Quantum Super-Integrable Systems as Exactly Solvable Models*

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Abstract. We consider some examples of quantum super-integrable systems and the associated nonlinear extensions of Lie algebras. The intimate relationship between super-integrability and exact solvability is illustrated. Eigenfunctions are constructed through the action of the commuting operators. Finite dimensional representations of the quadratic algebras are thus constructed in a way analogous to that of the highest weight representations of Lie algebras.

Key words: quantum integrability; super-integrability; exact solvability; Laplace–Beltrami

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Dedicated to the memory of Vadim Kuznetsov

1 Introduction

In classical mechanics, there is a well defined meaning to the term 'complete integrability'. If, in n degrees of freedom, we have n independent functions *in involution* (mutually Poisson commuting), then the system can be integrated, 'up to quadrature'. This is known as *Liouville's Theorem*. It is customary to consider one of these functions as 'the Hamiltonian' H, with the others being its *first integrals* and this Hamiltonian is said to be *completely integrable in the Liouville sense*. Whilst n is the maximal number of independent functions which can be *in involution*, it is possible to have further integrals of the Hamiltonian H, which necessarily generate a non-Abelian algebra of integrals of H. The maximal number of additional *independent* integrals is $n-1$, since the 'level surface' of $2n-1$ integrals (meaning the intersection of individual level surfaces) is just the (unparameterised) integral curve. Well known elementary examples are the isotropic harmonic oscillator, the Kepler system and the Calogero–Moser system. The quadratures of complete integrability are often achieved through the separation of variables of the Hamilton–Jacobi equation. The solution of a maximally super-integrable system can also be calculated purely algebraically (albeit implicitly), requiring just the solution of the equations $I_k = c_k$, $k = 1, \ldots, 2n-1$.

In n-dimensions, Quantum integrable systems are defined analogously by requiring the existence of n mutually commuting differential operators (usually with some requirements, such as self-adjointness). One of these will be of Schrödinger type, being the quantum version of the Hamiltonian H. However, we don't have a theorem analogous to Liouville's and we don't have anything resembling 'reduction to quadrature'. There *is* a notion of separation of variables (this time in the linear PDE sense), requiring a solution to be expressible as the product of n functions of a single variable. This reduces the problem of solving one Schrödinger equation in

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n-dimensions to that of solving n Schrödinger equations in 1-dimension. This achieves *less* than quadratures.

What we would like to achieve in quantum mechanics is to build the spectrum and corresponding eigenfunctions for Schrödinger's equation with some given boundary conditions. This class of Schrödinger equation is called *exactly solvable*. Even in 1-dimension exactly solvable Schrödinger equations are rare, so unlikely to be the end-product of applying the separation of variables method. One particular class of 1-dimension exactly solvable Schrödinger equations was presented by Infeld and Hull [8], related to the *factorisation of operators*. This can be related to Darboux transformations, which can then be generalised to higher dimensions (where factorisation is not possible). Darboux transformations can then be used to build exactly solvable Schrödinger equations in higher dimensions. In [4], Darboux transformations have been shown to be related to the symmetries of the Laplacian.

In the quantum case we also have a notion of *super-integrability*. By analogy with the classical case, super-integrability again involves having 'extra' commuting operators. Once again, we cannot have more than n such operators *in involution*, so the algebra is necessarily non-Abelian. Furthermore, if the operators are second order, their commutators are third order, with further commutators being fourth order and so on, so cannot be expected to generate a *finite dimensional Lie algebra*. However, they *can* generate a finite dimensional algebra with *nonlinear commutation relations* (see [2, 10, 13], for example). There are many papers on super-integrable systems (both classical and quantum). Particularly relevant to the present paper is a series of papers by Kalnins et al. (see [9, 10] and references therein), in which a classification is given of super-integrable systems in 2-dimensional flat and constant curvature spaces and in which a pair of quadratic first integrals, together with their quadratic commutation relations are derived. As is well known, super-integrable systems are separable in more than one coordinate system. In most of these papers the main emphasis is on the separation of variables, first by constructing and classifying coordinate systems and corresponding 'separable potentials' and then using this (in the quantum case) to build eigenfunctions.

