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q-Boson in Quantum Integrable Systems*

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Abstract. q-bosonic realization of the underlying Yang–Baxter algebra is identified for a series of quantum integrable systems, including some new models like two-mode q-bosonic model leading to a coupled two-component derivative NLS model, wide range of q-deformed matter-radiation models, q-anyon model etc. Result on a new exactly solvable interacting anyon gas, linked to q-anyons on the lattice is reported.

Key words: quantum integrable systems; Yang–Baxter algebra; quantum group, q-bosonic integrable models; q-deformed matter-radiation models; q-anyon; derivative-δ-function anyon gas

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1 Introduction

In a quantum system the related commutation rules determine the character of basic operators involved in the model, e.g. boson, fermion, spin, anyon etc. When such quantum systems are integrable the commutation rules are determined by the underlying Yang–Baxter (YB) algebra obtained from the quantum Yang–Baxter equation (QYBE)

$$R(\lambda-\mu)L_j(\lambda)\otimes L_j(\mu)=(I\otimes L_j(\mu))(L_j(\lambda)\otimes I)R(\lambda-\mu), \qquad j=1,2,\dots,N \tag{1.1}$$

together with the ultralocality condition $L_j(\lambda)\otimes L_k(\mu)=(I\otimes L_k(\mu))(L_j(\lambda)\otimes I)$, at $k\neq j$, which are at the same time sufficient for the integrability of the system. Here $L_j(\lambda)$ is the representative Lax operator of the lattice (or discretized) quantum integrable systems (QIS) at each lattice site $j=1,2,\dots,N$ and $R(\lambda-\mu)$ is the quantum R-matrix (c-number matrix) which determines the structure constants of the YB algebra. For rational solution of the R-matrix one usually gets the spin algebra or its bosonic realization, while the trigonometric solution yields q-spin and q-bosons, which are comparatively new entries in the field of quantum physics, discovered mainly from the study of integrable systems [1, 2, 3].

The concept of q-boson was formally introduced as q-parameter deformation of a standard boson, or through two-mode realization of quantum algebra [4, 5, 6]. Subsequently, the finding of more application oriented single mode q-bosonic realization of quantum algebra through q-deformation of Holstein–Primakov transformation [7, 8, 9] and linking it to QIS [10] was a significant step. However, it seems that this initial but important step has not been pursued with enough zest to explore the possibility of appearance of q-boson in other QIS and its applications in exactly solvable physical models. It is true that there were attempts to construct interesting q-bosonic and q-spin models in various fields and to study their physical effects [7, 8, 11, 12, 13, 14, 15]. However most of such models could not be related to the underlying

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YB algebra of a QIS permitting exact treatment by the Bethe ansatz (BA) method. Our aim here therefore, is mainly to identify the q-bosonic mode in a variety of QIS, which is deeply rooted in discrete or lattice regularized models allowing BA solutions. We discover also the appearance of such modes in some new QIS, e.g. physically relevant integrable matter-radiation models, a coupled double-mode q-bosonic model leading to a two-component derivative NLS model etc. and furthermore by introducing a q-anyon like concept find its application in another new exactly solvable derivative-δ function 1D anyon gas.

2 q-boson and q-spin

We review in brief the basic formulation of the q-boson and its relation to various other objects like q-spin, standard boson, canonical variables etc. q-boson can be defined through a deformation of the bosonic commutation relation (CR):

$$b_q b_q^\dagger - q^{-1} b_q^\dagger b_q = q^N, \qquad [b_q, N] = b_q, \qquad [b_q^\dagger, N] = -b_q^\dagger, \tag{2.1}$$

which is supposed to be invariant under a reflection $q \to q^{-1}$. Therefore combining such two relations we can define the q-boson in a more symmetric form

$$[b_q, b_q^\dagger] = \frac{\cos(\alpha(2N+1))}{\cos\alpha}, \qquad [b_q, N] = b_q, \qquad [b_q^\dagger, N] = -b_q^\dagger, \tag{2.2}$$

where $q = e^{2i\alpha}$. At $q \to 1$ the q-boson goes into a standard boson with both the above relations reducing to the bosonic CR:

$$[b, b^\dagger] = 1, \qquad [b, N] = b, \qquad [b^\dagger, N] = -b^\dagger. \tag{2.3}$$

One can find also a direct mapping between q-deformed and undeformed bosons as

$$b_q = b f(N), \qquad b_q^\dagger = f(N) b^\dagger, \qquad f(N) = \left(\frac{[N]_q}{N}\right)^{\frac{1}{2}}, \qquad N = b^\dagger b, \tag{2.4}$$

where $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$. Using these relations and the usual representation of bosonic operators it is easy therefore to construct a q-bosonic representation [4, 5, 6]. Initially the q-boson was introduced as a two-mode realization of the q-spin (q-deformed Schwinger transformation)

$$s_q^+ = b_{q1}^\dagger b_{q2}, \qquad s_q^- = b_{q2}^\dagger b_{q1}, \qquad s^3 = \frac{1}{2}(N_1 - N_2)$$

linked to quantum algebra $su_q(2)$

$$[s_q^+, s_q^-] = \epsilon [2s^3]_q, \qquad [s^3, s_q^\pm] = \pm s_q^\pm, \tag{2.5}$$

for $\epsilon = 1$. $\epsilon = -1$ on the other hand gives noncompact algebra $su_q(1,1)$. Subsequently, a single mode realization was found through q-Holstein–Primakov (q-HP) transformation [7, 8, 9]

$$s^3 = s \mp N, \qquad s_q^+ = g_\pm(N,q) b_q, \qquad s_q^- = b_q^\dagger g_\pm(N,q), \qquad g_\pm^2(N,q) = [2s \mp N]_q, \tag{2.6}$$

for $su_q(2)$ and $su_q(1,1)$, respectively, with s-being a spin parameter. One can find also mapping from q-boson to canonical variables: $[u, p] = i$ as

$$b_q = e^{-ip} f_c(u), \qquad b_q^\dagger = f_c(u) e^{ip}, \qquad f_c^2(u) = [c - u]_q \tag{2.7}$$

with $c = \text{const}$.

