Bethe Ansatz for the Ruijsenaars Model of BC_1-Type*

Oleg CHALYKH

School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom
E-mail: *o.chalykh@leeds.ac.uk*

Received December 14, 2006, in final form February 06, 2007; Published online February 22, 2007
Original article is available at `http://www.emis.de/journals/SIGMA/2007/028/`

Abstract. We consider one-dimensional elliptic Ruijsenaars model of type BC_1. It is given by a three-term difference Schrödinger operator L containing 8 coupling constants. We show that when all coupling constants are integers, L has meromorphic eigenfunctions expressed by a variant of Bethe ansatz. This result generalizes the Bethe ansatz formulas known in the A_1-case.

Key words: Heun equation; three-term difference operator; Bloch eigenfunction; spectral curve

2000 Mathematics Subject Classification: 33E30; 81U15

Dedicated to the memory of Vadim Kuznetsov

1 Introduction

The quantum Ruijsenaars model [14] in the simplest two-body case reduces to the following difference operator acting on functions of one variable:

$$L = \frac{\sigma(z-2\gamma m)}{\sigma(z)} T^{2\gamma} + \frac{\sigma(z+2\gamma m)}{\sigma(z)} T^{-2\gamma}, \tag{1}$$

where $\sigma(z)$ is the Weierstrass σ-function, m is the coupling parameter, and T^γ stands for the shift operator acting by $(T^\gamma f)(z) = f(z+\gamma)$. This operator, which first appeared in E. Sklyanin's work [15, 16], can be viewed as a difference version of the Lamé operator $-d^2/dz^2+m(m+1)\wp(z)$. It was observed by Krichever–Zabrodin [13] and by Felder–Varchenko [8, 7], that in the special case of integer coupling parameter the operator (1) shares many features with the Lamé operator. In particular, when $m \in \mathbb{Z}_+$ they both have meromorphic Bloch eigenfunctions which can be given explicitly by a suitable Bethe ansatz. This reflects the well-known fact that the Lamé operator is *finite-gap* for integer m (see e.g. [6] for a survey of the finite-gap theory).

The Lamé operator has the following generalization closely related to the Heun's equation:

$$H = -d^2/dz^2 + \sum_{p=0}^{3} g_p(g_p+1)\wp(z+\omega_p), \tag{2}$$

where ω_p are the half-periods of $\wp(z)$. It can be viewed as a BC_1-generalization of the Lamé operator. Again, for integer coupling parameters g_p this operator is finite-gap, as was discovered by Treibich–Verdier [19], see also [17, 18] for the detailed study of (2).

The operator (2) has a multivariable generalization known as the Inozemtsev model [9]. A relativistic version of the Inozemtsev model ($\equiv BC_n$ version of the Ruijsenaars model) was

*This paper is a contribution to the Vadim Kuznetsov Memorial Issue 'Integrable Systems and Related Topics'. The full collection is available at http://www.emis.de/journals/SIGMA/kuznetsov.html suggested by J.F. van Diejen [4, 5], see also [10, 11]. In the simplest one-variable case it takes the form of a three-term difference operator

$$L = a(z)T^{2\gamma} + b(z)T^{-2\gamma} + c(z), \tag{3}$$

where

$$a(z) = \prod_{p=0}^{3} \frac{\sigma_p(z-\mu_p)\sigma_p(z+\gamma-\mu'_p)}{\sigma_p(z)\sigma_p(z+\gamma)}, \qquad b(z) = a(-z), \tag{4}$$

$c(z)$ is given explicitly in (8) below, and the notations are explained at the beginning of the next section.

Therefore, the operator (3) should be viewed both as a difference analogue of (2) and a BC_1-version of the Ruijsenaars model (1). In the trigonometric limit it coincides with the Askew–Wilson difference operator [1]. (Notice that (3) contains eight parameters μ_p, μ'_p, compared to the four in the Askew–Wilson operator and in (2).) Therefore, it is natural to expect that (3) and (1) should have similar properties. This is indeed the case, as we will demonstrate below. Our main result says that in the case of integer coupling parameters

$$\mu_p = 2\gamma m_p, \qquad \mu'_p = 2\gamma m'_p, \qquad m_p, m'_p \in \mathbb{Z}_+ \qquad (p = 0, \ldots, 3), \tag{5}$$

the operator (3) has meromorphic Bloch eigenfunctions which can be given explicitly via a version of Bethe ansatz. Note that our derivation of the Bethe ansatz equations is very elementary: it only uses some simple facts about the operator (3).

