A Note on the Rotationally Symmetric $SO(4)$ Euler Rigid Body*

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Abstract. We consider an $SO(4)$ Euler rigid body with two 'inertia momenta' coinciding. We study it from the point of view of bihamiltonian geometry. We show how to algebraically integrate it by means of the method of separation of variables.

Key words: Euler top; separation of variables; bihamiltonian manifolds

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1 Introduction

It is fair to say that the problem of the free rigid body in $\mathbb{R}^4$ (i.e., the $SO(4)$ Euler or Euler–Manakov rigid body or top[1]) can be still considered as an interesting problem in the theory of Separation of Variables (SoV) for Hamilton–Jacobi equations.

Ingenious methods were devised to solve the very classical problem of the Euler–Manakov top, (see, e.g., [1, 2, 3, 5, 19, 21, 25, 27], but this list – as well as the bibliography of the present paper – is by far incomplete), in particular to characterise the solutions by means of suitable techniques coming from algebraic geometry.

In this paper we will consider, from the point of view of bihamiltonian geometry, a degenerate case of Euler top that can be called rotationally symmetric. Namely, we consider the case in which two of the four independent parameters that enter the construction (that, after Manakov's construction, can be termed generalised principal inertia moments) coincide. This is a particular case of the general $SO(4)$ system, but it possesses some interesting and peculiar features that deserve, in the author's opinion, to be spelled out.

It should be noticed that in [26] the SoV problem was implicitly considered in the study of SoV for elliptic Gaudin models, and basically solved, with a considerable amount of hard computations and ingenuity. Furthermore, in [20], interesting classes of solutions to the Hamilton–Jacobi equations associated with the $SO(4)$ Euler–Manakov systems (as well as other alike systems) were recently obtained. In the present paper we will follow a more direct and geometrical way to approach it, with the aim of giving a simple and explicit setting for the Separation of Variables problem of this symmetric SO(4) Euler top.

The framework we will use to study this system is the so-called bihamiltonian setting for Separation of Variables for Gel'fand–Zakharevich [12] systems that has been introduced a few years ago (see, e.g., [4, 8, 23], and references quoted therein) and formalised in [10]; we will sketch the content of this method in Section 2.

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[1]Actually, this model was studied, in the XIX century, by Schottky and Kötter.

After that, in Section 3 we will briefly resume those aspects of the $SO(4)$ Euler–Manakov system that are relevant in our analysis, and in the core of the paper (Section 4) we will consider the rotationally symmetric case and solve, applying the recipes described in Section 2, the SoV problem for this Hamiltonian system.

Separation of Variables, both in classical and quantum systems, was a research arena in which Vadim Kuznetsov obtained notable results. In particular, his papers [13, 15, 14, 16] were deeply influential on the study of the SoV problem (and more general integrability) from the point of view of bihamiltonian geometry, carried out by the author of the present paper in collaboration with F. Magri, M. Pedroni, and G. Tondo. We had the opportunity to discuss with him these and related subjects many times, and appreciate his scientific, as well as human talents. With deep sorrow I dedicate this work to his memory.

2 A bihamiltonian set-up for SoV

In this Section we will briefly review a scheme for solving the SoV problem in the Hamilton Jacobi equations, based on properties of bihamiltonian manifolds. We will discuss those features that are relevant for the case at hand, referring to [10] and to [9] for a comprehensive theoretical presentation, as well as for a wider list of references.

Let (M, P_1, P_2) be a bihamiltonian manifold, that is, a manifold endowed with a pair of compatible Poisson brackets $\{\cdot,\cdot\}_{P_i}$, $i = 1, 2$, or, equivalently, with two compatible Poisson bivectors (or operators) P_1, P_2, related to the brackets by the well-known formulas

$$\{f, g\}_{P_i} = \langle df, P_i dg\rangle \qquad \forall\, f, g \in C^\infty(M), \quad i = 1, 2.$$

We consider a Gel'fand–Zakharevich bihamiltonian system with one bihamiltonian chain. That is, we consider the datum, on the bihamiltonian manifold (M, P_1, P_2), of an 'anchored' Lenard–Magri chain of length $n > 0$,

$$P_1 dH_0 = 0, \qquad P_1 dH_i = P_2 dH_{i-1}, \quad i = 1, \dots, n, \qquad P_2 dH_n = 0, \tag{2.1}$$

that is, a family of bihamiltonian vector fields originating from a Casimir function of one Poisson operator and ending in a Casimir function of the other Poisson operator. We may suppose that p additional Casimir functions common to the two structures $C_1, \dots, C_p$ be also present[2], and we require that the system be complete, i.e., that $n = \dim M - 1 - p$ independent vector fields fill in the chain (2.1).

Such a system provides (families of) Liouville integrable systems as follows. One considers a (generic) symplectic leaf $\mathcal{S}$ of one of the two Poisson operators, say, P_1.

$\mathcal{S}$ is a submanifold of M defined by fixing the values of all Casimir functions $H_0, C_1, \dots, C_p$ of P_1. Any of the vector fields of the chain (2.1), say $X_1 = P_1 dH_1 (= P_2 dH_0))$, restricts to $\mathcal{S}$, is still Hamiltonian with respect to (the restriction of) the chosen Poisson operator P_1 (which becomes an ordinary symplectic operator on $\mathcal{S}$), and, thanks to the basic property of Lenard Magri chains, comes equipped with the right number of involutive integrals, namely the (restriction to $\mathcal{S}$ of) the other Hamiltonian functions of the vector fields X_i, $i = 2, \dots, N$ of (2.1).