3 q-deformed Yang–Baxter algebra

The Yang–Baxter algebra underlying QIS and providing guarantee for the integrability of a discrete or lattice regularized quantum system was found to be given in the q-deformed case by a general form [16, 17, 18]

$$[S_q^+, S_q^-] = \left(\hat{M}^+ \sin\left(2\alpha S^3\right) + \hat{M}^- \cos\left(2\alpha S^3\right)\right)\frac{1}{\sin\alpha},$$
$$[S^3, S_q^\pm] = \pm S_q^\pm, \qquad [\hat{M}^\pm, \cdot] = 0. \tag{3.1}$$

Note that this is a novel type of deformed algebra, which apart from an usual parameter q-deformation, also involves additional operator-deformation by central operators $\hat{M}^\pm$. Operator deformed quantum Yang–Baxter algebra (3.1) is a bialgebra and exhibits noncommutative Hopf algebra properties with the coproduct structures

$$\Delta(S_q^+) = S_q^+ \otimes \hat{c}_1^- q^{-S^3} + \hat{c}_2^- q^{S^3} \otimes S_q^+, \qquad \Delta(S_q^-) = S_q^- \otimes \hat{c}_2^+ q^{-S^3} + \hat{c}_1^+ q^{S^3} \otimes S_q^-,$$
$$\Delta(S^3) = I \otimes S^3 + S^3 \otimes I, \qquad \Delta(\hat{c}_a^\pm) = \hat{c}_a^\pm \otimes \hat{c}_a^\pm, \quad a = 1, 2, \tag{3.2}$$

where $\hat{c}_a^\pm$, $a = 1, 2$ are related to central operators as $\hat{M}^\pm = \frac{1}{2}(\hat{c}_1^+ \hat{c}_2^- \pm \hat{c}_1^- \hat{c}_2^+)$ and one redefines $q = e^{i\alpha}$. Note that unlike deforming parameter q, deforming central operators $\hat{c}_a^\pm$ and hence $\hat{M}^\pm$ have nontrivial coproduct.

The quantum Lax operator associated with this general YB algebra is

$$L_{\rm anc}(\xi) = \begin{pmatrix} \xi\hat{c}_1^+ e^{i\alpha S^3} + \xi^{-1}\hat{c}_1^- e^{-i\alpha S^3} & 2\sin\alpha S_q^- \\ 2\sin\alpha S_q^+ & \xi\hat{c}_2^+ e^{-i\alpha S^3} + \xi^{-1}\hat{c}_2^- e^{i\alpha S^3} \end{pmatrix}, \qquad \xi = e^{i\lambda}. \tag{3.3}$$

This ancestor model [16, 17, 18] seems to be general enough to generate all quantum integrable models with (2×2) Lax operator associated with the trigonometric $R^{\rm trig}$ matrix

$$R_{11}^{11} = R_{22}^{22} = \sin(\lambda + \alpha), \qquad R_{12}^{12} = R_{21}^{21} = \sin\lambda, \qquad R_{21}^{12} = R_{12}^{21} = \sin\alpha. \tag{3.4}$$

Therefore identifying the situations when the YB algebra (3.1) goes to different known algebras at different values of the deforming central operators, we can construct from the corresponding reductions of the Lax operator (3.3) a series of quantum integrable models solvable by the Bethe ansatz.

For example we may observe that in contrast to the well known q-deformed algebras (2.2), (2.5), algebra (3.1) has two different terms in the RHS of its main relation: the first one (sine-term) is similar to the quantum-spin algebra (2.5), while the second term (cosine-term) to the q-boson (2.2). Indeed both these q-deformed algebras can be obtained as subalgebras of (3.1). Notice that $\hat{M}^+ = 1$, $\hat{M}^- = 0$ yields quantum algebra $sl_q(2)$, while $\hat{M}^+ = -1$, $\hat{M}^- = 0$ gives the noncompact algebra $sl_q(1,1)$ as defined in (2.5). The Lax operators of the related integrable systems can be obtained directly from the general form (3.3), by using a compatible choice $\hat{c}_1^+ = \hat{c}_2^+ = 1$, $\hat{c}_1^- = \hat{c}_2^- = \epsilon$, $\epsilon = \pm 1$ as

$$L_{qs}(\xi) = \begin{pmatrix} \xi e^{i\alpha s^3} + \epsilon\xi^{-1} e^{-i\alpha s^3} & 2\sin\alpha s_q^- \\ 2\sin\alpha s_q^+ & \xi e^{-i\alpha s^3} + \epsilon\xi^{-1} e^{i\alpha s^3} \end{pmatrix}, \tag{3.5}$$

involving q-spin belonging to $sl_q(2)$ and $sl_q(1,1)$, for $\epsilon = \pm 1$, respectively.

On the other hand for the particular values of the deforming operators: $\hat{M}^+ = \sin\alpha$, $\hat{M}^- = i\cos\alpha$, one directly obtains the q-boson (2.2) as a subalgebra of (3.1) with a compatible choice $\hat{c}_1^+ = \hat{c}_2^+ = 1$, $\hat{c}_1^- = -iq$, $\hat{c}_2^- = \frac{i}{q}$. However note that such a choice of deforming operators, as seen from (3.2), seemingly does not respect the usual coproduct structure and hence the corresponding algebra representing q-boson exhibits the known difficulty in its formulation of the Hopf algebra. We get the q-bosonic Lax operator from the reduction of (3.3) as

$$L_{qb}(\xi) = \begin{pmatrix} \xi e^{i\alpha N} - i\xi^{-1} e^{-i\alpha(N-1)} & \kappa b_q^\dagger \\ \kappa b_q & \xi e^{-i\alpha N} + i\xi^{-1} e^{i\alpha(N-1)} \end{pmatrix}, \tag{3.6}$$

with a direct mapping:

$$S_q^+ = \frac{i\kappa}{\Omega} b_q, \qquad S_q^- = \frac{i\kappa}{\Omega} b_q^\dagger, \qquad S^3 = N, \qquad \Omega = q - q^{-1} \tag{3.7}$$

and $\kappa^2 = \frac{q^2 - q^{-2}}{4i}$. Interestingly if on the other hand we choose

$$\hat{c}_1^+ = \hat{c}_2^- = q^{\frac{1}{2}} \qquad \text{with} \qquad \hat{c}_1^- = \hat{c}_2^+ = 0 \tag{3.8}$$

or at least any one of $\hat{c}_1^-$, $\hat{c}_2^+$ is 0, by representing

$$S_q^+ = \tilde{\kappa} b_q q^{\frac{N}{2}}, \qquad S_q^- = \tilde{\kappa} q^{\frac{N}{2}} b_q^\dagger, \qquad S^3 = N, \qquad \tilde{\kappa}^2 = \frac{i}{\Omega}, \tag{3.9}$$

we can get a direct realization to the q-bosonic relations (2.1) as a subalgebra of (3.1). We construct also the corresponding Lax operator from the related reduction of (3.3). We see below that such q-bosonic models are intimately related to a series of interesting QIS including the quantum lattice Liouville model.

4 q-bosons in discrete QIS

We identify and present here mostly quantum integrable q-bosonic models which can be linked directly to Yang–Baxter algebra and solved exactly through algebraic Bethe ansatz.

4.1 q-Bose gas models

i) A quantum integrable model involving q-bosons constructed on a lattice was proposed in [10], which can be defined simply through Lax operator (3.5) involving q-spin operators $(s_q^\pm, s^3) \in sl_q(2)$. q-spin operators are in turn realized in single-mode q-boson b_q, $b_q^\dagger$ using q-HP transformation (2.6). This lattice q-Bose gas model associated with (3.5) and R^{trig} exactly satisfies the QYBE by construction (since a particular case of ancestor model (3.3)) and therefore the related eigenvalue problem can be solved exactly through algebraic Bethe ansatz (ABA). In analogy with the exact lattice model of nonlinear Schrödinger (NLS) equation [19] the q-Bose gas model on the lattice can be shown to have localized Hamiltonian [10], as obtained from the lattice NLS. However at the continuum limit the q-bosonic mode turns into bosonic field and the model goes to the same NLS field model.

ii) A q-harmonic oscillator model $H_q = \frac{\hbar\omega}{2}([N]_q + [N+1]_q)$ was considered in [21], motivated by the noncommutative quantum field theory, in analogy with that of the harmonic oscillator. Though this model does not belong to quantum integrable systems, it allows to calculate explicit result on different thermodynamic properties like free energy, specific heat, entropy etc.

4.2 Quantum lattice Liouville model as q-bosonic model

Quantum integrable exact lattice version of the Liouville model may be constructed as reduction of the above ancestor model, when the deforming operators are chosen as (3.8) as in the case of q-bosons and generators of algebra (3.1) are realized as

$$S^3 = iu, \qquad S_q^+ = e^{-i\epsilon p} f(u), \qquad S_q^- = f(u) e^{i\epsilon p}, \tag{4.1}$$

where $f(u)^2 = 1 - \epsilon^2 q^{-2iu+1}$. The Lax operator of this lattice Liouville model can be obtained directly from (3.3) and therefore its exact quantum integrability is guaranteed. If one relates the parameter with the lattice constant: $\epsilon = i\Delta$, then it is easy to show that the model goes to the Liouville QFT model at the continuum limit with the field Lax operator recovered from the present discrete Liouville Lax operator.

On the other hand, it is interesting to observe that, the q-boson realization (3.9), using its mapping (2.7) through the canonical variables turns unexpectedly to realization (4.1) linked to the lattice Liouville model. Therefore we may conclude that the lattice Liouville model is intimately related to and can be represented by the q-bosonic model.

4.3 Quantum derivative NLS through q-boson model

Reduction of the ancestor model to Lax operator (3.6) expressed directly in q-bosons gives another quantum integrable deformed Bose gas model on the lattice. L-operator (3.6) together with R^{trig}-matrix (3.4) satisfy the QYBE by construction and allow exact solution of this q-bosonic model by algebraic AB (ABA). Using mapping (2.4) the q-boson defined on a lattice can be linked to a bosonic operator with commutation relation $[\psi_i, \psi_j^\dagger] = \frac{\hbar\delta_{ij}}{\Delta}$ in the form

$$b_{qi} = \psi_i \left(\frac{[2N_i]_q}{2N_i \cos\alpha}\right)^{1/2}, \qquad N_i = \psi_i^\dagger \psi_i. \tag{4.2}$$

It is interesting to note that at the continuum limit $\Delta \to 0$, when the lattice boson ψ_i produces the bosonic field $\psi_i \to \sqrt{\Delta}\psi(x)$, the q-boson through realization (3.7) with the choice

$$\hat{c}_1^+ = \hat{c}_2^+ = 1, \qquad \hat{c}_1^- = -\frac{\Delta}{4} iq, \qquad \hat{c}_2^- = \frac{\Delta}{4}\frac{i}{q} \tag{4.3}$$

goes to a quantum integrable derivative NLS field model [20], given by the equation $i\psi_t = \psi_{xx} - i(\psi^\dagger\psi)\psi_x$. The lattice Lax operator (3.6) with mapping (4.2), (4.3) reduces at $\Delta \to 0$ to $L = I + \Delta U^{\mathrm{DNLS}}$, where

$$U^{\mathrm{DNLS}} = i \begin{pmatrix} c\psi^\dagger\psi - \frac{\lambda^2}{4} & \sqrt{c}\,\lambda\psi^\dagger \\ \sqrt{c}\,\lambda\psi & -c\psi^\dagger\psi + \frac{\lambda^2}{4} \end{pmatrix},$$

is the Lax operator of the DNLS field model [20]. Note that the quantum DNLS Hamiltonian in the N-particle sector is equivalent to the interacting Bose gas with *derivative* δ-function potential [45]. We introduce in Section 7 a q-deformed anyon on the lattice and subsequently construct an anyonic gas model interacting through δ'-function, which is solvable by exact Bethe ansatz.