In the present paper, there is no discussion of separation of variables and 'differential equations techniques' are kept to a minimum. Our main emphasis is on the *direct relationship* between *super-integrability* and *exactly solvability*. We use the commuting operators in a *direct and explicit manner* to build the eigenfunctions for our operators. This construction is the analogue (in the context of our quadratic algebras) to that of the *highest weight representation* of simple Lie algebras (see the Appendix). In particular, the polynomial eigenfunctions of the Krall–Sheffer operator of Example 9 can be *explicitly constructed* in this way. The issue of the relationship between super-integrability and exact solvability is not new and is discussed in [10, 11, 14]. In [14] it is conjectured that exactly solvability should hold quite widely for super-integrable systems. Whilst operator algebras are discussed in these papers they are not used to construct eigenfunctions.

In this paper we consider some particular super-integrable systems, associated with a particular Laplace–Beltrami operator in 2-dimensions. One of the examples is in the Krall–Sheffer classification of 'admissible operators'. The relationship between Krall–Sheffer operators and super-integrability was shown by Harnad, et al in [6, 15]. In these papers they reworked the Krall–Sheffer classification and showed that in each case the leading order terms in the operator correspond to the Laplace–Beltrami operator of either a zero or constant curvature, 2-dimensional space. They also show that each of the 9 cases is super-integrable by presenting pairs of commuting, second order operators. Example 9 of the current paper is *exactly* Case II of [6, 15]. In [15] it is pointed out that this case had been previously introduced and studied in [1, 10].

In the next section, we give some basic formulae related to our particular metric and its symmetries. We then present 3 super-integrable examples, with their operator algebras and finite representation spaces (families of eigenfunctions). We then give some concluding remarks.

2 A metric and its symmetries

For an n-dimensional (pseudo-)Riemannian space, with local coordinates $x^1, \dots, x^n$ and metric g_{ij}, the Laplace–Beltrami operator is defined by $L_b f = g^{ij}\nabla_i\nabla_j f$, which has explicit form

$$L_b f = \sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^i} \right), \tag{1}$$

where g is the determinant of the matrix g_{ij}. The coefficients of leading order terms in the Laplace–Beltrami operator are the coefficients of the *inverse metric* g^{ij}. For a metric with isometries, the infinitesimal generators (Killing vectors) are just first order differential operators which commute with the Laplace–Beltrami operator (1). When the space is either flat or constant curvature, it possesses the maximal group of isometries, which is of dimension $n(n+1)/2$. In this case, L_b is actually the second order *Casimir* function of the symmetry algebra (see [5]).

In this paper, we consider one particular constant curvature metric in 2-dimensions, with inverse

$$g^{ij} = \begin{pmatrix} x^2 & xy \\ xy & y^2 - y \end{pmatrix}. \tag{2}$$

The choice of coordinates here is motivated by the relationship to Krall–Sheffer operators [12], but otherwise quite arbitrary.

A convenient basis of Killing vectors is

$$\mathbf{H} = 4x\partial_x, \qquad \mathbf{E} = 2\sqrt{xy}\partial_y, \qquad \mathbf{F} = 4\sqrt{xy}\partial_x + 2(y-1)\sqrt{\frac{y}{x}}\partial_y, \tag{3}$$

satisfying the standard commutation relations of $sl(2,\mathbb{C})$:

$$[\mathbf{H},\mathbf{E}] = 2\mathbf{E}, \qquad [\mathbf{H},\mathbf{F}] = -2\mathbf{F}, \qquad [\mathbf{E},\mathbf{F}] = \mathbf{H}. \tag{4}$$

The Laplace–Beltrami operator for the metric (2) is proportional to the quadratic Casimir operator:

$$L_b = \frac{1}{16}(\mathbf{H}^2 + 2\mathbf{EF} + 2\mathbf{FE}) = x^2\partial_x^2 + 2xy\partial_x\partial_y + (y^2 - y)\partial_y^2 + \frac{3}{2}x\partial_x + \frac{1}{2}(3y-1)\partial_y. \tag{5}$$

Remark 1. The operator L_b is invariant under an **involution**,

$$\bar{x} = \frac{(y-1)^2}{x}, \qquad \bar{y} = y. \tag{6}$$

which, in fact, gives a concrete realisation of the Lie algebra automorphism

$$\mathbf{E} \leftrightarrow \mathbf{F}, \qquad \mathbf{H} \to -\mathbf{H}.$$

This will be useful in some of the calculations below.