What is lost, in general, in this procedure, is the bihamiltonian structure of the equations: indeed, the second Poisson operator does not restrict to $\mathcal{S}$, since it does not leave the function H_0, which is a Casimir of P_0 invariant, i.e., Hamitonian vector fields generated by P_2 do not leave the submanifolds $\mathcal{S}$ invariant.

However, as it is shown in [10], the manifold M may be provided with another suitable bihamiltonian structure. That is, along with P_1, a "new" second Poisson operator Q can be

[2]In the $SO(4)$ Euler top, indeed, we will have one of such Casimir functions.

defined deforming P_2; Q can be restricted to the symplectic leaves $\mathcal{S}$ and has those properties needed for integrating the vector fields of the chain (2.1).

For this new structure to exist, conditions are to be fulfilled. Namely, one has to find a vector field Z, defined on M, such that:

i) Z is a symmetry of P_1, transversal to the submanifolds $H_0 = \text{const}$, and leaving the common Casimir functions C_i invariant:

$$\mathrm{Lie}_Z(P_1) = 0, \qquad \mathrm{Lie}_Z(H_0) = 1, \qquad \mathrm{Lie}_Z(C_i) = 0, \quad i = 1, \dots, p.$$

ii) It deforms of the second Poisson tensor P_1 as follows:

$$\mathrm{Lie}_Z(P_2) = Y \wedge Z, \quad \text{for some vector field } Y.$$

Indeed, under these conditions it holds that the bivector $Q := P_2 - X_1 \wedge Z$ (where X_1 is the first vector field of the Lenard–Magri chain (2.1)) satisfies:

1. Q is a Poisson structure on M, compatible with P_1 which shares with P_1 all the Casimirs, and hence, together with P_1 induces a bihamiltonian structure on the symplectic leaves $\mathcal{S}$.

2. The Hamiltonians H_i do not form anymore Lenard–Magri sequences w.r.t. this new Poisson pair (P_1, Q), but still are in involution also w.r.t. the deformed (or new) structure $Q = P_2 - X_1 \wedge Z$.

These two properties are very important for our purposes; indeed, from the first one it follows [11, 17] that the pair (P_1, Q) defines, on each symplectic leaf $\mathcal{S}$, a special set of co-ordinates, called *Darboux–Nijenhuis* co-ordinates, associated with the eigenvalues of a torsionless 'recursion' operator.

From the second one, it follows [10] that the Hamiltonians H_i are separable in these DN co-ordinates, that is, that the Hamilton–Jacobi equation associated with any of the Hamiltonians H_i is separable in these co-ordinates. Hence, the first and basic step of the bihamiltonian recipe for SoV of Gel'fand–Zakharevich systems of type (2.1) essentially boils down to find/guess this vector field Z, which will be referred to as the *transversal* vector field.

For the reader's convenience, as well as to provide the necessary background to the calculations presented in Section 4, we discuss more in details the construction of DN co-ordinates. A preliminary remark is in order. As the two structures P_1 and Q share the same symplectic leaves, we will generically use the same letters P_1, Q also for their natural restrictions to the symplectic leaves to avoid cumbersome notations.

As already noticed, on any (generic) symplectic leaf of P_1 this operator is symplectic and thus invertible; hence, the compositions

$$N := QP^{-1} \colon T\mathcal{S} \to T\mathcal{S}, \qquad \text{and} \qquad N^* := P_1^{-1} Q \colon T^*\mathcal{S} \to T^*\mathcal{S}$$

are well defined. In the literature, N is called a Nijenhuis or recursion or hereditary operator; in our setting, its adjoint operator, N^* will play a more visible role.

Being the ratio of two antisymmetric operators, N^* has up to $m = (1/2) \dim \mathcal{S}$ distinct eigenvalues $\lambda_1, \dots, \lambda_m$. Under the assumption that the number of these distinct eigenvalues be exactly m, it follows, basically as a consequence of the compatibility condition between P_1 and Q, that, for each λ_i there is a pair of canonical (w.r.t. P_1) co-ordinates f_i, g_i that generates the eigenspaces of N^*, that is, it holds

$$N^* df_i = \lambda_i \, df_i; \qquad N^* dg_i, \qquad \{f_i, g_j\}_{P_1} = \delta_{ij}. \tag{2.2}$$

These co-ordinates are called Darboux–Nijenhuis co-ordinates associated with the pair P_1, Q. Notice that the 'eigenvalue' relations written above imply the Poisson bracket relations $\{f_i, g_j\}_Q = \delta_{ij} \lambda_i$ w.r.t. the second Poisson structure. Fortunately enough, in generic cases, there is no need to integrate (all the) two-dimensional distributions $\mathrm{Ker}(N^* - \lambda_i)$ to actually find DN co-ordinates, thanks to the following results.

It should be noticed that, since N^* depends on the point of $\mathcal{S}$, its eigenvalues are functions defined on $\mathcal{S}$. In particular, non-constant eigenvalues of N^* do provide DN co-ordinates. This means that, if $d\lambda_i \neq 0$, then one can choose the function λ_i as the co-ordinate function f_i of (2.2).

To find the missing canonical co-ordinate associated with the eigenvalue λ_i one can try to use a recipe, discussed in [10] that might be called method of deformation of the Hamiltonians, and goes as follows:

Consider the sum p_1 of all the eigenvalues of N^*, and the Hamiltonian vector field $Y = -P_1 dp_1$. Then collect the Hamiltonians filling the Lenard–Magri recursion relations (2.1) in the Gel'fand–Zakharevich polynomial

$$H(\lambda) := \lambda^n H_0 + \lambda^{n-1} H_1 + \cdots + \lambda H_{n-1} + H_n \tag{2.3}$$

and deform it repeatedly along the vector field Y, that is, consider the polynomials

$$H'(\lambda) = \mathrm{Lie}_Y H(\lambda), \qquad H''(\lambda) = \mathrm{Lie}_Y H'(\lambda), \quad \ldots.$$

If, for some $n \geq 0$ it happens that the polynomial $H^{(n+2)}(\lambda)$ identically vanishes in λ, (while $H^{(n+1)}(\lambda)$ is *not* identically vanishing), then evaluating the rational function

$$\frac{H^{(n)}(\lambda)}{H^{(n+1)}(\lambda)} \qquad \text{at} \qquad \lambda = \lambda_i$$

provides us with a DN co-ordinate g_i conjugate with $f_i \equiv \lambda_i$.