4.4 Coupled q-bosons and multi-component quantum integrable derivative NLS model

Note that the Yang–Baxter algebra (3.1) is invariant under the exchange of $\hat{c}_1^\pm \leftrightarrow \hat{c}_2^\mp$, since $\hat{M}^\pm$ do not change under such transformation. However, this is not the case with the associated Lax operator (3.3), which is transformed as $L(\xi) \to L^{-1}(\frac{1}{\xi})$. Therefore, using such a symmetry of algebra (3.1) we can introduce another Lax operator from (3.6) with a similar but different q-boson. Fusing these two Lax operators we can construct a novel quantum integrable two-mode coupled q-bosonic lattice model given by the Lax operator $L^{2qb}(\xi) = L^{qb}(\xi, b_{q1})(L^{qb}(\frac{1}{\xi}, b_{q2}))^{-1}$ with explicit expression of its matrix elements as [22]

$$L_{11}^{2qb} = q^{-N_1+N_2} + i\frac{\Delta}{4}\left(\frac{1}{\xi^2} q^{-(N_1+N_2+1)} + \xi^2 q^{N_1+N_2+1}\right) + \frac{\Delta^2}{16} q^{N_1-N_2} + \kappa^2 b_{q1}^\dagger b_{q2},$$

$$L_{12}^{2qb} = \kappa \left(\frac{1}{\xi} \left(q^{-N_1} b_{q2}^\dagger - i\frac{\Delta}{4} b_{q1}^\dagger q^{N_2+1} \right) + \xi (b_{q1}^\dagger q^{-N_2}) - i\frac{\Delta}{4} b_{q2}^\dagger q^{N_1+1} \right), \tag{4.4}$$

with $L_{22}^{2qb} = (L_{11}^{2qb})^\dagger$ and $L_{21}^{2qb} = (L_{12}^{2qb})^\dagger$. By combining N^+ number of q-boson of the first kind and N^- number from the second kind one can construct also a multi-mode generalization of this q-bosonic model. The mutually commuting conserved operators C_k, $k = 0, \pm 1, \pm 2, \ldots.$ in the simplest case of two-mode lattice q-bosonic model may be generated by

$$\tau(\xi) = tr \left(\prod_i^N L^{qb}(\xi, b_{q1i}) L^{qb} \left(\frac{1}{\xi}, b_{q2i} \right) \right)$$

through expansion in spectral parameter: $\tau(\xi) = \sum_k C_k \xi^k$. This long-range interacting lattice two-mode q-bosonic model is a new quantum integrable model and should be solved through algebraic Bethe ansatz for exact result. The possibility of obtaining its local Hamiltonian with few nearest neighbor interactions by applying the method developed in [19] and used for q-Bose gas in [10] is also an interesting problem to explore. Considering Δ to be the lattice constant, we can go to the continuum limit by taking $\Delta \to 0$, where the two-mode q-bosonic lattice Lax operator (4.4) turns into $L^{2qb}(\xi) \to I + i\Delta U^{\mathrm{MTM}}(\xi)$. Interestingly, Lax operator $U^{\mathrm{MTM}}(\xi)$ thus obtained is the same field Lax operator associated with the *bosonic* massive Thirring model as well as with a two-component derivative NLS model for different choices for the Hamiltonian. The elements of $U^{\mathrm{MTM}}(\xi)$ are given by [23, 20]

$$U_{11}^{\mathrm{MTM}} = -U_{22}^{\mathrm{MTM}} = \frac{1}{4} \left(\frac{1}{\xi^2} - \xi^2 \right) + \kappa_- \phi_1^\dagger \phi_1 - \kappa_+ \phi_2^\dagger \phi_2,$$
$$U_{12}^{\mathrm{MTM}} = \left(U_{21}^{\mathrm{MTM}} \right)^\dagger = \xi \phi_1^\dagger + \frac{1}{\xi} \phi_2^\dagger.$$

We can evaluate the commuting conserved quantities of this integrable field model at the classical limit (putting $\kappa_\pm \to 1$) using the Riccati equation:

$$U_{12} \left(\frac{\nu}{U_{12}} \right)_x + 2U_{11}\nu + \nu^2 = U_{12} U_{21}.$$

We solve this equation by expanding $\nu = \sum_k C_{2k} \xi^{2k}$ in spectral parameter at $\xi \to 0$ and at $\xi \to \infty$, obtaining the conserved charges $C_0, C_2, C_4, \ldots$ and $C_{-0}, C_{-2}, C_{-4}, \ldots$, respectively. In explicit form they are given as

$$C_0 = |\phi_2|^2,$$
$$C_2 = \phi_2^* \phi_1 - 4|\phi_2|^2 |\phi_1|^2 - 2\phi_2^* \phi_{2x},$$
$$C_4 = -C_{2x} - 4\phi_2 \phi_1^* |\phi_1|^2 + \frac{C_2}{\phi_2^*} \left(\phi_{2x}^* - \frac{1}{2}\phi_1^* \right) + (|\phi_1|^2 + |\phi_2|^2)(1 - 2C_2) + \phi_1^{*2} \frac{\phi_2}{\phi_2^*}$$

and

$$C_{-0} = -|\phi_1|^2,$$
$$C_{-2} = \phi_1^* \phi_2 - 4|\phi_2|^2 |\phi_1|^2 + 2\phi_1^* \phi_{1x},$$
$$C_{-4} = C_{-2x} + 4\phi_1 \phi_2^* |\phi_2|^2 - \frac{C_{-2}}{\phi_1^*} \left(\phi_{1x}^* + \frac{1}{2}\phi_2^* \right) - (|\phi_1|^2 + |\phi_2|^2)(1 + 2C_{-2}) - \phi_2^{*2} \frac{\phi_1}{\phi_1^*}.$$

