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Some Remarks on the KP System of the Camassa–Holm Hierarchy*

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Abstract. We study a Kadomtsev–Petviashvili system for the local Camassa–Holm hierarchy obtaining a candidate to the Baker–Akhiezer function for its first reduction generalizing the local Camassa–Holm. We focus our attention on the differences with the standard KdV-KP case.

Key words: KP hierarchy; CH hierarchy; Sato Grassmannian

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1 Introduction

The Camassa–Holm (CH) equation with zero critical velocity [2]

$$u_t - u_{xxt} = 6u_x u - 4u_{xx}u_x - 2uu_{xxx}$$

and his multi fields extensions [1, 4, 12, 14, 15, 20, 21, 22, 26] are widely studied integrable systems because they show a number of properties different from the standard Gelfand–Dikii systems such as Korteweg–de Vries (KdV). Its study is important to a deep understanding of infinite-dimensional integrability.

One of the most important differences between CH and KdV is that, even if CH has the same dispersionless part of the KdV equation

$$u_t = -u_{xxx} + 6uu_x,$$

it does not admit tau structure in the sense of Dubrovin's classification scheme [10]. Classically, for the Kadomtsev–Petviashivili (KP) equation and its reductions, the tau structure is related to the fact that the Baker–Akhiezer function ψ of the hierarchy satisfies a bilinear relation [8, 27, 28].

The bi-Hamiltonian method allows the interpretation of the KP system as a generalization of the conservation laws for the corresponding Noether currents. This construction [13] starts from the Riccati equation related to the system. In the KdV case the equation

$$h_x + h^2 = u + z^2, \qquad \text{with} \qquad h(z) := z + \sum_{i\geq -1} \frac{h_i}{z^i}, \tag{1}$$

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is the relation satisfied by the conserved densities h_i of KdV. Using the Noether theorem one can associate to every conserved density h_i a current $H_i^{(3)}$ such that $\partial_t h_i = \partial_x H_i^{(3)}$. Every commuting symmetry in the KdV hierarchy is related to a different current and then, for a symmetry of KdV involving the time $\partial/\partial t_s = \partial_s$, it holds $\partial_s h_i = \partial_x H_i^{(s)}$. Using the generator $h(z)$ we can collect the currents into a set of generators of currents satisfying

$$\partial_s h(z) = \partial_x H^{(s)}(z). \tag{2}$$

At this stage this set of equations is simply a way to rewrite the equations of the KdV hierarchy.

The amazing property of the currents $H^{(s)}$ is that they belong to a very particular space called H_+. This is the space generated by the linear span on $C^\infty(S^1,\mathbb{R})$ of the Faà di Bruno polynomials $h^{(n)} := (\partial_x + h)^n \cdot 1$ of the conserved density h. There is a unique way to write every current $H^{(s)}$ as $H^{(s)}(x,z) = \sum_{i=0}^{s} c_i(x) h^{(i)}(x,z)$.

Without account of (1), the equation (2) becomes an infinite differential system in an infinite number of fields. This is equivalent to the KP equation presented by the Japanese school. This system is a linear flow on a suitable Grassmannian whose H_+ is the positive part. A key property [24] of H_+ for the relation between KP and Grassmannian flows is that this space is invariant under the action of the operators $(\partial_s + H^{(s)})$:

$$(\partial_s + H^{(s)})H_+ \subset H_+. \tag{3}$$

Thanks to (3) the KP system can be written as

$$(\partial_s + H^{(s)})H^{(r)} = H^{(s+r)} + \sum_{i=1}^{s} H_i^r H^{(s-i)} + \sum_{i=1}^{r} H_i^s H^{(r-i)}. \tag{4}$$

If we forget the Faà di Bruno rule for construction of the space H_+, the currents $H^{(s)}$ are simply Laurent series in z and a collection $\{H^{(s)}\}_{s\geq 0}$ is related ([13]) to a point of the positive part H_+ of the Sato Grassmannian $\mathcal{W}$ defined e.g. in [28]. The equation (4) describes now a flow on $\mathcal{W}$ which admits as reduction the KP system imposing the relation between the currents and the Faà di Bruno polynomials. It is called the *central system* for KP. The time t_1 of the central system can be identified with the variable x of KP because the conservation of the linear momentum of KP equation.