Remarks:

1) When the eigenvalues λ_i are no more functionally independent, no general algorithms/recipes are known (at least to the author of the present paper) to find Darboux–Nijenhuis co-ordinates. However, when the Nijenhuis operator has a constant eigenvalue, say, λ_n, a case that happens for the symmetric $SO(4)$ top herewith considered, one can proceed as follows. One can at first apply the method briefly recalled above to construct a set of $(n-1)$ pairs of Darboux–Nijenhuis co-ordinates, associated with the $n-1$ co-ordinates $\{\lambda_\alpha\}_{\alpha=1}^{n-1}$ defined by the Nijenhuis tensor. Then one has to consider the distribution associated with the constant eigenvalue λ_n, and try to integrate it (that is, to find the functions f_n and g_n of equation (2.2)) by *ad hoc* methods.

2) It should be stressed that, with respect to the pair P_1, Q, the Hamiltonians do not fill in a standard Lenard–Magri chain, but rather a generalised one (see [8, 10, 18] for further details) of the form

$$Q dH_i = P_0 dH_{i+1} + p_i dH_1, \qquad i = 1, \ldots, n,$$

where p_i are (up to signs) the elementary symmetric polynomials associated with the eigenvalues λ_i. However, this invariance relation is sufficient to ensure separability of the Hamilton–Jacobi equations.

3) Contrary to other methods for integrating of Hamilton equations, and notably the method of Lax pairs, the bihamiltonian setting herewith briefly sketched provides, on general grounds, somewhat poor information on the Jacobi separation relations – that is, the relations tying pairs of separation co-ordinates with the Hamiltonians $H_0, H_1, H_2, \ldots, H_n$ and the common Casimir functions $\mathbf{C} = (C_1, \ldots, C_p)$.

However, the functional form of the separation relations can be sometimes ascertained from bihamiltonian geometry. Indeed, if the second Lie derivative of the GZ polynomial (2.3) with respect to the transversal vector field Z vanishes, then these relations will be *affine* functions of the Hamiltonians and the Casimirs, that is, they will be given by expressions of the form

$$F_1^i(\lambda_i,\xi_i)H_1+\cdots+F_n^i(\lambda_i,\xi_i)H_n+G(\lambda_i,\xi_i;\mathbf{C})=0,\qquad i=1,\ldots,m.$$

Separation relations of this kind are often referred to as *generalised* Stäckel separation relations.

3 The Euler–Manakov model

In this section we will briefly review the basic features of the $SO(4)$ Euler–Manakov top.

The phase space is the (dual of) the Lie Algebra $\mathfrak{so}(4)$, identified[3] with 4×4 antisymmetric matrices

$$M=\sum_{i<j=1}^{4} m_{ij}(\mathsf{E}_{ij}-\mathsf{E}_{ji}),$$

where E_{ij} is the elementary matrix with 1 at the (i,j)-th place.

This six dimensional manifold is naturally endowed with the Lie Poisson structure, that, in the natural variables $\mathbf{m}=\{m_{1,2},m_{1,3},m_{1,4},m_{2,3},m_{2,4},m_{3,4}\}$ is represented by the matrix

$$P_1=\begin{bmatrix} 0 & -m_{2,3} & -m_{2,4} & m_{1,3} & m_{1,4} & 0\\ m_{2,3} & 0 & -m_{3,4} & -m_{1,2} & 0 & m_{1,4}\\ m_{2,4} & m_{3,4} & 0 & 0 & -m_{1,2} & -m_{1,3}\\ -m_{1,3} & m_{1,2} & 0 & 0 & -m_{3,4} & m_{2,4}\\ -m_{1,4} & 0 & m_{1,2} & m_{3,4} & 0 & -m_{2,3}\\ 0 & -m_{1,4} & m_{1,3} & -m_{2,4} & m_{2,3} & 0 \end{bmatrix}.$$

The Hamiltonian is the quadratic function

$$H_{\mathcal{E}}=\frac{1}{2}\sum_{i<j=1}^{4} a_{ij}m_{ij}^2, \tag{3.1}$$

where the coefficients a_{ij} can be written as

$$a_{ij}=J_l^2+J_k^2,\qquad \text{with}\qquad \{i,j,l,k\}\text{ a permutation of }\{1,2,3,4\}.$$

The Hamilton equations of motion (that is, the Euler equations for the $SO(4)$ rigid body), are quadratic equations in the variables m_{ij} that depend parametrically on the coefficients J_l^2. For instance,

$$\frac{d}{dt}m_{12}={J_1}^2(m_{1,3}m_{2,3}+m_{1,4}m_{2,4})-{J_2}^2(m_{1,3}m_{2,3}+m_{1,4}m_{2,4})$$

and so on and so forth.

Complete Liouville integrability of the model is ensured following well known facts.

[3] We are actually identifying $\mathfrak{so}(4)$ and its dual.