It is interesting to observe that two different combinations of the above conserved operators of this integrable system can generate two important field models, which can also be raised to the quantum level as quantum integrable systems solvable by the algebraic Bethe ansatz. For example the field Hamiltonian given by $H_{\mathrm{MTM}} = \int dx(C_2 + C_{-2})$ and the momentum $P = \int dx(C_2 - C_{-2})$, with canonical PB relations $\{\phi_k(x), \phi_l^\dagger(y)\}_{\mathrm{PB}} = i\delta_{kl}\delta(x-y)$ would yield the relativistic massive Thirring model (MTM), rather its bosonic version. On the other hand the higher Hamiltonian $H_{2\mathrm{DNLS}} = \int dx(C_4 - C_{-4})$ would generate an integrable 2-component DNLS field model. At quantum level it would be an integrable QFT model, an exact lattice regularized version of which we have constructed already through two-mode q-bosonic model. It is desirable to investigate this new quantum integrable model both at the discrete and the continuum limit, through exact Bethe ansatz formalism.

4.5 q-boson and Ablowitz–Ladik model

The celebrated Ablowitz–Ladik (AL) model was discovered as a discretized version of the NLS field model [24] much before the discovery of q-boson. However surprisingly one identifies that the underlying YB algebra of the quantum integrable AL model is actually given by q-bosons. We observe first that twisting transformation $R_{ij}^{kl}(\lambda) \to e^{i\theta(j-k)}R_{ij}^{kl}(\lambda)$, can give a new R-matrix solution, where θ is some free parameter. Using this twisted R^{trig}-matrix one gets a θ-deformation of the quantum algebra (3.1), where the commutator is deformed as $pS_q^+S_q^- - p^{-1}S_q^-S_q^+$, $p = e^{i\theta}$, (similar to p,q-deformation) [16, 17, 18]. As a result one gets a θ-deformed q-boson. For a special value $\theta = -\alpha$ and with a redefinition of q-boson as $\tilde{b}_q \equiv b_q q^{-\frac{N}{2}}\Omega^{1/2}$ and $\tilde{b}_q^\dagger \equiv q^{-\frac{N}{2}} b_q^\dagger \Omega^{1/2}$, one gets another form of q-boson with algebra $q^2\tilde{b}_q\tilde{b}_q^\dagger - \tilde{b}_q^\dagger\tilde{b}_q = q^2 - 1$ as introduced by Macfarlane [4]. The Lax operator of the corresponding integrable model as reduced from (3.3) is given simply by

$$L_n = \begin{pmatrix} \xi^{-1} & \tilde{b}_{qn}^\dagger \\ \tilde{b}_{qn} & \xi \end{pmatrix}, \qquad [\tilde{b}_{qm}, \tilde{b}_{qn}^\dagger] = \hbar(1 - \tilde{b}_{qn}^\dagger\tilde{b}_{qn})\delta_{m,n}$$

with $\hbar = 1 - q^{-2}$, which turns out to be the lattice model proposed by Ablowitz and Ladik [24] and its quantum generalization discussed in [25].

We present another novel class of important models involving q-bosons, e.g. q-deformed matter-radiation models, in the next separate section.

5 Integrable matter-radiation models with q-bosons

Matter-radiation (MR) models represented by atoms interacting with radiation are usually described by two-level spin operators interacting with single mode boson. The well known and simplest models of this type are the Jaynes–Cummings (JC) [26], and the Buck–Sukumar (BS) model [27]. The basic physics underlying a variety of important phenomena in interacting MR systems, like those in quantum optics induced by resonance interaction between atom and a quantized laser field, in cavity QED, in trapped ion interacting with its center of mass motion irradiated by a laser beam [28] etc., seems to be nicely captured by such simple models. Many theoretical predictions based on these models, like vacuum Rabi splitting, Rabi oscillation and its quantum collapse and revival etc. have been verified in maser and laser experiments. However, for describing physical situations more accurately generalizations of these basic models, like q-deformed BS and JC model [7, 8, 29, 30], q-boson model interacting with q-spins [31] or its classical variant [32], trapped ion (TI) with nonlinear coupling, multi-atom models [33, 34] etc. have been proposed. However most of the above generalizations, except a few [31], go beyond exact solutions and quantum integrability, especially for interacting multi-atom models. Various q-bosonic models and quantum group symmetries were discussed in [35, 36, 37].

We propose new quantum integrable generalization of MR models involving q-bosons and q-spins. A general approach for constructing integrable MR model has been reported in our earlier work [38]. The physical significance of q-deformed MR models in comparison with their undeformed counterparts is given by their stronger nonlinear interactions between atomic excitations and the radiation mode, as well as the possible presence of nonlocal and asymmetric interactions between atoms due to nontrivial coproduct structure of q-spins. For constructing these models in a unified way and for ensuring their quantum integrability we may start from the integrable ancestor model (3.3) or more precisely by forming their Lax operators through a combination of (3.5) involving q-spins and (3.6) linked to the q-boson. The general form of such integrable q-deformed MR models may be given by the Lax operator

$$L = L_{\rm anc}(\xi, S_q^{\pm}, S^3) L_{qs}(\xi, s_q^{\pm}, s^3), \tag{5.1}$$

where $L_{\rm anc}(\xi, S_q^{\pm}, S^3)$ is given by (3.3) and $L_{qs}(\xi, s_q^{\pm}, s^3)$ by (3.5) with $\epsilon = +1$. Therefore by construction it would exactly satisfy the QYBE with $\operatorname{tr}(L) = \sum_n C_n \xi^n$ defining the mutually commuting conserved operators C_n, $n = 0, 1, 2, \ldots$ of this quantum integrable system. The Hamiltonian of the model given by C_0 takes the form:

$$H_{q\rm MR} = H_d + (s_q^+ S_q^- + s_q^- S_q^+)\eta, \qquad H_d = c^+ \cos\alpha(S^3 - s^3) + ic^- \sin\alpha(S^3 - s^3), \tag{5.2}$$

where $\eta = 4\sin^2\alpha$, and $c^{\pm}$ are related to the values of the deforming operators appearing in the Lax operators. This is a quantum integrable system with another commuting conserved operator $C_2 = S^3 + s^3$. Note that the quantum-spin operator can be expressed through N-number of spin-$\frac{1}{2}$ operators using the coproduct structure defined in the tensor product of N vector spaces as

$$s_q^{\pm} = \sum_j^s q^{-\sum\limits_{k<j} \sigma_k^3} \sigma_j^{\pm} q^{\sum\limits_{l>j} \sigma_l^3}, \qquad s^3 = \sum_j^s \sigma_j^3. \tag{5.3}$$

This describes an asymmetric as well as nonlocal interaction between the two-level atoms mediated by the q-bosonic radiation mode.