This construction is a hint for the existence of bilinear relations [13] and one wonders whether the same scheme can be applied to generalizations of CH hierarchy.

The study of KP equation for CH is interesting in this direction.

In [3] the authors show the existence of a KP equation constructed starting from local symmetries of the CH equation. In this paper we plan to pursue these ideas further. After a review of the results presented in [3], we study the KP-CH system obtaining a candidate for the Baker–Akhiezer function of the integrable reductions of the system. Next we consider the problem of the reduction from KP-CH to the local Camassa–Holm hierarchy. By means of this process we obtain a 3-field integrable system which generalizes the local symmetries for CH.

2 The full CH hierarchy

It was known [2, 16] that CH is bi-Hamiltonian on the vector space $C^\infty(S^1,\mathbb{R})$ by means of the Poisson pencil

$$P_{z^2} = P_0 - z^2 P_1 = \frac{1}{2}\partial_x - \frac{1}{2}\partial_x^3 - z^2\left(-m\partial_x - \partial_x m\right), \qquad z \in \mathbb{R},$$

where $m = (1-\partial_x^2)u$ and we call z^2 the pencil parameter for later convenience. The theory of bi-Hamiltonian systems (see e.g. [23]) shows that the conserved quantities for CH are the coefficients of the potential of an exact 1-form v in the kernel of the Poisson pencil. If $H(z) = \sum_{k=-\infty}^{+\infty} H_k z^{-k}$ satisfy $P_{z^2} dH(z) = 0$ then one can construct the Lenard–Magri recursion $P_0 dH_k = P_1 dH_{k+2}$. This chain implies that all H_k are in involution w.r.t. both the Poisson pencils.

Every element $v(z)$ in $\mathrm{Ker} P_{z^2}$ satisfies the equation:

$$\frac{1}{4}v^2 - \frac{1}{2}v_{xx}v + \frac{1}{4}v_x^2 + z^2 m v^2 = f(m,z), \tag{5}$$

where $f(m,z)$ satisfies $f_x = 0$. It turns out that v is an exact 1-form if f does not depend on m. Without loss of generality, we can put $f(z) = z^2/4$.

The equation (5) can be solved iteratively developing v in z. In order to find the potential $H(z)$ we evaluate v on the generic vector $\dot m$:

$$\begin{aligned}\langle v, \dot m\rangle &\overset{(5)}{=} \int_{S^1} v \frac{d}{dt}\left(\frac{z^2}{4v^2} + \frac{1}{2z^2}\frac{v_{xx}}{v} - \frac{1}{4z^2}\frac{v_x^2}{v^2} - \frac{1}{4z^2}\right) dx \\ &= \frac{d}{dt}\int_{S^1} \frac{1}{2v}\, dx + \int_{S^1} \partial_x\left(\dot{v_x} + \frac{v_x \dot v}{v}\right) dx = \frac{d}{dt}\int_{S^1} \frac{1}{2v}\, dx.\end{aligned}$$

Therefore $H(z) = \int_{S^1} (1/2v)\, dx$ is a generator of conserved quantities.

Let us focus now our attention on the density h of H. It is obviously defined up to a normalization and up to a total derivative. Paying attention to these 'gauge' choices one can put

$$h = z\frac{1}{2v} + \partial_x(\ln\sqrt{v})$$

and then the equation (5) becomes

$$h_x + h^2 = \frac{1}{4} + z^2 m. \tag{6}$$

By construction, all the solutions of this equation define conserved quantities for CH ([3, 19, 25] and, in the context of the inverse spectral problem, [5, 7]).

