Reduction of Symplectic Lie Algebroids by a Lie Subalgebroid and a Symmetry Lie Group*

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Abstract. We describe the reduction procedure for a symplectic Lie algebroid by a Lie subalgebroid and a symmetry Lie group. Moreover, given an invariant Hamiltonian function we obtain the corresponding reduced Hamiltonian dynamics. Several examples illustrate the generality of the theory.

Key words: Lie algebroids and subalgebroids; symplectic Lie algebroids; Hamiltonian dynamics; reduction procedure

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1 Introduction

As it is well known, symplectic manifolds play a fundamental role in the Hamiltonian formulation of Classical Mechanics. In fact, if M is the configuration space of a classical mechanical system, then the phase space is the cotangent bundle T^*M, which is endowed with a canonical symplectic structure, the main ingredient to develop Hamiltonian Mechanics. Indeed, given a Hamiltonian function H, the integral curves of the corresponding Hamiltonian vector field are determined by the Hamilton equations.

A method to obtain new examples of symplectic manifolds comes from different reduction procedures. One of these procedures is the classical Cartan symplectic reduction process: *If (M,ω) is a symplectic manifold and $i : C \to M$ is a coisotropic submanifold of M such that its characteristic foliation $\mathcal{F} = \ker(i^*\omega)$ is simple, then the quotient manifold $\pi : C \to C/\mathcal{F}$ carries a unique symplectic structure ω_r such that $\pi^*(\omega_r) = i^*(\omega)$.*

A particular situation of it is the well-known Marsden–Weinstein reduction in the presence of a G-equivariant momentum map [18]. In fact, in [7] it has been proved that, under mild assumptions, one can obtain any symplectic manifold as a result of applying a Cartan reduction of the canonical symplectic structure on $\mathbb{R}^{2n}$. On the other hand, an interesting application in Mechanics is the case when we have a Hamiltonian function on the symplectic manifold

*This paper is a contribution to the Proceedings of the Workshop on Geometric Aspects of Integrable Systems (July 17–19, 2006, University of Coimbra, Portugal). The full collection is available at http://www.emis.de/journals/SIGMA/Coimbra2006.html satisfying some natural conditions, since one can reduce the Hamiltonian vector field to the reduced symplectic manifold and therefore, obtain a reduced Hamiltonian dynamics.

A category which is closely related to symplectic and Poisson manifolds is that of Lie algebroids. A Lie algebroid is a notion which unifies tangent bundles and Lie algebras, which suggests its relation with Mechanics. In fact, there has been recently a lot of interest in the geometric description of Hamiltonian (and Lagrangian) Mechanics on Lie algebroids (see, for instance, [8, 11, 19, 21]). An important application of this Hamiltonian Mechanics, which also comes from reduction, is the following one: if we consider a principal G-bundle $\pi : Q \to M$ then one can prove that the solutions of the Hamilton-Poincaré equations for a G-invariant Hamiltonian function $H : T^*Q \to \mathbb{R}$ are just the solutions of the Hamilton equations for the reduced Hamiltonian $h : T^*Q/G \to \mathbb{R}$ on the dual vector bundle T^*Q/G of the Atiyah algebroid $\tau_{TQ/G} : TQ/G \to \mathbb{R}$ (see [11]).

Now, given a Lie algebroid $\tau_A : A \to M$, the role of the tangent of the cotangent bundle of the configuration manifold is played by A-tangent bundle to A^*, which is the subset of $A \times TA^*$ given by

$$\mathcal{T}^A A^* = \{(b, v) \in A \times TA^* / \rho_A(b) = (T\tau_{A^*})(v)\},$$

where ρ_A is the anchor map of A and $\tau_{A^*} : A^* \to M$ is the vector bundle projection of the dual bundle A^* to A. In this case, $\mathcal{T}^A A^*$ is a Lie algebroid over A^*. In fact, it is the pull-back Lie algebroid of A along the bundle projection $\tau_{A^*} : A^* \to M$ in the sense of Higgins and Mackenzie [9]. Moreover, $\mathcal{T}^A A^*$ is a symplectic Lie algebroid, that is, it is endowed with a nondegenerate and closed 2-section. Symplectic Lie algebroids are a natural generalization of symplectic manifolds since, the first example of a symplectic Lie algebroid is the tangent bundle of a symplectic manifold.

The main purpose of this paper is to describe a reduction procedure, analogous to Cartan reduction, for a symplectic Lie algebroid in the presence of a Lie subalgebroid and a symmetry Lie group. In addition, for Hamiltonian functions which satisfy some invariance properties, it is described the process to obtain the reduced Hamiltonian dynamics.

The paper is organized as follows. In Section 2, we recall the definition of a symplectic Lie algebroid and describe several examples which will be useful along the paper. Then, in addition, we describe how to obtain Hamilton equations for a symplectic Lie algebroid and a Hamiltonian function on it.

Now, consider a symplectic Lie algebroid $\tau_A : A \to M$ with symplectic 2-section Ω_A and $\tau_B : B \to N$ a Lie subalgebroid of A. Then, in Section 3 we obtain our main result. Suppose that a Lie group G acts properly and free on B by vector bundle automorphisms. Then, if Ω_B is the restriction to B of Ω_A, we obtain conditions for which the reduced vector bundle $\tau_{\widetilde{B}} : \widetilde{B} = (B/\ker \Omega_B)/G \to N/G$ is a symplectic Lie algebroid. In addition, if we have a Hamiltonian function $H_M : M \to \mathbb{R}$ we obtain, under some mild hypotheses, reduced Hamiltonian dynamics.

In the particular case when the Lie algebroid is the tangent bundle of a symplectic manifold, our reduction procedure is just the well known Cartan symplectic reduction in the presence of a symmetry Lie group. This example is shown in Section 4, along with other different interesting examples. A particular application of our results is a "symplectic description" of the Hamiltonian reduction process by stages in the Poisson setting (see Section 4.3) and the reduction of the Lagrange top (see Section 4.4).

In the last part of the paper, we include an Appendix where we describe how to induce, from a Lie algebroid structure on a vector bundle $\tau_A : A \to M$ and a linear epimorphism $\pi_A : A \to \widetilde{A}$ over $\pi_M : M \to \widetilde{M}$, a Lie algebroid structure on $\widetilde{A}$. An equivalent dual version of this result was proved in [3].

2 Hamiltonian Mechanics and symplectic Lie algebroids

2.1 Lie algebroids

Let A be a vector bundle of *rank* n over a manifold M of dimension m and $\tau_A : A \to M$ be the vector bundle projection. Denote by $\Gamma(A)$ the $C^\infty(M)$-module of sections of $\tau_A : A \to M$. A *Lie algebroid structure* $(\llbracket\cdot,\cdot\rrbracket_A, \rho_A)$ on A is a Lie bracket $\llbracket\cdot,\cdot\rrbracket_A$ on the space $\Gamma(A)$ and a bundle map $\rho_A : A \to TM$, called *the anchor map*, such that if we also denote by $\rho_A : \Gamma(A) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map then

$$\llbracket X, fY\rrbracket_A = f\llbracket X, Y\rrbracket_A + \rho_A(X)(f)Y,$$

for $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$. The triple $(A, \llbracket\cdot,\cdot\rrbracket_A, \rho_A)$ is called *a Lie algebroid over* M (see [13]).

If $(A, \llbracket\cdot,\cdot\rrbracket_A, \rho_A)$ is a Lie algebroid over M, then the anchor map $\rho_A : \Gamma(A) \to \mathfrak{X}(M)$ is a homomorphism between the Lie algebras $(\Gamma(A), \llbracket\cdot,\cdot\rrbracket_A)$ and $(\mathfrak{X}(M), [\cdot,\cdot])$.

Trivial examples of Lie algebroids are real Lie algebras of finite dimension and the tangent bundle TM of an arbitrary manifold M. Other examples of Lie algebroids are the following ones:

- **The Lie algebroid associated with an infinitesimal action.**

Let $\mathfrak{g}$ be a real Lie algebra of finite dimension and $\Phi : \mathfrak{g} \to \mathfrak{X}(M)$ an infinitesimal left action of $\mathfrak{g}$ on a manifold M, that is, Φ is a $\mathbb{R}$-linear map and

$$\Phi([\xi, \eta]_{\mathfrak{g}}) = -[\Phi(\xi), \Phi(\eta)], \qquad \text{for all } \ \xi, \eta \in \mathfrak{g},$$

where $[\cdot,\cdot]_{\mathfrak{g}}$ is the Lie bracket on $\mathfrak{g}$. Then, the trivial vector bundle $\tau_A : A = M \times \mathfrak{g} \to M$ admits a Lie algebroid structure. The anchor map $\rho_A : A \to TM$ of A is given by

$$\rho_A(x, \xi) = -\Phi(\xi)(x), \qquad \text{for } \ (x, \xi) \in M \times \mathfrak{g} = A.$$

On the other hand, if ξ and η are elements of $\mathfrak{g}$ then ξ and η induce constant sections of A which we will also denote by ξ and η. Moreover, the Lie bracket $\llbracket\xi, \eta\rrbracket_A$ of ξ and η in A is the constant section on A induced by $[\xi, \eta]_{\mathfrak{g}}$. In other words,

$$\llbracket\xi, \eta\rrbracket_A = [\xi, \eta]_{\mathfrak{g}}.$$

The resultant Lie algebroid is called the *Lie algebroid associated with the infinitesimal action* Φ.

Let $\{\xi_\alpha\}$ be a basis of $\mathfrak{g}$ and (x^i) be a system of local coordinates on an open subset U of M such that

$$[\xi_\alpha, \xi_\beta]_{\mathfrak{g}} = c^\gamma_{\alpha\beta}\xi_\gamma, \qquad \Phi(\xi_\alpha) = \Phi^i_\alpha \frac{\partial}{\partial x^i}.$$

If $\widetilde{\xi}_\alpha : U \to M \times \mathfrak{g}$ is the map defined by

$$\widetilde{\xi}_\alpha(x) = (x, \xi_\alpha), \qquad \text{for all } \ x \in U,$$

then $\{\widetilde{\xi}_\alpha\}$ is a local basis of sections of the action Lie algebroid. In addition, the corresponding local structure functions with respect to (x^i) and $\{\tilde{\xi}_\alpha\}$ are

$$C^\gamma_{\alpha\beta} = c^\gamma_{\alpha\beta}, \qquad \rho^i_\alpha = \Phi^i_\alpha.$$

- **The Atiyah (gauge) algebroid associated with a principal bundle.**

Let $\pi_P : P \to M$ be a principal left G-bundle. Denote by $\Psi : G \times P \to P$ the free action of G on P and by $T\Psi : G \times TP \to TP$ the tangent action of G on TP. The space TP/G of orbits of the action is a vector bundle over the manifold M with vector bundle projection $\tau_{TP}|G : TP/G \to P/G \cong M$ given by

$$(\tau_{TP}|G)([v_p]) = [\tau_{TP}(v_p)] = [p], \qquad \text{for } \ v_p \in T_pP,$$

$\tau_{TP} : TP \to P$ being the canonical projection. A section of the vector bundle $\tau_{TP}|G : TP/G \to P/G \cong M$ may be identified with a vector field on P which is G-invariant. Thus, using that every G-invariant vector field on P is π_P-projectable on a vector field on M and that the standard Lie bracket of two G-invariant vector fields is also a G-invariant vector field, we may induce a Lie algebroid structure $(\llbracket\cdot,\cdot\rrbracket_{TP/G}, \rho_{TP/G})$ on the vector bundle $\tau_{TP}|G : TP/G \to P/G \cong M$. The Lie algebroid $(TP/G, \llbracket\cdot,\cdot\rrbracket_{TP/G}, \rho_{TP/G})$ is called the *Atiyah (gauge) algebroid associated with the principal bundle* $\pi_P : P \to M$ (see [11, 13]).

Let $D : TP \to \mathfrak{g}$ be a connection in the principal bundle $\pi_P : P \to M$ and $R : TP \oplus TP \to \mathfrak{g}$ be the curvature of D. We choose a local trivialization of the principal bundle $\pi_P : P \to M$ to be $U \times G$, where U is an open subset of M. Suppose that e is the identity element of G, that (x^i) are local coordinates on U and that $\{\xi_a\}$ is a basis of $\mathfrak{g}$.

Denote by $\{\xi_a^L\}$ the corresponding left-invariant vector fields on G. If

$$D\left(\frac{\partial}{\partial x^i}_{|(x,e)}\right) = D_i^a(x)\xi_a, \qquad R\left(\frac{\partial}{\partial x^i}_{|(x,e)}, \frac{\partial}{\partial x^j}_{|(x,e)}\right) = R_{ij}^a(x)\xi_a,$$

for $x \in U$, then the horizontal lift of the vector field $\frac{\partial}{\partial x^i}$ is the vector field on $U \times G$ given by

$$\left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} - D_i^a \xi_a^L.$$

Therefore, the vector fields on $U \times G$ $\{e_i = \frac{\partial}{\partial x^i} - D_i^a \xi_a^L, e_b = \xi_b^L\}$ are G-invariant and they define a local basis $\{e_i', e_b'\}$ of $\Gamma(TP/G)$. The corresponding local structure functions of $\tau_{TP/G} : TP/G \to M$ are

$$\begin{aligned} &C_{ij}^k = C_{ia}^j = -C_{ai}^j = C_{ab}^i = 0, \qquad C_{ij}^a = -R_{ij}^a, \qquad C_{ia}^c = -C_{ai}^c = c_{ab}^c D_i^b, \qquad C_{ab}^c = c_{ab}^c, \\ &\rho_i^j = \delta_{ij}, \qquad \rho_i^a = \rho_a^i = \rho_a^b = 0 \end{aligned} \tag{1}$$

(for more details, see [11]).

An important operator associated with a Lie algebroid $(A, \llbracket\cdot,\cdot\rrbracket_A, \rho_A)$ over a manifold M is the *differential* $d^A : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$ *of* A which is defined as follows

$$\begin{aligned} d^A\mu(X_0, \dots, X_k) = &\sum_{i=0}^k (-1)^i \rho_A(X_i)(\mu(X_0, \dots, \widehat{X_i}, \dots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \mu(\llbracket X_i, X_j\rrbracket_A, X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k), \end{aligned}$$

for $\mu \in \Gamma(\wedge^k A^*)$ and $X_0, \dots, X_k \in \Gamma(A)$. It follows that $(d^A)^2 = 0$. Note that if $A = TM$ then d^{TM} is the standard differential exterior for the manifold M.

On the other hand, if $(A, \llbracket\cdot,\cdot\rrbracket_A, \rho_A)$ and $(A', \llbracket\cdot,\cdot\rrbracket_{A'}, \rho_{A'})$ are Lie algebroids over M and M', respectively, then a morphism of vector bundles $F : A \to A'$ is a *Lie algebroid morphism* if

$$d^A(F^*\phi') = F^*(d^{A'}\phi'), \qquad \text{for } \ \phi' \in \Gamma(\wedge^k (A')^*). \tag{2}$$

Note that $F^*\phi'$ is the section of the vector bundle $\wedge^k A^* \to M$ defined by

$$(F^*\phi')_x(a_1, \dots, a_k) = \phi'_{f(x)}(F(a_1), \dots, F(a_k)),$$

for $x \in M$ and $a_1, \dots, a_k \in A$, where $f : M \to M'$ is the mapping associated with F between M and M'. We remark that (2) holds if and only if

$$\begin{aligned} d^A(g' \circ f) &= F^*(d^{A'} g'), \qquad \text{for } \ g' \in C^\infty(M'), \\ d^A(F^*\alpha') &= F^*(d^{A'}\alpha'), \qquad \text{for } \ \alpha' \in \Gamma((A')^*). \end{aligned} \tag{3}$$

This definition of a Lie algebroid morphism is equivalent of the original one given in [9].