2 Ruijsenaars operator of type BC_1

2.1 Preliminaries

Let $\sigma(z) = \sigma(z; 2\omega_1, 2\omega_2)$ denote the Weierstrass σ-function with the half-periods ω_1, ω_2. Recall that $\sigma(z)$ is an entire odd function on the complex plane quasiperiodic with respect to $2\omega_1$, $2\omega_2$. It will be convenient to introduce the third half period as $\omega_3 = -\omega_1 - \omega_2$. One has

$$\sigma(z + 2\omega_s) = -\sigma(z)e^{2\eta_s(z+\omega_s)}, \qquad s = 1, 2, 3,$$

with $\eta_s = \zeta(\omega_s)$, where $\zeta(z) = \sigma'(z)/\sigma(z)$ denotes the Weierstrass ζ-function. Clearly, $\eta_1 + \eta_2 + \eta_3 = 0$. There is a relation between η_s and half-periods as follows:

$$\eta_1\omega_2 - \eta_2\omega_1 = \pi i/2\,.$$

It is known that $\sigma(z)$ has simple zeros at points of the period lattice 2Γ, where $\Gamma = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. Likewise, $\zeta(z)$ has simple poles with residue 1 at those points. Let us introduce the shifted versions of σ as follows:

$$\sigma_r(z) = e^{-\eta_r z}\sigma(z + \omega_r)/\sigma(\omega_r), \qquad r = 1, 2, 3.$$

These are even functions: $\sigma_r(-z) = \sigma_r(z)$. The quasiperiodicity properties of $\sigma_r(z)$ look as follows:

$$\sigma_r(z + 2\omega_s) = (-1)^{\delta_{r,s}}\sigma_r(z)e^{2\eta_s(z+\omega_s)}, \qquad r, s = 1, 2, 3.$$

Below we will use the convention that $\sigma_0(z) = \sigma(z)$ and $\omega_0 = \eta_0 = 0$. Note that

$$\sigma(2z) = 2\sigma_0(z)\sigma_1(z)\sigma_2(z)\sigma_3(z).$$

Let us remark on quasiperiodicity of the coefficients $a(z)$ and $b(z) = a(-z)$ of the operator (3). Let us use the subscript to indicate the dependence of $a(z) = a_\mu(z)$ and $b(z) = b_\mu(z)$ on the parameters $\mu = \{\mu_p, \mu'_p\}$. Using the translation properties of σ_p one checks directly that $a(z)$, $b(z)$ given by (4) have the following covariance with respect to the shift by a half-period ω_r $(r = 0, \dots, 3)$:

$$a_\mu(z+\omega_r) = a_{\widetilde{\mu}}(z) e^{\eta_r \sum\limits_{p=0}^{3}(\mu_p+\mu'_p)}, \qquad b_\mu(z+\omega_r) = b_{\widetilde{\mu}}(z) e^{-\eta_r \sum\limits_{p=0}^{3}(\mu_p+\mu'_p)}, \tag{6}$$

with $\widetilde{\mu}_p = \mu_{\pi_r(p)}$ and $\widetilde{\mu}'_p = \mu'_{\pi_r(p)}$, where π_r is one of the following permutations:

$$\pi_0 = \mathtt{id}, \qquad \pi_1 = (01)(23), \qquad \pi_2 = (02)(13), \qquad \pi_3 = (03)(12). \tag{7}$$

Note that these permutations form an Abelian subgroup of S_4, and $\pi_p(0) = p$ for $p = 0, \dots, 3$. It follows that $\pi_r \circ \pi_p = \pi_q$ whenever $q = \pi_r(p)$.

2.2 Ruijsenaars operator of type BC_1

Let L be the operator (3)–(4) with the coefficient $c(z)$ given by

$$c(z) = \sum_{p=0}^{3} c_p(\zeta_p(z+\gamma) - \zeta_p(z-\gamma)), \qquad \zeta_p(z) = \frac{\sigma'_p(z)}{\sigma_p(z)} = -\eta_p + \zeta(z+\omega_p), \tag{8}$$

where c_p looks as follows:

$$c_p = -\frac{2}{\sigma(2\gamma)} \prod_{s=0}^{3} \sigma_s(\gamma + \mu_{\pi_p(s)}) \sigma_s(\mu'_{\pi_p(s)}). \tag{9}$$

Here the permutations π_p are the same as in (7).

Remark 1. L is a one-dimensional counterpart of its two-variable version introduced in [4], see also [5, 10, 11]. Their precise relation is as follows: the BC_2 version in [4] involves one more parameter μ attached to the roots $e_1 \pm e_2$, and it decouples when $\mu = 0$. Namely, let D be as in [4], formulas (3.11), (3.19)–(3.20), (3.23)–(3.24). Then we have, for an appropriate constant α, that $\lim\limits_{\mu\to 0}(D - \alpha\mu^{-1}) = L_1 + L_2$, where L_1, L_2 act in $z = z_1$ and z_2, respectively, and are given by (3)–(4) and (8)–(9) above. Observe that in the special case $\mu_0 = 2\gamma m$, $\mu'_0 = \mu_p = \mu'_p = 0$ $(p = 1, 2, 3)$ the operator L reduces to (1).

