4.8) in the proof of Theorem 4.1 and the occurrence of Askey–Wilson polynomials with shifted parameters as eigenfunctions of D (see (3.17)–(3.19)) is probably much related to the one-variable case of the q-difference equations in Rains [10, Corollary 2.4].

From (3.21), (4.6) and (4.7) we obtain

$$E_{-n} = \frac{ab}{ab-1}(P_n - Q_n) \qquad (n = 1, 2, \ldots), \tag{4.9}$$

$$E_n = \frac{(1-q^nab)(1-q^{n-1}abcd)}{(1-ab)(1-q^{2n-1}abcd)} P_n - \frac{ab(1-q^n)(1-q^{n-1}cd)}{(1-ab)(1-q^{2n-1}abcd)} Q_n \qquad (n = 1, 2, \ldots). \tag{4.10}$$

Next, (4.9), (4.10) and (3.20) yield

$$T_1 E_{-n} = -\frac{1 + ab - abcdq^{n-1} - abq^n}{1 - abcdq^{2n-1}} E_{-n} - ab\, E_n \qquad (n = 1, 2, \ldots), \tag{4.11}$$

$$\begin{aligned} T_1 E_n &= \frac{(1-q^n)(1-abq^n)(1-cdq^{n-1})(1-abcdq^{n-1})}{(1-abcdq^{2n-1})^2} E_{-n} \\ &\quad - \frac{abq^{n-1}(cd + q - cdq^n - abcdq^n)}{1 - abcdq^{2n-1}} E_n \qquad (n = 1, 2, \ldots). \end{aligned} \tag{4.12}$$

5 A PBW-type theorem for $\tilde{\mathfrak{H}}$

In this section I will give two other sets of relations for $\tilde{\mathfrak{H}}$, both equivalent to (3.1)–(3.4) and both of PBW-type form. For the second set of relations we will see that the spanning set of elements of $\tilde{\mathfrak{H}}$, as implied by these relations, is indeed a basis. This is done by showing that this set of elements is linearly independent in the basic representation, which also shows that this representation is faithful. The faithfulness of the basic representation was first shown, in the more general n variable setting, by Sahi [11].

Proposition 5.1. *$\tilde{\mathfrak{H}}$ can equivalently be described as the algebra generated by T_1, T_0, Z, Z^{-1} with relations $ZZ^{-1} = 1 = Z^{-1}Z$ and*

$$T_1^2 = -(ab+1)T_1 - ab, \tag{5.1}$$
$$T_0^2 = -(q^{-1}cd+1)T_0 - q^{-1}cd, \tag{5.2}$$
$$T_1Z = Z^{-1}T_1 + (ab+1)Z^{-1} - (a+b), \tag{5.3}$$
$$T_1Z^{-1} = ZT_1 - (ab+1)Z^{-1} + (a+b), \tag{5.4}$$
$$T_0Z = qZ^{-1}T_0 - (q^{-1}cd+1)Z + (c+d), \tag{5.5}$$
$$T_0Z^{-1} = qZT_0 + q^{-1}(q^{-1}cd+1)Z - q^{-1}(c+d). \tag{5.6}$$

$\tilde{\mathfrak{H}}$ is spanned by the elements $Z^mT_0^iY^nT_1^j$, where $m \in \mathbb{Z}$, $n = 0, 1, 2, \ldots$, $i, j = 0, 1$.

Proof. (5.1), (5.3) are equivalent to (3.1), (3.3), and (5.2), (5.5) are equivalent to (3.2), (3.4). Furthermore, (5.3) is equivalent to (5.4), and (5.5) is equivalent to (5.6). Hence relations (5.1)–(5.6) are equivalent to relations (3.1)–(3.4).