If $M = M'$ and $f = id : M \to M$ then it is easy to prove that F is a Lie algebroid morphism if and only if

$$F[\![X, Y]\!]_A = [\![FX, FY]\!]_{A'}, \qquad \rho_{A'}(FX) = \rho_A(X),$$

for $X, Y \in \Gamma(A)$.

If F is a Lie algebroid morphism, f is an injective immersion and F is also injective, then the Lie algebroid $(A, [\![\cdot,\cdot]\!]_A, \rho_A)$ is a *Lie subalgebroid* of $(A', [\![\cdot,\cdot]\!]_{A'}, \rho_{A'})$.

2.2 Symplectic Lie algebroids

Let $(A, [\![\cdot,\cdot]\!]_A, \rho_A)$ be a Lie algebroid over a manifold M. Then A is said to be a *symplectic Lie algebroid* if there exists a section Ω_A of the vector bundle $\wedge^2 A^* \to M$ such that Ω_A is nondegenerate and $d^A\Omega_A = 0$ (see [11]). The first example of a symplectic Lie algebroid is the tangent bundle of a symplectic manifold.

It is clear that the rank of a symplectic Lie algebroid A is even. Moreover, if $f \in C^\infty(M)$ one may introduce the *Hamiltonian section* $\mathcal{H}_f^{\Omega_A} \in \Gamma(A)$ *of* f *with respect to* Ω_A which is characterized by the following condition

$$i(\mathcal{H}_f^{\Omega_A})\Omega_A = d^A f. \tag{4}$$

On the other hand, the map $\flat_{\Omega_A} : \Gamma(A) \to \Gamma(A^*)$ given by

$$\flat_{\Omega_A}(X) = i(X)\Omega_A, \qquad \text{for } \ X \in \Gamma(A),$$

is an isomorphism of $C^\infty(M)$-modules. Thus, one may define a section Π_{Ω_A} of the vector bundle $\wedge^2 A \to A$ as follows

$$\Pi_{\Omega_A}(\alpha, \beta) = \Omega_A(\flat_{\Omega_A}^{-1}(\alpha), \flat_{\Omega_A}^{-1}(\beta)), \qquad \text{for } \ \alpha, \beta \in \Gamma(A^*).$$

Π_{Ω_A} is a *triangular matrix* for the Lie algebroid A (Π_{Ω_A} is an A-Poisson bivector field on the Lie algebroid A in the terminology of [2]) and the pair (A, A^*) is a *triangular Lie bialgebroid* in the sense of Mackenzie and Xu [14]. Therefore, the base space M admits a Poisson structure, that is, a bracket of functions

$$\{\cdot,\cdot\}_M : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

which satisfies the following properties:

1. $\{\cdot,\cdot\}_M$ is $\mathbb{R}$-bilinear and skew-symmetric;
2. It is a derivation in each argument with respect to the standard product of functions, i.e.,

$$\{ff', g\}_M = f\{f', g\}_M + f'\{f, g\}_M;$$

3. It satisfies the Jacobi identity, that is,

$$\{f,\{g,h\}_M\}_M + \{g,\{h,f\}_M\}_M + \{h,\{f,g\}_M\}_M = 0.$$

In fact, we have that

$$\{f,g\}_M = \Omega_A(\mathcal{H}_f^{\Omega_A}, \mathcal{H}_g^{\Omega_A}), \qquad \text{for} \quad f,g \in C^\infty(M). \tag{5}$$

Using the Poisson bracket $\{\cdot,\cdot\}_M$ one may consider the *Hamiltonian vector field* of a real function $f \in C^\infty(M)$ as the vector field $\mathcal{H}_f^{\{\cdot,\cdot\}_M}$ on M defined by

$$\mathcal{H}_f^{\{\cdot,\cdot\}_M}(g) = -\{f,g\}_M, \qquad \text{for all} \ \ g \in C^\infty(M).$$

From (4) and (5), we deduce that $\rho_A(\mathcal{H}_f^{\Omega_A}) = \mathcal{H}_f^{\{\cdot,\cdot\}_M}$. Moreover, the flow of the vector field $\mathcal{H}_f^{\{\cdot,\cdot\}_M}$ on M is covered by a group of 1-parameter automorphisms of A: it is just the (infinitesimal flow) of the A-vector field $\mathcal{H}_f^{\Omega_A}$ (see [6]).

A symplectic Lie algebroid may be associated with an arbitrary Lie algebroid as follows (see [11]).

Let $(A, [\![\cdot,\cdot]\!]_A, \rho_A)$ be a Lie algebroid of rank n over a manifold M and $\tau_{A^*} : A^* \to M$ be the vector bundle projection of the dual bundle A^* to A. Then, we consider the A-tangent bundle to A^* as the subset of $A \times TA^*$ given by

$$\mathcal{T}^A A^* = \{(b,v) \in A \times TA^* / \rho_A(b) = (T\tau_{A^*})(v)\}.$$

$\mathcal{T}^A A^*$ is a vector bundle over A^* of rank $2n$ and the vector bundle projection $\tau_{\mathcal{T}^A A^*} : \mathcal{T}^A A^* \to A^*$ is defined by

$$\tau_{\mathcal{T}^A A^*}(b,v) = \tau_{TA^*}(v), \qquad \text{for} \ \ (b,v) \in \mathcal{T}^A A^*.$$

A section $\widetilde{X}$ of $\tau_{\mathcal{T}^A A^*} : \mathcal{T}^A A^* \to A^*$ is said to be *projectable* if there exists a section X of $\tau_A : A \to M$ and a vector field $S \in \mathfrak{X}(A^*)$ which is τ_{A^*}-projectable to the vector field $\rho_A(X)$ and such that $\tilde{X}(p) = (X(\tau_{A^*}(p)), S(p))$, for all $p \in A^*$. For such a projectable section $\tilde{X}$, we will use the following notation $\tilde{X} \equiv (X,S)$. It is easy to prove that one may choose a local basis of projectable sections of the space $\Gamma(\mathcal{T}^A A^*)$.

The vector bundle $\tau_{\mathcal{T}^A A^*} : \mathcal{T}^A A^* \to A^*$ admits a Lie algebroid structure $([\![\cdot,\cdot]\!]_{\mathcal{T}^A A^*}, \rho_{\mathcal{T}^A A^*})$. Indeed, if (X,S) and (Y,T) are projectable sections then

$$[\![(X,S),(Y,T)]\!]_{\mathcal{T}^A A^*} = ([\![X,Y]\!]_A, [S,T]), \qquad \rho_{\mathcal{T}^A A^*}(X,S) = S.$$

$(\mathcal{T}^A A^*, [\![\cdot,\cdot]\!]_{\mathcal{T}^A A^*}, \rho_{\mathcal{T}^A A^*})$ is the *A-tangent bundle to A^** or *the prolongation of A over the fibration* $\tau_{A^*} : A^* \to M$ (for more details, see [11]).

Moreover, one may introduce a canonical section $\Theta_{\mathcal{T}^A A^*}$ of the vector bundle $(\mathcal{T}^A A^*)^* \to A^*$ as follows

$$\Theta_{\mathcal{T}^A A^*}(\gamma)(b,v) = \gamma(b),$$

for $\gamma \in A^*$ and $(b,v) \in \mathcal{T}^A_\gamma A^*$. $\Theta_{\mathcal{T}^A A^*}$ is called the *Liouville section associated with* A and $\Omega_{\mathcal{T}^A A^*} = -d^{\mathcal{T}^A A^*}\Theta_{\mathcal{T}^A A^*}$ is the *canonical symplectic section associated with* A. $\Omega_{\mathcal{T}^A A^*}$ is a symplectic section for the Lie algebroid $\mathcal{T}^A A^*$.

Therefore, the base space A^* admits a Poisson structure $\{\cdot,\cdot\}_{A^*}$ which is characterized by the following conditions

$$\{f \circ \tau_{A^*}, g \circ \tau_{A^*}\}_{A^*} = 0, \qquad \{f \circ \tau_{A^*}, \widehat{X}\}_{A^*} = \rho_A(X)(f) \circ \tau_{A^*},$$

$$\{\widehat{X}, \widehat{Y}\}_{A^*} = -\widehat{[\![X, Y]\!]}_A,$$

for $f, g \in C^\infty(M)$ and $X, Y \in \Gamma(A)$. Here, $\widehat{Z}$ denotes the linear function on A^* induced by a section $Z \in \Gamma(A)$. The Poisson structure $\{\cdot,\cdot\}_{A^*}$ is called the *canonical linear Poisson structure on A^* associated with the Lie algebroid A.*

Next, suppose that (x^i) are local coordinates on an open subset U of M and that $\{e_\alpha\}$ is a local basis of $\Gamma(A)$ on U. Denote by (x^i, y_α) the corresponding local coordinates on A^* and by ρ^i_α, $C^\gamma_{\alpha\beta}$ the local structure functions of A with respect to the coordinates (x^i) and to the basis $\{e_\alpha\}$. Then, we may consider the local sections $\{\mathcal{X}_\alpha, \mathcal{P}^\alpha\}$ of the vector bundle $\tau_{\mathcal{T}^AA^*} : \mathcal{T}^AA^* \to A^*$ given by

$$\mathcal{X}_\alpha = \left(e_\alpha \circ \tau_{A^*}, \rho^i_\alpha \frac{\partial}{\partial x^i}\right), \qquad \mathcal{P}^\alpha = \left(0, \frac{\partial}{\partial y_\alpha}\right).$$

We have that $\{\mathcal{X}_\alpha, \mathcal{P}^\alpha\}$ is a local basis of $\Gamma(\mathcal{T}^AA^*)$ and

$$\begin{aligned}
&[\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!]_{\mathcal{T}^AA^*} = C^\gamma_{\alpha\beta}\mathcal{X}_\gamma, \qquad [\![\mathcal{X}_\alpha, \mathcal{P}^\beta]\!]_{\mathcal{T}^AA^*} = [\![\mathcal{P}^\alpha, \mathcal{P}^\beta]\!]_{\mathcal{T}^AA^*} = 0,\\
&\rho_{\mathcal{T}^AA^*}(\mathcal{X}_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}, \qquad \rho_{\mathcal{T}^AA^*}(\mathcal{P}^\alpha) = \frac{\partial}{\partial y_\alpha},\\
&\Theta_{\mathcal{T}^AA^*} = y_\alpha \mathcal{X}^\alpha, \qquad \Omega_{\mathcal{T}^AA^*} = \mathcal{X}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2} C^\gamma_{\alpha\beta} y_\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta,
\end{aligned}$$

$\{\mathcal{X}^\alpha, \mathcal{P}_\alpha\}$ being the dual basis of $\{\mathcal{X}_\alpha, \mathcal{P}^\alpha\}$. Moreover, if $H, H' : A^* \to \mathbb{R}$ are real functions on A^* it follows that

$$\begin{aligned}
\{H, H'\}_{A^*} &= -\frac{\partial H}{\partial y_\alpha}\frac{\partial H'}{\partial y_\beta} C^\gamma_{\alpha\beta} y_\gamma + \left(\frac{\partial H}{\partial x^i}\frac{\partial H'}{\partial y_\alpha} - \frac{\partial H}{\partial y_\alpha}\frac{\partial H'}{\partial x^i}\right)\rho^i_\alpha,\\
\mathcal{H}^{\Omega_{\mathcal{T}^AA^*}}_H &= \frac{\partial H}{\partial y_\alpha}\mathcal{X}_\alpha - \left(\rho^i_\alpha \frac{\partial H}{\partial x^i} + C^\gamma_{\alpha\beta} y_\gamma \frac{\partial H}{\partial y_\beta}\right)\mathcal{P}^\alpha,\\
\mathcal{H}^{\{\cdot,\cdot\}_{A^*}}_H &= \frac{\partial H}{\partial y_\alpha}\rho^i_\alpha \frac{\partial}{\partial x^i} - \left(\rho^i_\alpha \frac{\partial H}{\partial x^i} + C^\gamma_{\alpha\beta} y_\gamma \frac{\partial H}{\partial y_\beta}\right)\frac{\partial}{\partial y_\alpha},
\end{aligned} \tag{6}$$

(for more details, see [11]).

Example 1.

1. If A is the standard Lie algebroid TM then $\mathcal{T}^AA^* = T(T^*M)$, $\Omega_{\mathcal{T}^AA^*}$ is the canonical symplectic structure on $A^* = T^*M$ and $\{\cdot,\cdot\}_{T^*M}$ is the canonical Poisson bracket on T^*M induced by $\Omega_{\mathcal{T}^AA^*}$.
2. If A is the Lie algebroid associated with an infinitesimal action $\Phi : \mathfrak{g} \to \mathfrak{X}(M)$ of a Lie algebra $\mathfrak{g}$ on a manifold M (see Section 2.1), then the Lie algebroid $\mathcal{T}^AA^* \to A^*$ may be identified with the trivial vector bundle

   $$(M \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*) \to M \times \mathfrak{g}^*$$

   and, under this identification, the canonical symplectic section $\Omega_{\mathcal{T}^AA^*}$ is given by

   $$\Omega_{\mathcal{T}^AA^*}(x, \alpha)((\xi, \beta), (\xi', \beta')) = \beta'(\xi) - \beta(\xi') + \alpha[\xi, \xi']_{\mathfrak{g}},$$

   for $(x, \alpha) \in A^* = M \times \mathfrak{g}^*$ and $(\xi, \beta), (\xi', \beta') \in \mathfrak{g} \times \mathfrak{g}^*$, where $[\cdot,\cdot]_{\mathfrak{g}}$ is the Lie bracket on $\mathfrak{g}$.

   The anchor map $\rho_{\mathcal{T}^AA^*} : (M \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*) \to T(M \times \mathfrak{g}^*) \cong TM \times (\mathfrak{g}^* \times \mathfrak{g}^*)$ of the A-tangent bundle to A^* is

   $$\rho_{\mathcal{T}^AA^*}((x, \alpha), (\xi, \beta)) = (-\Phi(\xi)(x), \alpha, \beta)$$

and the Lie bracket of two constant sections (ξ,β) and (ξ',β') is just the constant section $([\xi,\xi']_{\mathfrak{g}},0)$.

Note that in the particular case when M is a single point (that is, A is the real algebra $\mathfrak{g}$) then the linear Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is just the (minus) Lie–Poisson bracket induced by the Lie algebra $\mathfrak{g}$.

3. Let $\pi_P : P \to M$ be a principal G-bundle and $A = TP/G$ be the Atiyah algebroid associated with $\pi_P : P \to M$. Then, the cotangent bundle T^*P to P is the total space of a principal G-bundle over $A^* \cong T^*P/G$ and the Lie algebroid $\mathcal{T}^A A^*$ may be identified with the Atiyah algebroid $T(T^*P)/G \to T^*P/G$ associated with this principal G-bundle (see [11]). Moreover, the canonical symplectic structure on T^*P is G-invariant and it induces a symplectic section $\widetilde{\Omega}$ on the Atiyah algebroid $T(T^*P)/G \to T^*P/G$. $\widetilde{\Omega}$ is just the canonical symplectic section of $\mathcal{T}^A A^* \cong T(T^*P)/G \to A^* \cong T^*P/G$. Finally, the linear Poisson bracket on $A^* \cong T^*P/G$ is characterized by the following property: if on T^*P we consider the Poisson bracket induced by the canonical symplectic structure then the canonical projection $\pi_{T^*P} : T^*P \to T^*P/G$ is a Poisson morphism.