2.3 Symmetries of L

It is obvious from the formula (8) for $c(z)$ that $c(-z) = c(z)$. As a result, L is invariant under $(z \leftrightarrow -z)$. Next, a direct computation shows that the function $c(z) = c_\mu(z)$ is covariant under the shifts by half-periods, namely:

$$c_\mu(z+\omega_r) = c_{\widetilde{\mu}}(z), \qquad \text{where} \quad \widetilde{\mu}_p = \mu_{\pi_r(p)}, \quad \widetilde{\mu}'_p = \mu'_{\pi_r(p)}. \tag{10}$$

Let us write $L = L_\mu$ to indicate dependence on μ. Combining (6) and (10), we obtain that

$$T^{\omega_r} \circ L_\mu \circ T^{-\omega_r} = e^{-\lambda_r z} \circ L_{\widetilde{\mu}} \circ e^{\lambda_r z}, \qquad \lambda_r = \eta_r (2\gamma)^{-1} \sum_{p=0}^{3} (\mu_p + \mu'_p),$$

where $\widetilde{\mu} = \pi_r(\mu)$ is the same as in (6), (10). This implies that for any $\omega \in \Gamma$

$$T^{\omega} \circ L_\mu \circ T^{-\omega} = e^{-\lambda(\omega) z} \circ L_{\widetilde{\mu}} \circ e^{\lambda(\omega) z}, \tag{11}$$

where $\lambda(\omega) := n_1\lambda_1 + n_2\lambda_2$ if $\omega = n_1\omega_1 + n_2\omega_2$, and $\widetilde{\mu}$ in the right-hand side is defined as $\widetilde{\mu} = \pi_s(\mu)$ if $\omega \equiv \omega_s \mod 2\Gamma$.

3 Bethe ansatz

Providing the coupling constants satisfy (5), put $m = \sum_{p=0}^{3}(m_p + m'_p)$ and consider the following function $\psi(z)$ depending on the parameters $t_1, \ldots, t_m, k \in \mathbb{C}$:

$$\psi(z) = e^{kz} \prod_{i=1}^{m} \sigma(z + t_i). \tag{12}$$

Let us impose m relations onto these parameters as follows:

$$\psi(\omega_s + 2j\gamma) = \psi(\omega_s - 2j\gamma) e^{4j\gamma m \eta_s} \qquad (j = 1, \ldots, m_s), \tag{13}$$

$$\psi(\omega_s + (2j-1)\gamma) = \psi(\omega_s - (2j-1)\gamma) e^{(4j-2)\gamma m \eta_s} \qquad (j = 1, \ldots, m'_s). \tag{14}$$

(Here $s = 0, \ldots, 3$.) We will refer to (13)–(14) as the *Bethe ansatz equations*, or simply the Bethe equations. Explicitly, they look as follows:

$$e^{4j\gamma m \eta_s} \prod_{i=1}^{m} \frac{\sigma(t_i + \omega_s - 2j\gamma)}{\sigma(t_i + \omega_s + 2j\gamma)} = e^{4j\gamma k} \qquad (j = 1, \ldots, m_s), \tag{15}$$

$$e^{(4j-2)\gamma m \eta_s} \prod_{i=1}^{m} \frac{\sigma(t_i + \omega_s - (2j-1)\gamma)}{\sigma(t_i + \omega_s + (2j-1)\gamma)} = e^{(4j-2)\gamma k} \qquad (j = 1, \ldots, m'_s). \tag{16}$$

Now we can formulate the main result of this paper.

Theorem 1. *Suppose the parameters $t_1, \ldots, t_m, k$ satisfy the Bethe equations* (13)–(14) *and the conditions $t_i + t_j \notin 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$ for $1 \le i \ne j \le m$. Then the corresponding function $\psi(z)$* (12) *is an eigenfunction of the operator* (3)–(5).

The proof will be given in the next section.

Remark 2. To compute the corresponding eigenvalue, one evaluates the expression $L\psi/\psi$ at any suitable point z. For instance, a convenient choice is $z = 2\gamma m_0$ (provided $m_0 > 0$), because then the first term in $L\psi$ vanishes.

Remark 3. If some of the coupling parameters m_p, m'_p vanish, then the corresponding sets of the Bethe equations are not present in (13)–(14). For example, in the case when the only nonzero parameter is $m_0 = m$, the Bethe equations take the form:

$$\psi(2j\gamma) = \psi(-2j\gamma) \qquad (j = 1, \ldots, m).$$

In that form (seemingly different from [13, 8]) they appeared in [21].