Different reductions of this generalized q-deformed MR model performed through possible realizations of the generators $S_q^{\pm}$, S^3 of the ancestor algebra, as we have listed above, can generate different physically relevant MR models in a unified way, as we describe below.

5.1 Integrable q-deformed Jaynes–Cummings model

This model is constructed as a quantum integrable system of interacting q-spins with q-boson representing atoms interacting nonlinearly with radiation as well as among themselves. The Lax operator of this integrable model can be constructed as a reduction of (5.1) by replacing the ancestor model by (3.6) using the q-bosonic realization (3.7). Consequently the Hamiltonian of the model can be constructed from (5.2) as

$$H_{q\rm JC} = c\sin\alpha(N - s^3 + \omega) + (b_q^{\dagger} s_q^+ + s_q^- b_q)\eta, \tag{5.4}$$

where c, η, ω are constant parameters (dependent on α), adjusted to simplify the expression. Notice that at $q \to 1$ limit, when $s_q^{\pm} \to s^{\pm}$, $b_q \to b$, the q-deformed Jaynes–Cummings (qJC) model (5.4) goes to the integrable multi-atom Jaynes–Cummings model (at α^2 order). In general this model describes nonlinear and nonlocal interactions between atoms and the radiation mode, since $s_q^{\pm}$ can be expressed through N-number of two-level atoms in a nonlocal way as (5.3) and the radiation represented by the q-bosonic mode b_q, $b_q^{\dagger}$ is expressed through standard boson b, $b^{\dagger}$ in a nonlinear way through the mapping (2.4).

Another similar multi-atom qJC model with explicit interatomic interactions may be given by defining the Lax operator as

$$L = L_{qb} \prod_{j}^{N} L_{xxz}^{(j)},$$

where $L_{xxz}^{(j)}$ is the Lax operator of anisotropic xxz spin-$\frac{1}{2}$ chain, which represents here the N interacting two-level atoms. The corresponding Hamiltonian of this model would include more complicated matter-radiation interactions as well as explicit atom-atom interactions.

Both the above qJC models are novel quantum integrable models solvable exactly by algebraic Bethe ansatz. Detailed analysis of these models with possible physical importance needs further pursuing.

5.2 Integrable q-deformed Buck–Sukumar model

This quantum integrable model also describes matter-radiation interaction through q-spins and a q-boson interacting in a stronger nonlinear way. The idea of construction is to start from the Lax operator (5.1) and take the realization of the ancestor model through generators of the noncompact quantum algebra $su_q(1,1)$ as given by (3.5) for $\epsilon = -1$. Subsequently such generators are realized through q-boson via q-Holstein–Primakov transformation:

$$s_q^+ = \sqrt{[N]_q} b_q^\dagger, \qquad s_q^- = b_q \sqrt{[N]_q}, \qquad s_q^3 = N + \frac{1}{2}$$

to represent the radiation mode. Consequently the Hamiltonian of this q-deformed Buck–Sukumar (qBS) model which can be reduced from the general qMR model (5.2) takes the form

$$H_{q\mathrm{BS}} = c \sin\alpha (N - s^3 + \omega) + \eta \left(\sqrt{[N]_q} b_q^\dagger s_q^- + s_q^+ b_q \sqrt{[N]_q} \right).$$

The q-deformed BS model gives an integrable version of an earlier model [7, 8], when q-spin operator is replaced by $\vec{\sigma}$-matrices by taking spin-$\frac{1}{2}$ representation.

This novel quantum integrable qBS model can be exactly solved using Bethe ansatz, which should produce physically interesting result generalizing that of the well known BS model, and therefore it deserves detailed investigation. At $q \to 0$, as is clearly seen, the qBS model goes to the integrable multi-atom BS model.

Note that the quantum integrable model of q-bosons interacting with q-spins proposed in [31] is similar in spirit to the present model, where however q-bosons were introduced not directly but as realization of $su_q(2)$ through q-HP transformation.

5.3 Integrable q-deformed trapped-ion model

Though this model does not involve q-bosons directly, we present it here since this novel quantum integrable MR model belongs to the same trigonometric class associated with the quantum R^{trig}-matrix and can be obtained again from the q-deformed MR model (5.2) under suitable realization.

Quantum Yang–Baxter algebra (3.1) under reduction $\hat{c}_2^\pm = 0$, when both $\hat{M}^\pm = 0$, simplifies to

$$[S_q^+, S_q^-] = 0, \qquad [S^3, S_q^\pm] = \pm S_q^\pm,$$

allowing realization through canonical operators: $[x,p] = i$, as

$$S_q^\pm = e^{\mp ix}, \qquad S^3 = p.$$

Note that the related reduction of (3.3) yields the Lax operator of the quantum integrable relativistic Toda chain [41, 42, 44]. Interestingly the same realization can reproduce from the general MR Hamiltonian (5.2) a q-deformation of the trapped ion (qTI) model given by

$$H_{q\mathrm{TI}} = c_+ \cos\alpha(p - s^3 + \omega) + c_- \sin\alpha(p - s^3 + \omega) + \eta(e^{-ix}s_q^- + e^{ix}s_q^+)$$

with highly nonlinear coupling between the atomic excitation and the vibration of the center of mass motion of the trapped ion, described by displacement x. The system also has another conserved operator $C_1 = p + s^3$. Usually for achieving exact solution such nonlinear oscillations are linearized through several approximations like Dicke approximation, rotating wave approximation etc. [28]. The present model however is solvable in principle with full exponential nonlinearity and without any approximation.