   We remark that, using a connection in the principal G-bundle $\pi_P : P \to M$, the space $A^* \cong T^*P/G$ may be identified with the Whitney sum $W = T^*M \oplus_M \widetilde{\mathfrak{g}}^*$, where $\widetilde{\mathfrak{g}}^*$ is the coadjoint bundle associated with the principal bundle $\pi_P : P \to M$. In addition, under the above identification, the Poisson bracket on $A^* \cong T^*P/G$ is just the so-called *Weinstein space Poisson bracket* (for more details, see [20]).

2.3 Hamilton equations and symplectic Lie algebroids

Let A be a Lie algebroid over a manifold M and Ω_A be a symplectic section of A. Then, as we know, Ω_A induces a Poisson bracket $\{\cdot,\cdot\}_M$ on M.

A Hamiltonian function for A is a real C^∞-function $H : M \to \mathbb{R}$ on M. If H is a Hamiltonian function one may consider the Hamiltonian section $\mathcal{H}_H^{\Omega_A}$ of H with respect to Ω_A and the Hamiltonian vector field $\mathcal{H}_H^{\{\cdot,\cdot\}_M}$ of H with respect to the Poisson bracket $\{\cdot,\cdot\}_M$ on M.

The solutions of the Hamilton equations for H in A are just the integral curves of the vector field $\mathcal{H}_H^{\{\cdot,\cdot\}_M}$.

Now, suppose that our symplectic Lie algebroid is the A-tangent bundle to A^*, $\mathcal{T}^A A^*$, where A is an arbitrary Lie algebroid. As we know, the base space of $\mathcal{T}^A A^*$ is A^* and the corresponding Poisson bracket $\{\cdot,\cdot\}_{A^*}$ on A^* is just the canonical linear Poisson bracket on A^* associated with the Lie algebroid A. Thus, if $H : A^* \to \mathbb{R}$ is a Hamiltonian function for $\mathcal{T}^A A^*$, we have that the solutions of the Hamilton equations for H in $\mathcal{T}^A A^*$ (or simply, the solutions of the Hamilton equations for H) are the integral curves of the Hamiltonian vector field $\mathcal{H}_H^{\{\cdot,\cdot\}_{A^*}}$ (see [11]).

If (x^i) is a system of local coordinates on an open subset U of M and $\{e_\alpha\}$ is a local basis of $\Gamma(A)$ on U, we may consider the corresponding system of local coordinates (x^i, y_α) on $\tau_{A^*}^{-1}(U) \subseteq A^*$. In addition, using (6), we deduce that a curve $t \to (x^i(t), y_\alpha(t))$ on $\tau_{A^*}^{-1}(U)$ is a solution of the Hamilton equations for H if and only if

$$\frac{dx^i}{dt} = \rho^i_\alpha \frac{\partial H}{\partial x^i}, \qquad \frac{dy_\alpha}{dt} = -\left(C^\gamma_{\alpha\beta} y_\gamma \frac{\partial H}{\partial y_\beta} + \rho^i_\alpha \frac{\partial H}{\partial x^i} \right), \tag{7}$$

(see [11]).

Example 2.

1. Let A be the Lie algebroid associated with an infinitesimal action $\Phi : \mathfrak{g} \to \mathfrak{X}(M)$ of a Lie algebra $\mathfrak{g}$ on the manifold M. If $H : A^* = M \times \mathfrak{g}^* \to \mathbb{R}$ is a Hamiltonian function, $\{\xi_\alpha\}$ is

a basis of $\mathfrak{g}$ and (x^i) is a system of local coordinates on M such that

$$[\xi_\alpha, \xi_\beta]_{\mathfrak{g}} = c^\gamma_{\alpha\beta}\xi_\gamma, \qquad \Phi(\xi_\alpha) = \Phi^i_\alpha \frac{\partial}{\partial x^i},$$

then the curve $t \to (x^i(t), y_\alpha(t))$ on $A^* = M \times \mathfrak{g}^*$ is a solution of the Hamilton equations for H if and only if

$$\frac{dx^i}{dt} = \Phi^i_\alpha \frac{\partial H}{\partial x^i}, \qquad \frac{dy_\alpha}{dt} = -\left(c^\gamma_{\alpha\beta} y_\gamma \frac{\partial H}{\partial y_\beta} + \Phi^i_\alpha \frac{\partial H}{\partial x^i}\right).$$

Note that if M is a single point (that is, A is the Lie algebra $\mathfrak{g}$) then the above equations are just *the (minus) Lie–Poisson equations on* $\mathfrak{g}^*$ *for the Hamiltonian function* $H : \mathfrak{g}^* \to \mathbb{R}$.

2. Let $\pi_P : P \to M$ be a principal G-bundle, $D : TP \to \mathfrak{g}$ be a principal connection and $R : TP \oplus TP \to \mathfrak{g}$ be the curvature of D. We choose a local trivialization of the principal bundle $\pi_P : P \to M$ to be $U \times G$, where U is an open subset of M. Suppose that (x^i) are local coordinates on U and that $\{\xi_a\}$ is a basis of $\mathfrak{g}$ such that $[\xi_a, \xi_b]_{\mathfrak{g}} = c^c_{ab}\xi_c$. Denote by D^a_i (respectively, R^a_{ij}) the components of D (respectively, R) with respect to the coordinates (x^i) and to the basis $\{\xi_a\}$ and by $(x^i, y_\alpha = p_i, \bar{p}_a)$ the corresponding local coordinates on $A^* \cong T^*P/G$ (see Section 2.1). If $h : T^*P/G \to \mathbb{R}$ is a Hamiltonian function and $c : t \to (x^i(t), p_i(t), \bar{p}_a(t))$ is a curve on $A^* \cong T^*P/G$ then, using (1) and (7), we conclude that c is a solution of the Hamilton equations for h if and only if

$$\begin{aligned} &\frac{dx^i}{dt} = \frac{\partial h}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial h}{\partial x^i} + R^a_{ij}\bar{p}_a \frac{\partial h}{\partial p_j} - c^c_{ab} D^b_i \bar{p}_c \frac{\partial h}{\partial \bar{p}_a}, \\ &\frac{d\bar{p}_a}{dt} = c^c_{ab} D^b_i \bar{p}_c \frac{\partial h}{\partial p_i} - c^c_{ab}\bar{p}_c \frac{\partial h}{\partial \bar{p}_b}. \end{aligned} \tag{8}$$

These equations are just *the Hamilton–Poincaré equations associated with the G-invariant Hamiltonian function $H = h \circ \pi_{T^*P}$*, $\pi_{T^*P} : T^*P \to T^*P/G$ being the canonical projection (see [11]).

Note that in the particular case when G is the trivial Lie group, equations (8) reduce to

$$\frac{dx^i}{dt} = \frac{\partial h}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial h}{\partial x^i},$$

which are the classical Hamilton equations for the Hamiltonian function $h : T^*P \cong T^*M \to \mathbb{R}$.

3 Reduction of the Hamiltonian dynamics on a symplectic Lie algebroid by a Lie subalgebroid and a symmetry Lie group

3.1 Reduction of the symplectic Lie algebroid

Let A be a Lie algebroid over a manifold M and Ω_A be a symplectic section of A. In addition, suppose that B is a Lie subalgebroid over the submanifold N of M and denote by $i_B : B \to A$ the corresponding monomorphism of Lie algebroids. The following commutative diagram illustrates the above situation

$$\begin{tikzcd} B \arrow[r, "i_B"] \arrow[d, "\tau_B"'] & A \arrow[d, "\tau_A"] \\ N \arrow[r, "i_N"] & M \end{tikzcd}$$

Ω_A induces, in a natural way, a section Ω_B of the real vector bundle $\wedge^2 B^* \to N$. In fact,

$$\Omega_B = i_B^* \Omega_A.$$

Since i_B is a Lie algebroid morphism, it follows that

$$d^B \Omega_B = 0. \tag{9}$$

However, Ω_B is not, in general, nondegenerate. In other words, Ω_B is a presymplectic section of the Lie subalgebroid $\tau_B : B \to N$.

For every point x of N, we will denote by $\ker \Omega_B(x)$ the vector subspace of B_x defined by

$$\ker \Omega_B(x) = \{a_x \in B_x / i(a_x)\Omega_B(x) = 0\} \subseteq B_x.$$

In what follows, *we will assume that the dimension of the subspace* $\ker \Omega_B(x)$ *is constant, for all* $x \in N$. Thus, the space $B/\ker \Omega_B$ is a quotient vector bundle over the submanifold N.

Moreover, we may consider the vector subbundle $\ker \Omega_B$ of B whose fiber over the point $x \in N$ is the vector space $\ker \Omega_B(x)$. In addition, we have that:

Lemma 1. $\ker \Omega_B$ *is a Lie subalgebroid over* N *of* B*.*

Proof. It is sufficient to prove that

$$X, Y \in \Gamma(\ker \Omega_B) \Rightarrow [\![X, Y]\!]_B \in \Gamma(\ker \Omega_B), \tag{10}$$

where $[\![\cdot, \cdot]\!]_B$ is the Lie bracket on $\Gamma(B)$.

Now, using (9), it follows that (10) holds. ∎

Next, we will suppose that there is *a proper and free action* $\Psi : G \times B \to B$ *of a Lie group* G *on* B *by vector bundle automorphisms*. Then, the following conditions are satisfied:

C1) N is the total space of a principal G-bundle over $\widetilde{N}$ with principal bundle projection $\pi_N : N \to \widetilde{N} \cong N/G$ and we will denote by $\psi : G \times N \to N$ the corresponding free action of G on N.

C2) $\Psi_g : B \to B$ is a vector bundle isomorphism over $\psi_g : N \to N$, for all $g \in G$.

The following commutative diagram illustrates the above situation

$$\begin{tikzcd}
B \arrow[rr, "\Psi_g"] \arrow[d, "\tau_B"'] & & B \arrow[d, "\tau_B"] \\
N \arrow[rr, "\psi_g"] \arrow[dr, "\pi_N"'] & & N \arrow[dl, "\pi_N"] \\
& \widetilde{N} \cong N/G &
\end{tikzcd}$$

In such a case, the action Ψ of the Lie group G on the presymplectic Lie algebroid (B, Ω_B) is said to be *presymplectic* if

$$\Psi_g^* \Omega_B = \Omega_B, \qquad \text{for all } \; g \in G. \tag{11}$$

Now, we will prove the following result.

Theorem 1. *Let $\Psi : G \times B \to B$ be a presymplectic action of the Lie group G on the presymplectic Lie algebroid (B, Ω_B). Then, the action of G on B induces an action of G on the quotient vector bundle $B/\ker\Omega_B$ such that:*

a) The space of orbits $\widetilde{B} = (B/\ker\Omega_B)/G$ of this action is a vector bundle over $\widetilde{N} = N/G$ and the diagram

$$\begin{tikzcd} B \arrow[r, "\widetilde{\pi}_B"] \arrow[d, "\tau_B"'] & \widetilde{B} = (B/\ker\Omega_B)/G \arrow[d, "\tau_{\widetilde{B}}"] \\ N \arrow[r, "\pi_N"] & \widetilde{N} = N/G \end{tikzcd}$$

defines an epimorphism of vector bundles, where $\widetilde{\pi}_B : B \to \widetilde{B}$ is the canonical projection.

b) There exists a unique section $\Omega_{\widetilde{B}}$ of $\wedge^2\widetilde{B}^* \to \widetilde{N}$ such that

$$\widetilde{\pi}_B^*\Omega_{\widetilde{B}} = \Omega_B.$$

c) $\Omega_{\widetilde{B}}$ is nondegenerate.

Proof. If $g \in G$ and $x \in M$ then $\Psi_g : B_x \to B_{\psi_g(x)}$ is a linear isomorphism and

$$\Psi_g(\ker\Omega_B(x)) = \ker\Omega_B(\psi_g(x)).$$

Thus, Ψ induces an action $\widetilde{\Psi} : G \times (B/\ker\Omega_B) \to (B/\ker\Omega_B)$ of G on $B/\ker\Omega_B$ and $\widetilde{\Psi}_g$ is a vector bundle isomorphism, for all $g \in G$. Moreover, the canonical projection $\pi_B : B \to B/\ker\Omega_B$ is equivariant with respect to the actions Ψ and $\widetilde{\Psi}$.

On the other hand, the vector bundle projection $\tau_{B/\ker\Omega_B} : B/\ker\Omega_B \to N$ is also equivariant with respect to the action $\widetilde{\Psi}$. Consequently, it induces a smooth map $\tau_{\widetilde{B}} : \widetilde{B} = (B/\ker\Omega_B)/G \to \widetilde{N} = N/G$ such that the following diagram is commutative

$$\begin{tikzcd} B/\ker\Omega_B \arrow[r, "\tau_{B/\ker\Omega_B}"] \arrow[d, "\pi_{B/\ker\Omega_B}"'] & N \arrow[d, "\pi_N"] \\ \widetilde{B} = (B/\ker\Omega_B)/G \arrow[r, "\tau_{\widetilde{B}}"] & \widetilde{N} = N/G \end{tikzcd}$$

Now, if $x \in N$ then, using that the action ψ is free, we deduce that the map $\pi_{B_x/\ker\Omega_B(x)} : B_x/\ker\Omega_B(x) \to \tau_{\widetilde{B}}^{-1}(\pi_N(x))$ is bijective. Thus, one may introduce a vector space structure on $\tau_{\widetilde{B}}^{-1}(\pi_N(x))$ in such a way that the map $\pi_{B_x/\ker\Omega_B(x)} : B_x/\ker\Omega_B(x) \to \tau_{\widetilde{B}}^{-1}(\pi_N(x))$ is a linear isomorphism. Furthermore, if $\pi_N(y) = \pi_N(x)$ then $\pi_{B_y/\ker\Omega_B(y)} = \pi_{B_x/\ker\Omega_B(x)} \circ \widetilde{\Psi}_g^{-1}$.

Therefore, $\widetilde{B}$ is a vector bundle over $\widetilde{N}$ with vector bundle projection $\tau_{\widetilde{B}} : \widetilde{B} \to \widetilde{N}$ and if $x \in N$ then the fiber of $\widetilde{B}$ over $\pi_N(x)$ is isomorphic to the vector space $B_x/\ker\Omega_B(x)$. This proves a).

On the other hand, it is clear that the section Ω_B induces a section $\Omega_{(B/\ker\Omega_B)}$ of the vector bundle $\wedge^2(B/\ker\Omega_B) \to N$ which is characterized by the condition

$$\pi_B^*(\Omega_{(B/\ker\Omega_B)}) = \Omega_B. \tag{12}$$

Then, using (11) and (12), we deduce that the section $\Omega_{(B/\ker\Omega_B)}$ is G-invariant, that is,

$$\widetilde{\Psi}_g^*(\Omega_{(B/\ker\Omega_B)}) = \Omega_{(B/\ker\Omega_B)}, \qquad \text{for all} \quad g \in G.$$

Thus, there exists a unique section $\Omega_{\widetilde{B}}$ of the vector bundle $\wedge^2\widetilde{B}^* \to \widetilde{N}$ such that

$$\pi^*_{B/\ker\Omega_B}(\Omega_{\widetilde{B}}) = \Omega_{(B/\ker\Omega_B)}. \tag{13}$$

This proves $b)$ (note that $\widetilde{\pi}_B = \pi_{B/\ker\Omega_B} \circ \pi_B$).