1. The rank of the $\mathfrak{so}(4)$ Lie Poisson structure is 4; its Casimir functions are

$$H_0 = \sum_{i<j} m_{ij}^2, \qquad C = m_{1,2}m_{3,4} + m_{1,4}m_{2,3} - m_{1,3}m_{2,4}.$$

2. The second independent non-trivial constant of the motion for $H_{\mathcal{E}}$ is provided by the quadratic function

$$K_{\mathcal{E}} = \sum_{i<j=1}^{4} b_{ij} m_{ij}^2, \quad b_{ij} = J_l^2 J_k^2, \qquad \text{with} \quad \{i,j,l,k\} \text{ a permutation of } \{1,2,3,4\}.$$

The Hamiltonian vector field X associated with $H_{\mathcal{E}}$ admits a Lax representation with parameter [3, 19]; indeed, if one considers the matrix $\mathsf{J} := \mathrm{diag}(J_1, J_2, J_3, J_4)$, and forms the matrix

$$L(\lambda) = \lambda\, \mathsf{J}^2 + M, \tag{3.2}$$

the Euler equations are equivalent to the Lax equations

$$\frac{d}{dt} L(\lambda) = [L(\lambda), B(\lambda)],$$

where $B(\lambda) = \Omega + \lambda \mathsf{J}$, and Ω is the 'matrix of angular velocities', i.e., defined by M via $M = \mathsf{J}\,\Omega + \Omega\,\mathsf{J}$.

As it is well known, the integrals of the motion (as well as the Casimirs of the Lie Poisson structure) are collected in the characteristic polynomial of $L(\lambda)$. In particular, if one uses the product $\rho\lambda$ as 'eigenvalue parameter', one gets

$$\mathrm{Det}(L(\lambda) - \rho\lambda \mathbf{1}) = \lambda^4 (P_4(\rho)) + \lambda^2 (\rho^2 (H_0) + \rho\, H_1 + H_2) + C^2,$$

where $P_4(\rho) = \prod_{i=1}^{4} (J_i^2 - \rho)$, $H_1 = -2\, H_{\mathcal{E}}$, $H_2 = K_{\mathcal{E}}$, and H_0, C are the two Casimirs of the Lie Poisson structure (the second one being the Pfaffian of M.

As it was discovered in [6, 7], and independently in [22], the Euler–Manakov equations of motion admit a bihamiltonian formulation that can be described as follows. The matrix J^2 defines a deformed commutator on the Lie algebra $\mathfrak{so}(4)$ as:

$$[M_1, M_1]_{\mathsf{J}^2} := [M_1\, \mathsf{J}^2\, M_2, M_2\, \mathsf{J}^2\, M_1] = M_1 \mathsf{J}^2 M_2 - M_2 \mathsf{J}^2 M_1. \tag{3.3}$$

The Lie Poisson structure associated with $[\cdot,\cdot]_{\mathsf{J}^2}$ provides a second Hamiltonian structure P_2 on $\mathfrak{so}(4)$. Compatibility with the standard one is assured by the method of augmented translations, i.e., by the fact that $[\cdot,\cdot]_\lambda = [\cdot,\cdot]_{\mathsf{J}^2} - \lambda[\cdot,\cdot]$ is a one-parameter family of commutators, that is, the Jacobi identity holds identically in λ. The 6×6 matrix representing the second Poisson structure in the phase space variables $\mathbf{m} = \{m_{ij}\}_{i<j=1,\dots,4}$ can be easily found to be

$$P_2 = \begin{bmatrix} 0 & -{J_1}^2 m_{2,3} & -{J_1}^2 m_{2,4} & {J_2}^2 m_{1,3} & {J_2}^2 m_{1,4} & 0 \\ {J_1}^2 m_{2,3} & 0 & -{J_1}^2 m_{3,4} & -m_{1,2}{J_3}^2 & 0 & m_{1,4}{J_3}^2 \\ {J_1}^2 m_{2,4} & {J_1}^2 m_{3,4} & 0 & 0 & -m_{1,2}{J_4}^2 & -m_{1,3}{J_4}^2 \\ -{J_2}^2 m_{1,3} & m_{1,2}{J_3}^2 & 0 & 0 & -{J_2}^2 m_{3,4} & m_{2,4}{J_3}^2 \\ -{J_2}^2 m_{1,4} & 0 & m_{1,2}{J_4}^2 & {J_2}^2 m_{3,4} & 0 & -m_{2,3}{J_4}^2 \\ 0 & -m_{1,4}{J_3}^2 & m_{1,3}{J_4}^2 & -m_{2,4}{J_3}^2 & m_{2,3}{J_4}^2 & 0 \end{bmatrix}.$$

In particular, the $\mathfrak{so}(4)$ Euler system is a Hamitonian vector field also w.r.t. the Poisson operator P_2, defined by the deformed commutator (3.3), with 'second' Hamiltonian the function $-(1/2)\,H_0 = -(1/2)\sum_{i<j} m_{ij}^2$. Moreover, a direct computation ensures the following:

Proposition 1. *The characteristic polynomial* (3.3) $\mathcal{P}(\lambda,\rho) = \mathrm{Det}(L(\lambda) - (\rho\lambda)\mathbf{1})$ *of the Lax matrix* $L(\lambda)$ *is a Casimir of the Poisson pencil* $P_\rho = P_2 - \rho P_1$, *that is,*

$$P_2\big(d\,\mathcal{P}(\lambda,\rho)\big) = \rho\, P_1\big(d\mathcal{P}(\lambda,\rho)\big) \quad \textit{identically in} \quad \rho,\ \lambda.$$

In other words, the Pfaffian C *is a common Casimir of the two structures, while the three functions* H_0, H_1, H_2 *satisfy the GZ recurrence relations*

$$P_1 d(H_0) = 0, \qquad P_2 d(H_0) = P_1 d(H_1), \qquad P_2 d(H_1) = P_1 d(H_2), \qquad P_2 d(H_2) = 0.$$

For the sequel of the paper, the following well known considerations will be useful.