3.1 Invariant subspaces

The idea of the proof of the theorem is that applying L to ψ will not destroy the conditions (13)–(14), cf. [2, 3]. We begin with two elementary results about a three-term difference operator with meromorphic coefficients:

$$D = a(z)T^{2\gamma} + b(z)T^{-2\gamma} + c(z).$$

Suppose that a, b, c are regular at $z \in 2\gamma\mathbb{Z}$, apart from $z = 0$ where a, b have simple poles. Furthermore, suppose that

$$\mathrm{res}_{z=0}(a + b) = 0 \tag{17}$$

and that for some $m \in \mathbb{Z}_+$ the following is true:

$$a(2\gamma m) = 0, \qquad a(2j\gamma) = b(-2j\gamma), \qquad c(2j\gamma) = c(-2j\gamma) \qquad (j = \pm 1, \ldots, \pm m). \tag{18}$$

Lemma 1 (cf. [2], Lemma 2.2). *Let D be as above and define Q_m as the space of meromorphic functions $f(z)$ which are regular at all points $z \in 2\gamma\mathbb{Z}$ and satisfy the conditions $f(2j\gamma) = f(-2j\gamma)$ for all $j = 1, \ldots, m$. Then $D(Q_m) \subseteq Q_m$.*

Proof. For $D' = aT^{2\gamma} + bT^{-2\gamma}$ this is precisely Lemma 2.2 from [2], thus $D'(Q_m) \subseteq Q_m$. On the other hand, the conditions on c in (18) imply trivially that $cQ_m \subseteq Q_m$. ■

Corollary 1. *Suppose that instead of* (17), (18) *we know that D is invariant under $(z \leftrightarrow -z)$ and that $a(2\gamma m) = 0$. Then $D(Q_m) \subseteq Q_m$.*

Proof. Indeed, in that case we know that $b(z) = a(-z)$ and $c(z) = c(-z)$ identically. This implies the conditions (17)–(18). ■

For the next lemma, we assume that: (1) a is regular at $z \in \gamma + 2\gamma\mathbb{Z}$ apart from a simple pole at $z = -\gamma$; (2) b is regular at $z \in \gamma + 2\gamma\mathbb{Z}$ apart from a simple pole at $z = \gamma$; (3) c is regular at $z \in \gamma + 2\gamma\mathbb{Z}$ apart from simple poles at $z = \pm\gamma$. Also, suppose that

$$\mathrm{res}_{z=-\gamma}(a + c) = \mathrm{res}_{z=\gamma}(b + c) = 0, \qquad \mathrm{res}_{z=-\gamma}a = \mathrm{res}_{z=\gamma}b, \tag{19}$$

$$(a + b + c)|_{z=-\gamma} = (a + b + c)|_{z=\gamma}. \tag{20}$$

(The last condition makes sense because (19) implies that $a + b + c$ is regular at $z = \pm\gamma$.) In addition to that , assume that for some $m \in \mathbb{Z}_+$ the following is true:

$$a((2m - 1)\gamma) = 0, \qquad a((2j - 1)\gamma) = b((-2j + 1)\gamma) \quad \text{for} \quad j = \pm 1, \ldots, \pm m, \tag{21}$$

$$c((2j + 1)\gamma) = c(-(2j + 1)\gamma) \quad \text{for } j = 1, \ldots, m - 1. \tag{22}$$

Lemma 2 (cf. [2], Lemma 2.3). *For D as above, define Q'_m as the space of meromorphic functions $f(z)$ which are regular at $z \in \gamma + 2\gamma\mathbb{Z}$ and satisfy the conditions $f((2j - 1)\gamma) = f((-2j + 1)\gamma)$ for $j = 1, \ldots, m$. Then $D(Q'_m) \subseteq Q'_m$.*

Proof. For $D' = a(T^{2\gamma} - 1) + b(T^{-2\gamma} - 1)$ the proof of the inclusion $D'(Q'_m) \subset Q'_m$ follows the proof of Lemma 2.3 in [2]. On the other hand, the conditions on a, b, c imply that $c' := c + a + b$ belongs to Q'_m. Thus, $c'Q'_m \subseteq Q'_m$. Therefore, the operator $D = D' + (a+b+c)$ preserves Q'_m. ■

Corollary 2. *The lemma above remains valid after replacing* (20)–(22) *by the invariance of D under $(z \leftrightarrow -z)$ and the condition that $a((2m - 1)\gamma) = 0$.*

Proof. Indeed, the conditions (20)–(22) follow easily from the fact that $b(z) = a(-z)$ and $c(z) = c(-z)$. ■