Now, using (12) and the fact $\ker\pi_B = \ker\Omega_B$, it follows that the section $\Omega_{(B/\ker\Omega_B)}$ is nondegenerate. Therefore, from (13), we conclude that $\Omega_{\widetilde{B}}$ is also nondegenerate (note that if $x \in M$ then $\pi_{B/\ker\Omega_B} : B_x/\ker\Omega_B(x) \to \widetilde{B}_{\pi_N(x)}$ is a linear isomorphism). ■

Next, we will describe the space of sections of $\widetilde{B} = (B/\ker\Omega_B)/G$. For this purpose, we will use some results contained in the Appendix of this paper.

Let $\Gamma(B)^p_{\widetilde{\pi}_B}$ be the space of $\widetilde{\pi}_B$-projectable sections of the vector bundle $\tau_B : B \to N$. As we know (see the Appendix), a section X of $\tau_B : B \to N$ is said to be $\widetilde{\pi}_B$-projectable if there exists a section $\widetilde{\pi}_B(X)$ of the vector bundle $\tau_{\widetilde{B}} : \widetilde{B} \to \widetilde{N}$ such that $\widetilde{\pi}_B \circ X = \widetilde{\pi}_B(X) \circ \pi_N$. Thus, it is clear that X is $\widetilde{\pi}_B$-projectable if and only if $\pi_B \circ X$ is a $\pi_{(B/\ker\Omega_B)}$-projectable section.

On the other hand, it is easy to prove that the section $\pi_B \circ X$ is $\pi_{(B/\ker\Omega_B)}$-projectable if and only if it is G-invariant. In other words, $(\pi_B \circ X)$ is $\pi_{(B/\ker\Omega_B)}$-projectable if and only if for every $g \in G$ there exists $Y_g \in \Gamma(\ker\pi_B)$ such that $\Psi_g \circ X = (X + Y_g) \circ \psi_g$, where $\psi : G \times N \to N$ is the corresponding free action of G on N. Note that

$$\ker\widetilde{\pi}_B = \ker\pi_B = \ker\Omega_B \tag{14}$$

and therefore, the above facts, imply that

$$\Gamma(B)^p_{\widetilde{\pi}_B} = \{X \in \Gamma(B)/\forall\, g \in G,\ \exists\, Y_g \in \Gamma(\ker\Omega_B) \text{ and } \Psi_g \circ X = (X + Y_g) \circ \psi_g\}. \tag{15}$$

Consequently, using some results of the Appendix (see (A.2) in the Appendix), we deduce that

$$\Gamma(\widetilde{B}) \cong \frac{\{X \in \Gamma(B)/\forall g \in G,\ \ \exists Y_g \in \Gamma(\ker\Omega_B) \text{ and } \Psi_g \circ X = (X + Y_g) \circ \psi_g\}}{\Gamma(\ker\Omega_B)}$$

as $C^\infty(\widetilde{N})$-modules.

Moreover, we may prove the following result

Theorem 2 (The reduced symplectic Lie algebroid). *Let $\Psi : G \times B \to B$ be a presympletic action of the Lie group G on the Lie algebroid (B, Ω_B). Then, the reduced vector bundle $\tau_{\widetilde{B}} : \widetilde{B} = (B/\ker\Omega_B)/G \to \widetilde{N} = N/G$ admits a unique Lie algebroid structure such that $\widetilde{\pi}_B : B \to \widetilde{B}$ is a Lie algebroid epimorphism if and only if the following conditions hold:*

i) The space $\Gamma(B)^p_{\widetilde{\pi}_B}$ is a Lie subalgebra of the Lie algebra $(\Gamma(B), [\![\cdot,\cdot]\!]_B)$.

ii) $\Gamma(\ker\Omega_B)$ is an ideal of this Lie subalgebra.

Furthermore, if the conditions i) and ii) hold, we get that there exists a short exact sequence of Lie algebroids

$$0 \to \ker\Omega_B \to B \to \widetilde{B} \to 0$$

and that the nondegenerate section $\Omega_{\widetilde{B}}$ induces a symplectic structure on the Lie algebroid $\tau_{\widetilde{B}} : \widetilde{B} \to \widetilde{N}$.

Proof. The first part of the Theorem follows from (14), (15) and Theorem A.1 in the Appendix.

On the other hand, if the conditions *i)* and *ii)* in the Theorem hold then, using Theorem 1 and the fact that $\widetilde{\pi}_B$ is a Lie algebroid epimorphism, we obtain that

$$\widetilde{\pi}_B^*(d^{\widetilde{B}}\Omega_{\widetilde{B}}) = 0,$$

which implies that $d^{\widetilde{B}}\Omega_{\widetilde{B}} = 0$. ■

Note that if G is the trivial group, the condition *ii)* of Theorem 2 is satisfied and if only if $\Gamma(\ker \Omega_B)$ is an ideal of $\Gamma(B)$.

Finally, we deduce the following corollary.

Corollary 1. *Let $\Psi : G \times B \to B$ be a presymplectic action of the Lie group G on the presymplectic Lie algebroid (B, Ω_B) and suppose that:*

i) *$\Psi_g : B \to B$ is a Lie algebroid isomorphism, for all $g \in G$.*

ii) *If $X \in \Gamma(B)^p_{\widetilde{\pi}_B}$ and $Y \in \Gamma(\ker \Omega_B)$, we have that*

$$[\![X, Y]\!]_B \in \Gamma(\ker \Omega_B).$$

Then, the reduced vector bundle $\tau_{\widetilde{B}} : \widetilde{B} \to \widetilde{N}$ admits a unique Lie algebroid structure such that $\widetilde{\pi}_B : B \to \widetilde{B}$ is a Lie algebroid epimorphism. Moreover, the nondegenerate section $\Omega_{\widetilde{B}}$ induces a symplectic structure on the Lie algebroid $\tau_{\widetilde{B}} : \widetilde{B} \to \widetilde{N}$.

Proof. If $X, Y \in \Gamma(B)^p_{\widetilde{\pi}_B}$ then

$$\Psi_g \circ X = (X + Z_g) \circ \psi_g, \qquad \Psi_g \circ Y = (Y + W_g) \circ \psi_g, \qquad \text{for all} \;\; g \in G,$$

where $Z_g, W_g \in \Gamma(\ker \Omega_B)$ and $\psi_g : N \to N$ is the diffeomorphism associated with $\Psi_g : B \to B$. Thus, using the fact that Ψ_g is a Lie algebroid isomorphism, it follows that

$$\Psi_g \circ [\![X, Y]\!]_B \circ \psi_{g^{-1}} = [\![X + Z_g, Y + W_g]\!]_B, \qquad \text{for all} \;\; g \in G.$$

Therefore, from condition *ii)* in the corollary, we deduce that

$$\Psi_g \circ [\![X, Y]\!]_B \circ \psi_{g^{-1}} - [\![X, Y]\!]_B \in \Gamma(\ker \Omega_B), \qquad \text{for all} \;\; g \in G,$$

which implies that $[\![X, Y]\!]_B \in \Gamma(B)^p_{\widetilde{\pi}_B}$. This proves that $\Gamma(B)^p_{\widetilde{\pi}_B}$ is a Lie subalgebra of the Lie algebra $(\Gamma(B), [\![\cdot, \cdot]\!]_B)$. ■

3.2 Reduction of the Hamiltonian dynamics

Let A be a Lie algebroid over a manifold M and Ω_A be a symplectic section of A. In addition, suppose that B is a Lie subalgebroid of A over the submanifold N of M and that G is a Lie group such that one may construct the symplectic reduction $\tau_{\widetilde{B}} : \widetilde{B} = (B/\ker \Omega_B)/G \to \widetilde{N} = N/G$ of A by B and G as in Section 3.1 (see Theorem 2).

We will also assume that B and N are subsets of A and M, respectively (that is, the corresponding immersions $i_B : B \to A$ and $i_N : N \to M$ are the canonical inclusions), and that N is a closed submanifold.

Theorem 3 (The reduction of the Hamiltonian dynamics). *Let $H_M : M \to \mathbb{R}$ be a Hamiltonian function for the symplectic Lie algebroid A such that:*

i) *The restriction H_N of H_M to N is G-invariant and* *ii) If $\mathcal{H}^{\Omega_A}_{H_M}$ is the Hamiltonian section of H_M with respect to the symplectic section Ω_A, we have that $\mathcal{H}^{\Omega_A}_{H_M}(N) \subseteq B$.*

Then:

a) H_N induces a real function $H_{\widetilde{N}} : \widetilde{N} \to \mathbb{R}$ such that $H_{\widetilde{N}} \circ \pi_N = H_N$.

b) The restriction of $\mathcal{H}^{\Omega_A}_{H_M}$ to N is $\widetilde{\pi}_B$-projectable over the Hamiltonian section of the function $H_{\widetilde{N}}$ with respect to the reduced symplectic structure $\Omega_{\widetilde{B}}$ and

c) If $\gamma : I \to M$ is a solution of the Hamilton equations for H_M in the symplectic Lie algebroid (A, Ω_A) such that $\gamma(t_0) \in N$, for some $t_0 \in I$, then $\gamma(I) \subseteq N$ and $\pi_N \circ \gamma : I \to \widetilde{N}$ is a solution of the Hamilton equations for $H_{\widetilde{N}}$ in the symplectic Lie algebroid $(\widetilde{B}, \Omega_{\widetilde{B}})$.

Proof. *a*) Using that the function H_N is G-invariant, we deduce *a*).

b) If $x \in N$ and $a_x \in B_x$ then, since $\widetilde{\pi}_B^* \Omega_{\widetilde{B}} = \Omega_B$ (see Theorem 2), $\mathcal{H}^{\Omega_A}_{H_M}(N) \subseteq B$ and $\tau_B : B \to N$ is a Lie subalgebroid of A, we have that

$$(i(\widetilde{\pi}_B(\mathcal{H}^{\Omega_A}_{H_M}(x)))\Omega_{\widetilde{B}}(\pi_N(x))(\widetilde{\pi}_B(a_x))) = (d^B H_N)(x)(a_x).$$

Thus, using that $\widetilde{\pi}_B$ is a Lie algebroid epimorphism and the fact that $H_{\widetilde{N}} \circ \pi_N = H_N$, it follows that

$$\begin{aligned}(i(\widetilde{\pi}_B(\mathcal{H}^{\Omega_A}_{H_M}(x)))\Omega_{\widetilde{B}}(\pi_N(x)))(\widetilde{\pi}_B(a_x))) &= (d^{\widetilde{B}} H_{\widetilde{N}})(\pi_N(x))(\widetilde{\pi}_B(a_x)) \\ &= (i(\mathcal{H}^{\Omega_{\widetilde{B}}}_{H_{\widetilde{N}}}(\pi_N(x)))\Omega_{\widetilde{B}}(\pi_N(x)))(\widetilde{\pi}_B(a_x))).\end{aligned}$$

This implies that

$$\widetilde{\pi}_B(\mathcal{H}^{\Omega_A}_{H_M}(x)) = \mathcal{H}^{\Omega_{\widetilde{B}}}_{H_{\widetilde{N}}}(\pi_N(x)).$$

c) Using *b*), we deduce that the vector field $\rho_B((\mathcal{H}^{\Omega_A}_{H_M})_{|N})$ is π_N-projectable on the vector field $\rho_{\widetilde{B}}(\mathcal{H}^{\Omega_{\widetilde{B}}}_{H_{\widetilde{N}}})$ (it is a consequence of the equality $\rho_{\widetilde{B}} \circ \widetilde{\pi}_B = T\pi_N \circ \rho_B$). This proves *c*). Note that N is closed and that the integral curves of $\rho_B((\mathcal{H}^{\Omega_A}_{H_M})_{|N})$ (respectively, $\rho_{\widetilde{B}}(\mathcal{H}^{\Omega_{\widetilde{B}}}_{H_{\widetilde{N}}})$) are the solutions of the Hamilton equations for H_M in A (respectively, for $H_{\widetilde{N}}$ in $\widetilde{B}$) with initial condition in N (respectively, in $\widetilde{N}$). ■

4 Examples and applications

4.1 Cartan symplectic reduction in the presence of a symmetry Lie group

Let M be a symplectic manifold with symplectic 2-form Ω_{TM}. Suppose that N is a submanifold of M, that G is a Lie group and that N is the total space of a principal G-bundle over $\widetilde{N} = N/G$. We will denote $\psi : G \times N \to N$ the free action of G on N, by $\pi_N : N \to \widetilde{N} = N/G$ the principal bundle projection and by Ω_{TN} the 2-form on N given by

$$\Omega_{TN} = i_N^* \Omega_{TM}.$$

Here, $i_N : N \to M$ is the canonical inclusion.

Proposition 1. *If the vertical bundle to π_N is the kernel of the 2-form Ω_{TN}, that is,*

$$V\pi_N = \ker \Omega_{TN}, \tag{16}$$

then there exists a unique symplectic 2-form $\Omega_{T(N/G)}$ on $\widetilde{N} = N/G$ such that $\pi_N^* \Omega_{T(N/G)} = \Omega_{TN}$.

Proof. We will prove the result using Theorem 1 and Corollary 1. In fact, we apply Theorem 1 to the followings elements:

- The standard symplectic Lie algebroid $\tau_{TM} : TM \to M$,
- The Lie subalgebroid of TM, $\tau_{TN} : TN \to N$ and
- The presymplectic action $T\psi : G \times TN \to TN$, i.e., the tangent lift of the action ψ of G on N.

The corresponding reduced vector bundle $(\widetilde{TN} = (TN/\ker \Omega_{TN})/G, \Omega_{\widetilde{TN}})$ is isomorphic to the tangent bundle $T(N/G)$ in a natural way.

On the other hand, it is well-known that the Lie bracket of a π_N-projectable vector field and a π_N-vertical vector field is a π_N-vertical vector field. Therefore, we may use Corollary 1 and we deduce that the symplectic vector bundle $\tau_{\widetilde{TN}} : \widetilde{TN} \to \widetilde{N} = N/G$ admits a unique Lie algebroid structure such that the canonical projection $\widetilde{\pi}_{TN} : TN \to \widetilde{TN}$ is a Lie algebroid epimorphism. In addition, $\Omega_{\widetilde{TN}}$ is a symplectic section of the Lie algebroid $\tau_{\widetilde{TN}} : \widetilde{TN} \to \widetilde{N} = N/G$.

Finally, under the identification between the vector bundles $\widetilde{TN}$ and $T(N/G)$, one may see that the resultant Lie algebroid structure on the vector bundle $\tau_{T(N/G)} : T(N/G) \to N/G$ is the just standard Lie algebroid structure. ■

Remark 1. The reduction process described in Proposition 1 is just the classical Cartan symplectic reduction process (see, for example, [20]) for the particular case when the kernel of the presymplectic 2-form on the submanifold N of the original symplectic manifold is just the union of the tangent spaces of the G-orbits associated with a principal G-bundle with total space N.