The possible solutions of (6) are two functions, depending on the essential singularity point of $h(z)$. The first one, $h(z) = \sum_{i=-1}^{+\infty} h_i z^{-i}$, has its essential singularity in 0. It corresponds to the Lenard–Magri recursion starting from the Casimir of P_1 and all the quantities are local in m.

The second one, $k(z) = \sum_{i=0}^{+\infty} k_i z^i$, contains CH and it has an essential singularity at infinity. It corresponds to the Lenard–Magri recursion starting from the Casimir of P_0, and it has an infinite number of quantities nonlocal in m.

The calculation of these solutions is not difficult but a little bit involved. It is performed explicitly in [3]. We now simply list and comment the results.

The local hierarchy

The local hierarchy is related to the solutions of (6) of the form $h(z) = h_{-1}z + h_0 + h_1/z + h_2/z^2 + \cdots$:

$$h_{-1} = \sqrt{m}, \qquad h_0 = (\ln(m^{-1/4}))_x,$$

$$h_1 = \frac{1}{8\sqrt{m}} - \frac{1}{8}\frac{m_{xx}}{\sqrt{m^3}} + \frac{5}{32}\frac{m_x^2}{\sqrt{m^5}},$$
$$h_2 = \left(-\frac{1}{16m} + \frac{1}{16}\frac{m_{xx}}{m^2} + \frac{5}{64}\frac{m_x^2}{m^3}\right)_x, \quad \dots.$$

The conserved quantities $H_i = \int h_{i-1}\,dx$ for $i = 0, 1, 2, \dots$ are nontrivial only when h_{i-1} are not a total derivative, i.e. when i is even. Using only the even conserved quantities we obtain $H(z^2) = \sum_{i\geq 0} H_{2i} z^{-2i}$.

We can iterate the Lenard–Magri chain defining an infinite number of vector fields $X_{2i} = P_0 dH_{2i} = P_1 dH_{2i+2}$ pairwise commuting and we identify $X_{2i} = \partial_{2i} m$. The first one, already presented in [2], is the *local Camassa–Holm equation*

$$\partial_0 m = (\partial_x - \partial_x^3)\frac{1}{4\sqrt{m}}. \tag{7}$$

This part of the hierarchy generates the negative flows of CH.

The nonlocal hierarchy

The nonlocal hierarchy is related to the solutions of (6) of the form $k(z) = k_0 + k_{-1}z + k_{-2}z^2 + \cdots$:

$$\begin{aligned}
k_0 &= \frac{1}{2},\\
k_{-1} &= 0,\\
k_{-2} &= (1+\partial_x)^{-1} m = u - u_x,\\
k_{-3} &= 0,\\
k_{-4} &= -(1+\partial_x)^{-1}((1+\partial_x)^{-1} m)^2 = -u^2 - u_x^2 + \text{total derivatives},\\
k_{-5} &= 0,\\
k_{-6} &= 2(1+\partial_x)^{-1}((1+\partial_x)^{-1}((1+\partial_x)^{-1} m)^2 (1+\partial_x)^{-1} m)\\
&= u^3 + u u_x^2 + \text{total derivatives}, \quad \dots.
\end{aligned}$$

We remark that one can continue the iteration thanks to the invertibility of the operator $1 + \partial_x$ in the space of smooth periodic functions [3]. The densities are increasingly non-local in m, but the first three are local in the field $u = (1 - \partial_x^2)^{-1} m$.

As in the local case one can define conserved quantities $K_{-i} = \int k_{-i}\,dx$ for $i = 0, 1, 2, \dots$. Taking only the nontrivial ones, $i = 2, 4, 6, \dots$, we obtain $K(z^2) = \sum_{i\geq 1} K_{-2i} z^{2i}$.