The Lie algebra $\mathfrak{so}(4)$ is isomorphic to the direct sum $\mathfrak{so}(3)\oplus\mathfrak{so}(3)$; a linear change of variables that explicitly realises this isomorphism is the following:

$$\begin{aligned}
x_1 &= \frac{1}{\sqrt{2}}(m_{1,2} - m_{3,4}), &\qquad y_1 &= \frac{1}{\sqrt{2}}(m_{1,3} + m_{2,4}), &\qquad z_1 &= \frac{1}{\sqrt{2}}(m_{1,4} - m_{2,3}),\\
x_2 &= \frac{1}{\sqrt{2}}(m_{1,2} + m_{3,4}), &\qquad y_2 &= \frac{1}{\sqrt{2}}(m_{1,3} - m_{2,4}), &\qquad z_2 &= \frac{1}{\sqrt{2}}(m_{1,4} + m_{2,3}).
\end{aligned}$$

In particular, the variables $\{x_i, y_i, z_i\}_{i=1,2}$ satisfy, with respect to the standard Lie Poisson structure P_1 the $\mathfrak{so}(3)$ commutation relations:

$$\{x_i, y_i\}_{P_1} = \sqrt{2} z_i, \qquad \{x_i, z_i\}_{P_1} = -\sqrt{2} y_i, \qquad \{y_i, z_i\}_{P_1} = \sqrt{2} x_i, \qquad i = 1, 2,$$

while Poisson brackets involving co-ordinates from different $\mathfrak{so}(3)$ subalgebras vanish, e.g. $\{x_1, z_2\}_{P_1} = 0$ and so on and so forth.

Under this co-ordinate change, the Euler Hamiltonian (3.1) acquires the form

$$H_{\mathcal{E}} = 2\mu_4 x_1 x_2 + 2\mu_3 y_1 y_2 + 2\mu_2 z_1 z_2 + \mu_1\left({y_1}^2 + {y_2}^2 + {x_2}^2 + {x_1}^2 + {z_1}^2 + {z_2}^2\right), \tag{3.4}$$

where the new constants μ_i are related with the J_i's as follows:

$$\begin{aligned}
{J_1}^2 &= -\mu_4 + \mu_1 - \mu_3 - \mu_2, &\qquad {J_2}^2 &= \mu_3 - \mu_4 + \mu_1 + \mu_2,\\
{J_3}^2 &= \mu_1 - \mu_3 + \mu_4 + \mu_2, &\qquad {J_4}^2 &= -\mu_2 + \mu_1 + \mu_3 + \mu_4.
\end{aligned}$$

One can notice that the Hamiltonian (3.4) is the sum of a multiple of the Casimir function H_0 of the standard Lie–Poisson structure, and the classical analogue

$$H_{XYZ} = 2(\mu_4 x_2 x_1 + \mu_3 y_1 y_2 + \mu_2 z_1 z_2)$$

of the Hamiltonian of a (2-site) XYZ Heisenberg model.

We will, in the remainder of the present paper, consider a special case, namely the one that goes under the name of XXZ model, that is we will study the case $\mu_4 = \mu_3$. From the point of view of the Euler rigid body in $\mathfrak{so}(4)$, this is tantamount to consider a rigid body with two principal inertia moments (J_2^2 and J_3^2) that are equal.

4 The symmetric (or XXZ) Euler systems

In the case $\mu_4 = \mu_3$, the non-trivial part of the Hamiltonian reads

$$H_{XXZ} = 2\,\mu_3(x_2 x_1 + y_1 y_2) + 2\mu_2 z_1\, z_2.$$

This fact suggests to choose linear co-ordinates in $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ adapted to the symmetries of H_{XXZ}; the choice we will follow in the sequel will be to consider the sixtuple $\{u_1, v_1, z_1, u_2, v_2, z_2\}$ related with the standard $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ co-ordinates $\{x_k, y_k, z_k\}_{i=1,2}$ by

$$u_k = x_k + i y_k, \qquad v_k = x_k - i y_k, \qquad k = 1, 2.$$

In these co-ordinates, the characteristic polynomial of the Lax matrix $\hat{L}(\lambda)$ associated with the problem (that is, the one given in (3.2), with $\mu_4 = \mu_3$) has the expression

$$\lambda^4(\hat{P}_4(\rho)) + \lambda^2(\rho^2\mathcal{H}_0 + \rho\,\mathcal{H}_1 + \mathcal{H}_2) + \frac{1}{4}C_2^2,$$

where

$$\begin{aligned}
&\mathcal{H}_0 = u_1 v_1 + v_2 u_2 + {z_1}^2 + {z_2}^2, \qquad C_2 = u_2 v_2 + {z_2}^2 - u_1 v_1 - {z_1}^2,\\
&\mathcal{H}_1 = -2\mu_3(u_2 v_1 + v_2 u_1) - 4\mu_2 z_1 z_2 - 2\mu_1 \mathcal{H}_0,\\
&\mathcal{H}_2 = \mu_1^2 \mathcal{H}_0 + 4\mu_1\mu_2 z_1 z_2 + 2\mu_3(\mu_1 + \mu_2)(v_2 u_1 + u_2 v_1) + \mu_2^2({z_1}^2 + {z_2}^2 - v_1 u_1 - v_2 u_2)\\
&\qquad - 2\mu_3^2(z_1 - z_2 t)^2.
\end{aligned} \tag{4.1}$$