Let us apply these facts to the Ruijsenaars operator (3) with integer coupling parameters (6). Below we always assume that the step γ is irrational, i.e. $\gamma \notin \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma = \mathbb{Q}\omega_1 + \mathbb{Q}\omega_2$. We proceed by defining Q as the space of entire functions $\psi(z)$ satisfying the following conditions for every $\omega \in \omega_s + 2\Gamma$ $(s = 0, \ldots, 3)$:

$$\psi(\omega + 2j\gamma) = \psi(\omega - 2j\gamma)e^{4j\gamma m\eta(\omega)} \qquad (j = 1, \ldots, m_s), \tag{23}$$

$$\psi(\omega + (2j - 1)\gamma) = \psi(\omega - (2j - 1)\gamma)e^{(4j-2)\gamma m\eta(\omega)} \qquad (j = 1, \ldots, m'_s). \tag{24}$$

Here m stands as before for $m = \sum_{p=0}^{3}(m_p + m'_p)$, and the constant $\eta(\omega)$ is defined for $\omega = n_1\omega_1 + n_2\omega_2$ as $\eta(\omega) = n_1\eta_1 + n_2\eta_2$.

Proposition 1. *For integer coupling parameters* (6) *the Ruijsenaars operator* (3) *preserves the space Q of entire functions with the properties* (23)–(24)*: $L(Q) \subseteq Q$.*

Proof. First, by applying Corollaries 1, 2 to the Ruijsenaars operator, we obtain that L preserves the spaces Q_{m_0} and $Q'_{m'_0}$ (in the notations of Lemmas (1), (2)). Note that in doing so, we only have to check the vanishing of $a(z)$ as required in Corollaries 1, 2, and the conditions on the residues (19). This is where the formula (9) becomes crucial. Finally, in order to show that L preserves similar conditions at other points $\omega \in \Gamma$, one applies the formula (11). ■

Next, given $\alpha_1, \alpha_2 \in \mathbb{C}$ and $m \in \mathbb{Z}$, let us write $\mathcal{F}_m^{\alpha_1,\alpha_2}$ for the space of meromorphic functions $\psi(z)$ having the following quasiperiodicity properties:

$$\psi(z + 2\omega_s) = e^{m\eta_s z + \alpha_s}\psi(z), \qquad s = 1, 2.$$

Entire functions in $\mathcal{F}_m^{\alpha_1,\alpha_2}$ $(m > 0)$ are known as *theta-functions of order* m (with characteristics), each of them being a constant multiple of (12), for appropriate $t_1, \ldots, t_m, k$.

Now, a simple check shows that in the case (6) the Ruijsenaars operator (3) preserves these spaces corresponding to $m = \sum\limits_{p=0}^{3}(m_p + m'_p)$:

$$L(\mathcal{F}_m^{\alpha_1,\alpha_2}) \subseteq \mathcal{F}_m^{\alpha_1,\alpha_2}, \qquad \forall\, \alpha_1, \alpha_2.$$

Combining this with Proposition 1, we conclude that L preserves the space of theta-functions of order m satisfying the conditions (23)–(24):

$$L(\mathcal{F}_m^{\alpha_1,\alpha_2} \cap Q) \subseteq \mathcal{F}_m^{\alpha_1,\alpha_2} \cap Q\,, \quad \forall\, \alpha_1, \alpha_2\,. \tag{25}$$

Proof of the Theorem 1. Take a solution $(t_1, \ldots, t_m, k)$ to the Bethe equations and the corresponding function ψ (12). Clearly, ψ belongs to $\mathcal{F}_m^{\alpha_1,\alpha_2}$ for some α_1, α_2. The Bethe equations give the conditions (23)–(24) only for $\omega = \omega_s$, but the rest follows from the translation properties of ψ. Thus, ψ belongs to the space $\mathcal{F}_m^{\alpha_1,\alpha_2} \cap Q$. By (25), $\widetilde{\psi} := L\psi$ also belongs to this space. Now we use the following fact (whose proof will be given below):

Lemma 3. *For any two functions $\psi, \widetilde{\psi} \in F_m^{\alpha_1,\alpha_2} \cap Q$, their quotient $\widetilde{\psi}/\psi$ is an even elliptic function, i.e. it belongs to $\mathbb{C}(\wp(z))$.*

By this lemma, if $\widetilde{\psi}/\psi$ is not a constant, then its poles must be invariant under $z \mapsto -z$, thus there exist at least two of $t_1, \ldots, t_m$ such that their sum belongs to 2Γ. ■