We can iterate the Lenard–Magri chain defining an infinite number of vector fields $Y_{-2i} = P_0 dK_{-2i} = P_1 dK_{-2i+2} := \partial_{-2i} m$ pairwise commuting. The first nontrivial vector field is the conservation of the linear momentum $\partial_{-2} m = P_1 dK_{-2} = P_0 dK_{-4} = -m_x$. The second one is the standard Camassa–Holm equation with null critical velocity

$$\partial_{-4} m = P_1 dK_{-4} = -(m\partial_x + \partial_x m)(1 - \partial_x^2)^{-1}(-2u + 2u_{xx}) = 4mu_x + 2m_x u = P_0 dK_{-6}$$

that is, with $t_{-4} = t$,

$$u_t - u_{xxt} = 6u_x u - 4u_{xx}u_x - 2u u_{xxx}.$$

The conserved quantities H_i and K_{-i} commute with each other by the bi-Hamiltonian construction. Therefore we can collect the two iteration chain into a unique one defining the full CH hierarchy

$$
\begin{tikzcd}
\cdots \arrow[dr, "P_1"] & \\
& \cdots \\
dH_4 \arrow[ur, "P_0"] \arrow[dr, "P_1"] & \\
& (\partial_x - \partial_x^3)\dfrac{-4m^2 - 4m_{xx}m + 5m_x^2}{128m^{7/2}} \\
dH_2 \arrow[ur, "P_0"] \arrow[dr, "P_1"] & \\
& (\partial_x - \partial_x^3)\dfrac{1}{4\sqrt{m}} \\
dH_0 \arrow[ur, "P_0"] \arrow[dr, "P_1"] & \\
& 0 \\
dK_{-2} \arrow[ur, "P_0"] \arrow[dr, "P_1"] & \\
& -m_x \\
dK_{-4} \arrow[ur, "P_0"] \arrow[dr, "P_1"] & \\
& 4mu_x + 2m_x u \\
dK_{-6} \arrow[ur, "P_0"] \arrow[dr, "P_1"] & \\
& \cdots \\
\cdots \arrow[ur, "P_0"] &
\end{tikzcd}
$$

3 The Noether currents

In the KdV case the KP system has a simple interpretation as a generalization of the Noether relations (2) between conserved densities and related currents [13]. To understand if such a construction is still possible, we write down these currents in our case. In the case of local conserved quantities the calculation is already performed in [3] and gives the result:

$$J^{(s)} = \sum_{i=0}^{s} \left(-\frac{1}{2}\partial_x v_i(z^{s-i+2}) + v_i(z^{s-i+2}h) \right) = \frac{z^{s+3}}{2} + O(z), \qquad s \geq 0. \tag{8}$$

With similar computations, using

$$\partial_{-2s} m = -P_{z^2} \sum_{j=1}^{s} z^{2j-2s-2} dK_{-2j}$$

and $w(z) = zdK(z)$, one can show that, in the nonlocal case, the currents are:

$$J^{(-s)} = \sum_{j=1}^{s} \left(\frac{1}{2}\partial_x w_{-j}(z^{j-s-1}) - w_{-j}(z^{j-s-1}k) \right) = \frac{z^{-s-2}}{2} + O(1), \qquad s \geq 1.$$

Although it seems to possess the same structure, these two families of currents are really different. The local currents, as in the KdV case, are elements of a particular space called J_+. It is the span on the periodic smooth functions of the Faà di Bruno polynomials

$$h^{(n)} = (\partial_x + h)^n z^2, \qquad n \geq 0. \tag{9}$$

It holds

Proposition 1. *The currents $J^{(2s)}$, with $s \geq 0$, are elements of J_+.*

Proof. (See [3].) Thanks to the representation (8), it suffices to show that z^{2i} and $z^{2i}h$ are elements of J_+ for all $i \geq 1$. First of all, $z^2 = h^{(0)} \in J_+$ and $z^2 h = h^{(1)} \in J_+$ by definition of J_+. Moreover, the Riccati equation (6) multiplied by z^2,

$$h^{(2)} = \frac{z^2}{4} + z^4 m, \tag{10}$$

shows that $z^4 = 1/m(h^{(2)} - (1/4)h^{(0)}) \in H_h$. Acting with $(\partial_x + h)^n$ on both sides of (10) we prove the statement. ■

For every $s \geq 0$, the current $J^{(s)}$ has a essential singularity in zero and a pole of order $s+3$ at infinity. The space which contains these currents has to be generated by Laurent series with similar characteristics. Actually, in the proof of the previous proposition, a fundamental role is played by the fact that the conserved density $h(z)$, which is the seed for the Faà di Bruno polynomials, diverges both at 0 (essential singularity) and at ∞ (simple pole).