Now, assume that the submanifold N is closed. Then, using Theorem 3, we may prove the following result:

Corollary 2. *Let $H_M : M \to \mathbb{R}$ be a Hamiltonian function on the symplectic manifold M such that:*

1. *The restriction H_N of H_M to N is G-invariant and*
2. *The restriction to N of the Hamiltonian vector field $\mathcal{H}_{H_M}^{\Omega_{TM}}$ of H_M with respect to Ω_{TM} is tangent to N.*

Then:

a) *H_N induces a real function $H_{\widetilde{N}} : \widetilde{N} = N/G \to \mathbb{R}$ such that $H_{\widetilde{N}} \circ \pi_N = H_N$;*

b) *The vector field $(\mathcal{H}_{H_M}^{\Omega_{TM}})_{|N}$ on N is π_N-projectable on the Hamiltonian vector field of $H_{\widetilde{N}}$ with respect to the reduced symplectic 2-form $\Omega_{T(N/G)}$ on N/G and*

c) *If $\gamma : I \to M$ is a solution of the Hamiltonian equations for H_M in the symplectic manifold (M, Ω_{TM}) such that $\gamma(t_0) \in N$, for some $t_0 \in I$, then $\gamma(I) \subseteq N$ and $\pi_N \circ \gamma : I \to \widetilde{N}$ is a solution of the Hamiltonian equations for $H_{\widetilde{N}}$ in the symplectic manifold $(\widetilde{N}, \Omega_{\widetilde{TN}})$.*

4.2 Symplectic reduction of symplectic Lie algebroids by Lie subalgebroids

Let A be a symplectic Lie algebroid over the manifold M with symplectic section Ω_A and B be a Lie subalgebroid of A over the submanifold N of M.

Denote by $(\llbracket\cdot,\cdot\rrbracket_B, \rho_B)$ the Lie algebroid structure of B and by Ω_B the presymplectic section on B given by

$$\Omega_B = i_B^* \Omega_A,$$

where $i_B : B \to A$ is the canonical inclusion.

As in Section 2, we will assume that the dimension of the subspace $\ker \Omega_B(x)$ is constant, for all $x \in N$. Then, one may consider the vector bundle $\tau_{\ker \Omega_B} : \ker \Omega_B \to N$ which is a Lie subalgebroid of the Lie algebroid $\tau_B : B \to N$.

Corollary 3. *If the space $\Gamma(\ker \Omega_B)$ is an ideal of the Lie algebra $(\Gamma(B), [\![\cdot,\cdot]\!]_B)$ then the quotient vector bundle $\tau_{B/\ker \Omega_B} : \widetilde{B} = B/\ker \Omega_B \to N$ admits a unique Lie algebroid structure such that the canonical projection $\pi_B : B \to \widetilde{B} = B/\ker \Omega_B$ is a Lie algebroid epimorphism over the identity $Id : N \to N$ of N. Moreover, there exists a unique symplectic section $\Omega_{\widetilde{B}}$ on the Lie algebroid $\tau_{\widetilde{B}} : \widetilde{B} \to N$ which satisfies the condition*

$$\pi_B^* \Omega_{\widetilde{B}} = \Omega_B.$$

Proof. It follows using Theorem 2 (in this case, G is the trivial Lie group $G = \{e\}$). ∎

Next, we will consider the particular case when the manifold M is a single point.

Corollary 4. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a symplectic Lie algebra of finite dimension with symplectic 2-form $\Omega_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, $\Omega_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ is the restriction of $\Omega_{\mathfrak{g}}$ to $\mathfrak{h} \times \mathfrak{h}$ and $\ker \Omega_{\mathfrak{h}}$ is an ideal of the Lie algebra $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ then:*

1. *The quotient vector space $\widetilde{\mathfrak{h}} = \mathfrak{h}/\ker \Omega_{\mathfrak{h}}$ admits a unique Lie algebra structure such that the canonical projection $\pi_{\mathfrak{h}} : \mathfrak{h} \to \widetilde{\mathfrak{h}} = \mathfrak{h}/\ker \Omega_{\mathfrak{h}}$ is a Lie algebra epimorphism and*
2. *There exists a unique symplectic 2-form $\Omega_{\widetilde{\mathfrak{h}}} : \widetilde{\mathfrak{h}} \times \widetilde{\mathfrak{h}} \to \mathbb{R}$ on $\widetilde{\mathfrak{h}}$ which satisfies the condition*

$$\pi_{\mathfrak{h}}^* \Omega_{\widetilde{\mathfrak{h}}} = \Omega_{\mathfrak{h}}.$$

Proof. From Corollary 3 we deduce the result. ∎

The symplectic Lie algebra $(\widetilde{\mathfrak{h}}, \Omega_{\widetilde{\mathfrak{h}}})$ is called the *symplectic reduction of the Lie algebra* $(\mathfrak{g}, \Omega_{\mathfrak{g}})$ *by the Lie subalgebra* $\mathfrak{h}$.

This reduction process of symplectic Lie algebras plays an important role in the description of symplectic Lie groups (see, for instance, [5]).

4.3 Reduction of a symplectic Lie algebroid by a symplectic Lie subalgebroid and a symmetry Lie group

Let A be a symplectic Lie algebroid over the manifold M with symplectic section Ω_A and B be a Lie subalgebroid of A over the submanifold N of M.

Denote by $([\![\cdot,\cdot]\!]_B, \rho_B)$ the Lie algebroid structure on B and by Ω_B the presymplectic section on B given by

$$\Omega_B = i_B^* \Omega_A,$$

where $i_B : B \to A$ is the canonical inclusion.

We will assume that Ω_B is nondegenerate. Thus, Ω_B is a symplectic section on the Lie subalgebroid $\tau_B : B \to N$ and $\ker \Omega_B(x) = \{0\}$, for all $x \in N$.

Corollary 5. *Let $\Psi : G \times B \to B$ be a symplectic action of the Lie group G on the symplectic Lie algebroid (B, Ω_B) such that the space $\Gamma(B)^G$ of G-invariant sections of B,*

$$\Gamma(B)^G = \{X \in \Gamma(B) / \Psi_g \circ X = X \circ \psi_g, \ \textit{for all } g \in G\}$$

is a Lie subalgebra of $(\Gamma(B), [\![\cdot,\cdot]\!]_B)$, where $\psi : G \times N \to N$ the corresponding action on N. Then:

1. *The quotient vector bundle* $\tau_{\widetilde{B}} : \widetilde{B} = B/G \to \widetilde{N} = N/G$ *admits a unique Lie algebroid structure such that the canonical projection* $\widetilde{\pi}_B : B \to \widetilde{B}$ *is a Lie algebroid epimorphism and*

2. *There exists a unique symplectic structure* $\Omega_{\widetilde{B}}$ *on the Lie algebroid* $\tau_{\widetilde{B}} : \widetilde{B} \to \widetilde{N}$ *such that*

$$\widetilde{\pi}_B^* \Omega_{\widetilde{B}} = \Omega_B.$$

Proof. It follows using Theorem 2. ■

Remark 2. Let A be a Lie algebroid over a manifold M with Lie algebroid structure $(\llbracket\cdot,\cdot\rrbracket_A, \rho_A)$ and vector bundle projection $\tau_A : A \to M$.

Suppose that we have a principal action $\psi : G \times M \to M$ of the Lie group G on M and an action $\Psi : G \times A \to A$ of G on A such that $\tau_A : A \to M$ is G-equivariant and Ψ is a Lie algebroid action.

Then, we may consider the quotient vector bundle A/G over M/G. In fact, the canonical projection $\pi_A : A \to A/G$ is a vector bundle morphism and $(\pi_A)_{|A_x} : A_x \to (A/G)_{\pi_M(x)}$ is a linear isomorphism, for all $x \in M$. As we know, the space $\Gamma(A/G)$ of sections of the vector bundle $\tau_{A/G} : A/G \to M/G$ may be identified with the set $\Gamma(A)^G$ of G-invariant sections of A, that is,

$$\Gamma(A/G) \cong \Gamma(A)^G = \{X \in \Gamma(A) / \Psi_g \circ X = X \circ \psi_g, \ \forall\, g \in G\}.$$

Now, if $g \in G$ and $X, Y \in \Gamma(A)^G$ then, since Ψ_g is a Lie algebroid morphism, we deduce that

$$\Psi_g \circ \llbracket X, Y\rrbracket_A \circ \psi_{g^{-1}} = \llbracket \Psi_g \circ X \circ \psi_{g^{-1}}, \Psi_g \circ Y \circ \psi_{g^{-1}}\rrbracket_A = \llbracket X, Y\rrbracket_A,$$

i.e., $\llbracket X, Y\rrbracket_A \in \Gamma(A)^G$. Thus, $\Gamma(A)^G$ is a Lie subalgebra of $(\Gamma(A), \llbracket\cdot,\cdot\rrbracket_A)$ and therefore, using Theorem A.1 (see Appendix), it follows that the quotient vector bundle A/G admits a unique Lie algebroid structure such that π_A is a Lie algebroid morphism.

Next, we will apply Corollary 5 to the following example.

Example 3. Let A be a Lie algebroid over a manifold M and G be a Lie group as in Remark 2. Note that the action Ψ of G on A is principal and it induces an action Ψ^* of G on A^* which is also principal. In fact, $\Psi^*_g = \Psi^t_{g^{-1}}$, where $\Psi^t_{g^{-1}}$ is the dual map of $\Psi_{g^{-1}} : A \to A$. The space A^*/G of orbits of this action is a quotient vector bundle over M/G which is isomorphic to the dual vector bundle to $\tau_{A/G} : A/G \to M/G$. Moreover, the canonical projection $\pi_{A^*} : A^* \to A^*/G$ is a vector bundle morphism and $(\pi_{A^*})_{|A^*_x} : A^*_x \to (A^*/G)_{\pi_M(x)}$ is a linear isomorphism, for all $x \in M$.

Now, we consider the action $\mathcal{T}^*\Psi$ of G on the A-tangent bundle to A^*, $\mathcal{T}^A A^*$, given by

$$\mathcal{T}^*\Psi(g, (a_x, X_{\alpha_x})) = (\Psi_g(a_x), (T_{\alpha_x}\Psi^t_{g^{-1}})(X_{\alpha_x})),$$

for $g \in G$ and $(a_x, X_{\alpha_x}) \in \mathcal{T}^A_{\alpha_x} A^*$. Since Ψ_g and $T\Psi^t_{g^{-1}}$ are Lie algebroid isomorphisms, we have that $\mathcal{T}^*_g\Psi$ is also a Lie algebroid isomorphism. Therefore, from Remark 2, we have that the space

$$\Gamma(\mathcal{T}^A A^*)^G = \{\widetilde{X} \in \Gamma(\mathcal{T}^A A^*) / \mathcal{T}^*_g\Psi \circ \widetilde{X} = \widetilde{X} \circ \Psi^t_{g^{-1}}, \ \forall\, g \in G\}$$

of G-invariant sections of $\tau_{\mathcal{T}^A A^*} : \mathcal{T}^A A^* \to A^*$ is a Lie subalgebra of the Lie algebra $(\Gamma(\mathcal{T}^A A^*), \llbracket\cdot,\cdot\rrbracket_{\mathcal{T}^A A^*})$.

In addition, if $\Theta_{\mathcal{T}^A A^*} \in \Gamma((\mathcal{T}^A A^*)^*)$ is the Liouville section associated with A, it is easy to prove that

$$(\mathcal{T}^*_g \Psi)^*(\Theta_{\mathcal{T}^A A^*}) = \Theta_{\mathcal{T}^A A^*}, \qquad \text{for all } \ g \in G.$$

Thus, if $\Omega_{\mathcal{T}^A A^*} \in \Gamma(\wedge^2(\mathcal{T}^A A^*)^*)$ is the canonical symplectic section associated with A, it follows that

$$(\mathcal{T}^*_g \Psi)^*(\Omega_{\mathcal{T}^A A^*}) = \Omega_{\mathcal{T}^A A^*}, \qquad \text{for all } \ g \in G.$$

This implies that $\mathcal{T}^*\Psi$ is a symplectic action of G on $\mathcal{T}^A A^*$. Consequently, using Corollary 5, we deduce that the quotient vector bundle $\tau_{\mathcal{T}^A A^*/G} : \mathcal{T}^A A^*/G \to A^*/G$ admits a unique Lie algebroid structure such that the canonical projection $\pi_{\mathcal{T}^A A^*} : \mathcal{T}^A A^* \to \mathcal{T}^A A^*/G$ is a Lie algebroid epimorphism. Moreover, there exists a unique symplectic structure $\Omega_{(\mathcal{T}^A A^*)/G}$ on the Lie algebroid $\tau_{(\mathcal{T}^A A^*)/G} : (\mathcal{T}^A A^*)/G \to A^*/G$ such that

$$\pi^*_{\mathcal{T}^A A^*}(\Omega_{(\mathcal{T}^A A^*)/G}) = \Omega_{\mathcal{T}^A A^*}.$$

Next, we will prove that the symplectic Lie algebroid $(\mathcal{T}^A A^*/G, \Omega_{(\mathcal{T}^A A^*)/G})$ is isomorphic to the A/G-tangent bundle to A^*/G. For this purpose, we will introduce the map $(\pi_A, T\pi_{A^*}) : \mathcal{T}^A A^* \to \mathcal{T}^{A/G}(A^*/G)$ defined by

$$(\pi_A, T\pi_{A^*})(a_x, X_{\alpha_x}) = (\pi_A(a_x), (T_{\alpha_x}\pi_{A^*})(X_{\alpha_x})), \qquad \text{for } \ (a_x, X_{\alpha_x}) \in \mathcal{T}^A_{\alpha_x} A^*.$$

We have that $(\pi_A, T\pi_{A^*})$ is a vector bundle morphism. Furthermore, if $(a_x, X_{\alpha_x}) \in \mathcal{T}^A_{\alpha_x} A^*$ and

$$0 = (\pi_A, T\pi_{A^*})(a_x, X_{\alpha_x})$$

then $a_x = 0$ and, therefore, $X_{\alpha_x} \in T_{\alpha_x} A^*_x$ and $(T_{\alpha_x}\pi_{A^*})(X_{\alpha_x}) = 0$. But, as $(\pi_{A^*})_{|A^*_x} : A^*_x \to (A^*/G)_{\pi_M(x)}$ is a linear isomorphism, we conclude that $X_{\alpha_x} = 0$. Thus, $(\pi_A, T\pi_{A^*})_{|\mathcal{T}^A_{\alpha_x} A^*} : \mathcal{T}^A_{\alpha_x} A^* \to \mathcal{T}^{A/G}_{\pi_{A^*}(\alpha_x)}(A^*/G)$ is a linear isomorphism (note that $\dim \mathcal{T}^A_{\alpha_x} A^* = \dim \mathcal{T}^{A/G}_{\pi_{A^*}(\alpha)}(A^*/G)$).

On the other hand, using that π_A (respectively, $T\pi_{A^*}$) is a Lie algebroid morphism between the Lie algebroids $\tau_A : A \to M$ and $\tau_{A/G} : A/G \to M/G$ (respectively, $\tau_{TA^*} : TA^* \to A^*$ and $\tau_{T(A^*/G)} : T(A^*/G) \to A^*/G$) we deduce that $(\pi_A, T\pi_{A^*})$ is also a Lie algebroid morphism M.

