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Hamiltonian Structure of PI Hierarchy*

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Abstract. The string equation of type $(2, 2g + 1)$ may be thought of as a higher order analogue of the first Painlevé equation that corresponds to the case of $g = 1$. For $g > 1$, this equation is accompanied with a finite set of commuting isomonodromic deformations, and they altogether form a hierarchy called the PI hierarchy. This hierarchy gives an isomonodromic analogue of the well known Mumford system. The Hamiltonian structure of the Lax equations can be formulated by the same Poisson structure as the Mumford system. A set of Darboux coordinates, which have been used for the Mumford system, can be introduced in this hierarchy as well. The equations of motion in these Darboux coordinates turn out to take a Hamiltonian form, but the Hamiltonians are different from the Hamiltonians of the Lax equations (except for the lowest one that corresponds to the string equation itself).

Key words: Painlevé equations; KdV hierarchy; isomonodromic deformations; Hamiltonian structure; Darboux coordinates

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1 Introduction

The so called 'string equations' were introduced in the discovery of an exact solution of two-dimensional quantum gravity [1, 2, 3]. Since it was obvious that these equations are closely related to equations of the Painlevé and KdV type, this breakthrough in string theory soon yielded a number of studies from the point of view of integrable systems [4, 5, 6, 7, 8].

The string equations are classified by a pair (q, p) of coprime positive integers. The simplest case of $(q, p) = (2, 3)$ is nothing but the first Painlevé equation

$$\frac{1}{4}u_{xx} + \frac{3}{4}u^2 + x = 0,$$

and the equations of type $(2, p)$ for $p = 5, 7, \ldots$ may be thought of as higher order analogues thereof. Unlike the case of type $(2, 3)$, these higher order PI equations are accompanied with a finite number of commuting flows, which altogether form a kind of finite-dimensional 'hierarchy'. This hierarchy is referred to as 'the PI hierarchy' in this paper. ('PI' stands for the first Painlevé equation).

The PI hierarchy can be characterized as a reduction of the KdV (or KP) hierarchy. This is also the case for the string equations of all types. The role of the string equation in this reduction resembles that of the equation of commuting pair of differential operators [10]. It is well known that the equation of commuting pairs, also called 'the stationary Lax equation', characterizes algebro-geometric solutions of the KP hierarchy [11, 12]. The reduction by the string equation, however, is drastically different in its nature. Namely, whereas the commuting pair equation imposes translational symmetries to the KP hierarchy, the string equation is related to Virasoro

*This paper is a contribution to the Vadim Kuznetsov Memorial Issue 'Integrable Systems and Related Topics'. The full collection is available at http://www.emis.de/journals/SIGMA/kuznetsov.html (and even larger $W_{1+\infty}$) symmetries [6]. In the Lax formalism of the KP hierarchy [13, 14, 15], the latter symmetries are realized by the Orlov–Schulman operator [16], which turns out to be an extremely useful tool for formulating the string equation and the accompanied commuting flows [9].

The string equation and the accompanied commuting flows may be viewed as a system of isomonodromic deformations. This is achieved by reformulating the equations as Lax equations of a polynomial L-matrix (a 2×2 matrix in the case of the PI hierarchy). This Lax formalism may be compared with the Lax formalism of the well known Mumford system [17]. Both systems have substantially the same 2×2 matrix L-matrix, which is denoted by $V(\lambda)$ in this paper. λ is a spectral parameter on which $V(\lambda)$ depends polynomially. The difference of these systems lies in the structure of the Lax equations. The Lax equations of the Mumford system take such a form as

$$\partial_t V(\lambda) = [U(\lambda), V(\lambda)],$$

where $U(\lambda)$ is also a 2×2 matrix of polynomials in λ. (Note that we show just one of the Lax equations representatively.) Obviously, this is an isospectral system. On the other hand, the corresponding Lax equations of the PI hierarchy have an extra term on the right hand side:

$$\partial_t V(\lambda) = [U(\lambda), V(\lambda)] + U'(\lambda), \qquad U'(\lambda) = \partial_\lambda U(\lambda).$$

The extra term $U'(\lambda)$ breaks isospectrality. This is actually a common feature of Lax equations that describe isomonodromic deformations.

We are concerned with the Hamiltonian structure of this kind of isomonodromic systems. As it turns in this paper, the PI hierarchy exhibits some new aspects of this issue. Let us briefly show an outline.

The Hamiltonian structure of the Mumford system is more or less well known [18, 19]. The Poisson brackets of the matrix elements of the L-matrix take the form of ‘generalized linear brackets’ [20]. (Actually, this system has a multi-Hamiltonian structure [21, 22], but this is beyond the scope of this paper.) The Lax equations can be thereby expressed in the Hamiltonian form

$$\partial_t V(\lambda) = \{V(\lambda), H\}.$$

Since the Lax equations of the PI hierarchy have substantially the same L-matrix as the Mumford system, we can borrow its Poisson structure. In fact, the role of the Poisson structure is simply to give an identity of the form

$$[U(\lambda), V(\lambda)] = \{V(\lambda), H\}.$$

We can thus rewrite the Lax equations as

$$\partial_t V(\lambda) = \{V(\lambda), H\} + U'(\lambda),$$

leaving the extra term intact. This is a usual understanding of the Hamiltonian structure of isomonodromic systems such as the Schlesinger system (see, e.g., [23, Appendix 5]). This naive prescription, however, leads to a difficulty when we consider a set of Darboux coordinates called ‘spectral Darboux coordinates’ and attempt to rewrite the Lax equations to a Hamiltonian system in these coordinates.

The notion of spectral Darboux coordinates originates in the pioneering work of Flaschka and McLaughlin [24], which was later reformulated by Novikov and Veselov [25] in a more general form. As regards the Mumford system, this notion lies in the heart of the classical algebro-geometric approach [17]. The Montreal group [26] applied these coordinates to separation of variables of various isospectral systems with a rational L-matrix. Their idea was generalized by Sklyanin [27] to a wide range of integrable systems including quantum integrable systems. On the other hand, spectral Darboux coordinates were also applied to isomonodromic systems [28, 29, 30, 31].

We can consider spectral Darboux coordinates for the PI hierarchy in exactly the same way as the case of the Mumford system. It will be then natural to attempt to derive equations of motions in those Darboux coordinates. Naive expectation will be that those equations of motion become a Hamiltonian system with the same Hamiltonian H as the Lax equation. This, however, turns out to be wrong (except for the lowest part of the hierarchy, namely, the string equation itself). The fact is that the extra term $U'(\lambda)$ in the Lax equation gives rise to extra terms in the equations of motions in the Darboux coordinates. Thus, not only the naive expectation is negated, it is also not evident whether those equations of motion take a Hamiltonian form with a suitable Hamiltonian. This is the aforementioned difficulty.

The goal of this paper is to show that those equations of motion are indeed a Hamiltonian system. As it turns out, the correct Hamiltonian K can be obtained by adding a correction term ΔH to H as

$$K = H + \Delta H.$$

This is a main conclusion of our results. It is interesting that the Hamiltonian of the lowest flow of the hierarchy (namely, the string equation itself) is free from the correction term.

Let us mention that this kind of correction terms take place in some other isomonodromic systems as well. An example is the Garnier system (so named and) studied by Okamoto [32]. The Garnier system is a multi-dimensional generalization of the Painlevé equations (in particular, the sixth Painlevé equation), and has two different interpretations as isomonodromic deformations. One is based on a second order Fuchsian equation. Another interpretation is the 2×2 Schlesinger system, from which the Garnier system can be derived as a Hamiltonian system for a special set of Darboux coordinates. It is easy to see that these Darboux coordinates are nothing but spectral Darboux coordinates in the aforementioned sense [28, 29], and that the Hamiltonians are the Hamiltonians of the Schlesinger system [23, Appendix 5] plus correction terms. These observations on the Garnier system have been generalized by Dubrovin and Mazzocco [31] to the Schlesinger system of an arbitrary size. A similar structure of Hamiltonians can be found in a 'degenerate' version of the Garnier system studied by Kimura [33] and Shimomura [34]. Actually, this system coincides with the PI hierarchy associated with the string equation of type $(2, 5)$.

This paper is organized as follows. In Section 2, we introduce the string equations of type $(2, p)$, and explain why they can be viewed as a higher order analogues of the first Painlevé equations. In Section 3, these equations are cast into a 2×2 matrix Lax equation. Section 4 is a brief review of the KdV and KP hierarchies. In Section 5, the PI hierarchy is formulated as a reduction of the KP (or KdV) hierarchy, and converted to 2×2 matrix Lax equations. In Section 6, we introduce the notion of spectral curve and consider the structure of its defining equation in detail. Though the results of this section appear to be rather technical, they are crucial to the description of Hamiltonians in spectral Darboux coordinates. Section 7 deals with the Hamiltonian structure of the Lax equations. In Section 8, we introduce spectral Darboux coordinates, and in Section 9, identify the Hamiltonians in these coordinates. In Section 10, these results are illustrated for the first three cases of $(q, p) = (2, 3), (2, 5), (2, 7)$.

2 String equation as higher order PI equation

Let q and p be a pair of coprime positive integers. The string equation of type (q, p) takes the commutator form [4]

$$[Q, P] = 1 \qquad (2.1)$$

for a pair of ordinary differential operators

$$Q = \partial_x^q + g_2\partial_x^{q-2} + \cdots + g_q, \qquad P = \partial_x^p + f_2\partial_x^{p-2} + \cdots + f_p$$

of order q and p in one-dimensional spatial variable x ($\partial_x = \partial/\partial x$). In the following, we consider the equation of type $(q,p) = (2, 2g+1)$, $g = 1, 2, \ldots$. We shall see that g is equal to the genus of an underlying algebraic curve (spectral curve).

The simplest case, i.e., $(q,p) = (2,3)$, consists of operators of the form

$$Q = \partial_x^2 + u, \qquad P = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x,$$

where the subscript means a derivative as

$$u_x = \frac{\partial u}{\partial x}, \qquad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad \ldots.$$

The string equation (2.1) for these operators reduces to the third-order equation

$$\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x + 1 = 0.$$

We can integrate it once, eliminating the integration constant by shifting $x \to x + \mathrm{const}$, and obtain the first Painlevé equation

$$\frac{1}{4}u_{xx} + \frac{3}{4}u^2 + x = 0. \tag{2.2}$$

The setup for the general case of type $(2, 2g+1)$ relies on the techniques originally developed for the KdV hierarchy and its generalization [35, 36, 37, 38]. The basic tools are the fractional powers

$$Q^{n+1/2} = \partial_x^{2n+1} + \frac{2n+1}{2}u\partial_x^{2n-1} + \cdots + R_{n+1}\partial_x^{-1} + \cdots$$

of $Q = \partial_x^2 + u$. The fractional powers are realized as pseudo-differential operators. The coefficient R_{n+1} of ∂_x^{-1} is a differential polynomials of u called the Gelfand–Dickey polynomial:

$$R_0 = 1, \qquad R_1 = \frac{u}{2}, \qquad R_2 = \frac{1}{8}u_{xx} + \frac{3}{8}u^2,$$
$$R_3 = \frac{1}{32}u_{xxxx} + \frac{3}{16}uu_{xx} + \frac{5}{32}u_x^2 + \frac{5}{16}u^3, \quad \ldots.$$

For all n's, the highest order term in R_n is linear and proportional to $u^{(2n-2)}$:

$$R_n = \frac{1}{2^{2n+2}}u^{(2n-2)} + \cdots.$$

As in the construction of the KdV hierarchy, we introduce the differential operators

$$B_{2n+1} = \left(Q^{n+1/2}\right)_{\ge 0}, \qquad n = 0, 1, \ldots.$$

where $(\ )_{\ge 0}$ stands for the projection of a pseudo-differential operator to a differential operator. Similarly, we use the notation $(\ )_{<0}$ for the projection to the part of negative powers of ∂_x, i.e.,

$$\left(\sum_{j\in\mathbf{Z}} a_j\partial_x^j\right)_{\ge 0} = \sum_{j\ge 0} a_j\partial_x^j, \qquad \left(\sum_{j\in\mathbf{Z}} a_j\partial_x^j\right)_{<0} = \sum_{j<0} a_j\partial_x^j.$$

These differential operators have the special property that the commutator with Q is of order zero. More precisely, we have the identity

$$[B_{2n+1}, Q] = 2R_{n+1,x}. \tag{2.3}$$

Consequently, if we choose P to be a linear combination of these operators as

$$P = B_{2g+1} + c_1 B_{2g-1} + \cdots + c_g B_1 \tag{2.4}$$

with constant coefficients $c_1, \ldots, c_g$, the commutator with Q reads

$$[Q, P] = -2R_{g+1,x} - 2c_1 R_{g,x} - \cdots - 2c_g R_{1,x}.$$

The string equation (2.1) thus reduces to

$$2R_{g+1,x} + 2c_1 R_{g,x} + \cdots + 2c_g R_{1,x} + 1 = 0. \tag{2.5}$$

This equation can be integrated to become the equation

$$2R_{g+1} + 2c_1 R_g + \cdots + 2c_g R_1 + x = 0, \tag{2.6}$$

which gives a higher order analogue (of order $2g$) of the first Painlevé equation (2.2).