For every $s < 0$, the current $J^{(s)}$ has a essential singularity at infinity and a pole of order $-s-2$ at zero. However the related generator of conserved quantities $k(z)$ is regular at 0. Therefore $k(z)$ cannot generate the analogue of J_+ for the nonlocal currents by means of the previous method.

As we will see in the following, the existence of this space is a key ingredient for the construction of a KP-CH system.

4 The KP-CH system

Let us then concentrate on the local currents $J^{(s)}$. They can be characterized in a unique way by the following two properties:

$$1)\ J^{(s)} = \frac{1}{2}z^{s+3} + O(z), \qquad 2)\ J^{(s)} \in J_+.$$

These currents are not a basis of the J_+ space. To obtain a basis one has to add the constant z^2 so that $J_+ = \langle z^2, J^{(n)}\rangle_{n\geq 0}$. Assuming that h is an arbitrary Laurent series of the form

$$h(z) = h_{-1}z + \sum_{i=0}^{+\infty} \frac{h_i}{z^i},$$

where the coefficients h_i are not constrained by the Riccati equation, we can define the currents $J^{(s)}$, for all $s \geq 0$, imposing the two above-mentioned properties. We define the s-th equation of the *local KP-CH system* [3] as

$$\partial_s h = \partial_x J^{(s)}, \qquad s \geq 0. \tag{11}$$

It is an evolution equation in an infinite number of fields given by the coefficients h_{-1}, h_0, h_1, $\dots$ of h. The main difference between this system and the standard KP system [17] is that the density h is not an element of the space J_+. Nevertheless the equations (11) can be written using only elements of J_+

$$\partial_s h^{(1)} = z^2 \partial_x J^{(s)}.$$

These equations are nonlinear in the space J_+ because they involve the product of two elements of J_+. This nonlinearity in another difference between KP-CH and standard KP.

Directly from the equations of motion we easily see that this system possesses an infinite number of conserved quantities. Their densities are all the coefficients h_i of $h(z)$.

It is well known that the KP equation in 2+1D can be obtained simply combining in a suitable way the first equation of the KP system in infinite fields. The analogue of the $2+1$ dimensional KP equation can be obtained also in our case. The smallest closed differential subsystem of evolution equation involves the first 5 fields $h_{-1}, \dots, h_3$

$$\begin{aligned}
&\partial_0 h_{-1} = \partial_x \frac{h_1}{2h_{-1}}, \qquad \partial_0 h_0 = \partial_x \frac{h_2}{2h_{-1}}, \qquad \partial_0 h_1 = \partial_x \frac{h_3}{2h_{-1}},\\
&\partial_1 h_{-1} = \partial_x \left(\frac{h_2}{h_{-1}} + \frac{1}{h_{-1}} \partial_x \frac{h_1}{2h_{-1}} \right),\\
&\partial_1 h_0 = \partial_x \left(\frac{h_3}{h_{-1}} + \frac{h_1^2}{2h_{-1}^2} + \frac{1}{h_{-1}} \partial_x \frac{h_2}{2h_{-1}} \right).
\end{aligned}$$

After some simple algebraic manipulation one reduces this system to

$$\begin{aligned}
2u_t &= \left(\frac{w}{u}\right)_x,\\
u_y &= 2v_t + \left(\frac{u_t}{u}\right)_x,\\
v_y &= 2w_t + 2\frac{w}{u}u_t + \left(\frac{v_t}{u}\right)_x,
\end{aligned}$$

where $h_{-1} = u$, $h_0 = v$, $h_1 = w$ and $t_0 = t$, $t_1 = y$.