Proof of the lemma. Take any two functions $\psi, \widetilde{\psi}$ in $\mathcal{F}_m^{\alpha_1,\alpha_2} \cap Q$ and put $f := \widetilde{\psi}/\psi$. Note that f is an elliptic function of degree $\leq m$ (because its denominator and numerator have m zeros in the fundamental region). Let us label m pairs of points $\omega_s \pm 2j\gamma$, $\omega_s \pm (2j-1)\gamma$ as $P_l^{\pm}$ with $l = 1, \ldots, m$, then the properties of ψ, $\widetilde{\psi}$ imply that f satisfies the conditions

$$f(P_l^+) = f(P_l^-), \qquad l = 1, \ldots, m. \tag{26}$$

We may assume that f is regular in at least one of the half-periods ω_s, otherwise switch to $1/f = \psi/\widetilde{\psi}$. Let us anti-symmetrize f to get $g(z) := f(z) - f(-z)$, which will be odd elliptic, of degree $\leq 2m$. It is clear that g also satisfies the conditions (26). At the same time, it is anti-symmetric under any of the transformations $z \mapsto 2\omega_s - z$ $(s = 0, \ldots, 3)$. Altogether this implies that g must vanish at each of the $2m$ points $P_l^{\pm}$. Finally, it must vanish at one of the half-periods (where f was regular). So g has $\geq 2m + 1 > \deg(g)$ zeros, hence $g = 0$, $f(z) \equiv f(-z)$, and we are done.

The above argument, however, would not work if one or both of the functions ψ, $\widetilde{\psi}$ vanish at some of the points $P_l^{\pm}$. Indeed, then we cannot claim that f is regular at those points, so some of the conditions (26) would not hold for f. In that case, we can argue as follows. Let ψ_λ denote the linear combination $\psi_\lambda = \psi + \lambda\widetilde{\psi}$. Then ψ_λ, ψ_μ for generic λ, μ will have zero of the same multiplicity at any given point $P_l^\pm$. Thus, choosing λ, μ appropriately, we can always achieve that $\psi_\lambda/\psi_\mu \neq 0, \infty$ at every of these $2m$ points. Let r be the number of those pairs (P_l^+, P_l^-) where ψ_λ, ψ_μ vanish. Then we have that their ratio $f := \psi_\lambda/\psi_\mu$ still satisfies the conditions (26) at the remaining $m-r$ pairs of points and has degree $\leq m-r$ due to the cancelation of the zeros in the denominator and numerator of f. Thus, the previous argument applies and gives that f is even. Therefore, $\widetilde{\psi}/\psi$ is even. ■

3.2 Continuous limit

As remarked in [4], the operator (3) with the coupling parameters (6) in the continuous limit $\gamma \to 0$ turns into the BC_1-version (2) of the Lamé operator:

$$L = \text{const} + \gamma^2 w(z) \circ H \circ w^{-1}(z) + o(\gamma^2), \qquad \text{where} \quad w(z) = \prod_{p=0}^{3} (\sigma_p(z))^{g_p}, \tag{27}$$

and the coupling parameters g_p are given by $g_p := m_p + m_p'$.

To formulate a Bethe ansatz for the operator (2), we put $m = \sum\limits_{p=0}^{3} g_p$ and let $\psi(z) = \psi(z; k, t_1, \ldots, t_m)$ be the function (12). Let us impose the following m relations on the parameters $k, t_1, \ldots, t_m$:

$$\left[\frac{d^{2j-1}}{dz^{2j-1}} \left(\psi(z) e^{-m\eta_s z}\right)\right]_{z=\omega_s} = 0 \qquad \text{for} \quad j = 1, \ldots, g_s \quad \text{and} \quad s = 0, \ldots, 3. \tag{28}$$

Theorem 2. *Suppose the parameters $t_1, \ldots, t_m, k$ satisfy the Bethe equations* (28) *and the conditions $t_i + t_j \notin 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$ for $1 \leq i \neq j \leq m$. Then the function $w^{-1}(z)\psi(z)$ given by* (12) *with w as in* (27)*, is an eigenfunction of the operator* (2).

This theorem is proved analogously to Theorem 1.

Example 1. Let $g_p = 0$ for $p = 1, 2, 3$ and $g_0 = m$. Then the operator (2) becomes the Lamé operator $-d^2/dz^2 + m(m+1)\wp(z)$. Its eigenfunctions have the form $\psi(z)\sigma^{-m}(z)$ with $\psi(z) = e^{kz} \prod\limits_{j=1}^{m} \sigma(z + t_j)$. The Bethe ansatz equations (28) for $k, t_1, \ldots, t_m$ in this case reduce to:

$$\frac{d^{2j-1}\psi}{dz^{2j-1}}(0) = 0 \qquad \text{for} \quad j = 1, \ldots, m. \tag{29}$$

We should note that this particular form of Bethe equations differs from the classical result by Hermite [20]. For instance, in Hermite's equations one discards the points with $t_i = t_j \mod 2\Gamma$, while in Theorem 2 we discard the points with $t_i = -t_j \mod 2\Gamma$. Thus, comparing Theorem 2 with the Hermite's result, we conclude that (29) must be equivalent to Hermite's equations [20] provided $t_i \pm t_j \notin 2\Gamma$ for $i \neq j$. The same remark applies to the equations (28) when compared to the Bethe ansatz in, e.g., [18].