When we consider a hierarchy of commuting flows that preserve the string equation, the coefficients $c_1, \ldots, c_g$ play the role of time variables. For the moment, they are treated as constants.

3 Matrix Lax formalism of string equation

The string equation (2.1) is accompanied with the auxiliary linear problem

$$Q\psi = \lambda\psi, \qquad P\psi = \partial_\lambda \psi \tag{3.1}$$

with a spectral parameter λ ($\partial_\lambda = \partial/\partial\lambda$). We can rewrite this linear problem to a 2×2 matrix form [5] using various properties of the operators B_{2n+1} and the Gelfand–Dickey polynomials R_n [35, 36, 37, 38] as follows.

The first equation of (3.1) can be readily converted to the matrix form

$$\partial_x \boldsymbol{\psi} = U_0(\lambda)\boldsymbol{\psi}, \tag{3.2}$$

where

$$\boldsymbol{\psi} = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}, \qquad U_0(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}.$$

To rewrite the second equation of (3.1), we use the 'Q-adic' expansion formula

$$B_{2n+1} = \sum_{m=0}^{n} \left(R_m \partial_x - \frac{1}{2} R_{m,x} \right) Q^{n-m} \tag{3.3}$$

of B_{2n+1}'s. By this formula, $B_{2n+1}\psi$ becomes a linear combination of ψ and ψ_x as

$$B_{2n+1}\psi = \sum_{m=0}^{n} \left(R_m \psi_x - \frac{1}{2} R_{m,x} \psi \right) = R_m(\lambda)\psi_x - \frac{1}{2} R_m(\lambda)_x \psi,$$

where $R_n(\lambda)$ stand for the auxiliary polynomials

$$R_n(\lambda) = \lambda^n + R_1\lambda^{n-1} + \cdots + R_n, \qquad n = 0, 1, \ldots, \tag{3.4}$$

that play a central role throughout this paper. Since P is a linear combination of B_{2n+1}'s as (2.4) shows, we can express $P\psi$ as

$$P\psi = \alpha(\lambda)\psi + \beta(\lambda)\psi_x$$

with the coefficients

$$\beta(\lambda) = R_g(\lambda) + c_1R_{g-1}(\lambda) + \cdots + c_gR_0(\lambda), \qquad \alpha(\lambda) = -\frac{1}{2}\beta(\lambda)_x. \tag{3.5}$$

Moreover, differentiating this equation by x and substituting $\psi_{xx} = (\lambda - u)\psi$, we can express $P\psi_x$ as

$$P\psi_x = (P\psi)_x = \gamma(\lambda)\psi - \alpha(\lambda)_x\psi_x,$$

where

$$\gamma(\lambda) = \alpha(\lambda)_x + (\lambda - u)\beta(\lambda) = -\frac{1}{2}\beta(\lambda)_{xx} + (\lambda - u)\beta(\lambda). \tag{3.6}$$

The second equation of the auxiliary linear problem (3.1) can be thus converted to the matrix form

$$\partial_\lambda\boldsymbol{\psi} = V(\lambda)\boldsymbol{\psi} \tag{3.7}$$

with the coefficient matrix

$$V(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{pmatrix}.$$

The differential equation (3.2) in x may be thought of as defining isomonodromic deformations of the matrix ODE (3.7) in λ with polynomial coefficients. The associated Lax equation reads

$$\partial_x V(\lambda) = [U_0(\lambda), V(\lambda)] + U_0'(\lambda), \tag{3.8}$$

where the last term stands for the λ-derivative of $U_0(\lambda)$, i.e.,

$$U_0'(\lambda) = \partial_\lambda U_0(\lambda) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The presence of such an extra term other than a matrix commutator is a common characteristic of Lax equations for isomonodromic deformations in general.

In components, the Lax equation consists of the three equations

$$\partial_x\alpha(\lambda) = \gamma(\lambda) - (\lambda - u)\beta(\lambda), \qquad \partial_x\beta(\lambda) = -2\alpha(\lambda), \qquad \partial_x\gamma(\lambda) = 2(\lambda - u)\alpha(\lambda) + 1.$$

The first and second equations can be solved for $\alpha(\lambda)$ and $\gamma(\lambda)$; the outcome is just the definition of these polynomials in (3.5) and (3.6). Upon eliminating $\alpha(\lambda)$ and $\gamma(\lambda)$ by these equations, the third equation turns into the equation

$$\frac{1}{2}\beta(\lambda)_{xxx} - 2(\lambda - u)\beta(\lambda)_x + u_x\beta(\lambda) + 1 = 0 \tag{3.9}$$

for $\beta(\lambda)$ only.

Let us examine (3.9) in more detail. By the definition in (3.5), $\beta(\lambda)$ is a polynomial of the form

$$\beta(\lambda) = \lambda^g + \beta_1 \lambda^{g-1} + \cdots + \beta_g$$

with the coefficients

$$\beta_n = R_n + c_1 R_{n-1} + \cdots + c_n R_0, \qquad n = 1, \ldots, g. \tag{3.10}$$

Upon expanded into powers of λ, (3.9) yield the equations

$$\frac{1}{2}\beta_{n,xxx} + 2u\beta_{n,x} + u_x\beta_n - 2\beta_{n+1,x} = 0, \qquad n = 1, \ldots, g-1,$$

and

$$\frac{1}{2}\beta_{g,xxx} + 2u\beta_{g,x} + u_x\beta_g + 1 = 0.$$

Recalling here the well known Lenard recursion formula

$$R_{x,n+1} = \frac{1}{4}R_{n,xxx} + uR_{n,x} + \frac{1}{2}u_x R_n, \tag{3.11}$$

one can see that the forgoing equations for $n = 1, \ldots, g-1$ are identities, and that the latter one is equivalent to (2.5).

4 KdV and KP hierarchies

We shall derive the PI hierarchy from the KdV or KP hierarchy. Let us briefly review some basic stuff of these hierarchies [13, 14, 15]. Of particular importance is the notion of the Orlov–Schulman operator [16] that supplements the usual Lax formalism of these hierarchies.

4.1 KdV hierarchy from KP hierarchy

Let L denote the Lax operator

$$L = \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots$$

of the KP hierarchy. L obeys the Lax equations

$$\partial_n L = [B_n, L], \qquad B_n = (L^n)_{\geq 0},$$

in an infinite number of time variables $t_1, t_2, \ldots$ ($\partial_n = \partial/\partial t_n$). As usual, we identify t_1 with x.

The KdV hierarchy can be derived from the KP hierarchy by imposing the constraint

$$(L^2)_{<0} = 0.$$

Under this constraint, $Q = L^2$ becomes a differential operator of the form

$$Q = \partial_x^2 + u, \qquad u = 2u_2,$$

all even flows are trivial in the sense that

$$\partial_{2n} L = [L^{2n}, L] = 0,$$

and the whole hierarchy reduces to the Lax equations

$$\partial_{2n+1} Q = [B_{2n+1}, Q], \qquad B_{2n+1} = (Q^{n+1/2})_{\geq 0}, \tag{4.1}$$

of the KdV hierarchy. By (2.3), these Lax equations further reduce to the evolution equations

$$\partial_{2n+1} u = 2R_{n+1,x} \tag{4.2}$$

for u.

4.2 Orlov–Schulman operator

The Orlov–Schulman operator is a pseudo-differential operator (of infinite order) of the form

$$M = \sum_{n=2}^{\infty} n t_n L^{n-1} + x + \sum_{n=1}^{\infty} v_n L^{-n-1},$$

that obeys the Lax equations

$$\partial_n M = [B_n, M] \tag{4.3}$$

and the commutation relation

$$[L, M] = 1. \tag{4.4}$$

The existence of such an operator can be explained in the language of the auxiliary linear system

$$L\psi = z\psi, \qquad \partial_n \psi = B_n \psi, \tag{4.5}$$

of the KP hierarchy. This linear system has a (formal) solution of the form

$$\psi = \left(1 + \sum_{j=1}^{\infty} w_j z^{-j}\right) \exp\left(xz + \sum_{n=2}^{\infty} t_n z^n\right), \tag{4.6}$$

One can rewrite this solution as

$$\psi = W \exp\left(xz + \sum_{n=2}^{\infty} t_n z^n\right),$$

where W is a pseudo-differential operator (called the 'dressing operator' or the Sato–Wilson operator) of the form

$$W = 1 + \sum_{j=1}^{\infty} w_j \partial_x^{-j}$$

that satisfies the evolution equations

$$\partial_n W = -(W \cdot \partial_x^n \cdot W^{-1})_{<0} W \tag{4.7}$$

and the algebraic relations

$$L = W \cdot \partial_x \cdot W^{-1}, \qquad B_n = (W \cdot \partial_x^n \cdot W^{-1})_{\geq 0}.$$

One can now define M as

$$M = W \left(\sum_{n=2}^{\infty} n t_n \partial_x^{n-1} + x\right) W^{-1},$$

which turns out to satisfy the foregoing equations (4.3), (4.4) and the auxiliary linear equation

$$M\psi = \partial_z \psi. \tag{4.8}$$

Remark 1. In the reduction to the KdV hierarchy, the ∂_x^{-1}-part of (4.7) gives the equation

$$\partial_{2n+1} w_1 = -R_{n+1}. \tag{4.9}$$

Since $u = -2w_{1,x}$, this equation may be thought of as a once-integrated form of (4.2).

5 PI hierarchy

We now formulate the PI hierarchy as a reduction of the KP hierarchy, and rewrite it to a 2×2 matrix Lax equation, as we have done for the string equation itself. The string equation and the commuting flows of the PI hierarchy are thus unified to a system of multi-time isomonodromic deformations.

5.1 PI hierarchy from KP hierarchy

To derive the string equation from the KP hierarchy, we impose the constraints [9]

$$(Q)_{<0} = 0, \qquad (P)_{<0} = 0 \tag{5.1}$$

on the operators

$$Q = L^2, \qquad P = \frac{1}{2} M L^{-1} = \sum_{n=1}^{\infty} n t_n L^{n-2} + \sum_{n=1}^{\infty} v_n L^{-n-2}.$$

This leads to the following consequences.

1. As it follows from the commutation relation (4.4) of L and M, these operators obey the commutation relation $[Q, P] = 1$.
2. Under the first constraint of (5.1), Q becomes the Lax operator $\partial_x^2 + u$ of the KdV hierarchy. In the following, as usual, we suppress the time variables with even indices, i.e.,

$$t_2 = t_4 = \cdots = 0.$$

3. The second constraint of (5.1) implies that P is a differential operator (of infinite order) of the form

$$P = \frac{1}{2}\left(M L^{-1}\right)_{\geq 0} = \sum_{n=1}^{\infty} \frac{2n+1}{2} t_{2n+1} B_{2n+1}.$$

Therefore, if we set

$$t_{2g+3} = \frac{2}{2g+1}, \; t_{2g+5} = t_{2g+7} = \cdots = 0, \tag{5.2}$$

we are left with a differential operator of the form (2.4) with the coefficients $c_1, \ldots, c_g$ depending on the time variables as

$$c_n = c_n(t) = \frac{2n+1}{2} t_{2n+1}, \qquad n = 1, \ldots, g. \tag{5.3}$$

4. The auxiliary linear equations (4.5) and (4.8) imply the linear equations

$$Q\psi = z^2 \psi, \qquad P\psi = \frac{1}{2} z^{-1} \partial_z \psi.$$

These linear equations can be identified with the auxiliary linear equations (3.1) of the string equation if we define λ as

$$\lambda = z^2.$$

We can thus recover, from the KP hierarchy, the string equation (2.1) of type $(2, 2g+1)$ along with g extra commuting flows

$$\partial_{2n+1}Q = [B_{2n+1}, Q], \qquad \partial_{2n+1}P = [B_{2n+1}, P], \qquad n = 1, \dots, g.$$

We call this system the PI hierarchy. Let us mention that this hierarchy was first discovered in a more direct way [4]. Compared with that approach, the approach from the KP hierarchy [9] is more transparent.