For the second statement note that (5.1)–(5.6) imply that each word in $\tilde{\mathfrak{H}}$ can be written as a linear combination of words $Z^mT_0^i(T_1T_0)^nT_1^j$, where $m \in \mathbb{Z}$, $n = 0, 1, 2, \ldots$, $i, j = 0, 1$. Then substitute $Y = T_1T_0$. ■

Proposition 5.2. *$\tilde{\mathfrak{H}}$ can equivalently be described as the algebra generated by T_1, Y, Y^{-1}, Z, Z^{-1} with relations $YY^{-1} = 1 = Y^{-1}Y$, $ZZ^{-1} = 1 = Z^{-1}Z$ and*

$$\begin{aligned}
&T_1^2 = -(ab+1)T_1 - ab,\\
&T_1Z = Z^{-1}T_1 + (ab+1)Z^{-1} - (a+b),\\
&T_1Z^{-1} = ZT_1 - (ab+1)Z^{-1} + (a+b),\\
&T_1Y = q^{-1}abcdY^{-1}T_1 - (ab+1)Y + ab(1+q^{-1}cd),\\
&T_1Y^{-1} = q(abcd)^{-1}YT_1 + q(abcd)^{-1}(1+ab)Y - q(cd)^{-1}(1+q^{-1}cd),\\
&YZ = qZY + (1+ab)cd\,Z^{-1}Y^{-1}T_1 - (a+b)cd\,Y^{-1}T_1 - (1+q^{-1}cd)Z^{-1}T_1\\
&\qquad - (1-q)(1+ab)(1+q^{-1}cd)Z^{-1} + (c+d)T_1 + (1-q)(a+b)(1+q^{-1}cd),\\
&YZ^{-1} = q^{-1}Z^{-1}Y - q^{-2}(1+ab)cd\,Z^{-1}Y^{-1}T_1 + q^{-2}(a+b)cd\,Y^{-1}T_1\\
&\qquad + q^{-1}(1+q^{-1}cd)Z^{-1}T_1 - q^{-1}(c+d)T_1,\\
&Y^{-1}Z = q^{-1}ZY^{-1} - q(ab)^{-1}(1+ab)Z^{-1}Y^{-1}T_1 + (ab)^{-1}(a+b)Y^{-1}T_1\\
&\qquad + q(abcd)^{-1}(1+q^{-1}cd)Z^{-1}T_1 + q(abcd)^{-1}(1-q)(1+ab)(1+q^{-1}cd)Z^{-1}\\
&\qquad - (abcd)^{-1}(c+d)T_1 - (abcd)^{-1}(1-q)(1+ab)(c+d),\\
&Y^{-1}Z^{-1} = qZ^{-1}Y^{-1} + q(ab)^{-1}(1+ab)Z^{-1}Y^{-1}T_1 - (ab)^{-1}(a+b)Y^{-1}T_1\\
&\qquad - q^2(abcd)^{-1}(1+q^{-1}cd)Z^{-1}T_1 + q(abcd)^{-1}(c+d)T_1.
\end{aligned} \tag{5.7}$$

$\tilde{\mathfrak{H}}$ is spanned by the elements $Z^mY^nT_1^i$, where $m, n \in \mathbb{Z}$, $i = 0, 1$.

Proof. First we start with relations (5.1)–(5.6). Then (5.1), (5.2) give (3.5), (3.6). Next put $Y := T_1T_0$, $Y^{-1} := T_0^{-1}T_1^{-1}$. Then verify relations (5.7) from relations (5.1)–(5.6), most conveniently with the aid of computer algebra package, for instance by using [8].

Conversely we start with relations (5.7). Then the first of these relations gives (3.5). Put $T_0 := T_1^{-1}Y$. Then verify relations (5.1)–(5.6) from relations (5.7), where again computer algebra may be used.