Moreover, if $\Theta_{\mathcal{T}^{A/G}(A^*/G)} \in \Gamma(\mathcal{T}^{A/G}(A^*/G))^*$ is the Liouville section associated with the Lie algebroid $\tau_{A/G} : A/G \to M/G$, we have that

$$(\pi_A, T\pi_{A^*})^*(\Theta_{\mathcal{T}^{A/G}(A^*/G)}) = \Theta_{\mathcal{T}^A A^*}$$

which implies that

$$(\pi_A, T\pi_{A^*})^*(\Omega_{\mathcal{T}^{A/G}(A^*/G)}) = \Omega_{\mathcal{T}^A A^*},$$

where $\Omega_{\mathcal{T}^{A/G}(A^*/G)}$ is the canonical symplectic section associated with the Lie algebroid $\tau_{A/G} : A/G \to M/G$.

Now, since $\pi_A \circ \Psi_g = \pi_A$ and $\pi_{A^*} \circ \Psi^*_g = \pi_{A^*}$ for all $g \in G$, we deduce that the map $(\pi_A, T\pi_{A^*}) : \mathcal{T}^A A^* \to \mathcal{T}^{A/G}(A^*/G)$ induces a map $\widetilde{(\pi_A, T\pi_{A^*})} : (\mathcal{T}^A A^*)/G \to \mathcal{T}^{A/G}(A^*/G)$ such that

$$\widetilde{(\pi_A, T\pi_{A^*})} \circ \pi_{\mathcal{T}^A A^*} = (\pi_A, T\pi_{A^*}).$$

Finally, using the above results, we obtain that $\widetilde{(\pi_A, T\pi_{A^*})}$ is a Lie algebroid isomorphism and, in addition,

$$\widetilde{(\pi_A, T\pi_{A^*})}^* \Omega_{\mathcal{T}^{A/G}(A^*/G)} = \Omega_{(\mathcal{T}^A A^*)/G}.$$

In conclusion, we have proved that the symplectic Lie algebroids $((\mathcal{T}^A A^*)/G, \Omega_{(\mathcal{T}^A A^*)/G})$ and $(\mathcal{T}^{A/G}(A^*/G), \Omega_{\mathcal{T}^{A/G}(A^*/G)})$ are isomorphic.

Next, we will prove the following result.

Corollary 6. *Under the same hypotheses as in Corollary 3, if the submanifold N is closed and $H_M : M \to \mathbb{R}$ is a Hamiltonian function for the symplectic Lie algebroid (A, Ω_A) such that the restriction H_N of H_M is G-invariant then:*

1. *H_N induces a real function $H_{\widetilde{N}} : \widetilde{N} = N/G \to \mathbb{R}$ such that $H_{\widetilde{N}} \circ \pi_N = H_N$;*
2. *The Hamiltonian section $\mathcal{H}_{H_M}^{\Omega_A}$ of H_M in the symplectic Lie algebroid (A, Ω_A) satisfies the condition*

   $$\mathcal{H}_{H_M}^{\Omega_A}(N) \subseteq B.$$

   Moreover, $(\mathcal{H}_{H_M}^{\Omega_A})_{|N}$ is $\widetilde{\pi}_B$-projectable on the Hamiltonian section of the function $H_{\widetilde{N}}$ with respect to the reduced symplectic section $\Omega_{\widetilde{B}}$ and
3. *If $\gamma : I \to M$ is a solution of the Hamilton equations for H_M in the symplectic Lie algebroid (A, Ω_A) such that $\gamma(t_0) \in N$ for some $t_0 \in I$, then $\gamma(I) \subseteq N$ and $\pi_N \circ \gamma : I \to \widetilde{N}$ is a solution of the Hamilton equations for $H_{\widetilde{N}}$ in the symplectic Lie algebroid $(\widetilde{B}, \Omega_{\widetilde{B}})$.*

Proof. Let $\mathcal{H}_{H_N}^{\Omega_B}$ be the Hamiltonian section of the Hamiltonian function $H_N : N \to \mathbb{R}$ in the symplectic Lie algebroid (B, Ω_B). Then, we have that $(\mathcal{H}_{H_M}^{\Omega_A})_{|N} = \mathcal{H}_N^{\Omega_B}$ and, thus, $\mathcal{H}_{H_M}^{\Omega_A}(N) \subseteq B$. Therefore, using Theorem 3, we deduce the result. ■

Next, we will apply the above results in order to give a "symplectic description" of the Hamiltonian reduction process by stages in the Poisson setting.

This reduction process may be described as follows (see, for instance, [16, 20]).

Let Q be the total space of a principal G_1-bundle and the configuration space of an standard Hamiltonian system with Hamiltonian function $H : T^*Q \to \mathbb{R}$. We will assume that H is G_1-invariant. Then, the quotient manifold T^*Q/G_1 is a Poisson manifold (see Section 2.2) and if $\pi^1_{T^*Q} : T^*Q \to T^*Q/G_1$ is the canonical projection, we have that the Hamiltonian system (T^*Q, H) is $\pi^1_{T^*Q}$-projectable over a Hamiltonian system on T^*Q/G_1. This means that $\pi^1_{T^*Q} : T^*Q \to T^*Q/G_1$ is a Poisson morphism (see Section 2.2) and that there exists a real function $\widetilde{H}_1 : T^*Q/G_1 \to \mathbb{R}$ such that $\widetilde{H}_1 \circ \pi^1_{T^*Q} = H$. Thus, if $\gamma : I \to T^*Q$ is a solution of the Hamilton equations for H in T^*Q then $\pi^1_{T^*Q} \circ \gamma : I \to T^*Q/G_1$ is a solution of the Hamilton equations for $\widetilde{H}_1$ in T^*Q/G_1.

Now, suppose that G_2 is a closed normal subgroup of G_1 such that the action of G_2 on Q induces a principal G_2-bundle. Then, it is clear that the construction of the reduced Hamiltonian system $(T^*Q/G_1, \widetilde{H}_1)$ can be carried out in two steps.

First step: The Hamiltonian function H is G_2-invariant and, therefore, the Hamiltonian system (T^*Q, H) may be reduced to a Hamiltonian system in the Poisson manifold T^*Q/G_2 with Hamiltonian function $\widetilde{H}_2 : T^*Q/G_2 \to \mathbb{R}$.

Second step: The cotangent lift of the action of G_1 on Q induces a Poisson action of the quotient Lie group G_1/G_2 on the Poisson manifold T^*Q/G_2 and the space $(T^*Q/G_2)/(G_1/G_2)$ of orbits of this action is isomorphic to the full reduced space T^*Q/G_1. Moreover, the function $\widetilde{H}_2$ is (G_1/G_2)-invariant and, consequently, it induces a Hamiltonian function $\widetilde{\widetilde{H}}_2$ on the space $(T^*Q/G_2)/(G_1/G_2)$. Under the identification between $(T^*Q/G_2)/(G_1/G_2)$ and T^*Q/G_1, $\widetilde{\widetilde{H}}_2$ is just the Hamiltonian function $\widetilde{H}_1$. In conclusion, in this second step, we see that the Hamiltonian system $(T^*Q/G_2, \widetilde{H}_2)$ may be reduced to the Hamiltonian system $(T^*Q/G_1, \widetilde{H}_1)$.

Next, we will give a "symplectic description" of the above reduction process.

For this purpose, we will consider the Atiyah algebroids $\tau_{TQ/G_2} : TQ/G_2 \to \widetilde{Q}_2 = Q/G_2$ and $\tau_{TQ/G_1} : TQ/G_1 \to \widetilde{Q}_1 = Q/G_1$ associated with the principal bundles $\pi_Q^2 : Q \to \widetilde{Q}_2 = Q/G_2$ and $\pi_Q^1 : Q \to \widetilde{Q}_1 = Q/G_1$. We also consider the corresponding dual vector bundles $\tau_{T^*Q/G_2} : T^*Q/G_2 \to \widetilde{Q}_2 = Q/G_2$ and $\tau_{T^*Q/G_1} : T^*Q/G_1 \to \widetilde{Q}_1 = Q/G_1$.

Then, the original Hamiltonian system (T^*Q, H) may be considered as a Hamiltonian system, with Hamiltonian function H, in the standard symplectic Lie algebroid

$$\tau_{T(T^*Q)} : T(T^*Q) \to T^*Q.$$

Note that this symplectic Lie algebroid is isomorphic to the TQ-tangent bundle to T^*Q.

On the other hand, the reduced Poisson Hamiltonian system $(T^*Q/G_1, \widetilde{H}_1)$ may be considered as a Hamiltonian system, with Hamiltonian function $\widetilde{H}_1 : T^*Q/G_1 \to \mathbb{R}$ in the symplectic Lie algebroid

$$\tau_{\mathcal{T}^{TQ/G_1}(T^*Q/G_1)} : \mathcal{T}^{TQ/G_1}(T^*Q/G_1) \to T^*Q/G_1.$$

Now, in order to give the symplectic description of the two steps of the reduction process, we proceed as follows:

First step: The action of G_2 on Q induces a symplectic action of G_2 on the symplectic Lie algebroid $\tau_{\mathcal{T}^{TQ}(T^*Q)} : \mathcal{T}^{TQ}(T^*Q) \cong T(T^*Q) \to T^*Q$. Furthermore (see Example 3), the symplectic reduction of this Lie algebroid by G_2 is just the TQ/G_2-tangent bundle to T^*Q/G_2, that is, $\mathcal{T}^{TQ/G_2}(T^*Q/G_2)$ (note that the standard Lie bracket of two G_2-invariant vector fields on Q is again a G_2-invariant vector field). Therefore, the Hamiltonian system, in the symplectic Lie algebroid $\tau_{T(T^*Q)} : T(T^*Q) \to T^*Q$, with Hamiltonian function $H : T^*Q \to \mathbb{R}$ may be reduced to a Hamiltonian system, in the symplectic Lie algebroid $\tau_{\mathcal{T}^{TQ/G_2}(T^*Q/G_2)} : \mathcal{T}^{TQ/G_2}(T^*Q/G_2) \to T^*Q/G_2$, with Hamiltonian function $\widetilde{H}_2 : T^*Q/G_2 \to \mathbb{R}$.

Second step: The action of the Lie group G_1 on Q induces a Lie algebroid action of the quotient Lie group G_1/G_2 on the Atiyah algebroid $\tau_{TQ/G_2} : TQ/G_2 \to \widetilde{Q}_2 = Q/G_2$. Thus (see Remark 2), the quotient vector bundle $(TQ/G_2)/(G_1/G_2) \to (Q/G_2)/(G_1/G_2)$ admits a quotient Lie algebroid structure. If follows that this Lie algebroid is isomorphic to the Atiyah algebroid $\tau_{TQ/G_1} : TQ/G_1 \to \widetilde{Q}_1 = Q/G_1$. On the other hand, the Lie algebroid action of G_1/G_2 on the Atiyah algebroid $\tau_{TQ/G_2} : TQ/G_2 \to \widetilde{Q}_2 = Q/G_2$ induces a symplectic action of G_1/G_2 on the symplectic Lie algebroid $\mathcal{T}^{TQ/G_2}(T^*Q/G_2) \to T^*Q/G_2$ which is also a Lie algebroid action (see Example 3). Moreover (see again Example 3), the symplectic reduction of $\tau_{\mathcal{T}^{TQ/G_2}(T^*Q/G_2)} : \mathcal{T}^{TQ/G_2}(T^*Q/G_2) \to T^*Q/G_2$ by G_1/G_2 is the $(TQ/G_2)/(G_1/G_2)$-tangent bundle to $(T^*Q/G_2)/(G_1/G_2)$, that is, the symplectic Lie algebroid $\tau_{\mathcal{T}^{TQ/G_1}(T^*Q/G_1)} : \mathcal{T}^{TQ/G_1}(T^*Q/G_1) \to T^*Q/G_1$. Therefore, the Hamiltonian system in the symplectic Lie algebroid $\tau_{\mathcal{T}^{TQ/G_2}(T^*Q/G_2)} : \mathcal{T}^{TQ/G_2}(T^*Q/G_2) \to T^*Q/G_2$, with Hamiltonian function $\widetilde{H}_2 : T^*Q/G_2 \to \mathbb{R}$, may be reduced to the Hamiltonian system in the symplectic Lie algebroid $\tau_{\mathcal{T}^{TQ/G_1}(T^*Q/G_1)} : \mathcal{T}^{TQ/G_1}(T^*Q/G_1) \to T^*Q/G_1$, with Hamiltonian function $\widetilde{H}_1 : T^*Q/G_1 \to \mathbb{R}$.

4.4 A Lagrange top

In Sections 4.2 and 4.3, we discussed the reduction of a symplectic Lie algebroid A by a Lie subalgebroid B and a symmetry Lie group G for the particular case when G is the trivial group and for the particular case when B is symplectic, respectively.

Next, we will consider an example such that the Lie subalgebroid B is not symplectic and the symmetry Lie group G is not trivial: a Lagrange top.

A Lagrange top is a classical mechanical system which consists of a symmetric rigid body with a fixed point moving in a gravitational field (see, for instance, [4, 15]).

A Lagrange top may be described as a Hamiltonian system on the $\widetilde{A}$-tangent bundle to $\widetilde{A}^*$, where $\widetilde{A}$ is an action Lie algebroid over the sphere S^2 in $\mathbb{R}^3$

$$S^2 = \{\vec{x} \in \mathbb{R}^3 / \|\vec{x}\| = 1\}.$$

The Lie algebroid structure on $\widetilde{A}$ is defined as follows.

Let $SO(3)$ be the special orthogonal group of order 3. As we know, the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ may identified with $\mathbb{R}^3$ and, under this identification, the Lie bracket $[\cdot,\cdot]_{\mathfrak{so}(3)}$ is just the cross product on $\mathbb{R}^3$.

We have the standard left action of $SO(3)$ on S^2 and the corresponding infinitesimal left action $\Phi : \mathfrak{so}(3) \cong \mathbb{R}^3 \to \mathfrak{X}(S^2)$ given by

$$\Phi(\xi)(\vec{x}) = \xi \times \vec{x}, \qquad \text{for } \ \xi \in \mathfrak{so}(3) \cong \mathbb{R}^3 \ \text{ and } \ \vec{x} \in S^2.$$

$\widetilde{A}$ is the Lie algebroid over S^2 associated with the infinitesimal left action Φ. Thus, $\widetilde{A} = S^2 \times \mathfrak{so}(3) \cong S^2 \times \mathbb{R}^3$ and if $[\![\cdot,\cdot]\!]_{\widetilde{A}}$ is the Lie bracket on the space $\Gamma(\widetilde{A})$ we can choose a global basis $\{e_1, e_2, e_3\}$ of $\Gamma(\widetilde{A})$ such that

$$[\![e_1, e_2]\!]_{\widetilde{A}} = e_3, \qquad [\![e_3, e_1]\!]_{\widetilde{A}} = e_2, \qquad [\![e_2, e_3]\!]_{\widetilde{A}} = e_1.$$

The dual vector bundle $\widetilde{A}^*$ to $\widetilde{A}$ is the trivial vector bundle $\tau_{\widetilde{A}^*} : \widetilde{A}^* \cong S^2 \times \mathbb{R}^3 \to S^2$ and the Hamiltonian function $\widetilde{H} : \widetilde{A}^* \cong S^2 \times \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$\widetilde{H}(\vec{x}, \pi) = \frac{1}{2}\left(\frac{\pi_1^2}{I} + \frac{\pi_2^2}{I} + \frac{\pi_3^2}{J}\right) + (mgl)z,$$

for $\vec{x} = (x, y, z) \in S^2$ and $\pi = (\pi_1, \pi_2, \pi_3) \in \mathbb{R}^3$, where (I, I, J) are the (positive) eigenvalues of the inertia tensor, m is the mass, g is the gravity and l is the distance from the fixed point of the body to the centre of mass.