Remark 2. This hierarchy thus contains $g+1$ flows with time variables $t_1(= x)$, t_3, $\dots$, t_{2g+1}. The last flow in t_{2g+1}, however, turns out to be spurious, i.e., can be absorbed by other flows (see below). We shall suppress this flow when we consider the Hamiltonian structure.

5.2 Matrix Lax formalism of commuting flows

We can again use the Q-adic expansion formula (3.3) of B_{2n+1} to rewrite the auxiliary linear equations

$$\partial_{2n+1}\psi = B_{2n+1}\psi, \qquad n = 1, \dots, g,$$

of the PI hierarchy to linear equations

$$\partial_{2n+1}\boldsymbol{\psi} = U_n(\lambda)\boldsymbol{\psi} \tag{5.4}$$

for $\boldsymbol{\psi}$. The matrix elements of the coefficient matrix

$$U_n(\lambda) = \begin{pmatrix} \mathrm{A}_n(\lambda) & \mathrm{B}_n(\lambda) \\ \Gamma_n(\lambda) & -\mathrm{A}_n(\lambda) \end{pmatrix}$$

are the following polynomials in λ:

$$\begin{aligned} &\mathrm{B}_n(\lambda) = R_n(\lambda), \qquad \mathrm{A}_n(\lambda) = -\frac{1}{2}R_n(\lambda)_x, \\ &\Gamma_n(\lambda) = -\frac{1}{2}R_n(\lambda)_{xx} + (\lambda - u)R_n(\lambda). \end{aligned} \tag{5.5}$$

(3.2) can be included in these linear equations as a special case with $n = 0$ ($t_1 = x$).

As one can see by comparing these matrix elements with those of $U_n(\lambda)$'s defined in (3.5) and (3.6), $V(\lambda)$ is a linear combination of $U_n(\lambda)$'s:

$$V(\lambda) = U_g(\lambda) + c_1(t)U_{g-1}(\lambda) + \cdots + c_g(t)U_0(\lambda). \tag{5.6}$$

We shall consider implications of this linear relation later on.

Having obtained the full set of auxiliary linear equations (3.2), (3.7), (5.4) in a 2×2 matrix form, we can now reformulate the commuting flows of the PI hierarchy as the isomonodromic matrix Lax equations

$$\partial_{2n+1}V(\lambda) = [U_n(\lambda), V(\lambda)] + U_n'(\lambda), \qquad n = 0, 1, \dots, g, \tag{5.7}$$

including (3.8) as the case with $n = 0$ ($t_1 = x$). Note that the Lax equations of the higher flows, too, have an extra term

$$U_n'(\lambda) = \partial_\lambda U_n(\lambda)$$

other than the matrix commutator $[U_n(\lambda), V(\lambda)]$.

5.3 Variant of self-similarity

Nowadays it is widely known that many isomonodromic equations can be derived from soliton equations as 'self-similar reduction' [39, 40]. The transition from the KP hierarchy to the PI hierarchy is actually a variant of self-similar reduction.

Recall that P is a linear combination of B_n's as (2.4) shows. This implies that the left hand side of the auxiliary linear equation $P\psi = \partial_\lambda\psi$ can be expressed as

$$P\psi = (\partial_{2g+1} + c_1(t)\partial_{2g-1} + \cdots + c_g(t)\partial_1)\psi.$$

The auxiliary linear equation thus turns into a kind of linear constraint:

$$(\partial_{2g+1} + c_1(t)\partial_{2g-1} + \cdots + c_g(t)\partial_1)\psi = \partial_\lambda\psi. \tag{5.8}$$

If we were considering the equation

$$[Q, P] = 0$$

of commuting pair of differential operators [10, 11, 12], all c_n's would be constants, and (5.8) would mean the existence of stationary directions (in other words, translational symmetries) in the whole time evolutions of the KdV hierarchy. It is well known that this condition characterizes algebro-geometric solutions of the KdV hierarchy.

In the present setting, where c_n's are not constant but variables as (5.3) shows, (5.8) may be thought of as a variant of self-similarity condition. We can see some more manifestation of this condition. For instance, $V(\lambda)$ satisfies the linear equation

$$(\partial_{2g+1} + c_1(t)\partial_{2g-1} + \cdots + c_g(t)\partial_1)V(\lambda) = V'(\lambda)$$

as one can deduce from the Lax equations (5.7) and the linear relation (5.6) among their coefficients. Similarly, u satisfies a similar equation

$$(\partial_{2g+1} + c_1(t)\partial_{2g-1} + \cdots + c_g(t)\partial_1)u = 1$$

as a consequence of (2.5) and (4.2). These equations show, in particular, that the t_{g+1}-flow is spurious.

Remark 3. (5.8) stems from the Virasoro constraints associated with the string equation. The Virasoro constraints are usually formulated in the language of the τ-function [6]. Reformulated in terms of ψ, the lowest one (the L_{-1}-constraint) of those constraints becomes the linear equation

$$\sum_{n=0}^{\infty} \frac{2n+1}{2} t_{2n+1}\partial_{2n-1}\psi = \partial_\lambda\psi.$$

This turns into the aforementioned constraint (5.8) if the higher time variables $t_{2g+3}, t_{2g+5}, \ldots$ are set to the special values of (5.2).

6 Building blocks of spectral curve

6.1 Equation of spectral curve

We now consider the spectral curve of the matrix $V(\lambda)$. This curve is defined by the characteristic equation

$$\det(\mu I - V(\lambda)) = \mu^2 + \det V(\lambda) = 0$$

or, more explicitly, by the equation

$$\mu^2 = h(\lambda) = \alpha(\lambda)^2 + \beta(\lambda)\gamma(\lambda).$$

Since $\alpha(\lambda), \beta(\lambda), \gamma(\lambda)$ are polynomials of the form

$$\alpha(\lambda) = \alpha_1\lambda^{g-1} + \cdots + \alpha_g,$$
$$\beta(\lambda) = \lambda^g + \beta_1\lambda^{g-1} + \cdots + \beta_g,$$
$$\gamma(\lambda) = \lambda^{g+1} + \gamma_1\lambda^g + \cdots + \gamma_{g+1},$$

$h(\lambda)$ is a polynomial of the form $h(\lambda) = \lambda^{2g+1} + \cdots$, and the spectral curve is a hyperelliptic curve of genus g.

The spectral curve plays a central role in the algebro-geometric theory of commuting pairs [11, 12]. In that case, the flows of the KdV hierarchy can be translated to the *isospectral* Lax equations

$$\partial_{2n+1}V(\lambda) = [U_n(\lambda), V(\lambda)] \tag{6.1}$$

of the same matrix $V(\lambda)$ as we have used thus far (except that c_n's are genuine constant). In particular, the polynomial $h(\lambda)$ (hence the spectral curve itself) is invariant under time evolutions:

$$\partial_{2n+1}h(\lambda) = 0.$$

This system (6.1) is called 'the Mumford system' [17].

In contrast, the polynomial $h(\lambda)$ for the matrix Lax equations (5.7) of the PI hierarchy is *not* constant in time evolutions. Straightforward calculations show that the t-derivatives of $h(\lambda)$ take non-zero values as

$$\partial_{2n+1}h(\lambda) = \mathrm{Tr}\, U_n'(\lambda)V(\lambda).$$

For instance, the lowest equation for $n = 0$ ($t_1 = x$) reads

$$\partial_x h(\lambda) = \beta(\lambda).$$

The spectral curve thus deforms as x and t_{2n+1}'s vary. Obviously, the extra terms $U_n'(\lambda)$ are responsible for this phenomena. This is a common feature of isomonodromic Lax equations.

6.2 Some technical lemmas

As we shall show later on, the coefficients of terms of higher degrees in the polynomial $h(\lambda)$ have a rather special structure. We present here a few technical lemmas that are used to explain this fact.

The following lemmas are concerned with the KdV hierarchy rather than the PI hierarchy. In the case of the KdV hierarchy, the matrices $U_n(\lambda)$ are defined for all nonnegative integers $n = 0, 1, 2, \ldots$.

Lemma 1. *There is a* 2×2 *matrix*

$$\Phi(\lambda) = \begin{pmatrix} 1 + O(\lambda^{-1}) & O(\lambda^{-1}) \\ w_1 + O(\lambda^{-1}) & 1 + O(\lambda^{-1}) \end{pmatrix}$$

of Laurent series of λ *that satisfies the equations*

$$\partial_{2n+1}\Phi(\lambda) = U_n(\lambda)\Phi(\lambda) - \Phi(\lambda)\lambda^n\Lambda, \qquad n = 0, 1, 2, \ldots, \tag{6.2}$$

where

$$\Lambda = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}.$$

Proof. Let $\psi(z)$ be the special solution, (4.6), of the auxiliary linear equations. In the case of the KdV hierarchy, this is a function of the form $\psi(z) = w(z)e^{\xi(z)}$, where

$$w(z) = 1 + \sum_{j=1}^{\infty} w_j z^{-j}, \qquad \xi(z) = \sum_{n=0}^{\infty} t_{2n+1} z^{2n+1} \qquad (t_1 = x).$$

The associated vector-valued function

$$\boldsymbol{\psi}(z) = \begin{pmatrix} \psi(z) \\ \psi(z)_x \end{pmatrix} = \begin{pmatrix} w(z) \\ zw(z) + w(z)_x \end{pmatrix} e^{\xi(z)}$$

satisfies the auxiliary linear equations

$$\partial_{2n+1}\boldsymbol{\psi}(z) = U_n(\lambda)\boldsymbol{\psi}(z) \qquad (\lambda = z^2)$$

with the same coefficients $U_n(\lambda)$ as in (5.4) but now defined by (5.5) for all nonnegative integers $n = 0, 1, 2, \ldots$.

Moreover, since λ remains invariant by substituting $z \to -z$, $\boldsymbol{\psi}(-z)$ is also a solution of these linear equations. Thus we actually have a 2×2 matrix-valued solution

$$\big(\boldsymbol{\psi}(z)\ \boldsymbol{\psi}(-z)\big) = \begin{pmatrix} w(z) & w(-z) \\ zw(z) + w(z)_x & -zw(-z) + w(-z)_x \end{pmatrix} \begin{pmatrix} e^{\xi(z)} & 0 \\ 0 & e^{-\xi(z)} \end{pmatrix}.$$

We now consider

$$\Psi(\lambda) = \big(\boldsymbol{\psi}(z)\ \boldsymbol{\psi}(-z)\big) \begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1} = \left(\frac{\boldsymbol{\psi}(z) + \boldsymbol{\psi}(-z)}{2}\ \ \frac{\boldsymbol{\psi}(z) - \boldsymbol{\psi}(-z)}{2z} \right),$$

which is also a matrix-valued solution of the foregoing linear equations. Note that the matrix elements are even functions of z (hence functions of λ). Moreover, $\Psi(\lambda)$ can be factorized to the product of

$$\Phi(\lambda) = \begin{pmatrix} w(z) & w(-z) \\ zw(z) + w(z)_x & -zw(-z) + w(-z)_x \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1}$$

and

$$\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix} \begin{pmatrix} e^{\xi(z)} & 0 \\ 0 & e^{-\xi(z)} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1} = \exp\left(\sum_{n=0}^{\infty} t_{2n+1} \Lambda^{2n+1} \right).$$

By plugging

$$\Psi(\lambda) = \Phi(\lambda) \exp\left(\sum_{n=0}^{\infty} t_{2n+1} \Lambda^{2n+1} \right)$$

into the auxiliary linear equations, $\Phi(\lambda)$ turns out to satisfy the equations

$$\partial_{2n+1}\Phi(\lambda) = U_n(\lambda)\Phi(\lambda) - \Phi(\lambda)\Lambda^{2n+1},$$

which are nothing but (6.2) because of the identity $\Lambda^{2n+1} = \lambda^n \Lambda$. ■

Now let us introduce the matrix

$$U(\lambda) = \Phi(\lambda)\Lambda\Phi(\lambda)^{-1}.$$

As a consequence of (6.2), it satisfies the Lax equations

$$\partial_{2n+1}U(\lambda) = [U_n(\lambda), U(\lambda)], \qquad n = 0, 1, 2, \ldots. \tag{6.3}$$

The following lemma shows that we can use $U(\lambda)$ as a kind of generating function for $U_n(\lambda)$'s