The explicit expressions of the Poisson tensors (in the new co-ordinates) are, respectively,

$$P_1 = \begin{bmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & -A_2 \end{bmatrix} \qquad \text{with} \qquad A_i = \begin{bmatrix} 0 & 2z_i & -u_i \\ -2z_i & 0 & v_i \\ u_i & -v_i & 0 \end{bmatrix} \tag{4.2}$$

and $P_2 = \mu_1\, P_1 + \Delta$, with P_1 still given by (4.2), and

$$\Delta = \mu_2 \begin{bmatrix} 0 & 2z_2 & 0 & 0 & 0 & u_1 \\ & 0 & 0 & 0 & 0 & -v_1 \\ & & 0 & u_2 & -v_2 & 0 \\ & & & 0 & -2z_1 & 0 \\ & & * & & 0 & 0 \\ & & & & & 0 \end{bmatrix} + \mu_3 \begin{bmatrix} 0 & 0 & -u_2 & 0 & 2(z_2 - z_1) & -u_2 \\ & 0 & v_2 & 2(z_1 - z_2) & 0 & v_2 \\ & & 0 & -u_1 & v_1 & 0 \\ & & & 0 & 0 & u_1 \\ & & * & & 0 & -v_1 \\ & & & & & 0 \end{bmatrix}.$$

(We indicate with a $*$ the lower diagonal part of these antisymmetric tensors.)

The Hamiltonian vector field X_1 (i.e., up to a numeric factor, the Euler–Manakov $SO(4)$ field in the rotationally symmetric case), generated under P_1 by the Hamiltonian $\mathcal{H}_1$ is explicitly given, in these new co-ordinates, by

$$\begin{aligned}
X_1 = {} & 4(\mu_2 u_1 z_2 - \mu_3 u_2 z_1)\frac{\partial}{\partial u_1} - 4(\mu_2 v_1 z_2 - \mu_3 v_2 z_1)\frac{\partial}{\partial v_1} - 4(\mu_2 u_2 z_1 - \mu_3 u_1 z_2)\frac{\partial}{\partial u_2}\\
& + 4(\mu_2 v_2 z_1 - \mu_3 v_1 z_2)\frac{\partial}{\partial v_2} + 2\mu_3(u_2 v_1 - u_1 v_2)\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right).
\end{aligned} \tag{4.3}$$

We will study the SoV problem for this Hamiltonian vector field, within the scheme of SoV for GZ systems of [10], resumed in Section 2. Namely we have to:

1. Find a suitable transversal vector field Z;
2. Consider the Poisson operators $P_1, Q := P_2 - X_1 \wedge Z$, as well as their restrictions to the generic symplectic leaves;
3. Find the DN co-ordinates associated with these restrictions;
4. Find the Jacobi separation relations linking pairs of DN co-ordinates and the Hamiltonians.

One can check that a suitable transversal vector field for the problem is given by

$$Z = \frac{1}{2u_1}\frac{\partial}{\partial v_1} + \frac{1}{2u_2}\frac{\partial}{\partial v_2}.$$

Namely, Z is a symmetry of P_1, that is, $\mathrm{Lie}_Z(P_1) = 0$. Moreover,

$$\mathrm{Lie}_Z(\mathcal{H}_0) = 1, \qquad \mathrm{Lie}_Z(C_2) = 0,$$

and one indeed can check that the bivector

$$Q = P_2 - X_1 \wedge Z$$

turns out to be a (generically rank 4) Poisson operator compatible with P_1 that admits $\mathcal{H}_1$ and C_2 as Casimir functions, that is, shares with the Poisson tensor P_1, associated with the standard Lie algebra structure of $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ the same symplectic leaves.

Remark. For further use, we notice that a direct computation shows that the second Lie derivative of the characteristic polynomial $\mathrm{Det}(L(\lambda)-(\rho\lambda)\mathbf{1})$ w.r.t. the transversal vector field Z vanishes as well,

$$\mathrm{Lie}_Z\left(\mathrm{Lie}_Z(\mathrm{Det}(L(\lambda) - (\rho\lambda)\mathbf{1})\right) = 0, \tag{4.4}$$

that is, the condition for Stäckel separability is fulfilled.

From the theoretical framework recalled in Section 2 we know that the symplectic leaves of P_1 are four dimensional manifolds endowed with a pair of compatible Poisson structure, i.e., the restrictions of P_1 and of Q.

These four dimensional symplectic leaves $\mathcal{S}$ are obtained fixing the values $\{\mathcal{H}_0, C_2\}$ of these common Casimir function. Furthermore, thanks to the explicit expressions given in the first line of (4.1), in (open sets of) these symplectic leaves one can use, as co-ordinates, the four parameters

$$\mathsf{u} = \{u_1, z_1, u_2, z_2\},$$

since one can express the co-ordinates v_1, v_2 as follows:

$$v_1 = \frac{1}{2}\frac{\mathcal{H}_0 - C_2 - 2\,{z_1}^2}{u_1}, \qquad v_2 = \frac{1}{2}\frac{\mathcal{H}_0 + C_2 - 2z_2^2}{u_2}.$$

The restrictions P and Q of the Poisson structures P_1 and Q to the leaf $\mathcal{S}$ are represented by 4×4 matrices that have, in these co-ordinates, the explicit expressions

$$\mathsf{P} = \begin{bmatrix} 0 & -u_1 & 0 & 0 \\ & 0 & 0 & 0 \\ & * & 0 & u_2 \\ & & & 0 \end{bmatrix}, \qquad \mathsf{Q} = \begin{bmatrix} 0 & -(\mu_3 u_2 + \mu_1 u_1) & 0 & \mu_2 u_1 - \mu_3 u_2 \\ & 0 & \mu_2 u_2 - \mu_3 u_1 & 0 \\ & * & 0 & \mu_1 u_2 + \mu_3 u_1 \\ & & & 0 \end{bmatrix}.$$