The $\widetilde{A}$-tangent bundle to $\widetilde{A}^*$, $\mathcal{T}^{\widetilde{A}}\widetilde{A}^*$, is isomorphic to the trivial vector bundle over $\widetilde{A}^* \cong S^2 \times \mathbb{R}^3$

$$\tau_{\mathcal{T}^{\tilde{A}}\widetilde{A}^*} : \mathcal{T}^{\widetilde{A}}\widetilde{A}^* \cong (S^2 \times \mathbb{R}^3) \times (\mathbb{R}^3 \times \mathbb{R}^3) \to \widetilde{A}^* \cong S^2 \times \mathbb{R}^3.$$

Under the canonical identification $T\widetilde{A}^* \cong TS^2 \times (\mathbb{R}^3 \times \mathbb{R}^3)$, the anchor map $\rho_{\mathcal{T}^{\tilde{A}}\widetilde{A}^*} : \mathcal{T}^{\widetilde{A}}\widetilde{A}^* \to T\widetilde{A}^*$ of $\mathcal{T}^{\widetilde{A}}\widetilde{A}^*$ is given by

$$\rho_{\mathcal{T}^{\tilde{A}}\widetilde{A}^*}((\vec{x}, \pi), (\xi, \alpha)) = ((\vec{x}, -\xi \times \vec{x}), (\pi, \alpha))$$

for $((\vec{x}, \pi), (\xi, \alpha)) \in \mathcal{T}^{\widetilde{A}}\widetilde{A}^* \cong (S^2 \times \mathbb{R}^3) \times (\mathbb{R}^3 \times \mathbb{R}^3)$. Moreover, we may choose, in a natural way, a global basis $\{\widetilde{e}_i, \widetilde{f}_i\}_{i=1,2,3}$ of $\Gamma(\mathcal{T}^{\widetilde{A}}\widetilde{A}^*)$ such that

$$[\![\widetilde{e}_1, \widetilde{e}_2]\!]_{\mathcal{T}^{\tilde{A}}\widetilde{A}^*} = \widetilde{e}_3, \qquad [\![\widetilde{e}_3, \widetilde{e}_1]\!]_{\mathcal{T}^{\tilde{A}}\widetilde{A}^*} = \widetilde{e}_2, \qquad [\![\widetilde{e}_2, \widetilde{e}_3]\!]_{\mathcal{T}^{\tilde{A}}\widetilde{A}^*} = \widetilde{e}_1,$$

and the rest of the fundamental Lie brackets are zero.

The symplectic section $\Omega_{\mathcal{T}^{\tilde{A}}\widetilde{A}^*}$ is given by

$$\Omega_{\mathcal{T}^{\tilde{A}}\widetilde{A}^*}(\vec{x}, \pi)((\xi, \alpha), (\xi', \alpha')) = \alpha'(\xi) - \alpha(\xi') + \pi(\xi \times \xi'),$$

for $(\vec{x}, \pi) \in \widetilde{A}^* \cong S^2 \times \mathbb{R}^3$ and $(\xi, \alpha), (\xi', \alpha') \in \mathbb{R}^3 \times \mathbb{R}^3$.

Now, we consider the Lie subgroup G of $SO(3)$ of rotations about the z-axis. The elements of G are of the form

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{with } \theta \in \mathbb{R}.$$

It is clear that G is isomorphic to $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$.

Note that the standard action of $G \cong S^1$ on S^2 is not free (the north and the south poles are fixed points). Therefore, we must restrict this action to the open subset M of S^2

$$M = S^2 - \{(0,0,1),(0,0,-1)\}.$$

We have that the map $\mu : S^1 \times \mathbb{R} \cong \mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R} \to M$ given by

$$\mu([\theta],t) = \left(\frac{\cos\theta}{\cosh t}, \frac{\sin\theta}{\cosh t}, \tanh t\right), \qquad \text{for } ([\theta],t) \in S^1 \times \mathbb{R}$$

is a diffeomorphism. Under this identification between M and $S^1 \times \mathbb{R}$, the action ψ of $G \cong S^1$ on $M \cong S^1 \times \mathbb{R}$ is defined by

$$\psi([\theta'],([\theta],t)) = ([\theta+\theta'],t), \qquad \text{for } [\theta'] \in S^1 \text{ and } ([\theta],t) \in S^1 \times \mathbb{R}.$$

Next, we consider the open subset of $\widetilde{A}$

$$A = \tau_{\widetilde{A}}^{-1}(M) \cong (S^1 \times \mathbb{R}) \times \mathbb{R}^3.$$

It is clear that A is an action Lie algebroid over $M \cong S^1 \times \mathbb{R}$ and the dual bundle to A is the trivial vector bundle $\tau_{A^*} : A^* \cong (S^1 \times \mathbb{R}) \times \mathbb{R}^3 \to S^1 \times \mathbb{R}$. The restriction of $\widetilde{H}$ to A^* is the Hamiltonian function $H : A^* \cong (S^1 \times \mathbb{R}) \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$H(([\theta],t),\pi) = \frac{1}{2}\left(\frac{\pi_1^2}{I} + \frac{\pi_2^2}{I} + \frac{\pi_3^2}{J}\right) + (mgl)\tanh t.$$

On the other hand, the action of $G \cong S^1$ on M and the standard action of G on $\mathbb{R}^3$ induce an action Ψ of G on A in such a way Ψ is a Lie algebroid action of G on A. Thus, we may consider the corresponding symplectic action $\mathcal{T}^*\Psi$ (see Example 3) of G on the symplectic Lie algebroid

$$\tau_{\mathcal{T}^A A^*} : \mathcal{T}^A A^* \cong (M \times \mathbb{R}^3) \times (\mathbb{R}^3 \times \mathbb{R}^3) \to M \times \mathbb{R}^3.$$

Note that A^* is isomorphic to the open subset $\tau_{\widetilde{A}^*}^{-1}(M)$ of $\widetilde{A}^*$ and that

$$\mathcal{T}^A A^* \cong \tau_{\mathcal{T}^{\widetilde{A}}\widetilde{A}^*}^{-1}(\tau_{\widetilde{A}^*}^{-1}(M)).$$

So, the basis $\{\widetilde{e}_i, \widetilde{f}_i\}_{i=1,2,3}$ of $\Gamma(\mathcal{T}^{\widetilde{A}}\widetilde{A}^*)$ induces a basis of $\Gamma(\mathcal{T}^A A^*)$ which we will denote by $\{\bar{e}_i, \bar{f}_i\}_{i=1,2,3}$. The Hamiltonian section $\mathcal{H}_H^{\Omega_{\mathcal{T}^A A^*}}$ of H in the symplectic Lie algebroid $(\mathcal{T}^A A^*, \Omega_{\mathcal{T}^A A^*})$ is given by

$$\begin{aligned}\mathcal{H}_H^{\Omega_{\mathcal{T}^A A^*}}(([\theta],t),\pi) = \frac{\pi_1}{I}\bar{e}_1 + \frac{\pi_2}{I}\bar{e}_2 + \frac{\pi_3}{J}\bar{e}_3 &+ \left(\frac{(I-J)\pi_2\pi_3}{IJ} - \frac{mgl}{(\cosh t)}\sin\theta\right)\bar{f}_1 \\ &- \left(\frac{(I-J)\pi_1\pi_3}{IJ} - \frac{mgl}{(\cosh t)}\cos\theta\right)\bar{f}_2\end{aligned}$$

for $(([\theta],t),\pi) \in A^* \cong (S^1 \times \mathbb{R}) \times \mathbb{R}^3$.

Now, we consider the submanifold N of $A^* \cong M \times \mathbb{R}^3$

$$N = \{(([\theta], t), \pi) \in A^*/\pi_3 = 0\}$$

and the Lie subalgebroid B (over N) of $\mathcal{T}^A A^*$

$$B = \{((([\theta], t), \pi), (\xi, \alpha)) \in \mathcal{T}^A A^*/\pi_3 = \alpha_3 = 0\}.$$

If Ω_B is the restriction of $\Omega_{\mathcal{T}^A A^*}$ to the Lie subalgebroid B then

$$\dim(\ker \Omega_B(([\theta], t), \pi)) = 1, \qquad \text{for all } (([\theta], t), \pi) \in N$$

and the section s of $\tau_B : B \to N$ defined by

$$s = (\bar{e}_3 + \pi_2 \bar{f}_1 - \pi_1 \bar{f}_2)_{|N}$$

is a global basis of $\Gamma(\ker \Omega_B)$. Moreover, the restriction of the symplectic action $\mathcal{T}^*\Psi$ to the Lie algebroid $\tau_B : B \to N$ is a presymplectic action. In fact, the action of G on N is given by

$$([\theta'], (([\theta], t), \pi)) \to (([\theta + \theta'], t), A_{\theta'}\pi),$$

for $[\theta'] \in G \cong S^1$ and $(([\theta], t), \pi) \in N$, and the action of G on B is

$$([\theta'], ((([\theta], t), \pi), (\xi, \alpha))) \to ((([\theta + \theta'], t), A_{\theta'}\pi), (A_{\theta'}\xi, A_{\theta'}\alpha)).$$

Thus, using Theorem 1, we may consider the reduced vector bundle $\tau_{\widetilde{B}} : \widetilde{B} = (B/\ker \Omega_B)/G \to \widetilde{N} = N/G$. Furthermore, if $\widetilde{\pi}_B : B \to \widetilde{B}$ and $\pi_N : N \to \widetilde{N} = N/G$ are the canonical projections then a basis of the space $\Gamma(B)^p_{\widetilde{\pi}_B}$ of $\widetilde{\pi}_B$-projectable sections of $\tau_B : B \to N$ is $\{e'_1, e'_2, s, f'_1, f'_2\}$, where

$$\begin{aligned} e'_1 &= (\cos\theta)\bar{e}_{1|N} + (\sin\theta)\bar{e}_{2|N}, \qquad & e'_2 &= -(\sin\theta)\bar{e}_{1|N} + \cos(\theta)\bar{e}_{2|N}, \\ f'_1 &= (\cos\theta)\bar{f}_{1|N} + (\sin\theta)\bar{f}_{2|N}, \qquad & f'_2 &= -(\sin\theta)\bar{f}_{1|N} + (\cos\theta)\bar{f}_{2|N}. \end{aligned}$$

We have that

$$\begin{aligned} &[\![e'_1, e'_2]\!]_B = (\sinh t)e'_1 + s - (\pi_2\cos\theta - \pi_1\sin\theta)f'_1 + (\pi_1\cos\theta + \pi_2\sin\theta)f'_2, \\ &[\![e'_1, f'_1]\!]_B = -(\sinh t)f'_2, \qquad [\![e'_1, f'_2]\!]_B = (\sinh t)f'_1, \end{aligned}$$

and the rest of the Lie brackets between the elements of the basis $\{e'_1, e'_2, s, f'_1, f'_2\}$ are zero.

Note that the vertical bundle to π_N is generated by the vector field on N

$$\frac{\partial}{\partial\theta} - \pi_2\frac{\partial}{\partial\pi_1} + \pi_1\frac{\partial}{\partial\pi_2}$$

and therefore, $[\![e'_1, e'_2]\!]_B,\ [\![e'_1, f'_1]\!]_B,\ [\![e'_1, f'_2]\!]_B \in \Gamma(B)^p_{\widetilde{\pi}_B}$.

Consequently, $\Gamma(B)^p_{\widetilde{\pi}_B}$ is a Lie subalgebra of $(\Gamma(B), [\![\cdot,\cdot]\!]_B)$ and $\Gamma(\ker \Omega_B)$ is an ideal of $\Gamma(B)^p_{\widetilde{\pi}_B}$. This implies that the vector bundle $\tau_{\widetilde{B}} : \widetilde{B} \to \widetilde{N}$ admits a unique Lie algebroid structure $([\![\cdot,\cdot]\!]_{\widetilde{B}}, \rho_{\widetilde{B}})$ such that $\widetilde{\pi}_B$ is a Lie algebroid epimorphism. In addition, Ω_B induces a symplectic section $\Omega_{\widetilde{B}}$ on the Lie algebroid $\tau_{\widetilde{B}} : \widetilde{B} \to \widetilde{N}$ such that $\widetilde{\pi}_B^*\Omega_{\widetilde{B}} = \Omega_B$ (see Theorem 2).

On the other hand, using Theorem 3, we deduce that the original Hamiltonian system, with Hamiltonian function $H : A^* \to \mathbb{R}$ in the symplectic Lie algebroid $\tau_{\mathcal{T}^A A^*} : \mathcal{T}^A A^* \to A^*$, may be reduced to a Hamiltonian system, with Hamiltonian function $\widetilde{H} : \widetilde{N} \to \mathbb{R}$, in the symplectic Lie algebroid $\tau_{\widetilde{B}} : \widetilde{B} \to \widetilde{N}$.

Next, we will give an explicit description of the reduced symplectic Lie algebroid and the reduced Hamiltonian system on it.

In fact, one may prove that the Lie algebroid $\tau_{\widetilde{B}} : \widetilde{B} \to \widetilde{N}$ is isomorphic to the trivial vector bundle $\tau_{\mathbb{R}^3\times\mathbb{R}^4} : \mathbb{R}^3 \times \mathbb{R}^4 \to \mathbb{R}^3$ with basis $\mathbb{R}^3$ and fiber $\mathbb{R}^4$ and, under this identification, the anchor map $\rho_{\widetilde{B}}$ is given by

$$\rho_{\widetilde{B}}((t,v),(\eta,\beta)) = -\eta_2(\cosh t)\frac{\partial}{\partial t} + (\beta_1 - \eta_1\nu_2 \sinh t)\frac{\partial}{\partial \nu_1} + (\beta_2 + \eta_1\nu_1 \sinh t)\frac{\partial}{\partial \nu_2},$$

for $(t, \nu \equiv (\nu_1,\nu_2)) \in \mathbb{R}\times\mathbb{R}^2 \cong \mathbb{R}^3$ and $(\eta \equiv (\eta_1,\eta_2), \beta \equiv (\beta_1,\beta_2)) \in \mathbb{R}^2\times\mathbb{R}^2 \cong \mathbb{R}^4$.

Moreover, if $\{e_i\}_{i=1,\dots,4}$ is the canonical basis of $\Gamma(\widetilde{B}) \cong \Gamma(\mathbb{R}^3\times\mathbb{R}^4)$ we have that

$$\begin{aligned}
&[\![e_1,e_2]\!]_{\widetilde{B}} = (\sinh t)e_1 - \nu_2 e_3 + \nu_1 e_4,\\
&[\![e_1,e_3]\!]_{\widetilde{B}} = -(\sinh t)e_4, \qquad [\![e_1,e_4]\!]_{\widetilde{B}} = (\sinh t)e_3
\end{aligned}$$

and the rest of the Lie brackets between the elements of the basis is zero.