Second order operators, commuting with L_b, are just symmetric quadratic forms of Killing vectors. Suppose $\mathbf{K}$ is such an operator. Then we may seek functions u and v, such that

$$[L_b + u, \mathbf{K} + v] = 0,$$

which constitutes a coupled system of partial differential equations for u and v. The solution depends upon a pair of arbitrary functions, each of one variable, typical of separable systems. Requiring that there exist two of these second order commuting operators, $I_j = \mathbf{K}_j + v_j$, strongly constrains these (formerly) arbitrary functions, which reduce to rational functions depending upon only a finite number of parameters. The coefficients of the second order derivatives in such an operator $\mathbf{K}$ define a contravariant, rank-two Killing tensor. For brevity, we will refer to these operators as Killing tensors in what follows.

The examples discussed below were presented in [4], in the context of Darboux transformations, which will not be discussed in this paper. Instead we discuss how to use the commuting second order operators to build eigenfunctions of $L = L_b + u$ in a way analogous to the highest weight representation of a Lie algebra.

For some calculations, such as those of [4], it is convenient to use separation coordinates. In our case we can choose 2 (or even 3) such coordinate systems, but the current calculations are as easily done in terms of the $x - y$ coordinates.

3 The system with Killing tensors $\mathbf{H}^2$ and $\mathbf{E}^2$

Requiring *both* $I_1 = \mathbf{H}^2 + v_1$ and $I_2 = \mathbf{E}^2 + v_2$ to commute with L leads to the specific forms of $u(x, y)$, $v_i(x, y)$ in the operators below

$$\begin{aligned}
L &= L_b + \frac{c_0}{x} + \frac{c_1}{y} + c_2\frac{y-1}{x^2},\\
I_1 &= \mathbf{E}^2 - 4\left(c_1\frac{x}{y} + c_2\frac{y}{x}\right),\\
I_2 &= \mathbf{H}^2 - 16\left(c_0\left(\frac{y-1}{x}\right) + c_2\left(\frac{y-1}{x}\right)^2\right).
\end{aligned} \tag{7}$$

In order to build eigenfunctions of L it is convenient to change gauge:

$$L \mapsto \tilde{L} = G^{-1}LG \qquad \text{with} \qquad G = \exp\left(\frac{\alpha}{2}\left(\frac{y-1}{x}\right)\right) x^{(\beta+\gamma-1)/2} y^{-(2\gamma+1)/4}, \tag{8}$$

which simultaneously reduces the 3 potential terms to constants. The specific coefficients in G were chosen to simplify the final expressions for the operators. Removing the additive constants and adjusting overall multiplicative factors (and dropping the 'tilde') we arrive at the following operators:

$$\begin{aligned}
L &= x^2\partial_x^2 + 2xy\partial_x\partial_y + \left(y^2 - y\right)\partial_y^2 + (\beta x + \alpha)\partial_x + (\beta y + \gamma)\partial_y,\\
I_1 &= xy\partial_y^2 + (\alpha y - \gamma x)\partial_y,\\
I_2 &= x^2\partial_x^2 + ((\beta+\gamma)x + \alpha(1-y))\partial_x,
\end{aligned} \tag{9}$$

where $c_0 = \alpha(2 - \beta - \gamma)/2$, $c_1 = (2\gamma + 1)(2\gamma + 3)/16$, $c_2 = \alpha^2/4$. This is an example of *admissible operator*, as defined by Krall and Sheffer [12]. The connection between Krall–Sheffer operators and super-integrability was first pointed out in [6, 15]. Indeed, Example 9 above is *exactly* Case II of [6, 15]. As pointed out in [15], this case was previously introduced and studied in [1, 10].