The last statement follows from the PBW-type structure of the relations (5.7). Observe that by the first five relations together with the trivial relations, every word in T_1, Y, Y^{-1}, Z, Z^{-1} can be written as a linear combination of words with at most one occurrence of T_1 in each word and only on the right, and with no substrings YY^{-1}, $Y^{-1}Y$, ZZ^{-1}, $Z^{-1}Z$, and with no more occurrences of Y, Y^{-1}, Z, Z^{-1} in each word than in the original word. If in one of these terms there are misplacements (Y or Y^{-1} before Z or Z^{-1}) then apply one of the last four relations followed by the previous step in order to reduce the number of misplacements. ■

Theorem 5.3. *The basic representation* (3.10)–(3.12) *of* $\tilde{\mathfrak{H}}$ *is faithful. A basis of* $\tilde{\mathfrak{H}}$ *is provided by the elements* $Z^mY^nT_1^i$*, where* $m,n\in\mathbb{Z}$*,* $i=0,1$*.*

Proof. For $j>0$ we have

$$\begin{aligned}
&Z^mY^n\,E_{-j} = q^{-jn}\,z^{m-j} + \cdots + \text{const}\,z^{m+j-1},\\
&Z^mY^nT_1\,E_{-j} = \text{const}\,z^{m-j} + \cdots - ab(q^{j-1}abcd)^n\,z^{m+j},\\
&Z^mY^nT_1^{-1}\,E_{-j} = \text{const}\,z^{m-j} + \cdots + (q^{j-1}abcd)^n\,z^{m+j}.
\end{aligned}\tag{5.8}$$

This follows from (4.4)–(4.7), (4.11) and (3.5). Suppose that some linear combination

$$\sum_{m,n} a_{m,n}Z^mY^n + \sum_{m,n} b_{m,n}Z^mY^nT_1 \tag{5.9}$$

acts as the zero operator in the basic representation, while not all coefficients $a_{m,n}$, $b_{m,n}$ are zero. Then there is a maximal r for which $a_{r,n}$ or $b_{r,n}$ is nonzero for some n. If $b_{r,n}\neq 0$ for some n then let the operator (5.9) act on E_{-j}. By (5.8) we have that for all $j\geq 1$

$$\sum_n b_{r,n}(q^{j-1}abcd)^n\,z^{r+j} = 0, \qquad \text{hence} \qquad \sum_n b_{r,n}(q^{j-1}abcd)^n = 0.$$

By assumption (1.1) we see that $\sum_n b_{r,n}w^n = 0$. Hence $b_{r,n}=0$ for all n, which is a contradiction.

So $a_{r,n}\neq 0$ for some n. Let the operator (5.9) act on $T_1^{-1}E_{-j}$. By (5.8) we have that for all $j\geq 1$

$$\sum_n a_{r,n}(q^{j-1}abcd)^n\,z^{r+j} = 0, \qquad \text{hence} \qquad \sum_n a_{r,n}(q^{j-1}abcd)^n = 0.$$

Again we arrive at the contradiction that $a_{r,n}=0$ for all n. ■

6 The embedding of a central extension of $AW(3,Q_0)$ in $\tilde{\mathfrak{H}}$

Let us now examine whether the representation (2.5) of $AW(3)$ on $\mathcal{A}_{\text{sym}}$ extends to a representation on $\mathcal{A}$ if we let K_0 act as D instead of D_{sym}. It will turn out that this is only true for certain specializations of a, b, c, d, but that a suitable central extension $\widetilde{AW}(3)$ of $AW(3)$ involving T_1 will realize what we desire.

Definition 6.1. $\widetilde{AW}(3)$ is the algebra generated by K_0, K_1, T_1 with relations

$$\begin{aligned}
&T_1K_0 = K_0T_1, \qquad T_1K_1 = K_1T_1, \qquad (T_1+ab)(T_1+1) = 0,\\
&(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2
\end{aligned}\tag{6.1}$$

$$
\begin{aligned}
&= B\,K_1 + C_0\,K_0 + D_0 + E\,K_1(T_1+ab) + F_0(T_1+ab), && (6.2)\\
&(q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2\\
&= B\,K_0 + C_1\,K_1 + D_1 + E\,K_0(T_1+ab) + F_1(T_1+ab), && (6.3)
\end{aligned}
$$

where the structure constants are given by (2.4) together with

$$
\begin{aligned}
&E := -q^{-2}(1-q)^3(c+d),\\
&F_0 := q^{-3}(1-q)^3(1+q)(cd+q), && (6.4)\\
&F_1 := q^{-3}(1-q)^3(1+q)(a+b)cd.
\end{aligned}
$$