Lemma 2. *The matrix elements of*

$$U(\lambda) = \begin{pmatrix} \mathrm{A}(\lambda) & \mathrm{B}(\lambda) \\ \Gamma(\lambda) & -\mathrm{A}(\lambda) \end{pmatrix},$$

are Laurent series of λ of the form

$$\mathrm{A}(\lambda) = O(\lambda^{-1}), \qquad \mathrm{B}(\lambda) = 1 + O(\lambda^{-1}), \qquad \Gamma(\lambda) = \lambda + O(\lambda^0),$$

that satisfy the following algebraic conditions:

$$\begin{aligned} &\mathrm{A}_n(\lambda) = \left(\lambda^n \mathrm{A}(\lambda)\right)_{\geq 0}, \qquad \mathrm{B}_n(\lambda) = \left(\lambda^n \mathrm{B}(\lambda)\right)_{\geq 0}, \\ &\Gamma_n(\lambda) = \left(\lambda^n \Gamma(\lambda)\right)_{\geq 0} - R_{n+1}, \qquad n = 0, 1, \ldots, \end{aligned} \tag{6.4}$$

$$\mathrm{A}(\lambda)^2 + \mathrm{B}(\lambda)\Gamma(\lambda) = \lambda, \tag{6.5}$$

where $(\ )_{\geq 0}$ stands for the polynomial part of a Laurent series:

$$\left(\sum_{j\in\mathbf{Z}} a_j \lambda^j\right)_{\geq 0} = \sum_{j\geq 0} a_j \lambda^j.$$

Proof. Rewrite (6.2) as

$$U_n(\lambda) = \partial_{2n+1}\Phi(\lambda)\cdot\Phi(\lambda)^{-1} + \Phi(\lambda)\lambda^n\Lambda\Phi(\lambda)^{-1} = \partial_{2n+1}\Phi(\lambda)\cdot\Phi(\lambda)^{-1} + \lambda^n U(\lambda),$$

and compare the polynomial part of both hand sides. $U_n(\lambda)$ is a matrix of polynomials, and the first term on the right hand side is a matrix of the form

$$\partial_{2n+1}\Phi(\lambda)\cdot\Phi(\lambda)^{-1} = \begin{pmatrix} O(\lambda^{-1}) & O(\lambda^{-1} \\ \partial_{2n+1}w_1 + O(\lambda^{-1}) & O(\lambda^{-1}) \end{pmatrix}.$$

Thus, also by recalling (4.9), (6.4) turns out to hold. (6.5) is an immediate consequence of the definition of $U(\lambda)$ and the identity $\det\Lambda = \lambda$. ■

Remark 4. Since $\mathrm{B}_n(\lambda)$ is equal to the auxiliary polynomial $R_n(\lambda)$ defined in (3.4), the second equation of (6.4) implies that $\mathrm{B}(\lambda)$ is a generating function of all R_n's:

$$\mathrm{B}(\lambda) = 1 + R_1\lambda^{-1} + R_2\lambda^{-2} + \cdots.$$

The lowest ($n = 0$) part of (6.3) is not a genuine evolution equation. In components, this equation reads

$$\begin{aligned} &\partial_x \mathrm{A}(\lambda) = \Gamma(\lambda) - (\lambda - u)\mathrm{B}(\lambda), \qquad \partial_x \mathrm{B}(\lambda) = -2\mathrm{A}(\lambda), \\ &\partial_x \Gamma(\lambda) = 2(\lambda - u)\mathrm{A}(\lambda). \end{aligned} \tag{6.6}$$

The first two equations can be solved for $\mathrm{A}(\lambda)$ and $\Gamma(\lambda)$ as

$$\mathrm{A}(\lambda) = -\frac{1}{2}\mathrm{B}(\lambda)_x, \qquad \Gamma(\lambda) = -\frac{1}{2}\mathrm{B}(\lambda)_{xx} + (\lambda - u)\mathrm{B}(\lambda).$$

The third equation thereby reduces to

$$\frac{1}{2}\mathrm{B}(\lambda)_{xx} - 2(\lambda - u)\mathrm{B}(\lambda)_x + u_x\mathrm{B}(\lambda) = 0. \tag{6.7}$$

It is easy to see that this is a generating functional form of the Lenard relations (3.11).

Remark 5. If $\mathrm{A}(\lambda)$ and $\Gamma(\lambda)$ are eliminated by (6.6), (6.5) becomes another generating functional formula

$$\frac{1}{4}\big(\mathrm{B}(\lambda)_x\big)^2 + \mathrm{B}(\lambda)\left(-\frac{1}{2}\mathrm{B}(\lambda)_{xx} + (\lambda - u)\mathrm{B}(\lambda)\right) = \lambda \tag{6.8}$$

of relations among R_n's. (6.8) may be thought of as a once-integrated form of (6.7). If expanded in powers of λ, (6.8) becomes a sequence of relations that determine R_n recursively *without integration procedure*. This is an alternative and more practical way for calculating R_n's.

6.3 Detailed structure of $h(\lambda)$

We now turn to the issue of $h(\lambda)$. Let us recall (5.6). In components, it reads

$$\begin{aligned}
\alpha(\lambda) &= \mathrm{A}_g(\lambda) + c_1(t)\mathrm{A}_{g-1}(\lambda) + \cdots + c_g(t)\mathrm{A}_0(\lambda),\\
\beta(\lambda) &= \mathrm{B}_g(\lambda) + c_1(t)\mathrm{B}_{g-1}(\lambda) + \cdots + c_g(t)\mathrm{B}_0(\lambda),\\
\gamma(\lambda) &= \Gamma_g(\lambda) + c_1(t)\Gamma_{g-1}(\lambda) + \cdots + c_g(t)\Gamma_0(\lambda).
\end{aligned}$$

We can thereby express $h(\lambda)$ as

$$h(\lambda) = \sum_{m,n=0}^{g} c_m(t)c_n(t)(\mathrm{A}_{g-m}(\lambda)\mathrm{A}_{g-n}(\lambda) + \mathrm{B}_{g-m}(\lambda)\Gamma_{g-n}(\lambda)),$$

where it is understood that $c_0(t) = 1$. Moreover, since (6.4) implies that

$$\begin{aligned}
&\mathrm{A}_n(\lambda) = \lambda^n\mathrm{A}(\lambda) + O(\lambda^{-1}), \qquad \mathrm{B}_n(\lambda) = \lambda^n\mathrm{B}(\lambda) + O(\lambda^{-1}),\\
&\Gamma_n(\lambda) = \lambda^n\Gamma(\lambda) - R_{n+1} + O(\lambda^{-1}),
\end{aligned}$$

we can further rewrite $h(\lambda)$ as

$$\begin{aligned}
h(\lambda) = &\sum_{m,n=0}^{g} c_m(t)c_n(t)\lambda^{2g-m-n}(\mathrm{A}(\lambda)^2 + \mathrm{B}(\lambda)\Gamma(\lambda))\\
&- 2\sum_{m=0}^{g} c_m(t)R_{g+1-m}\lambda^g + O(\lambda^{g-1}).
\end{aligned}$$

We can now use (6.5) and (2.6) on the right hand side. This leads to the following final result.

Theorem 1. *Up to terms of $O(\lambda^{g-1})$, $h(\lambda)$ can be expressed as*

$$\begin{aligned}
h(\lambda) = &\lambda^{2g+1} + 2c_1(t)\lambda^{2g} + (2c_2(t) + c_1(t)^2)\lambda^{2g-1} + (2c_3(t) + 2c_1(t)c_2(t))\lambda^{2g-2} + \cdots\\
&+ \sum_{m=0}^{g} c_m(t)c_{g-m}(t)\lambda^{g+1} + \left(\sum_{m=1}^{g-1} c_m(t)c_{g+1-m}(t) + x\right)\lambda^g + O(\lambda^{g-1}).
\end{aligned} \tag{6.9}$$

In particular, the coefficients of $\lambda^{2g+1}, \ldots, \lambda^g$ in $h(\lambda)$ do not contain $u, u_x, \ldots$.

Let $I_0(\lambda)$ denote the part of $h(\lambda)$ consisting of $\lambda^{2g+1}, \ldots, \lambda^g$, and $I_1, \ldots, I_g$ the coefficients of $\lambda^{g-1}, \ldots, 1$:

$$h(\lambda) = I_0(\lambda) + I_1\lambda^{g-1} + \cdots + I_g.$$

We have seen above that $I_0(\lambda)$ is a kinematical quantity that is independent of the solution of the PI hierarchy in question. In contrast, the remaining coefficients $I_1, \ldots, I_g$ are genuine dynamical quantities.

In the case of the Mumford system (6.1), these coefficients $I_1, \ldots, I_g$ are Hamiltonians of commuting flows. More precisely, it is not these coefficients but their suitable linear combinations $H_1, \ldots, H_g$ that exactly correspond to the flows in $t_1, t_3, \ldots$. We shall encounter the same problem in the case of the PI hierarchy.

7 Hamiltonian structure of Lax equations

We use the same Poisson structure as used for the Mumford system [18, 19]. This Poisson structure is defined on the $3g+1$-dimensional moduli space of the matrix $V(\lambda)$ with coordinates $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{g+1}$. It is customary to use the tensor notation

$$\{V(\lambda) \overset{\otimes}{,} V(\mu)\} = \sum_{a,b,c,d=1,2} \{V_{ab}(\lambda), V_{cd}(\lambda)\} E_{ab} \otimes E_{cd},$$

where E_{ab} denote the usual basis of 2×2 matrices. The Poisson brackets of the matrix elements of $V(\lambda)$ can be thereby written in a compact form as

$$\{V(\lambda) \overset{\otimes}{,} V(\mu)\} = [V(\lambda) \otimes I + I \otimes V(\mu), r(\lambda-\mu)] + [V(\lambda) \otimes I - I \otimes V(\mu), E_{21} \otimes E_{21}],$$

where $r(\lambda - \mu)$ is the standard rational r-matrix

$$r(\lambda - \mu) = \frac{P}{\lambda - \mu}, \qquad P = \sum_{a,b=1,2} E_{ab} \otimes E_{ba}.$$

This is a version of the 'generalized linear brackets' [20]. More explicitly,

$$\begin{aligned}
&\{\alpha(\lambda), \alpha(\mu)\} = 0, \qquad \{\beta(\lambda), \beta(\mu)\} = 0,\\
&\{\alpha(\lambda), \beta(\mu)\} = \frac{\beta(\lambda) - \beta(\mu)}{\lambda - \beta}, \qquad \{\alpha(\lambda), \gamma(\mu)\} = -\frac{\gamma(\lambda) - \gamma(\mu)}{\lambda - \mu},\\
&\{\beta(\lambda), \gamma(\mu)\} = 2\frac{\alpha(\lambda) - \alpha(\mu)}{\lambda - \mu}, \qquad \{\gamma(\lambda), \gamma(\mu)\} = -2\alpha(\lambda) + 2\alpha(\mu).
\end{aligned} \tag{7.1}$$

We can convert the Lax equations (5.7) of the PI hierarchy to a Hamiltonian form with respect to this Poisson structure. This procedure is fully parallel to the case of the Mumford system.

A clue is the Poisson commutation relation

$$\{V(\lambda), h(\mu)\} = \left[V(\lambda), \frac{V(\mu)}{\lambda - \mu} + \beta(\mu) E_{21}\right], \tag{7.2}$$

which can be derived from (7.1) by straightforward calculations. One can derive from this relation the Poisson brackets $\{V(\lambda), I_{n+1}\}$ as follows.