Finally, one may prove that the reduced symplectic section $\Omega_{\widetilde{B}}$, the reduced Hamiltonian function $\widetilde{H}$ and the corresponding reduced Hamiltonian vector field $\mathcal{H}_H^{\{\cdot,\cdot\}_{\widetilde{N}}}$ on $\widetilde{N} \cong \mathbb{R}^3$ are given by the following expressions

$$\begin{aligned}
&\Omega_{\widetilde{B}}(t,\nu)((\eta,\beta),(\eta',\beta')) = \beta'(\eta) - \beta(\eta'),\\
&\widetilde{H}(t,\nu) = \frac{1}{2}\left(\frac{\nu_1^2}{I} + \frac{\nu_2^2}{I}\right) + mgl(\tanh t),\\
&\mathcal{H}_{\widetilde{H}}^{\{\cdot,\cdot\}_{\widetilde{N}}} = -\frac{\nu_2}{I}\cosh t\frac{\partial}{\partial t} - \frac{\nu_1\nu_2}{I}\sinh t\frac{\partial}{\partial \nu_1} + \left(\frac{mgl}{\cosh t} + \frac{\nu_1^2}{I}\sinh t\right)\frac{\partial}{\partial \nu_2}.
\end{aligned}$$

5 Conclusions and future work

In this paper we have generalized the Cartan symplectic reduction by a submanifold in the presence of a symmetry Lie group to the Lie algebroid setting. More precisely, we develop a procedure to reduce a symplectic Lie algebroid to a certain quotient symplectic Lie algebroid, using a Lie subalgebroid and a symmetry Lie group. In addition, under some mild assumptions we are able to reduce the Hamiltonian dynamics. Several examples illustrate our theory.

As we already mentioned in the introduction, a particular example of reduction of a symplectic manifold is the well known Marsden–Weinstein reduction of a symmetric Hamiltonian dynamical system (M,Ω,H) in the presence of a G-equivariant momentum map $J : M \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G which acts on M by $\Phi : G\times M \to M$. In this situation, the submanifold is the inverse image $J^{-1}(\mu)$ of a regular value $\mu \in \mathfrak{g}^*$ of J, and the Lie group acting on $J^{-1}(\mu)$ is the isotropy group of μ. It would be interesting to generalize this procedure to the setting of symplectic Lie algebroids. In this case, we would have to define a proper notion of a momentum map for this framework. This appropiate moment map would allow us to get the right submanifold in order to apply the reduction procedure we have developed in this paper. Moreover, under these hypotheses, we should be able to reduce also the Hamiltonian dynamics. These topics are the subject of a forthcoming paper (see [10]).

A natural generalization of symplectic manifolds is that of Poisson manifolds. This type of manifolds play also a prominent role in the Hamiltonian description of Mechanics, in particular for systems with constraints or with symmetry. For Poisson manifolds, one can also develop a reduction procedure (see [17]). On the other hand, as we mentioned in Section 2.2, a symplectic section on a Lie algebroid A can be seen as a particular example of a triangular matrix for A, that is, a Poisson structure in the Lie algebroid setting. Therefore, it should be interesting to generalize our reduction constructions to Poisson structures on Lie algebroids and, more generally, to obtain a reduction procedure for Lie bialgebroids (we remark that triangular matrices on Lie algebroids are a particular example of Lie bialgebroids, the so-called triangular Lie bialgebroids).

On the other hand, recently, in [1] it has been described a reduction procedure for Courant algebroids and Dirac structures. We recall that there is a relation between Courant algebroids and Lie bialgebroids: there is a one-to-one correspondence between Lie bialgebroids and pairs of complementary Dirac structures for a Courant algebroid (see [12]). Since symplectic Lie algebroids are a particular example of this situation, it would be interesting to compare both approaches. This is the subject of a forthcoming paper.

Appendix

Suppose that $\tau_A : A \to M$ and $\tau_{\widetilde{A}} : \widetilde{A} \to \widetilde{M}$ are real vector bundles and that $\pi_A : A \to \widetilde{A}$ is a vector bundle epimorphism over the surjective submersion $\pi_M : M \to \widetilde{M}$.

Thus, one may consider the vector subbundle $\ker \pi_A$ of A whose fiber over the point $x \in M$ is the kernel of the linear epimorphism $(\pi_A)_{|A_x} : A_x \to \widetilde{A}_{\pi_M(x)}$.

On the other hand, a section X of the vector bundle $\tau_A : A \to M$ is said to be π_A-projectable if there exists a section $\pi_A(X)$ of $\tau_{\widetilde{A}} : \widetilde{A} \to \widetilde{M}$ such that

$$\pi_A \circ X = \pi_A(X) \circ \pi_M. \tag{A.1}$$

We will denote by $\Gamma(A)^p_{\pi_A}$ the set of π_A-projectable sections. $\Gamma(A)^p_{\pi_A}$ is a $C^\infty(\widetilde{M})$-module and it is clear that $\Gamma(\ker \pi_A)$ is a $C^\infty(\widetilde{M})$-submodule of $\Gamma(A)^p_{\pi_A}$.

Note that if $X \in \Gamma(A)^p_{\pi_A}$ then there exists a unique section $\pi_A(X)$ of the vector bundle such that (A.1) holds.

Thus, we may define a map

$$\Pi_A : \Gamma(A)^p_{\pi_A} \to \Gamma(\widetilde{A})$$

and it follows that Π_A is an epimorphism of $C^\infty(\widetilde{M})$-modules and that $\ker \Pi_A = \Gamma(\ker \pi_A)$. Therefore, we have that the $C^\infty(\widetilde{M})$-modules $\Gamma(\widetilde{A})$ and $\dfrac{\Gamma(A)^p_{\pi_A}}{\Gamma(\ker \pi_A)}$ are isomorphic, that is,

$$\Gamma(\widetilde{A}) \cong \frac{\Gamma(A)^p_{\pi_A}}{\Gamma(\ker \pi_A)}. \tag{A.2}$$

Moreover, one may prove that following result.

Theorem A.1. *Let $\pi_A : A \to \widetilde{A}$ be a vector bundle epimorphism between the vector bundles $\tau_A : A \to M$ and $\tau_{\widetilde{A}} : \widetilde{A} \to \widetilde{M}$ over the surjective submersion $\pi_M : M \to \widetilde{M}$ and suppose that A is a Lie algebroid with Lie algebroid structure $(\llbracket \cdot, \cdot \rrbracket_A, \rho_A)$. Then, there exists a unique Lie algebroid structure on the vector bundle $\tau_{\widetilde{A}} : \widetilde{A} \to \widetilde{M}$ such that π_A is a Lie algebroid epimorphism if and only if the following conditions hold:*

i) The space $\Gamma(A)^p_{\pi_A}$ is a Lie subalgebra of the Lie algebra $(\Gamma(A), \llbracket \cdot, \cdot \rrbracket_A)$ and

ii) $\Gamma(\ker \pi_A)$ is an ideal of the Lie algebra $\Gamma(A)^p_{\pi_A}$.

Proof. Assume that $(\llbracket \cdot, \cdot \rrbracket_{\widetilde{A}}, \rho_{\widetilde{A}})$ is a Lie algebroid structure on the vector bundle $\tau_{\widetilde{A}} : \widetilde{A} \to \widetilde{M}$ such that π_A is a Lie algebroid epimorphism.

If $X \in \Gamma(A)^p_{\pi_A}$ then, using that

$$d^A(\widetilde{f} \circ \pi_M) = (\pi_A)^*(d^{\widetilde{A}}\widetilde{f}), \qquad \forall\, \widetilde{f} \in C^\infty(\widetilde{M}),$$

we directly conclude that the vector field $\rho_A(X)$ on M is π_M-projectable on the vector field $\rho_{\widetilde{A}}(\pi_A(X))$ on $\widetilde{M}$.

Thus, if $Y \in \Gamma(A)^p_{\pi_A}$ and $\widetilde{\alpha} \in \Gamma(\widetilde{A}^*)$, we have that

$$\begin{aligned}&\widetilde{\alpha}(\pi_M(x))([\![\pi_A(X), \pi_A(Y)]\!]_{\widetilde{A}}(\pi_M(x)))\\ &\qquad = -(\pi_A^*(d^{\widetilde{A}}\widetilde{\alpha}))(x)(X(x), Y(x)) + \rho_A(X)(x)(\pi_A^*\widetilde{\alpha})(Y)) - \rho_A(Y)(x)((\pi_A^*\widetilde{\alpha})(X)),\end{aligned}$$

for all $x \in M$. Therefore, since $(\pi_A^*(d^{\widetilde{A}}\widetilde{\alpha})) = d^A(\pi_A^*\widetilde{\alpha})$ it follows that

$$\begin{aligned}&\widetilde{\alpha}(\pi_M(x))([\![\pi_A(X), \pi_A(Y)]\!]_{\widetilde{A}}(\pi_M(x)))\\ &\qquad = (\pi_A^*\widetilde{\alpha})(x)([\![X, Y]\!]_A(x)) = \widetilde{\alpha}(\pi_M(x))(\pi_A[\![X, Y]\!]_A(x)).\end{aligned}$$

Consequently, we deduce the following result

$$X, Y \in \Gamma(A)^p_{\pi_A} \Rightarrow \pi_A \circ [\![X, Y]\!]_A = [\![\pi_A(X), \pi_A(Y)]\!]_{\widetilde{A}} \circ \pi_M.$$

This proves (i) and (ii).

Conversely, suppose that conditions (i) and (ii) hold. Then, we will define a Lie bracket $[\![\cdot,\cdot]\!]_{\widetilde{A}}$ on the space $\Gamma(\widetilde{A})$ as follows. If $\widetilde{X}$ and $\widetilde{Y}$ are sections of $\tau_{\widetilde{A}} : \widetilde{A} \to \widetilde{M}$, we can choose $X, Y \in \Gamma(A)^p_{\pi_A}$ such that

$$\pi_A(X) = \widetilde{X}, \qquad \pi_A(Y) = \widetilde{Y}.$$

Now, we have that $[\![X, Y]\!]_A \in \Gamma(A)^p_{\pi_A}$ and we define

$$[\![\widetilde{X}, \widetilde{Y}]\!]_{\widetilde{A}} = [\![\pi_A(X), \pi_A(Y)]\!]_{\widetilde{A}} = \pi_A[\![X, Y]\!]_A. \tag{A.3}$$

Using condition (ii), it follows that the map $[\![\cdot,\cdot]\!]_{\widetilde{A}} : \Gamma(\widetilde{A}) \times \Gamma(\widetilde{A}) \to \Gamma(\widetilde{A})$ is well-defined. Moreover, since $[\![\cdot,\cdot]\!]_A$ is a Lie bracket on $\Gamma(A)$, we conclude that $[\![\cdot,\cdot]\!]_{\widetilde{A}}$ is also a Lie bracket on $\Gamma(\widetilde{A})$.

Next, will see that

$$Z \in \Gamma(\ker \pi_A) \;\Rightarrow\; \rho_A(Z) \text{ is a } \pi_M\text{-vertical vector field.} \tag{A.4}$$

In fact, if $\widetilde{f} \in C^\infty(\widetilde{M})$ we have that

$$[\![Z, (\widetilde{f} \circ \pi_M)Y]\!]_A \in \Gamma(\ker \pi_A), \qquad \text{for all } \; Y \in \Gamma(A)^p_{\pi_A}.$$

On the other hand,

$$[\![Z, (\widetilde{f} \circ \pi_M)Y]\!]_A = (\widetilde{f} \circ \pi_M)[\![Z, Y]\!]_A + \rho_A(Z)(\widetilde{f} \circ \pi_M)Y$$

which implies that $\rho_A(Z)(\widetilde{f} \circ \pi_M) = 0$ and, thus, the vector field $\rho_A(Z)$ is vertical with respect to the map π_M.

Now, let $\widetilde{X}$ be a section of $\tau_{\widetilde{A}} : \widetilde{A} \to \widetilde{M}$ and X be a section of $\tau_A : A \to M$ such that $\pi_A(X) = \widetilde{X}$. If $\widetilde{f} \in C^\infty(\widetilde{M})$ it follows that

$$[\![X, (\widetilde{f} \circ \pi_M)Y]\!]_A \in \Gamma(A)^p_{\pi_A}, \qquad \text{for all } \; Y \in \Gamma(A)^p_{\pi_A}$$

and, since

$$[\![X, (\widetilde{f} \circ \pi)Y]\!]_A = (\widetilde{f} \circ \pi_M)[\![X, Y]\!]_A + \rho_A(X)(\widetilde{f} \circ \pi_M)Y$$

we obtain that $\rho_A(X)(\widetilde{f} \circ \pi_M)$ is a basic function with respect to the map π_M. Therefore, the vector field $\rho_A(X)$ is π_M-projectable to a vector field $\rho_{\widetilde{A}}(\widetilde{X})$ on $\widetilde{M}$.

Using (A.4), we deduce that the map $\rho_{\widetilde{A}} : \Gamma(\widetilde{A}) \to \mathfrak{X}(\widetilde{M})$ is well-defined. In fact, $\rho_{\widetilde{A}}$ is a homomorphism of $C^\infty(\widetilde{M})$-modules and if we also denote by $\rho_{\widetilde{A}} : \widetilde{A} \to T\widetilde{M}$ the corresponding bundle map then

$$T\pi_M \circ \rho_A = \rho_{\widetilde{A}} \circ \pi_A. \tag{A.5}$$

In addition, since $([\![\cdot,\cdot]\!]_A, \rho_A)$ is a Lie algebroid structure on A, we conclude that the pair $([\![\cdot,\cdot]\!]_{\widetilde{A}}, \rho_{\widetilde{A}})$ is also a Lie algebroid structure on $\widetilde{A}$.

On the other hand, from (A.5), it follows that

$$\pi_A^*(d^{\widetilde{A}}\widetilde{f}) = d^A(\widetilde{f} \circ \pi_M), \qquad \text{for} \quad \widetilde{f} \in C^\infty(\widetilde{M}).$$

Furthermore, using (A.3) and (A.5), we obtain that

$$\pi_A^*(d^A\widetilde{\alpha}) = d^A(\pi_A^*\widetilde{\alpha}), \qquad \text{for} \quad \widetilde{\alpha} \in \Gamma(\widetilde{A}^*).$$

Consequently, π_A is a Lie algebroid epimorphism.

Now, suppose that $([\![\cdot,\cdot]\!]'_{\widetilde{A}}, \rho'_{\widetilde{A}})$ is another Lie algebroid structure on the vector bundle $\tau_{\widetilde{A}} : \widetilde{A} \to \widetilde{M}$ such that π_A is a Lie algebroid epimorphism. Denote by $d'^{\widetilde{A}}$ the differential of the Lie algebroid $(\widetilde{A}, [\![\cdot,\cdot]\!]'_{\widetilde{A}}, \rho'_{\widetilde{A}})$. Then, the condition

$$\pi_A^*(d'^{\widetilde{A}}\widetilde{f}) = d^A(\widetilde{f} \circ \pi_M), \qquad \text{for} \quad \widetilde{f} \in C^\infty(\widetilde{M}),$$

implies that $T\pi_M \circ \rho_A = \rho'_{\widetilde{A}} \circ \pi_A$. Thus, from (A.5), we deduce that $\rho_{\widetilde{A}} = \rho'_{\widetilde{A}}$.

Therefore, the condition

$$\pi_A^*(d'^{\widetilde{A}}\widetilde{\alpha}) = d^A(\pi_A^*\widetilde{\alpha}), \qquad \text{for} \quad \widetilde{\alpha} \in \Gamma(\widetilde{A}^*),$$

implies that

$$[\![\pi_A(X), \pi_A(Y)]\!]'_{\widetilde{A}} = \pi_A([\![X, Y]\!]_A), \qquad \text{for} \quad X, Y \in \Gamma(A)^p_{\pi_A}$$

and, using (A.3), we conclude that $[\![\cdot,\cdot]\!]_{\widetilde{A}} = [\![\cdot,\cdot]\!]'_{\widetilde{A}}$. ■

Remark A.2. An equivalent dual version of Theorem A.1 was proved in [3].

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