4.1 The evolution of the currents

We study now the evolution of the currents $J^{(s)}$. In the standard KP case they are related to the Sato linear flows on the Grassmannian [13]. For the KP-CH equation the situation is somewhat different. By definition, the space J_+ is invariant w.r.t. $\partial_x + h$ but the invariance is lost w.r.t. $\partial_s + J^{(s)}$. Actually the action of $\partial_s + J^{(s)}$ of the generic element of the basis of J_+ is

$$(\partial_s + J^{(s)})h^{(k)} = (\partial_s + J^{(s)})(\partial_x + h)^k z^2 = z^2(\partial_x + h)^k(\partial_s + J^{(s)}) \subset z^2 J_+. \tag{12}$$

Remark 1. As we have seen in the introduction, the analogue for KP of the space J_+ is the space H_+. The main difference between these two spaces is that in the KP-CH case the space J_+ is not invariant under the action of the operators $(\partial_s + J^{(s)})$.

Using arguments similar to those used in [13] it is easy to show, for example in the case $z^2 J_+ \subset J_+$ studied in the next paragraph, the commutativity of the flows generated by $\{\partial_s\}_{s\geq 0}$. Therefore, in the same case, we can introduce the function ψ defined by $\partial_s \ln\psi = J^{(s)}$ and by $\partial_x \ln\psi = h$. Such a function exist also in the KP case and, in that case [24], it is the Baker–Akhiezer function.

We conjecture that the generalizations of ψ are the Baker–Akhiezer functions for the integrable reductions of KP-CH.

5 On the KP-CH reductions

To recover the local CH hierarchy from (11), one has to impose on h the constraint given by the Riccati equation (6). Therefore all the fields h_i can be written in terms of m and its x-derivatives. Thus the local KP-CH system (11) reduces to the local CH hierarchy. As in the Gelfand–Dikii cases this reduction is a stationary reduction because the Riccati equation imply $J^{(1)} = \frac{z^4}{2}$ and the triviality of the evolution along t_1. From Proposition 1 it also follows that all t_{2s+1} are stationary. However, contrarily to the KdV case, the constraint given by Riccati equation (6) is not equivalent to $J^{(1)} = \frac{z^4}{2}$. The local Camassa–Holm equation is a strange reduction of a system obtained in its turn as a reduction from the KP-CH. Under the constraint

$$J^{(1)} = \frac{z^4}{2} \tag{13}$$

the evolution w.r.t. the time t_0 becomes

$$\begin{aligned}
\partial_0 h_{-1} &= -\frac{\partial_x(h_{-1})h_1}{2{h_{-1}}^2} + \frac{\partial_x h_1}{2h_{-1}},\\
\partial_0 h_0 &= -\frac{3}{4}\frac{(\partial_x h_{-1})^2 h_1}{{h_{-1}}^4} + \frac{3}{4}\frac{(\partial_x h_{-1})(\partial_x h_1)}{{h_{-1}}^3} + \frac{1}{4}\frac{(\partial_x^2 h_{-1})h_1}{{h_{-1}}^3} - \frac{1}{4}\frac{\partial_x^2 h_1}{{h_{-1}}^2},\\
\partial_0 h_1 &= \frac{(\partial_x h_{-1}){h_1}^2}{2{h_{-1}}^3} - \frac{15}{8}\frac{h_1(\partial_x h_{-1})^3}{{h_{-1}}^6} + \frac{15}{8}\frac{(\partial_x h_{-1})^2\partial_x h_1}{{h_{-1}}^5} + \frac{5}{4}\frac{h_1(\partial_x^2 h_{-1})\partial_x h_{-1}}{{h_{-1}}^5}\\
&\quad - \frac{3}{4}\frac{(\partial_x h_{-1})\partial_x^2 h_1}{{h_{-1}}^4} - \frac{h_1\partial_x h_1}{2{h_{-1}}^2} - \frac{(\partial_x^2 h_{-1})\partial_x h_1}{2{h_{-1}}^4} - \frac{h_1\partial_x^3 h_{-1}}{8{h_{-1}}^4} + \frac{\partial_x^3 h_1}{8{h_{-1}}^3}.
\end{aligned} \tag{14}$$

One can see that the field h_0 does not affect the evolution of the system. This is true also for the following times of the reduction because the currents themselves do not depend on h_0. Actually, by direct computation, we can show that the current $J^{(0)}$ do not depend on h_0. By (12) all the successive currents do not depend on h_0.