Lemma 3.

$$\{V(\lambda), I_{n+1}\} = [V_n(\lambda), V(\lambda)], \tag{7.3}$$

where $V_n(\lambda)$ is a matrix of the form

$$V_n(\lambda) = \begin{pmatrix} \alpha_n(\lambda) & \beta_n(\lambda) \\ \gamma_n(\lambda) & -\alpha_n(\lambda) \end{pmatrix}$$

with the matrix elements

$$\alpha_n(\lambda) = \big(\lambda^{n-g}\alpha(\lambda)\big)_{\geq 0}, \qquad \beta_n(\lambda) = \big(\lambda^{n-g}\beta(\lambda)\big)_{\geq 0},$$
$$\gamma_n(\lambda) = \big(\lambda^{n-g}\gamma(\lambda)\big)_{\geq 0} - \beta_{n+1}, \qquad n = 0, 1, \ldots, g-1.$$

Proof. I_{n+1} can be extracted from $h(\mu)$ by a contour integral of the form

$$I_{n+1} = \oint \frac{d\mu}{2\pi i}\mu^{n-g}h(\mu),$$

where the contour is understood to be a circle around $\mu = \infty$. The same contour integral applied to (7.2) yields the Poisson bracket in question:

$$\{V(\lambda), I_{n+1}\} = \oint \frac{d\mu}{2\pi i}\mu^{n-g}\{V(\lambda), h(\mu)\} = \left[V(\lambda), \oint \frac{d\mu}{2\pi i}\frac{\mu^{n-g}V(\mu)}{\lambda-\mu} + \beta_{n+1}E_{21}\right].$$

Let us examine the matrix

$$V_n(\lambda) = -\oint \frac{d\mu}{2\pi i}\frac{\mu^{n-g}V(\mu)}{\lambda-\mu} - \beta_{n+1}E_{21}.$$

Since this type of contour integral gives, up to signature, the polynomial part of a Laurent series $f(\lambda)$ as

$$\oint \frac{d\mu}{2\pi i}\frac{f(\mu)}{\lambda-\mu} = -\big(f(\lambda)\big)_{\geq 0},$$

$V_n(\lambda)$ can be expressed as

$$V_n(\lambda) = \big(\lambda^{n-g}V(\lambda)\big)_{\geq 0} - \beta_{n+1}E_{21}.$$

The statement of the lemma follows from this expression of $V_n(\lambda)$. ■

Actually, this result is not what we really want – we have to derive $[U_n(\lambda), V(\lambda)]$ rather than $[V_n(\lambda), V(\lambda)]$. Here we need another clue, which is the following linear relations among $U_n(\lambda)$'s and $V_n(\lambda)$'s.

Lemma 4.

$$V_0(\lambda) = U_0(\lambda),$$
$$V_n(\lambda) = U_n(\lambda) + c_1(t)U_{n-1}(\lambda) + \cdots + c_n(t)U_0(\lambda), \qquad n = 1, \ldots, g-1. \tag{7.4}$$

Proof. The linear relations (3.10) among β_n's and R_n's imply the linear relations

$$\begin{aligned}\beta_n(\lambda) &= R_n(\lambda) + c_1(t)R_{n-1}(\lambda) + \cdots + c_n(t)R_0(\lambda)\\ &= \mathrm{B}_n(\lambda) + c_1(t)\mathrm{B}_{n-1}(\lambda) + \cdots + c_n(t)\mathrm{B}_0(\lambda)\end{aligned}$$

of the auxiliary polynomials. On the other hand, $\alpha(\lambda)$ and $\gamma(\lambda)$ s are connected with $\beta(\lambda)$ as

$$\alpha(\lambda) = -\frac{1}{2}\beta(\lambda)_x, \qquad \gamma(\lambda) = -\frac{1}{2}\beta(\lambda)_{xx} + (\lambda - u)\beta(\lambda),$$

Expanding these relations in powers of λ and picking out the terms contained $\alpha_n(\lambda)$ and $\gamma_n(\lambda)$, one can see that $\alpha_n(\lambda)$ and $\gamma_n(\lambda)$ are linearly related to $\mathrm{A}_n(\lambda)$'s and $\Gamma_n(\lambda)$'s with the same coefficients as

$$\alpha_n(\lambda) = \mathrm{A}_n(\lambda) + c_1(t)\mathrm{A}_{n-1}(\lambda) + \cdots + c_n(t)\mathrm{A}_0(\lambda),$$
$$\gamma_n(\lambda) = \Gamma_n(\lambda) + c_1(t)\Gamma_{n-1}(\lambda) + \cdots + c_n(t)\Gamma_0(\lambda).$$

These are exactly the linear relations presented in (7.4) in a matrix form. ■

In view of this lemma, we define new Hamiltonians $H_1, \dots, H_g$ by the (triangular) linear equations

$$I_1 = H_1, \qquad I_{n+1} = H_{n+1} + c_1(t)H_n + \cdots + c_n(t)H_1, \qquad n = 1, \dots, g-1. \tag{7.5}$$

Note that H_{g+1} is not defined (because I_{g+1} does not exist). The foregoing formula (7.3) of the Poisson brackets of $V(\lambda)$ and I_n's can be thereby converted to the form

$$\{V(\lambda), H_{n+1}\} = [U_n(\lambda), V(\lambda)]$$

that we have sought for. We thus eventually obtain the following result.

Theorem 2. *Except for the t_{2g+1}-flow, the matrix Lax equations* (5.7) *of the PI hierarchy can be cast into the Hamiltonian form*

$$\partial_{2n+1}V(\lambda) = \{V(\lambda), H_{n+1}\} + U_n'(\lambda), \qquad n = 0, 1, \dots, g-1, \tag{7.6}$$

with the Hamiltonians defined by (7.5)*.*

Remark 6. As regards the excluded t_{2g+1}-flow, the polynomial $h(\lambda)$ obviously contains no candidate of Hamiltonian. If we naively extrapolate (7.5) to $n = g$, we end up with the linear relation

$$0 = I_{g+1} = H_{g+1} + c_1(t)H_g + \cdots + c_g(t)H_1$$

of the Hamiltonians. In a sense, this is a correct statement, which says that H_{g+1} is not an independent Hamiltonian.

Remark 7. $I_0(\lambda)$ is a central element (i.e., a Casimir function) of the Poisson algebra. To see this, note that the right hand side of (7.2) is of order $O(\mu^{g-1})$ as $\mu \to \infty$. This implies that the terms of degree greater than g in $h(\mu)$ have no contribution to $\{V(\lambda), h(\mu)\}$, in other words,

$$\{V(\lambda), I_0(\mu)\} = 0. \tag{7.7}$$

This means that $I_0(\mu)$ is a Casimir function. This fact is in accord with the observation in the last section that $I_0(\lambda)$ does not contain genuine dynamical variables.

Remark 8. I_n's are Poisson-commuting, i.e., $\{I_j, I_k\} = 0$. This is a consequence of another basic Poisson relation

$$\{h(\lambda), h(\mu)\} = 0,$$

which, too, can be derived from (7.1) by straightforward calculations (or by a standard r-matrix technique).

8 Spectral Darboux coordinates

The construction of 'Spectral Darboux coordinates' is also parallel to the case of the Mumford system. These coordinates consist of the roots $\lambda_1, \dots, \lambda_g$ of $\beta(\lambda)$ and the values $\mu_1, \dots, \mu_g$ of $\alpha(\lambda)$ at these roots of $\beta(\lambda)$:

$$\beta(\lambda) = \prod_{j=1}^{g}(\lambda - \lambda_j), \qquad \mu_j = \alpha(\lambda_j), \qquad j = 1, \dots, g.$$

To avoid delicate problems, the following consideration is limited to a domain of the phase space where λ_j's are distinct.

λ_j and μ_j satisfy the equation of the spectral curve:

$$\mu_j^2 = h(\lambda_j).$$

We thus have a g-tuple $(\lambda_j, \mu_j)_{j=1}^g$ of points of the spectral curve (in other words, an effective divisor of degree g) that represents a point of the Jacobi variety of the spectral curve. In the case of the Mumford system, the commuting flows are thereby mapped to linear flows on the Jacobi variety [11, 12, 17]. The case of the PI hierarchy is more complicated because the spectral curve itself is dynamical. If one wishes to pursue this approach, one has to consider the coupled dynamics of both the divisor and the underlying spectral curve; unlike the case of isospectral problems, this does not reduce the complexity of dynamics of the original nonlinear problem. Actually, this is not what we seek for. We simply borrow the idea of spectral Darboux coordinates to describe the Hamiltonian structure of the system in question.

As it follows from (7.1) by a standard procedure [20, 26, 27], these new variables satisfy the canonical Poisson relations

$$\{\lambda_j, \lambda_k\} = 0, \qquad \{\mu_j, \mu_k\} = 0, \qquad \{\lambda_j, \mu_k\} = \delta_{jk}.$$

On the other hand, they Poisson-commute with $I_0(\lambda)$,

$$\{\lambda_j, I_0(\lambda)\} = 0, \qquad \{\mu_j, I_0(\lambda)\} = 0,$$

because $I_0(\lambda)$ is a Casimir function as (7.7) shows. Thus λ_j's and μ_j's may be literally called 'Darboux coordinates'.

These Darboux coordinates λ_j, μ_j and the coefficients of $\lambda^{2g}, \ldots, \lambda^g$ in $I_0(\lambda) = \lambda^{2g+1} + \cdots$ give an alternative (local) coordinate system of the $3g+1$-dimensional Poisson structure on the space of L-matrices with the original $g + g + (g + 1)$ coordinates γ_j, β_j, γ_j. To reconstruct the L-matrix $V(\lambda)$ from these new coordinates, we use the familiar Lagrange interpolation formula

$$f(\lambda) = \sum_{j=1}^{g} \frac{f(\lambda_j)}{\beta'(\lambda_j)} \frac{\beta(\lambda)}{\lambda - \lambda_j} \tag{8.1}$$

that holds for any polynomial $f(\lambda) = f_1\lambda^{g-1} + \cdots + f_g$ of degree less than g. Since

$$\frac{\beta(\lambda)}{\lambda - \lambda_j} = -\frac{\partial \beta(\lambda)}{\partial \lambda_j} = -\sum_{n=1}^{g} \frac{\partial \beta_n}{\partial \lambda_j} \lambda^{g-n},$$

this formula implies the formula

$$f_n = -\sum_{j=1}^{g} \frac{f(\lambda_j)}{\beta'(\lambda_j)} \frac{\partial \beta_n}{\partial \lambda_j} \tag{8.2}$$

for the coefficients of $f(\lambda)$ as well. Note that β_n's are understood here to be functions of λ_j's (in fact, they are elementary symmetric functions). We apply this formula (8.2) to $\alpha(\lambda)$ and obtain the explicit formula

$$\alpha_n = -\sum_{j=1}^{g} \frac{\mu_j}{\beta'(\lambda_j)} \frac{\partial \beta_n}{\partial \lambda_j}$$

that recovers α_n's from λ_j's and μ_j's. In a similar way, we apply (8.2) to the case where

$$f(\lambda) = \sum_{n=1}^{n} I_n \lambda^{g-n} = h(\lambda) - I_0(\lambda),$$

and find the expression

$$I_n = -\sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} \frac{\partial \beta_n}{\partial \lambda_j} \tag{8.3}$$

of I_n's in terms of λ_j's, μ_j's and $I_0(\lambda)$. For instance,

$$I_1 = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)}.$$

In these formulas, $I_0(\lambda)$ is understood to be the polynomial

$$\begin{aligned} I_0(\lambda) &= \lambda^{2g+1} + 2c_1(t)\lambda^{2g} + (2c_2(t) + c_1(t)^2)\lambda^{2g-1} + (2c_3(t) + 2c_1(t)c_2(t))\lambda^{2g-2} + \cdots \\ &\quad + \sum_{m=0}^{g} c_m(t)c_{g-m}(t)\lambda^{g+1} + \left(\sum_{m=1}^{g-1} c_m(t)c_{g+1-m}(t) + x\right)\lambda^g. \end{aligned}$$

Once $\alpha(\lambda)$ and $h(\lambda)$ are thus reconstructed, we can recover $\gamma(\lambda)$ as

$$\gamma(\lambda) = \frac{h(\lambda) - \alpha(\lambda)^2}{\beta(\lambda)}.$$

It is convenient to rewrite the foregoing formula (8.3) slightly. Recall the auxiliary polynomials

$$\beta_n(\lambda) = \lambda^n + \beta_1\lambda^{n-1} + \cdots + \beta_n.$$

Lemma 5.

$$\frac{\partial \beta_n}{\partial \lambda_j} = -\beta_{n-1}(\lambda_j), \qquad n = 1, \ldots, g. \tag{8.4}$$

Proof. Start from the identity

$$\frac{\partial \beta(\lambda)}{\partial \lambda_j} = -\frac{\beta(\lambda)}{\lambda - \lambda_j} = -\frac{\beta(\lambda) - \beta(\lambda_j)}{\lambda - \lambda_j}$$

and do substitution

$$-\beta_n \frac{\lambda^n - \lambda_j^n}{\lambda - \lambda_j} = -\beta_n(\lambda^{n-1} + \lambda_j\lambda^{n-2} + \cdots + \lambda_j^{n-2}\lambda + \lambda_j^{n-1})$$

for each term on the right hand side. This leads to the identity

$$\begin{aligned} \frac{\partial \beta(\lambda)}{\partial \lambda_j} &= -\lambda^{g-1} - (\lambda_j + \beta_1)\lambda^{g-2} - \cdots - (\lambda_j^{g-1} + \beta_1\lambda_j^{g-2} + \cdots + \beta_{g-1}) \\ &= -\lambda^{g-1} - \beta_1(\lambda_j)\lambda^{g-2} - \cdots - \beta_{g-1}(\lambda_j), \end{aligned}$$

which implies (8.4). ■

By these identities, we can rewrite (8.3) as

$$I_{n+1} = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} \beta_n(\lambda_j), \qquad n = 0, 1, \ldots, g-1.$$

(For notational convenience, n is shifted by one). We can derive, from these formulas, a similar expression of the Hamiltonians H_{n+1} introduced in the last section. Recall that β_n's are connected with R_n's by the linear relation (3.10). It is easy to see that the auxiliary polynomials $\beta_n(\lambda)$, too, are linearly related to the auxiliary polynomials $R_n(\lambda)$ as

$$\beta_n(\lambda) = R_n(\lambda) + c_1(t)R_{n-1}(\lambda) + \cdots + c_n(t)R_0(\lambda).$$

Comparing this linear relation with the linear relation (7.5) among I_n's and H_n's, we find that the Hamiltonians H_{n+1} can be expressed as

$$H_{n+1} = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} R_n(\lambda_j), \qquad n = 0, 1, \ldots, g-1.$$

Note that R_n's in this formula have to be redefined as functions of λ_j that satisfy the linear relations (3.10):

$$R_1 = \beta_1 - c_1(t), \qquad R_2 = \beta_2 - c_1(t)\beta_1 + c_1(t)^2 - c_2(t), \quad \ldots.$$

In particular, these Hamiltonians are time-dependent, the time-dependence stemming from both $I_0(\lambda)$ and $R_n(\lambda)$.