The transpose Nijenhuis operator $N^* = \mathsf{P}^{-1}\mathsf{Q}$ is given by

$$\mathsf{N}^* = \begin{bmatrix} \mu_3\dfrac{u_2}{u_1}+\mu_1 & 0 & +\mu_2\dfrac{u_2}{u_1}-\mu_3 & 0 \\ 0 & \mu_3\dfrac{u_2}{u_1}+\mu_1 & 0 & \mu_3\dfrac{u_2}{u_1}-\mu_2 \\ \mu_2\dfrac{u_1}{u_2}-\mu_3 & 0 & \mu_3\dfrac{u_1}{u_2}+\mu_1 & 0 \\ 0 & \mu_3\dfrac{u_1}{u_2}-\mu_2 & 0 & \mu_3\dfrac{u_1}{u_2}+\mu_1 \end{bmatrix}. \tag{4.5}$$

Its eigenvalues are

$$\lambda_1 = \mu_1+\mu_2, \qquad \lambda_2 = \mu_1-\mu_2+\mu_3\left(\frac{u_1}{u_2}+\frac{u_2}{u_1}\right).$$

From the general theory, we know that λ_2 is one of the DN co-ordinates we are looking for (which we need to complement with its conjugate co-ordinate ξ_2), while we need to find both canonical co-ordinates relative to the constant eigenvalue λ_1.

The problem of finding the canonical co-ordinate conjugated to the non–constant eigenvalue λ_2 can be dealt with the idea of deforming the Hamiltonian polynomial. Indeed we consider the sum p_1 of the eigevalues of N, $p_1 = 2\mu_1+\mu_3(u_1/u_2+u_2/u_1)$, and the vector field $Y = -\mathsf{P}dp_1$; it is given by

$$Y = \mu_3 G(\mathsf{u})\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right),$$

where the function $G(\mathsf{u})$ is given by

$$G(\mathsf{u}) = \frac{u_2}{u_1}-\frac{u_1}{u_2}, \tag{4.6}$$

and is connected with the eigenvalue λ_2 by

$$G(\mathsf{u})^2 = 4+\left(\frac{\lambda_2-\mu_1+\mu_2}{\mu_3}\right)^2. \tag{4.7}$$

Now we iteratively apply the vector field Y to the polynomial containing the relevant Hamiltonians, that is

$$\mathcal{H}(\rho) = \rho^2\,\mathcal{H}_0+\rho\,\mathcal{H}_1+\mathcal{H}_2.$$

By means of direct computation one can check that $\mathrm{Lie}_Y(\mathcal{H})$ factors as

$$\mathrm{Lie}_Y(\mathcal{H}) = \frac{4\mu_3(\rho-\mu_1-\mu_2)}{u_1\,u_2}\,G(\mathsf{u})\,L(\mathsf{u})$$

with

$$L(\mathsf{u}) = \mu_3\left(z_2{u_1}^2+z_1{u_2}^2\right)-\mu_2u_1u_2\left(z_1+z_2\right).$$

Since, quite obviously $\mathrm{Lie}_Y(u_1\,u_2)=0$, and, thanks to (4.7),

$$\mathrm{Lie}_Y(G(\mathsf{u})) = \{G(\mathsf{u}),\lambda_2\}_{\mathsf{P}} = 0,$$

we are left, for the computation of the second Lie derivative of $\mathcal{H}$, with the computation of $\mathrm{Lie}_Y(L(\mathsf{u}))$. It gives

$$\mathrm{Lie}_Y(L(\mathsf{u})) = \mu_3 G(\mathsf{u})\big(u_1\,u_2\,F(\mathsf{u})\big),$$

where

$$F(\mathsf{u}) = -2\mu_2 + \frac{({u_1}^2 + {u_2}^2)\mu_3}{u_1 u_2} = \lambda_2 - (\mu_1 + \mu_2).$$

So, the third Lie derivative of $\mathcal{H}$ w.r.t. Y vanishes, and so the function

$$\xi_2 = \left.\frac{\mathrm{Lie}_Y(\mathcal{H})}{\mathrm{Lie}_Y(\mathrm{Lie}_Y(\mathcal{H}))}\right|_{\rho=\lambda_2} = -\frac{1}{\mu_3\, u_1\, u_2}\left(\frac{L(\mathsf{u})}{G(\mathsf{u})\, F(\mathsf{u})}\right)$$

is the DN co-ordinate conjugated to λ_2 we were looking for.

The two functions $F(\mathsf{u})$ and $G(\mathsf{u})$ will play a role in the last task we will deal with, that is, the determination of the Jacobi separation relations.

Next we turn to consider the problem of finding DN co-ordinates associated with the constant eigenvalue $\lambda_1 = \mu_1 + \mu_2$ of the tensor N^* of equation (4.5). Being this a constant eigenvalue, we have to find 'by hands' the associated DN co-ordinates. Fortunately enough the expression of the operator $N^* - \lambda_1$ is given by

$$\begin{bmatrix} -\mu_2 + \dfrac{\mu_3 u_2}{u_1} & 0 & \dfrac{\mu_2 u_2}{u_1} - \mu_3 & 0 \\ 0 & -\mu_2 + \dfrac{\mu_3 u_2}{u_1} & 0 & -\mu_2 + \dfrac{\mu_3 u_2}{u_1} \\ \dfrac{\mu_2 u_1}{u_2} - \mu_3 & 0 & \dfrac{\mu_3 u_1}{u_2} - \mu_2 & 0 \\ 0 & \dfrac{\mu_3 u_1}{u_2} - \mu_2 & 0 & \dfrac{\mu_3 u_1}{u_2} - \mu_2 \end{bmatrix}.$$