The more compact way to write the system is

$$\partial_0\alpha = D_\alpha\frac{\gamma}{\alpha}, \qquad \partial_0\beta = -D_\alpha^2\frac{\gamma}{\alpha}, \qquad \partial_0\gamma = \alpha D_\alpha^3\frac{\gamma}{\alpha}, \tag{15}$$

where $\alpha = h_{-1}$, $\gamma = h_{-1}h_1$, $\beta = h_0$ and $D_\alpha = \partial_x \cdot (1/2\alpha)$.

The conserved densities for (15) are given by the condition (13) written in terms of the generator h of the Faà di Bruno polynomials (9):

$$\frac{1}{2\alpha^2}h^{(2)} - \left(\frac{\alpha_x}{2\alpha^3} + \frac{\beta}{\alpha^2}\right)h^{(1)} + \left(\frac{\beta_x}{2\alpha^2} + \frac{\gamma}{\alpha^2} - \frac{\beta\alpha_x}{2\alpha^3} - \frac{\beta^2}{\alpha^2}\right)h^{(0)} = \frac{z^4}{2}. \tag{16}$$

The local CH can be obtained from this equation system under the constraint:

$$\gamma = \frac{m_{xx}}{8m} - \frac{5m_x^2}{32m^2} + \frac{1}{8}, \qquad \beta = -\partial_x \ln(m^{1/4}), \qquad \alpha = \sqrt{m}$$

which reduces (15) to (7), i.e.:

$$\partial_0 m = \frac{1}{4}\left(\partial_x - \partial_x^3\right)\frac{1}{\sqrt{m}}.$$

The same reduction transforms (16) into (6). An important property of (15) is related to its associated dispersionless system:

$$\partial_0\alpha = \frac{1}{2}\partial_x\frac{y}{\alpha^2}, \qquad \partial_0\beta = 0, \qquad \partial_0\gamma = 0.$$

This trivial system is bi-Hamiltonian w.r.t. the Poisson tensors:

$$P_1^d = \frac{1}{2}\begin{pmatrix}\partial_x & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad P_0^d = \frac{1}{2}\begin{pmatrix}0 & 0 & \partial_x\alpha\\ 0 & 0 & 0\\ \alpha\partial_x & 0 & \gamma\partial_x + \partial_x\gamma\end{pmatrix}$$

and the Hamiltonians $H_1 = -\int(\gamma/\alpha)\,dx$ and $H_0 = \int(\gamma^2/2\alpha^3)\,dx$. The metric associated to P_1^d is obviously degenerate, then this dispersionless system is not of "Dubrovin–Novikov type" [9]. This is an important difference between this reduction and the KdV equation because it is not possible to include that system into the Dubrovin's classification scheme.

6 Conclusion

In this paper we have studied the KP system related to the Camassa–Holm hierarchy [3]. This system seems very different from the standard KP case. The first difference is in the fact that the evolution of the currents (12) is not contained in the space J_+ which plays the role of the projection on the positive part of the Sato Grassmannian naturally defined by the currents.

The second difference is that the first reduction, obtained fixing the first current, is not an integrable system whose dispersionless part is related to a non-degenerate metric. This could be seen as a motivation why they do not enter in the Dubrovin classification scheme [10].

In the paper we have defined, in a restricted case, a function ψ, and we conjecture that its generalizations are the Baker–Akhiezer functions for the KP-CH integrable reductions. We are studying also whether in our case ψ is an eigenfunction of a suitable Lax operator which can be used to construct a zero curvature equation. The open question that we will address is as follows: Are the local symmetries of CH bilinear?

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