It can be shown that the following adaptation of (2.3) is a Casimir operator for $\widetilde{AW}(3)$, commuting with K_0, K_1, T_1:

$$
\begin{aligned}
\widetilde{Q} :={}& (K_1K_0)^2 - (q^2+1+q^{-2})K_0(K_1K_0)K_1 + (q+q^{-1})K_0^2K_1^2\\
&+ (q+q^{-1})(C_0K_0^2 + C_1K_1^2) + \big(B + E(T_1+ab)\big)\big((q+1+q^{-1})K_0K_1 + K_1K_0\big)\\
&+ (q+1+q^{-1})\big(D_0 + F_0(T_1+ab)\big)K_0 + (q+1+q^{-1})\big(D_1 + F_1(T_1+ab)\big)K_1\\
&+ G(T_1+ab), \qquad (6.5)
\end{aligned}
$$

where

$$
\begin{aligned}
G :={}& -q^{-4}(1-q)^3\Big((a+b)(c+d)\big(cd(q^2+1)+q\big) - q(ab+1)\big((c^2+d^2)(q+1)-cd\big)\\
&+ (cd+e_4)(q^2+1) + (e_2+e_4-ab)q^3\Big). \qquad (6.6)
\end{aligned}
$$

Let $\widetilde{AW}(3,Q_0)$ be the algebra generated by K_0, K_1, T_1 with relations (6.1)–(6.3) and additional relation

$$\widetilde{Q} = Q_0, \tag{6.7}$$

where $\widetilde{Q}$ is given by (6.5) and Q_0 by (2.8).

Theorem 6.2. *There is a representation of the algebra $\widetilde{AW}(3,Q_0)$ on the space $\mathcal{A}$ of Laurent polynomials $f[z]$ such that K_0 acts as D, K_1 acts by multiplication by $z+z^{-1}$, and the action of T_1 is given by* (3.11)*. This representation is faithful.*

Proof. It follows by straightforward computation, possibly using computer algebra, that this is a representation of $\widetilde{AW}(3,Q_0)$. In the same way as for Lemma 2.1 it can be shown that $\widetilde{AW}(3,Q_0)$ is spanned by the elements

$$K_0^n(K_1K_0)^iK_1^mT_1^j \qquad (m,n=0,1,2,\ldots,\quad i,j=0,1). \tag{6.8}$$

Now we will prove that the representation is faithful. Suppose that for certain coefficients $a_{k,l}$, $b_{k,l}$, $c_{k,l}$, $d_{k,l}$ we have

$$
\begin{aligned}
&\sum_{k,l} a_{k,l}\,D^l\,(Z+Z^{-1})^k + \sum_{k,l} b_{k,l}\,D^{l-1}\,(Z+Z^{-1})\,D\,(Z+Z^{-1})^{k-1}\\
&+\left(\sum_{k,l} c_{k,l}\,D^l\,(Z+Z^{-1})^k + \sum_{k,l} d_{k,l}\,D^{l-1}\,(Z+Z^{-1})\,D\,(Z+Z^{-1})^{k-1}\right)(T_1+ab) = 0 \qquad (6.9)
\end{aligned}
$$

while acting on $\mathcal{A}$. Then, since $T_1 P_j = -abP_j$ (see (3.20)), we have for all $j \ge 0$ that

$$\sum_{k,l} a_{k,l}\, D^l_{\rm sym}\,(Z+Z^{-1})^k\, P_j[z] + \sum_{k,l} b_{k,l}\, D^{l-1}_{\rm sym}\,(Z+Z^{-1})\, D\,(Z+Z^{-1})^{k-1}\, P_j[z] = 0.$$

Then by the proof of Theorem 2.2 if follows that all coefficients $a_{k,l}$, $b_{k,l}$ vanish.