3.3 Spectral curve

Let us say few words about the structure of the solution set $X \subset \mathbb{C}^m \times \mathbb{C}$ to the Bethe equations (13)–(14). We will skip the details, since the considerations here are parallel to those in [8, 7, 13, 21].

First, using the properties of $\sigma(z)$, one observes that $\psi(z)$ acquires a constant factor under the transformations

$$(t_1,\dots,t_m,k)\mapsto(t_1,\dots,t_j+2\omega_s,\dots,t_m,k-2\eta_s)\qquad(s=1,2).$$

As a result, X is invariant under these transformations. Also, multiplying $\psi(z)$ by $e^{\pi iz/\gamma}$ does not affect the Bethe equations, because such an exponential factor is (anti)periodic under the shifting of z by multiples of γ. Therefore, X is invariant under the shifts of k by $\pi i/\gamma$:

$$(t_1,\dots,t_m,k)\mapsto(t_1,\dots,t_m,k+\pi i/\gamma).$$

Finally, ψ does not change under permutation of $t_1,\dots,t_m$, so X is invariant under such permutations.

Let $\widetilde{X}$ denote the quotient of X by the group generated by all of the above transformations. Explicitly, let $b_{s,j}(t_1,\dots,t_m)$ and $b'_{s,j}(t_1,\dots,t_m)$ denote the left-hand side of equations (15) and (16). (Here $s=0,\dots,3$ and $j=1,\dots,m_s$ or $j=1,\dots,m'_s$, respectively.) Introduce the variable $q:=e^{2\gamma k}$. Then X is described by the equations

$$b_{s,j}(t_1,\dots,t_m)=q^{2j},\qquad b'_{s,j}(t_1,\dots,t_m)=q^{2j-1}.\tag{30}$$

Excluding the q-variable from the equations (30), we may think of $\widetilde{X}$ as an algebraic subvariety in the symmetric product $S^m\mathcal{E}$ of m copies of the elliptic curve $\mathcal{E}=\mathbb{C}/2\Gamma$ where $\Gamma=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$. (See, however, Remark 4 below.) Counting the number of equations, we conclude that every irreducible component of $\widetilde{X}$ has dimension ≥ 1. Since we are interested (cf. Theorem 1) in those points $(t_1,\dots,t_m)$ of $\widetilde{X}$ where $t_i+t_j\notin 2\Gamma$, we should restrict ourselves to the open part $Y\subset\widetilde{X}$, lying in

$$S^m\mathcal{E}\setminus\cup_{i<j}\{t_i+t_j\equiv 0 \bmod 2\Gamma\}.$$

We need to show that Y is nonempty and one-dimensional. To this end, one easily observes from the equations (15)–(16) that the closure $\overline{Y}$ of Y in $S^m\mathcal{E}$ contains the points $P^+=(P_1^+,\dots,P_m^+)$ and $P^-=(P_1^-,\dots,P_m^-)$, in the notations of the proof of the Lemma 3. These 'infinite' points correspond to $q\to 0,\infty$ in (30). Similarly to [7], lemma 3.2, one shows that near $P^\pm$ the variety $\overline{Y}$ looks like a smooth curve, with $q^{\pm1}$ being a local parameter. One can show that $\overline{Y}$ is an irreducible, projective curve, and it should be regarded as the 'spectral curve' for the operator L. For every $(t_1,\dots,t_m)\in\overline{Y}\setminus\{P^+,P^-\}$, the corresponding value of $q=e^{2\gamma k}$ is determined from (30), and the corresponding $\psi(z)$ is unique, up to a factor of the form $e^{\pi iNz/\gamma}$. We have $L\psi=\epsilon\psi$, with the eigenvalue ϵ being a single-valued function on $\overline{Y}$ which has two simple poles at $P^\pm$. There is an involution ν on $\overline{Y}$, which sends $(t_1,\dots,t_m)$ to $(-t_1,\dots,-t_m)$ and the corresponding $\psi(z)$ to $\psi(-z)$; note that $\nu(P^+)=P^-$. The function ϵ is ν-invariant, and takes each its value exactly twice on $\overline{Y}$. It is straightforward to compute the asymptotics of ϵ and ψ near $P^\pm$. Finally, for generic value of ϵ, the eigenspace of meromorphic functions $\{f : Lf=\epsilon f\}$ is spanned by the corresponding $\psi(z)$, $\psi(-z)$ over the field K of 2γ-periodic meromorphic functions of z.