This is, however, not the end of the story. As it turns out below, these Hamiltonians (except $H_1 = I_1$) do *not* give correct equations of motion in the Darboux coordinates λ_j, μ_j. Correct Hamiltonians are obtained by adding correction terms to H_n's.

9 Hamiltonians in Darboux coordinates

9.1 Equations of motion in Darboux coordinates

Let us derive equations of motion for λ_j's and μ_j's from the Lax equations (5.7). In components, the Lax equations take the following form:

$$\begin{aligned}
\partial_{2n+1}\alpha(\lambda) &= \mathrm{B}_n(\lambda)\gamma(\lambda) - \beta(\lambda)\Gamma_n(\lambda) + \mathrm{A}'_n(\lambda),\\
\partial_{2n+1}\beta(\lambda) &= 2\mathrm{A}_n(\lambda)\beta(\lambda) - 2\mathrm{B}_n(\lambda)\alpha(\lambda) + \mathrm{B}'_n(\lambda),\\
\partial_{2n+1}\gamma(\lambda) &= 2\Gamma_n(\lambda)\alpha(\lambda) - 2\mathrm{A}_n(\lambda)\gamma(\lambda) + \Gamma'_n(\lambda).
\end{aligned} \tag{9.1}$$

To derive equations of motion for λ_j's, we differentiate the identity $\beta(\lambda_j) = 0$ by t_{2n+1}. By the chain rule, this yields the equation

$$\partial_{2n+1}\beta(\lambda)|_{\lambda=\lambda_j} + \beta'(\lambda_j)\partial_{2n+1}\lambda_j = 0.$$

By the second equation of (9.1), the first term on the right hand side can be expressed as

$$\partial_{2n+1}\beta(\lambda)|_{\lambda=\lambda_j} = -2\mathrm{B}_n(\lambda)\alpha(\lambda_j) + \mathrm{B}'_n(\lambda_j) = -2\mu_j\mathrm{B}_n(\lambda) + \mathrm{B}'_n(\lambda_j).$$

Thus the following equations are obtained for λ_j's:

$$\partial_{2n+1}\lambda_j = \frac{2\mu_j\mathrm{B}_n(\lambda_j)}{\beta'(\lambda_j)} - \frac{\mathrm{B}'_n(\lambda_j)}{\beta'(\lambda_j)}. \tag{9.2}$$

To derive equations of motion for μ_j's, we differentiate $\mu_j = \alpha(\lambda_j)$ by t_{2n+1}. The outcome is the equation

$$\partial_{2n+1}\mu_j = \partial_{2n+1}\alpha(\lambda)|_{\lambda=\lambda_j} + \alpha'(\lambda_j)\partial_{2n+1}\lambda_j.$$

By the first equation of (9.1), the first term on the right hand side can be expressed as

$$\partial_{2n+1}\alpha(\lambda)|_{\lambda=\lambda_j} = \mathrm{B}_n(\lambda_j)\gamma(\lambda_j) + \mathrm{A}'_n(\lambda_j).$$

The derivative $\partial_{2n+1}\lambda_j$ in the second term can be eliminated by (9.2). We can thus rewrite the foregoing equation as

$$\partial_{2n+1}\mu_j = \frac{2\mu_j\alpha'(\lambda_j) + \beta'(\lambda_j)\gamma(\lambda_j)}{\beta'(\lambda_j)}\mathrm{B}_n(\lambda_j) - \frac{\alpha'(\lambda_j)\mathrm{B}'_n(\lambda_j)}{\beta'(\lambda_j)} + \mathrm{A}'_n(\lambda_j).$$

Note here that the numerator of the first term on the right hand side is just the value of $h'(\lambda)$ at $\lambda = \lambda_j$:

$$h'(\lambda_j) = 2\alpha(\lambda_j)\alpha'(\lambda_j) + \beta'(\lambda_j)\gamma(\lambda_j) + \beta(\lambda_j)\gamma'(\lambda_j) = 2\mu_j\alpha'(\lambda_j) + \beta'(\lambda_j)\gamma(\lambda_j).$$

Thus the following equations of motion are obtained for μ_j's:

$$\partial_{2n+1}\mu_j = \frac{h'(\lambda_j)\mathrm{B}_n(\lambda_j)}{\beta'(\lambda_j)} - \frac{\alpha'(\lambda_j)\mathrm{B}'_n(\lambda_j)}{\beta'(\lambda_j)} + \mathrm{A}'_n(\lambda_j). \tag{9.3}$$

9.2 Why Hamiltonians need corrections

We now start from the Hamiltonian form (7.6) of the Lax equations and repeat similar calculations. In the case of isospectral Lax equations, such as the Mumford system, this procedure should lead to a Hamiltonian form of equations of motion for the spectral Darboux coordinates. In components, (7.6) consist of the following three sets of equations:

$$\begin{aligned}
\partial_{2n+1}\alpha(\lambda) &= \{\alpha(\lambda), H_{n+1}\} + \mathrm{A}'_n(\lambda),\\
\partial_{2n+1}\beta(\lambda) &= \{\beta(\lambda), H_{n+1}\} + \mathrm{B}'_n(\lambda),\\
\partial_{2n+1}\gamma(\lambda) &= \{\gamma(\lambda), H_{n+1}\} + \Gamma'_n(\lambda).
\end{aligned} \tag{9.4}$$

We again start from the identity

$$0 = \partial_{2n+1}\beta(\lambda_j) = \partial_{2n+1}\beta(\lambda)|_{\lambda=\lambda_j} + \beta'(\lambda_j)\partial_{2n+1}\lambda_j$$

and consider the second equation of (9.4), which implies that

$$\partial_{2n+1}\beta(\lambda)|_{\lambda=\lambda_j} = \{\beta(\lambda), H_{n+1}\}|_{\lambda=\lambda_j} + \mathrm{B}'_n(\lambda_j).$$

To calculate the first term on the right hand side, we use the identity

$$0 = \{\beta(\lambda_j), H_{n+1}\} = \{\beta(\lambda), H_{n+1}\}|_{\lambda=\lambda_j} + \beta'(\lambda_j)\{\lambda_j, H_{n+1}\}.$$

Thus the following equations of motion are obtained for λ_j's:

$$\partial_{2n+1}\lambda_j = \{\lambda_j, H_{n+1}\} - \frac{\mathrm{B}'_n(\lambda_j)}{\beta'(\lambda_j)} \tag{9.5}$$

In much the same way, we can derive the following equations of motion for μ_j's:

$$\partial_{2n+1}\mu_j = \{\mu_j, H_{n+1}\} - \frac{\alpha'(\lambda_j)\mathrm{B}'_n(\lambda_j)}{\beta'(\lambda_j)} + \mathrm{A}'_n(\lambda_j) \tag{9.6}$$

These results clearly show that H_{n+1} is not a correct Hamiltonian. If we could find a correct Hamiltonian, say K_{n+1}, the equations of motion would take the canonical form

$$\partial_{2n+1}\lambda_j = \{\lambda_j, K_{n+1}\}, \qquad \partial_{2n+1}\mu_j = \{\mu_j, K_{n+1}\}.$$

(9.5) and (9.6) fail to take this form because of extra terms on the right hand side. These terms stem from the extra term $U_n'(\lambda)$ in the Lax equations (5.7). When we say (5.7) is 'Hamiltonian', we ignore the presence of this term. It, however, cannot be ignored if we attempt to formulate the Hamiltonian structure in the language of the spectral Darboux coordinates λ_j and μ_j.

The case of $n = 0$ is exceptional. Since $\mathrm{B}_0(\lambda) = 1$, the extra terms on the right hand side of (9.2) and (9.3) disappear. Therefore

$$H_1 = I_1 = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)}$$

is a correct Hamiltonian for the equations of motion for the t_1-flow (namely, the higher order PI equation itself). Note, however, that the extra term $U_0'(\lambda)$ in the Lax equation still persist.

Remark 9. Another consequence of the foregoing calculations is that the Poisson brackets of H_{n+1} and the Darboux coordinates (in other words, the components of the Hamiltonian vector field of H_{n+1}) are given by

$$\{\lambda_j, H_{n+1}\} = \frac{2\mu_j \mathrm{B}_n(\lambda_j)}{\beta'(\lambda_j)}, \qquad \{\mu_j, H_{n+1}\} = \frac{h'(\lambda_j)\mathrm{B}_n(\lambda_j)}{\beta'(\lambda_j)}.$$

Consequently, the Poisson brackets of I_{n+1} and the Darboux coordinates turn out to be given by

$$\{\lambda_j, I_{n+1}\} = \frac{2\mu_j \beta_n(\lambda_j)}{\beta'(\lambda_j)}, \qquad \{\mu_j, I_{n+1}\} = \frac{h'(\lambda_j)\beta_n(\lambda_j)}{\beta'(\lambda_j)}.$$

One can derive these results directly from the Poisson brackets (7.1) of the matrix elements of $V(\lambda)$ as well.

9.3 Corrected Hamiltonians

We now seek for correct Hamiltonians in such a form as

$$K_{n+1} = H_{n+1} + \Delta H_{n+1}.$$

Correction terms ΔH_{n+1} have to be chosen to satisfy the conditions

$$\{\lambda_j, \Delta H_{n+1}\} = -\frac{\mathrm{B}_n'(\lambda_j)}{\beta'(\lambda_j)}, \qquad \{\mu_j, \Delta H_{n+1}\} = -\frac{\alpha'(\lambda_j)\mathrm{B}_n'(\lambda_j)}{\beta'(\lambda_j)} + \mathrm{A}_n'(\lambda_j).$$

We can convert this problem to that of I_{n+1}'s, namely, the problem to identify the correction term to I_{n+1}'s:

$$\Delta I_{n+1} = \Delta H_{n+1} + c_1(t)\Delta H_n + \cdots + c_{n-1}(t)\Delta H_1. \tag{9.7}$$

The conditions for to ΔI_{n+1} read

$$\{\lambda_j, \Delta I_{n+1}\} = -\frac{\beta_n'(\lambda_j)}{\beta'(\lambda_j)}, \qquad \{\mu_j, \Delta I_{n+1}\} = -\frac{\alpha'(\lambda_j)\beta_n'(\lambda_j)}{\beta'(\lambda_j)} + \alpha_n'(\lambda_j)$$

or, equivalently,

$$\frac{\partial \Delta I_{n+1}}{\partial \mu_j} = -\frac{\beta_n'(\lambda_j)}{\beta'(\lambda_j)}, \qquad \frac{\partial \Delta I_{n+1}}{\partial \lambda_j} = \frac{\alpha'(\lambda_j)\beta_n'(\lambda_j)}{\beta'(\lambda_j)} - \alpha_n'(\lambda_j). \tag{9.8}$$

This slightly simplifies the nature of the problem.

The goal of the subsequent consideration is to prove that a correct answer to this question is given by

$$\Delta I_{n+1} = -\sum_{k=1}^{g} \frac{\mu_k \beta_n'(\lambda_k)}{\beta'(\lambda_k)}. \tag{9.9}$$

Obviously, the first half of (9.8) is satisfied; what remains is to check the second half.

We can use the following lemma to reduce the problem to each term of the sum in (9.9).