It follows from (4.9) and (3.20) that $(T_1+ab)E_{-n} = -abQ_n$ (also if $ab=1$). Hence, if we let (6.9), with vanishing $a_{k,l}$, $b_{k,l}$, act on $E_{-j}[z]$, and divide by $-ab$, then:

$$\left(\sum_{k,l} c_{k,l}\, D^l\,(Z+Z^{-1})^k + \sum_{k,l} d_{k,l}\, D^{l-1}\,(Z+Z^{-1})\, D\,(Z+Z^{-1})^{k-1}\right) Q_j[z] = 0.$$

From (3.17) we see that the three-term recurrence relation (2.13) for $P_n[z]$ has an analogue for $Q_n[z]$:

$$(z+z^{-1})Q_n[z] = Q_{n+1}[z] + \tilde\beta_n Q_n[z] + \tilde\gamma_n Q_{n-1}[z] \qquad (n \ge 2),$$

where $\tilde\beta_n$ and $\tilde\gamma_n$ are obtained from the corresponding β_n and γ_n ((2.14) and (2.15)) by replacing a, b, n by qa, qb, $n-1$, respectively. Hence (2.18) remains valid if we replace each P by Q. Again, similarly as in the proof of Theorem 2.2, it follows that all coefficients $c_{k,l}$, $d_{k,l}$ vanish. ■

Corollary 6.3. *The algebra $\widetilde{AW}(3,Q_0)$ can be isomorphically embedded into $\tilde{\mathfrak{H}}$ by the mapping*

$$K_0 \mapsto Y + q^{-1}abcdY^{-1}, \qquad K_1 \mapsto Z+Z^{-1}, \qquad T_1 \mapsto T_1. \tag{6.10}$$

Proof. The embedding is valid for $\widetilde{AW}(3,Q_0)$ and $\tilde{\mathfrak{H}}$ acting on $\mathcal{A}$. Now use the faithfulness of the representations of $\widetilde{AW}(3,Q_0)$ and $\tilde{\mathfrak{H}}$ on $\mathcal{A}$. ■

Remark 6.4. By Corollary 6.3 the relations (6.1)–(6.3) and (6.7) are valid identities in $\tilde{\mathfrak{H}}$ after substitution by (6.10). These identities can also be immediately verified within $\tilde{\mathfrak{H}}$, for instance by usage of the package [8].

Remark 6.5. If a, b, c, d are such that $E, F_0, F_1 = 0$ in (6.4) then we have already a homomorphism of the original algebra $AW(3)$ into $\tilde{\mathfrak{H}}$ under the substitutions $K_0 := D$, $K_1 := Z + Z^{-1}$ in (2.1), (2.2). This is the case iff $c = -d = q^{\frac12}$ (or $-q^{\frac12}$) and $a = -b$. For these parameters the Askey–Wilson polynomials become the continuous q-ultraspherical polynomials (see [3, (7.5.25), (7.5.34)]):

$$P_n\bigl[z; a, -a, q^{\frac12}, -q^{\frac12} \mid q\bigr] = \mathrm{const}\, C_n\bigl(\tfrac12(z+z^{-1}); a^2 \mid q^2\bigr).$$

However, for these specializations of a, b, c, d we see from (2.3) and (6.5) that $\widetilde{Q}$ still slightly differs from Q: it is obtained from Q by adding the term $(q^{-1}-q)^3(1-a^2)(T_1-a^2)$. So $\widetilde{AW}(3,Q_0)$ then still differs from $AW(3,Q_0)$.

For such a, b, c, d the operator T_0 acting on $\mathcal{A}$ (formula (3.12)) simplifies to $(T_0 f)[z] = f[qz^{-1}]$. We then have the specialization of parameters in $\tilde{\mathfrak{H}}$ to the one-parameter double affine Hecke algebra of type A_1 (see [7, § 6.1–6.3]). Explicit formulas for the non-symmetric q-ultraspherical polynomials become much nicer than in the general four-parameter Askey–Wilson case, see [7, (6.2.7), (6.2.8)].

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