Remark 4. Note that in the case when all $m'_s=0$, the second set (14) of the Bethe equations is absent, thus a shift $k\mapsto k+\frac{\pi i}{2\gamma}$ is also allowed. In that case the coefficient $c(z)$ (8) vanishes, so L has two terms only, and it is easy to see that the transformation $\psi\mapsto e^{\frac{\pi iz}{2\gamma}}\psi$ changes the sign of the eigenvalue ϵ. As a result, the subvariety of $S^m\mathcal{E}$ which was obtained by excluding q from (30), will be a quotient of $\widetilde{X}$ by $\mathbb{Z}_2$, rather than $\widetilde{X}$ itself (cf. [8, 7, 13, 21]).

References

[1] Askey R., Wilson J., Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, *Mem. Amer. Math. Soc.* **319** (1985), 1–53.

[2] Chalykh O., Macdonald polynomials and algebraic integrability, *Adv. Math.* **166** (2002), 193–259, math.QA/0212313.

[3] Chalykh O., Etingof P., Oblomkov A., Generalized Lamé operators, *Comm. Math. Phys.* **239** (2003), 115–153, math.QA/0212029.

[4] van Diejen J.F., Integrability of difference Calogero–Moser systems, *J. Math. Phys.* **35** (1994), 2983–3004.

[5] van Diejen J.F., Difference Calogero–Moser systems and finite Toda chains, *J. Math. Phys.* **36** (1995), 1299–1323.

[6] Dubrovin B.A., Krichever I.M., Novikov S.P., Integrable systems I, Dynamical systems IV, *Encyclopaedia Math. Sci.*, Vol. 4, Springer, Berlin, 2001, 177–332.

[7] Felder G., Varchenko A., Algebraic integrability of the two-body Ruijsenaars operator, *Funktsional Anal. i Prilozhen.* **32** (1998), 8–25, q-alg/9610024.

[8] Felder G., Varchenko A., Algebraic Bethe ansatz for the elliptic quantum group $\mathrm{E}_{\tau,\eta}(\mathrm{sl}_2)$, *Nuclear Phys. B* **480** (1996), 485–503, q-alg/9605024.

[9] Inozemtsev V.I., Lax representation with spectral parameter on a torus for integrable particle systems, *Lett. Math. Phys.* **17** (1989), 11–17.

[10] Komori Y., Hikami K., Quantum integrability of the generalized elliptic Ruijsenaars models, *J. Phys. A: Math. Gen.* **30** (1997), 4341–4364.

[11] Komori Y., Hikami K., Conserved operators of the generalized elliptic Ruijsenaars models, *J. Math. Phys.* **39** (1998), 6175–6190.

[12] Koornwinder T.H., Askey–Wilson polynomials for root systems of type BC, in Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications, Editor D.St.P. Richards, *Contemp. Math.*, **138** (1992), 189–204.

[13] Krichever I., Zabrodin A., Spin generalization of the Ruijsenaars–Schneider model, the nonabelian 2D Toda lattice, and representations of the Sklyanin algebra, *Uspekhi Mat. Nauk* **50** (1995), 3–56, hep-th/9505039.

[14] Ruijsenaars S.N.M., Complete integrability of relativistic Calogero–Moser systems and elliptic functions identities, *Comm. Math. Phys.* **110** (1987), 191–213.

[15] Sklyanin E.K., Some algebraic structures connected with the Yang–Baxter equation (Russian), *Funktsional. Anal. i Prilozhen.* **16** (1982), no. 4, 27–34.

[16] Sklyanin E.K., Some algebraic structures connected with the Yang–Baxter equation, Representations of a quantum algebra (Russian), *Funktsional. Anal. i Prilozhen.* **17** (1983), no. 4, 34–48.

[17] Smirnov A.O., Elliptic solitons and Heun's equation, in The Kowalevski Property (Leeds, 2000), *CRM Proc. Lecture Notes*, Vol. 32, Editor V.B. Kuznetsov, American Mathematical Society, Providence, RI, 2002, 287–305, math.CA/0109149.

[18] Takemura K., The Heun equation and the Calogero–Moser–Sutherland system I, The Bethe ansatz method, *Comm. Math. Phys* **235** (2003), 467–494, math.CA/0103077.

[19] Treibich A., Verdier J.-L., Solitons elliptiques, in The Grothendieck Festschrift, Vol. III, Editots P. Cartier et al., *Progr. Math.*, Vol. 88, Birkhäuser, Boston, 1990.

[20] Whittaker E.T., Watson G.N., A course of modern analysis, Cambridge University Press, New York, 1962.

[21] Zabrodin A., On the spectral curve of the difference Lamé operator, *Int. Math. Res. Not.* (1999), no. 11, 589–614, math.QA/9812161.