Lemma 6.

$$\alpha'(\lambda_j) = -\sum_{k=1}^{g} \frac{\mu_k}{\beta'(\lambda_k)} \frac{\partial \beta'(\lambda)}{\partial \lambda_k}\bigg|_{\lambda=\lambda_j}, \qquad \alpha_n'(\lambda_j) = -\sum_{k=1}^{g} \frac{\mu_k}{\beta_n'(\lambda_k)} \frac{\partial \beta_n'(\lambda)}{\partial \lambda_k}\bigg|_{\lambda=\lambda_j}.$$

Proof. By the Lagrange interpolation formula (8.1), $\alpha(\lambda)$ can be expressed as

$$\alpha(\lambda) = -\sum_{k=1}^{g} \frac{\mu_k}{\beta'(\lambda_k)} \frac{\partial \beta(\lambda)}{\partial \lambda_k}.$$

Applying the projection operator $(\lambda^{n-g} \cdot\,)$ to both hand sides yields another identity

$$\alpha_n(\lambda) = -\sum_{k=1}^{g} \frac{\mu_k}{\beta_n'(\lambda_k)} \frac{\partial \beta_n(\lambda)}{\partial \lambda_k}.$$

The statement of the lemma follows by differentiating both hand sides of these identities and by setting $\lambda = \lambda_j$. ■

Because of this lemma, checking the second half of (9.8) can be reduced to proving the following identity:

$$\frac{\partial}{\partial \lambda_j} \frac{\beta_n'(\lambda_k)}{\beta'(\lambda_k)} = \frac{1}{\beta'(\lambda_k)} \left(\frac{\beta_n'(\lambda_j)}{\beta'(\lambda_j)} \frac{\partial \beta'(\lambda)}{\partial \lambda_k} - \frac{\partial \beta_n'(\lambda)}{\partial \lambda_k} \right)\bigg|_{\lambda=\lambda_j} \tag{9.10}$$

Note that this is a genuinely algebraic problem (related to elementary symmetric functions). We prepare some technical lemmas.

Lemma 7.

$$\frac{\partial^2 \beta_n}{\partial \lambda_j \partial \lambda_k} = -\frac{1}{\lambda_j - \lambda_k} \left(\frac{\partial \beta_n}{\partial \lambda_j} - \frac{\partial \beta_n}{\partial \lambda_k} \right) \qquad (j \neq k), \qquad \frac{\partial^2 \beta_n}{\partial \lambda_k^2} = 0. \tag{9.11}$$

Proof. Differentiate

$$\frac{\partial \beta(\lambda)}{\partial \lambda_k} = -\prod_{l \neq k} (\lambda - \lambda_l)$$

once again by λ_j. If $j \neq k$, this yields the identity

$$\frac{\partial^2 \beta(\lambda)}{\partial \lambda_j \partial \lambda_k} = \prod_{l \neq j,k} (\lambda - \lambda_l) = \frac{\beta(\lambda)}{(\lambda - \lambda_j)(\lambda - \lambda_k)} = -\frac{1}{\lambda_j - \lambda_k} \left(\frac{\partial \beta(\lambda)}{\partial \lambda_j} - \frac{\partial \beta(\lambda)}{\partial \lambda_k} \right),$$

proving the first part of (9.11). Similarly, if $j = k$, the outcome is the identity $\partial^2 \beta(\lambda)/\partial \lambda_k^2 = 0$, which implies the rest of (9.11). ■

Lemma 8.

$$\beta_n'(\lambda_k) = -\frac{\partial\beta_n}{\partial\lambda_k} - \frac{\partial\beta_{n-1}}{\partial\lambda_k}\lambda_k - \cdots - \frac{\partial\beta_1}{\partial\lambda_k}\lambda_k^{n-1}. \tag{9.12}$$

Proof. $\beta_n'(\lambda)$ can be expressed as

$$\begin{aligned}\beta_n'(\lambda) &= n\lambda^{n-1} + (n-1)\beta_1\lambda^{n-2} + \cdots + \beta_{n-1}\\ &= (\lambda^{n-1} + \beta_1\lambda^{n-2} + \cdots + \beta_{n-1}) + \cdots + (\lambda + \beta_1)\lambda^{n-2} + \lambda^{n-1}\\ &= \beta_{n-1}(\lambda) + \beta_{n-2}(\lambda)\lambda + \cdots + \beta_0(\lambda).\end{aligned}$$

Upon substituting $\lambda = \lambda_k$ and recalling (8.4), (9.12) follows. ■

Lemma 9.

$$\begin{aligned}\frac{\partial\beta_n'(\lambda_k)}{\partial\lambda_j} &= \frac{1}{\lambda_j - \lambda_k}\left(\frac{\partial\beta_n(\lambda_k)}{\partial\lambda_j} + \beta_n'(\lambda_k)\right) \qquad (j \neq k),\\ \frac{\partial\beta_n'(\lambda_k)}{\partial\lambda_k} &= \frac{1}{2}\beta_n''(\lambda_k).\end{aligned} \tag{9.13}$$

Proof. If $j \neq k$, (9.12) and (9.11) imply that

$$\begin{aligned}\frac{\partial\beta_n'(\lambda_k)}{\partial\lambda_j} &= -\frac{\partial^2\beta_n}{\partial\lambda_j\partial\lambda_k} - \frac{\partial^2\beta_{n-1}}{\partial\lambda_j\partial\lambda_k}\lambda_k - \cdots - \frac{\partial^2\beta_1}{\partial\lambda_j\partial\lambda_k}\lambda_k^{n-1}\\ &= \frac{1}{\lambda_j - \lambda_k}\left(\frac{\partial\beta_n}{\partial\lambda_j} - \frac{\partial\beta_n}{\partial\lambda_k}\right) + \cdots + \left(\frac{\partial\beta_1}{\partial\lambda_j} - \frac{\partial\beta_1}{\partial\lambda_k}\right)\lambda_k^{n-1}\\ &= \frac{1}{\lambda_j - \lambda_k}\left(\frac{\partial\beta_n}{\partial\lambda_j} + \frac{\partial\beta_{n-1}}{\partial\lambda_j}\lambda_k + \cdots + \frac{\partial\beta_1}{\partial\lambda_j}\lambda_k^{n-1}\right)\\ &\quad - \frac{1}{\lambda_j - \lambda_k}\left(\frac{\partial\beta_n}{\partial\lambda_k} + \frac{\partial\beta_{n-1}}{\partial\lambda_k}\lambda_k + \cdots + \frac{\partial\beta_1}{\partial\lambda_k}\lambda_k^{n-1}\right).\end{aligned}$$

Obviously,

$$\frac{\partial\beta_n}{\partial\lambda_j} + \frac{\partial\beta_{n-1}}{\partial\lambda_j}\lambda_k + \cdots + \frac{\partial\beta_1}{\partial\lambda_j}\lambda_k^{n-1} = \frac{\partial\beta_n(\lambda_k)}{\partial\lambda_j},$$

and by (9.12),

$$\frac{\partial\beta_n}{\partial\lambda_k} + \frac{\partial\beta_{n-1}}{\partial\lambda_k}\lambda_k + \cdots + \frac{\partial\beta_1}{\partial\lambda_k}\lambda_k^{n-1} = -\beta_n'(\lambda_k).$$

Thus the first part of (9.13) follows. If $j = k$, (9.12) and (9.11) imply that

$$\frac{\partial\beta_n'(\lambda_k)}{\partial\lambda_k} = -\frac{\partial\beta_{n-1}}{\partial\lambda_k} - 2\frac{\partial\beta_{n-2}}{\partial\lambda_k}\lambda_k - \cdots - (n-1)\frac{\partial\beta_1}{\partial\lambda_k}\lambda_k^{n-1}.$$

On the other hand, differentiating the identity

$$\beta_n'(\lambda) = \beta_{n-1}(\lambda) + \beta_{n-2}(\lambda)\lambda + \cdots + \beta_0(\lambda)\lambda^{n-1}$$

(which has been used in the proof of (9.12)) yields

$$\begin{aligned}\beta_n''(\lambda) &= \beta_{n-1}'(\lambda) + \beta_{n-2}'(\lambda)\lambda + \cdots + \beta_1'(\lambda)\lambda^{n-1}\\ &\quad + \beta_{n-2}(\lambda) + 2\beta_{n-3}(\lambda)\lambda + \cdots + (n-1)\beta_0(\lambda)\lambda^{n-2}.\end{aligned}$$

One can eliminate the derivatives $\beta'_{n-1}(\lambda), \dots, \beta'_1(\lambda)$ by the preceding identity itself. The outcome reads

$$\beta''_n(\lambda) = 2\big(\beta_{n-2}(\lambda) + 2\beta_{n-3}(\lambda)\lambda + \cdots + (n-1)\beta_0(\lambda)\lambda^{n-2}\big).$$

Upon substituting $\lambda = \lambda_k$ and using (8.4), one finds that

$$\begin{aligned}\beta''_n(\lambda_k) &= 2\big(\beta_{n-2}(\lambda_k) + 2\beta_{n-3}(\lambda_k)\lambda_k + \cdots + (n-1)\beta_0(\lambda_k)\lambda_k^{n-2}\big)\\ &= -2\left(\frac{\partial\beta_{n-1}}{\partial\lambda_k} + 2\frac{\partial\beta_{n-2}}{\partial\lambda_k}\lambda_k + \cdots + (n-1)\frac{\partial\beta_1}{\partial\lambda_k}\lambda_k^{n-2}\right).\end{aligned}$$

The last identity and the foregoing expression of $\partial\beta'_n(\lambda_k)/\partial\lambda_k$ lead to the second part of (9.13). ■

Using these lemmas, we can calculate both hand sides of (9.10).

Let us first consider the case of $j \neq k$. The left hand side of (9.10) can be calculated by the Leibniz rule and (9.13). Note here that the formula (9.13) for $n = g$ takes such a form as

$$\frac{\partial\beta'(\lambda_k)}{\partial\lambda_j} = \frac{\beta'(\lambda_j)}{\lambda_j - \lambda_k}$$

because $\beta_g(\lambda_k) = \beta(\lambda_k) = 0$. The outcome of this calculation reads

$$\frac{\partial}{\partial\lambda_j}\frac{\beta'_n(\lambda_k)}{\beta'(\lambda_k)} = \frac{1}{\beta'(\lambda_k)(\lambda_j - \lambda_k)}\frac{\partial\beta_n(\lambda_k)}{\partial\lambda_j}.$$

As regards the right hand side of (9.10), we can use (9.13) to calculate the derivatives in the parentheses as

$$\begin{aligned}\left.\frac{\partial\beta'(\lambda)}{\partial\lambda_k}\right|_{\lambda=\lambda_j} &= \frac{\partial\beta'(\lambda_j)}{\partial\lambda_k} = \frac{\beta'(\lambda_j)}{\lambda_k - \lambda_j},\\ \left.\frac{\partial\beta'_n(\lambda)}{\partial\lambda_k}\right|_{\lambda=\lambda_j} &= \frac{\partial\beta'_n(\lambda_j)}{\partial\lambda_k} = \frac{1}{\lambda_k - \lambda_j}\left(\frac{\partial\beta_n(\lambda_j)}{\partial\lambda_k} + \beta'_n(\lambda_j)\right).\end{aligned}$$

Consequently,

$$\frac{1}{\beta'(\lambda_k)}\left.\left(\frac{\beta'_n(\lambda_j)}{\beta'(\lambda_j)}\frac{\partial\beta'(\lambda)}{\partial\lambda_k} - \frac{\partial\beta'_n(\lambda)}{\partial\lambda_k}\right)\right|_{\lambda=\lambda_j} = \frac{1}{\beta'(\lambda_k)(\lambda_j - \lambda_k)}\frac{\partial\beta_n(\lambda_j)}{\partial\lambda_k}.$$

Since (8.4) implies that

$$\frac{\partial\beta_n(\lambda_j)}{\partial\lambda_k} = -\frac{\partial^2\beta_{n+1}}{\partial\lambda_j\partial\lambda_k} = \frac{\partial\beta_n(\lambda_k)}{\partial\lambda_j},$$

we eventually find that (9.10) holds for the case of $j \neq k$.