6 Exact solution through algebraic Bethe ansatz

Almost all q-deformed models presented here similarly to their unified construction allow exact ABA solutions also in a unified and vastly in a model-independent way.

Note that from the local QYBE taking the tensor product $T(\lambda) = \prod\limits_j L_j(\lambda)$ one can go to its global form:

$$R(\lambda-\mu)T(\lambda)\otimes T(\mu) = (I \otimes T(\mu))(T(\lambda)\otimes I)R(\lambda-\mu), \qquad j = 1,2,\ldots,N \tag{6.1}$$

reflecting the Hopf algebra structure of the underlying YB algebra. Taking trace from both the sides of (6.1) and defining $\tau(\lambda) = \operatorname{tr} T(\lambda)$ we get $[\tau(\lambda), \tau(\mu)] = 0$ and expanding further in spectral parameter: $\tau(\lambda) = \sum\limits_n C_n\lambda^n$ derive finally the quantum integrability condition for the conserved operators: $[C_n, C_m] = 0$.

ABA formalism aims to solve exactly the eigenvalue problem for all conserved operators simultaneously. The diagonal entries $\tau(\lambda) = T_{11}(\lambda) + T_{22}(\lambda)$ produce the conserved operators, while the off-diagonal elements $T_{21}(\lambda) \equiv B(\lambda)$ and $T_{12}(\lambda) \equiv C(\lambda)$ act like creation and annihilation operators of pseudoparticles. The M-particle state is defined as $|M\rangle_B = B(\lambda_1)\cdots B(\lambda_M)|0\rangle$ and the pseudovacuum $|0\rangle$ is defined through $C(\lambda)|0\rangle = 0$. The basic idea of algebraic BA [43] is to find the eigenvalue solution: $\tau(\lambda)|M\rangle_B = \Lambda(\lambda, \{\lambda_a\})|M\rangle_B$, for which diagonal elements $T_{ii}(\lambda)$, $i = 1, 2$ are pushed through the string of $B(\lambda_a)$'s toward $|0\rangle$, using the commutation relations obtainable from the QYBE (6.1). Considering further the actions $T_{11}(\lambda)|0\rangle = \alpha(\lambda)|0\rangle$, $T_{22}(\lambda)|0\rangle = \beta(\lambda)|0\rangle$, one arrives finally at the eigenvalue expression

$$\Lambda(\lambda, \{\lambda_a\}) = \alpha(\lambda)\prod_{a=1}^{M} f(\lambda-\lambda_a) + \beta(\lambda)\prod_{a=1}^{M} f(\lambda_a-\lambda), \tag{6.2}$$

where $f(\lambda)$ is defined through the elements of the R_{trig}-matrix as $\frac{\sin(\lambda+\alpha)}{\sin\lambda}$. Expanding $\Lambda(\lambda, \{\lambda_a\})$ in powers of λ we obtain the eigenvalues for all conserved operators including the Hamiltonian, where the rapidity parameters $\{\lambda_a\}$ involved can be determined from the Bethe equations

$$\frac{\alpha(\lambda_a)}{\beta(\lambda_a)} = \prod_{b\neq a} \frac{f(\lambda_b-\lambda_a)}{f(\lambda_a-\lambda_b)}, \qquad a = 1, 2, \ldots, M, \tag{6.3}$$

which follow in turn from the requirement of $|M\rangle_B$ to be an eigenvector. Returning to our models we find that, the major parts in key algebraic BA relations (6.2) and (6.3), described by R-matrix elements $f(\lambda)$ is model-independent and hence same for all q-bosonic and q-spin models belonging to the trigonometric class.

The only model-dependent parts in these equations, expressed through $\alpha(\lambda)$ and $\beta(\lambda)$ are determined from the Lax operator and therefore can be constructed in a unified way starting from (3.3) as

$$\alpha(\lambda) = c_1^+ \xi q^m + c_1^- \frac{1}{\xi} q^{-m}, \qquad \beta(\lambda) = c_2^+ \xi q^{-m} + c_2^- \frac{1}{\xi} q^m, \tag{6.4}$$

where $m = -\langle 0|S^3|0\rangle$ denotes the q-spin projection.

Since the q-bosonic or q-spin models are obtained as various reductions of this ancestor model one can obtain the corresponding exact result using the proper reduction of (6.4). Note however that for some models like relativistic Toda chain, q-deformed trapped ion model etc. since there is no easy pseudovacuum construction, the standard ABA method is not applicable to them and more generalized functional BA has to be implemented [41].

7 q-deformed anyon and δ' anyon gas

We have found in Section 4.3, that a quantum integrable q-bosonic model on a discrete lattice goes to an integrable derivative NLS quantum field model involving bosonic field operators $\psi(x)$, $\psi^\dagger(x)$. The Hamiltonian of this integrable QFT model may be given by

$$H = \int dx \big(\psi_x^\dagger \psi_x + i\kappa \psi^\dagger \psi (\psi^\dagger \psi_x - \psi_x^\dagger \psi)\big), \tag{7.1}$$

which at the N-particle sector can be shown to be equivalent to a derivative δ-function Bose gas model

$$H_N = -\sum_k^N \partial_{x_k}^2 + i\kappa \sum_{\langle k,l\rangle} \delta(x_k - x_l)\,(\partial_{x_k} + \partial_{x_l}) \tag{7.2}$$

which is exactly solvable by the coordinate Bethe ansatz [45].