Its kernel can be easily found to be generated by the two 1-forms

$$\alpha_1 = dz_2 - dz_1, \qquad \alpha_2 = (\mu_3 u_1 - \mu_2 u_2)du_1 + (\mu_3 u_2 - \mu_2 u_1)du_2$$

that integrate, respectively, to the functions

$$\zeta_1 = z_2 - z_1, \qquad \theta_1 = \frac{1}{2}\mu_3\, {u_1}^2 - \mu_2\, u_1 u_2 + \frac{1}{2}\mu_3\, {u_2}^2.$$

A direct computation shows that $\{\zeta_1, \theta_1\}_{\mathsf{P}} = -2\theta_1$, and so the DN co-ordinate conjugated to ζ_1 is $\xi_1 = -(1/2)\log\theta_1$. Thus, we have found, on the generic common symplectic leaf $\mathcal{S}$ of P_1 and Q, the desired set of DN co-ordinates $(\zeta_1, \xi_1 = -(1/2)\log\theta_1, \lambda_2, \xi_2)$. It can be noticed that, along with the two Casimirs C_1, C_2, they provide a set of co-ordinates in a Zariski open subset of the phase space $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

What we are left with the determination of the Jacobi separation relations, namely we have to seek for two relations of the form

$$\Phi_1(\mathbf{H}; \zeta_1, \xi_1) = 0, \qquad \Phi_2(\mathbf{H}; \lambda_2, \xi_2) = 0, \tag{4.8}$$

linking pairs of DN co-ordinates and the conserved quantities $\mathbf{H} = \{C_2, \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2\}$ of (4.1).

Owing to the different ways the separation co-ordinates were found, and, especially, the fact that one of the separating momenta is an additional constant of the motion, we expect that Φ_1 and Φ_2 have different functional dependence on their variables; so, instead of trying to use the spectral curve relations we directly seek for the Jacobi relations (4.8), by means of explicit calculations. Since the characteristic condition for Stäckel separability is verified in our case (see equation (4.4)), we can look for Jacobi relations that are affine functions in the Casimirs and the Hamiltonians.

Let us first consider $\Phi_1(\mathbf{H}; \zeta_1, \xi_1) = 0$; in this respect, one can notice that ζ_1 is an additional constant of the motion. Indeed, from the form of the Euler vector field (4.3), we easily ascertain that $\{\mathcal{H}_1, \zeta_1\}_{P_1} = 0$ and a direct computation (or a careful examination of the generalised Lenard relations associated with P, Q) shows that ζ_1 commutes with $\mathcal{H}_2$ as well.

So, there must be a functional relation between $\mathcal{H}_0$, C_2, $\mathcal{H}_1$, $\mathcal{H}_2$ and $\zeta_1 = z_1 - z_2$ alone, that is, we expect Φ_1 to be independent of ξ_1. Taking into account that the elements $\mathbf{H}$ are quadratic functions of z_1, z_2, we look for a relation of the form

$$\Phi_1 = \alpha\zeta_1^2 + \mathcal{H}_1 + \beta\mathcal{H}_2 + \gamma_1\mathcal{H}_0 + \gamma_2 C_2,$$

for some unknown *constants* α, β_i, γ_i, $i = 1, 2$. Indeed such a relation can be found with, respectively,

$$\alpha = 2\frac{{\mu_3}^2 - {\mu_2}^2}{\mu_1 + \mu_2}, \qquad \beta = \frac{1}{\mu_1 + \mu_2}, \qquad \gamma_1 = \mu_1 + \mu_2, \qquad \gamma_2 = 0.$$

To find the second separation relation is slightly more involved; still the idea is to look for a relation quadratic in ξ_2, and affine in $\mathcal{H}_0$, C_2, $\mathcal{H}_1$, $\mathcal{H}_2$, with coefficients that may depend non trivially on the co-ordinate λ_2, i.e. a relation of the form

$$\Phi_2 = p(\lambda_2)\xi_2^2 + q_1(\lambda_2)\mathcal{H}_1 + q_2(\lambda_2)\mathcal{H}_2 - \Psi(\lambda_2, \mathcal{H}_0, C_2), \tag{4.9}$$

for some functions p, q_1, q_2 that depend only on λ_2, and for an unknown function $\Psi(\lambda_2; \mathcal{H}_0, C_2)$, affine in the C_i's. After a direct computation one sees that the problem can be solved, and that the second separation relation has the form (4.9), with

$$q_1 = \lambda_2, \qquad q_2 = 1, \qquad p = -2\mu_3^2(F(\mathsf{u})^2 G(\mathsf{u})^2), \qquad \Psi = \lambda_2^2\mathcal{H}_0 - \mu_3 F(\mathsf{u})G(\mathsf{u})\, C_2, \tag{4.10}$$

where the functions $G(\mathsf{u})$ and $F(\mathsf{u})$ are given respectively by (4.6) and (4.7).

It can be noticed that the relation (4.10) is quadratic in the momentum ξ_2, and is algebraic – rather that polynomial – in the co-ordinate λ_2, owing to the relation (4.7).

The question whether these techniques might be useful in the study of the general $\mathfrak{so}(4)$ Euler–Manakov top, that is, the case $\mu_3 \neq \mu_4$, is still under investigation. What is still lacking, in this general case, is the determination of the suitable deformation vector field Z; it is conceivable that a careful use of the theory of elliptic function may provide the answer.

Also, the connection of the results herewith presented with the Clebsh model, according to an isomorphism described in [5] might be worth of further investigations[4], as well as the link with the setting, within the bihamiltonian theory, of the Separation of Variables problem for Lagrange tops presented in [24].

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