The case of $j = k$ can be treated in much the same way, and turns out to be simpler. The left hand side of (9.10) can be calculated as

$$\frac{\partial}{\partial\lambda_k}\frac{\beta'_n(\lambda_k)}{\beta'(\lambda_k)} = \frac{1}{2}\frac{\beta''_n(\lambda_k)}{\beta'(\lambda_k)} - \frac{1}{2}\frac{\beta'_n(\lambda_k)\beta''(\lambda_k)}{\beta'(\lambda_k)^2}.$$

The derivatives in the parentheses on the right hand side of (9.10) can be expressed as

$$\left.\frac{\partial\beta'(\lambda)}{\partial\lambda_k}\right|_{\lambda=\lambda_k} = \frac{\partial\beta'(\lambda_k)}{\partial\lambda_k} - \beta''(\lambda_k) = -\frac{1}{2}\beta''(\lambda_k),$$

$$\left.\frac{\partial\beta_n'(\lambda)}{\partial\lambda_k}\right|_{\lambda=\lambda_k} = \frac{\partial\beta_n'(\lambda)}{\partial\lambda_k} - \beta_n''(\lambda_k) = -\frac{1}{2}\beta_n''(\lambda_k).$$

Thus (9.10) turns out to hold in this case, too.

We have thus confirmed that (9.9) does satisfy (9.8). Note that (9.9) corresponds to the correction terms

$$\Delta H_{n+1} = -\sum_{j=1}^{g} \frac{\mu_j R_n'(\lambda_j)}{\beta'(\lambda_j)}$$

for H_{n+1} by the linear relation (9.7), because $\beta_n'(\lambda)$'s and $R_n'(\lambda)$'s are linearly related with the same coefficients as $\beta_n(\lambda)$'s and $R_n(\lambda)$'s. These results can be summarized as follows.

Theorem 3. *Equations of motion* (9.2) *and* (9.3) *can be cast into the Hamiltonian form*

$$\partial_{2n+1}\lambda_j = \{\lambda_j, K_{n+1}\}, \qquad \partial_{2n+1}\mu_j = \{\mu_j, K_{n+1}\}.$$

The Hamiltonians are given by

$$K_{n+1} = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} R_n(\lambda_j) - \sum_{j=1}^{g} \frac{\mu_j R_n'(\lambda_j)}{\beta'(\lambda_j)}.$$

10 Examples

We illustrate the results of the preceding section for the cases of $g = 1, 2, 3$. For notational simplicity, we set $t_{2g+1} = 0$. Consequently, $c_1(t)$ disappears from various formulas.

10.1 $g = 1$

The case of $g = 1$ corresponds to the first Painlevé equation itself. There is no higher flow (other than the excluded exceptional time t_3). In this case, everything can be presented explicitly as follows.

1) $\beta(\lambda)$ is linear and $\alpha(\lambda)$ does not depend on λ:

$$\beta(\lambda) = \lambda + \beta_1, \qquad \alpha(\lambda) = \alpha_1,$$
$$\beta_1 = R_1 = \frac{u}{2}, \qquad \alpha_1 = -\frac{\beta_{1,x}}{2} = -\frac{1}{4}u_x.$$

2) The Darboux coordinates λ_1, μ_1 are given by

$$\lambda_1 = -\beta_1 = -\frac{u}{2}, \qquad \mu_1 = \alpha(\lambda_1) = -\frac{1}{4}u_x.$$

3) $h(\lambda)$ is a cubic polynomial of the form

$$h(\lambda) = I_0(\lambda) + I_1, \qquad I_0(\lambda) = \lambda^3 + x\lambda.$$

4) The Hamiltonian K_1 is equal to $H_1 = I_1$. As a function of the Darboux coordinates, I_1 can be expressed as

$$I_1 = \mu_1^2 - \lambda_1^3 - x\lambda_1,$$

which coincides with the well known Hamiltonian of the first Painlevé equation.

10.2 $g = 2$

This case corresponds to the two-dimensional 'degenerate Garnier system' studied by Kimura [33] and Shimomura [34]. A higher flow with time variables t_3 now enters the game. This variable t_3 shows up in the description of relevant quantities through

$$c_2(t) = \frac{3}{2}t_3.$$

For instance, the linear relations between β_n's and R_n' now read

$$\beta_1 = R_1, \ \beta_2 = R_2 + c_2(t).$$

Though slightly more complicated than the previous case, this case, too, can be treated explicitly.

1) $\beta(\lambda)$ and $\alpha(\lambda)$ are quadratic and linear, respectively:

$$\beta(\lambda) = \lambda^2 + \beta_1\lambda + \beta_2, \qquad \alpha(\lambda) = \alpha_1\lambda + \alpha_2.$$

2) The Darboux coordinates $\lambda_1, \lambda_2, \mu_1, \mu_2$ are defined as

$$\beta(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2), \qquad \mu_1 = \alpha(\lambda_1), \qquad \mu_2 = \alpha(\lambda_2).$$

3) $h(\lambda)$ is a quintic polynomial of the form

$$h(\lambda) = I_0(\lambda) + I_1\lambda + I_2, \qquad I_0(\lambda) = \lambda^5 + 2c_2(t)\lambda + x\lambda.$$

4) We still have the simple relations $H_1 = I_1$ and $H_2 = I_2$ between H_n's and I_n's. They are redefined as functions of the Darboux coordinates by the linear equations

$$I_1\lambda_1 + I_2 = \mu_1^2 - I_0(\lambda_1), \qquad I_2\lambda_2 + I_2 = \mu_2^2 - I_0(\lambda_2).$$

More explicitly,

$$I_1 = \frac{\mu_1^2 - I_0(\lambda_1)}{\lambda_1 - \lambda_2} + \frac{\mu_2^2 - I_0(\lambda_2)}{\lambda_2 - \lambda_2}, \qquad I_2 = \frac{\mu_1^2 - I_0(\lambda_1)}{\lambda_1 - \lambda_2}\lambda_2 - \frac{\mu_2^2 - I_0(\lambda_2)}{\lambda_2 - \lambda_2}\lambda_1.$$

The correct Hamiltonians K_1, K_2 are given by

$$K_1 = I_1, \qquad K_2 = I_2 - \frac{\mu_1}{\lambda_1 - \lambda_2} - \frac{\mu_2}{\lambda_2 - \lambda_1}.$$

Remark 10. Kimura and Shimomura studied this system as isomonodromic deformations of a second order scalar ODE rather than the 2×2 matrix system. In the present setting, their scalar ODE corresponds to the equation

$$\frac{\partial^2\psi}{\partial\lambda^2} + p_1(\lambda)\frac{\partial\psi}{\partial\lambda} + p_2(\lambda)\psi = 0$$

that can be obtained from (3.7) by eliminating the second component of $\boldsymbol{\psi}$. The coefficients $p_1(\lambda)$ and $p_2(\lambda)$ are given by

$$p_1(\lambda) = \frac{\beta'(\lambda)}{\beta(\lambda)}, \qquad p_2(\lambda) = -h(\lambda) - \alpha'(\lambda) + \alpha\frac{\beta'(\lambda)}{\beta(\lambda)}.$$

Remark 11. Actually, Kimura and Shimomura considered two Hamiltonian forms for their degenerate Garnier system. One of them is defined by the aforementioned Hamiltonians K_1, K_2. The other one is derived therefrom by a canonical transformation, and has polynomial Hamiltonians.

10.3 $g = 3$

The case of $g = 3$ is more complicated than the preceding two cases. Here we have two higher flows with time variables t_3, t_5. They are connected with $c_2(t)$ and $c_3(t)$ as

$$c_2(t) = \frac{5}{2}t_3, \qquad c_3(t) = \frac{3}{2}t_5.$$

$h(\lambda)$ is a sextic polynomial of the form

$$h(\lambda) = I_0(\lambda) + I_1\lambda^2 + I_2\lambda + I_3.$$

A new feature of this case is the structure of $I_0(\lambda)$:

$$I_0(\lambda) = \lambda^7 + 2c_2(t)\lambda^5 + 2c_3(t)\lambda^4 + (c_2(t)^2 + x)\lambda^3.$$

Note that the coefficient of λ^3 is now a quadratic polynomial of the time variables. Of course, if we consider the general case (6.9), this is a rather common situation; the first two cases ($g = 1$ and $g = 2$) are exceptional.

11 Conclusion

We have thus elucidated the Hamiltonian structure of the PI hierarchy for both the Lax equations and the equations of motion in the spectral Darboux coordinates. Though the extra terms $U'_{n+1}(\lambda)$ in the Lax equations give rise to extra terms in the equations of motion for the Darboux coordinates, these terms eventually boil down (somewhat miraculously) to the correction terms ΔH_{n+1} in the Hamiltonian.

The correction terms ΔH_{n+1} are identified by brute force calculations. It is highly desirable to derive this result in a more systematic way. As regards the Garnier system, such a systematic explanation is implicit in the work of the Montreal group [28, 29], and presented (in a more general form) by Dubrovin and Mazzocco [31]. Let us recall its essence.

As mentioned in Introduction, the Garnier system is equivalent to the 2×2 Schlesinger system. The L-matrix of the Schlesinger system is a 2×2 matrix of rational functions of the form

$$V(\lambda) = \sum_{j=1}^{N} \frac{A_j}{\lambda - t_j}.$$

The matrix A_j takes values in a two-dimensional coadjoint orbit of $\mathrm{sl}(2, \mathbf{C})$. This orbit, as a symplectic leaf, carries special Darboux coordinates ξ_j, η_j. A_j can be thereby written as

$$A_j = \frac{1}{2}\begin{pmatrix} \xi_j\eta_j & \xi_j^2 \\ -\eta_j^2 + \theta_j^2\xi_j^{-2} & -\xi_j\eta_j \end{pmatrix},$$

where θ_j is a constant that determines the orbit, and may be interpreted as a monodromy exponent at the regular singular point at $\lambda = t_j$. The Lax equations can be converted to a Hamiltonian system in these Darboux coordinates ξ_j, η_j. Since the spectral Darboux coordinates λ_j, μ_j are connected with these Darboux coordinates by a *time-dependent* canonical transformation, the Hamiltonians in the latter coordinates have extra terms.

Unfortunately, this beautiful explanation of extra terms does not literally apply to the present setting. The relevant Lie algebra for this case is not $\mathrm{sl}(2, \mathbf{C})$ but its loop algebra $\mathrm{sl}(2, \mathbf{C})[\lambda, \lambda^{-1}]$, coadjoint orbits of which are more complicated.

A similar idea, however, can be found in the recent work of Mazzocco and Mo [41] on an isomonodromic hierarchy related to the second Painlevé equation. They start from the Lie-Poisson structure of a loop algebra, and convert the Hamiltonian structure on a coadjoint orbit to a Hamiltonian system in Darboux coordinates by a time-dependent canonical transformation. It will be interesting to reconsider the present setting from that point of view.

Another remarkable aspect of the work of Mazzocco and Mo is that they present another set of Darboux coordinates alongside the spectral Darboux coordinates. Unlike the spectral Darboux coordinates, these coordinates are rational functions of the dynamical variables in the Lax equations; this is a desirable property in view of the Painlevé property of the system.

Actually, borrowing their idea, we can find a similar set of Darboux coordinates $\mathcal{Q}_n$, $\mathcal{P}_n$ for the PI hierarchy as

$$\mathcal{Q}_n = \beta_{g+1-n}, \qquad \mathcal{P}_n = \sum_{k=1}^{g} \frac{\partial p_n}{\partial \beta_k} \frac{\alpha_k}{n},$$

where p_k stands for the k-th power sum

$$p_k = \sum_{j=1}^{g} \lambda_j^k.$$

(Note that p_k's and β_j's are related by the generating functional relation

$$\sum_{k=1}^{\infty} \frac{p_k}{k\lambda^k} = -\log \frac{\beta(\lambda)}{\lambda^g},$$

so that p_k's may be thought of as polynomial functions of β_j's.) One can prove, in the same way as the case of Mazzocco and Mo, that $\mathcal{Q}_n$, $\mathcal{P}_n$ do satisfy the canonical Poisson relations. In fact, they turn out to satisfy the stronger relation

$$\sum_{j=1}^{g} d\lambda_j \wedge d\mu_j = \sum_{n=1}^{g} d\mathcal{Q}_n \wedge d\mathcal{P}_n,$$

which implies that $\mathcal{Q}_n$, $\mathcal{P}_n$ are connected with the spectral Darboux coordinates λ_j, μ_j by a *time-independent* canonical transformation. In particular, the canonical transformation yields no correction term to the transformed Hamiltonians. Namely, K_{n+1}'s persist to be correct Hamiltonians in the new coordinates $\mathcal{Q}_n$, $\mathcal{P}_n$ as well. Moreover, employing the Lagrange interpolation formula, one can see that K_{n+1}'s are polynomials in these coordinates (and the time variables). We thus obtain a generalization of Kimura's polynomial Hamiltonians for the degenerate Garnier systems [33].

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