Defining a new notion of q-anyon on the lattice we propose a similar model of 1d anyon gas interacting through derivative δ-function and show that the model is also exactly solvable by coordinate BA. Let us consider a lattice of N-sites and let q-bosons: b_{qj}, $b_{qj}^\dagger$, $j = 1, 2, \ldots, N$ satisfying commutation relations (2.2) for the same site: $[b_{qj}, b_{qk}^\dagger] = \delta_{jk} \frac{\cos\alpha(2N_j+1)}{\cos\alpha}$, etc., while all operators commute at different sites with $j \neq k$. We define another set of nonlocal operators as

$$A_{qj} = e^{i\theta \sum\limits_{k=1}^{J} N_k} b_{qj}, \qquad A_{qj}^\dagger = b_{qj}^\dagger e^{-i\theta \sum\limits_{k=1}^{J} N_k} \tag{7.3}$$

and easily check that at the same site these new operators behave exactly like q-bosons with relations

$$[A_{qj}, A_{qj}^\dagger] = \frac{\cos\alpha(2N_j+1)}{\cos\alpha}, \qquad [A_{qj}, A_{qj}] = 0, \qquad \text{etc.} \tag{7.4}$$

However for different sites $j > l$, separated by any distance we get an additional phase:

$$A_{qj}A_{ql} = e^{i\theta} A_{ql}A_{qj}, \qquad A_{qj}^\dagger A_{ql}^\dagger = e^{i\theta} A_{ql}^\dagger A_{qj}^\dagger, \qquad A_{qj}A_{ql}^\dagger = e^{-i\theta} A_{ql}^\dagger A_{qj}, \qquad \text{etc.} \tag{7.5}$$

$\theta = 0$ recovers obviously the q-bosonic case, while $\theta = \pi$ gives anti-commutator at different sites similar to fermions. Coupling now q-anyon to q-boson mapping (7.3) with that from q-boson to boson (2.4) and defining the boson- anyon transformation as

$$A_j = e^{i\theta \sum_1^J N_k} b_j, \qquad A_j^\dagger = b_j^\dagger e^{-i\theta \sum_1^J N_k}, \qquad N_j = b_j^\dagger b_j = A_j^\dagger A_j^\dagger \tag{7.6}$$

we can derive a similar mapping from q-anyon to anyon as

$$A_{qj} = A_j f(N_j), \qquad A^\dagger_{qj} = f(N_j) A^\dagger_j, \qquad f(N) = \left(\frac{[N]_q}{N}\right)^{1/2}, \qquad N = A^\dagger A. \tag{7.7}$$

The commutation relations for the anyons A_j, $A^\dagger_j$ at the same site are same as that of bosons (2.3), while at different sites they are same as (7.5).

At the continuum limit $\Delta \to 0$, introducing a scaling through $\sqrt{\Delta}$ the q-boson reduces to a bosonic field $\psi(x)$:

$$b_{qi} \to \sqrt{\Delta}\psi(x), \qquad N_j \to \Delta \psi^\dagger(x)\psi(x).$$

Consequently the q-anyon on the lattice is reduced to 1d anyon field $\tilde{\psi}(x)$:

$$A_{qi} \to \sqrt{\Delta}\tilde{\psi}(x), \qquad N_j \to \Delta \tilde{\psi}^\dagger(x)\tilde{\psi}(x),$$

where the anyonic field satisfies the commutation relations:

$$\begin{aligned}
&\tilde{\psi}^\dagger(x_1)\tilde{\psi}^\dagger(x_2) = e^{i\theta\epsilon(x_1-x_2)}\tilde{\psi}^\dagger(x_2)\tilde{\psi}^\dagger(x_1), \\
&\tilde{\psi}(x_1)\tilde{\psi}^\dagger(x_2) = e^{-i\theta\epsilon(x_1-x_2)}\tilde{\psi}^\dagger(x_2)\tilde{\psi}(x_1) + \delta(x_1 - x_2), \qquad \text{etc.,}
\end{aligned} \tag{7.8}$$

where

$$\epsilon(x-y) = \begin{cases} \pm 1 & \text{for} \quad x > y, \quad x < y, \\ 0 & \text{for} \quad x = y. \end{cases} \tag{7.9}$$

From the introduction of the above discrete and 1d anyonic and q-anyonic operators it is clear that by replacing q-bosons in (3.6) by q-anyons one can construct formally a q-anyonic model. However unfortunately due to the nonlocal commutation rules (7.5) the model turns into a nonultralocal model not satisfying the QYBE and hence becomes nonintegrable. In a similar way if we consider the fields in the DNLS model (7.1) to be anyonic instead of bosonic, its quantum integrability would be lost immediately.

However we find interestingly that, if instead of Bose gas we consider (7.2) as the anyonic gas interacting through δ'-function the model remains exactly solvable through coordinate Bethe ansatz, though unlike the Bose gas the anyonic model is not a quantum integrable system with mutually commuting higher conserved operators. Without giving the details of this novel exactly solvable interacting anyon gas model, which we reserve for a separate publication, we mention only that the exact BA result of this model closely follow that of the Bose gas [45] with a redefinition of the coupling constant κ involving the anyon parameter θ, as happens also in the case of δ-function anyon gas [39, 40].

8 Concluding remarks

We have identified the appearance of q-bosons in quantum integrable systems, exploring from the well known to new models in a unified way. Some models, like Ablowitz–Ladik model, though well known, the underlying q-bosonic connection of this model was not obvious and even unexpected. In some cases involvement of q-bosons is more direct, like in the exact lattice version of the quantum DNLS or the massive Thirring model and in some others, like in the q-Bose gas model of [10], the connection to q-boson is only through realization of q-spin operators. Many models presented here, e.g. two-mode q-boson model and related two-component DNLS quantum field model, q-deformed JC, BS and TI models, as well as δ'-function anyon gas, are new quantum integrable systems with rich possibilities and deserve detailed investigation. Since our objective here is to focus on various integrable models and identify the role of q-bosons in them, we could not concentrate on any individual model in detail, which we plan to do elsewhere.

Finally since this article is dedicated to the memory of Vadim Kuznetsov, I would like to mention that Vadim's interest was closely linked to the present investigation, related to quantum algebra and quantum integrable systems. Though unfortunately I did not have any personal interaction with Vadim, I remember having intense discussion with him through email regarding our common interest in formulating quantum relativistic Toda chain model [41, 42, 44]*, which is intimately related to the q-deformed trapped ion model presented here.*

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