The operator $I_3 = [I_1, I_2]$ is third order, so cannot be written as a polynomial (in particular, *linear*, Lie algebraic) expression in L, I_1, I_2, but does satisfy the polynomial equation:

$$\begin{aligned}
I_3^2 = {} & \frac{4}{3}(I_1^2I_2 + I_1I_2I_1 + I_2I_1^2 - 2I_1^2) + \alpha(\beta - \gamma - 2)\left(I_1I_2 + I_2I_1 - \frac{4}{3}I_1\right)\\
& + (\beta + \gamma - 2)((\beta+\gamma)I_1^2 - 2\alpha I_1 L) + \alpha^2(I_2 - L)^2 + \frac{2}{3}\alpha^2(I_2 - (3\gamma + 1)L).
\end{aligned}$$

These 4 operators satisfy the commutation relations:

$$\begin{aligned}
&[L, I_1] = [L, I_2] = [L, I_3] = 0, \qquad [I_1, I_2] = I_3,\\
&[I_1, I_3] = 2I_1^2 + \alpha(\beta - \gamma - 2)I_1 + \alpha^2(I_2 - L),\\
&[I_2, I_3] = -2(I_1I_2 + I_2I_1) + (\beta + \gamma - 2)(\alpha L - (\beta + \gamma)I_1) - \alpha(\beta - \gamma - 2)I_2.
\end{aligned} \tag{10}$$

By definition (see [12]), an *admissible operator* possesses a sequence of eigenvalues λ_n (with $\lambda_n \neq \lambda_m$ for $m \neq n$)[1], such that, for each $n \geq 0$, there exist $n+1$ linearly independent polynomial eigenfunctions of degree n, with eigenvalue λ_n. The *degeneracy* (for $n > 0$) of λ_n stems from the super-integrability in a very explicit way. We show below how to use the operators I_1 and I_2 to build sequences of polynomial eigenfunctions of L, using a construction which is analogous to that of the 'highest weight representation' of a simple Lie algebra.

We start with a polynomial eigenfunction

$$P_{n,0} = x^n + \sum_{i=1}^{n} \rho_i x^{n-i},$$

for constants ρ_i, which is independent of y and therefore in the Kernel of I_1. This satisfies the 1-dimensional eigenvalue problem

$$L_x P_{n,0} \equiv (x^2\partial_x^2 + (\beta x + \alpha)\partial_x)P_{n,0} = \lambda_n P_{n,0}, \qquad \lambda_n = n(n + \beta - 1).$$

This value of λ_n is easily determined by looking at the coefficient of x^n. It is easy to show that

$$P_{1,0} = x + \frac{\alpha}{\beta}, \qquad P_{2,0} = x^2 + \frac{2\alpha x}{\beta + 2} + \frac{\alpha^2}{(\beta + 2)(\beta + 1)},$$

and that this sequence satisfies the 3-point recursion relation

$$\mathbf{r}_x : \quad P_{n+1,0} = (x + A_n)P_{n,0} + B_n P_{n-1,0},$$

with

$$A_n = \frac{\alpha(\beta - 2)}{(\beta + 2n)(\beta + 2n - 2)}, \qquad B_n = \frac{\alpha^2 n(n + \beta - 2)}{(\beta + 2n - 1)(\beta + 2n - 2)^2(\beta + 2n - 3)}.$$

We now use I_2 (for $\alpha \neq 0$) to build a sequence of (*monic*) eigenfunctions

$$P_{n-k,k} = x^{n-k}y^k + \text{ lower order terms}, \qquad k = 1, \dots, n.$$

The case of $\alpha = 0$ is the reduction $c_0 = c_2 = 0$, which has a different algebra and is dealt with later (see the system (12)). Since I_2 commutes with L, all such eigenfunctions will have the same eigenvalue λ_n, which means (important for $P_{0,n}$, below) that the degree of each polynomial must be n. The polynomials $P_{n-k,k}$ are defined recursively by

$$I_2 P_{n-k,k} = (n-k)(n-k-1+\beta+\gamma)P_{n-k,k} - \alpha(n-k)P_{n-k-1,k+1}, \qquad k = 0, \dots, n-1, \tag{11}$$

as can be seen by inspecting the form of I_2 and by inspecting the coefficients of $x^{n-k}y^k$ and $x^{n-k-1}y^{k+1}$. It can be seen that $I_2 P_{0,n} = 0$, so the polynomial $P_{0,n}$ must be independent of x.[2] The polynomials $P_{0,n}(y)$ also satisfy a 3-point recursion relation [3]

$$\mathbf{r}_y : \quad P_{0,n+1} = (y + \tilde{A}_n)P_{0,n} + \tilde{B}_n P_{0,n-1},$$

[1] The condition (3.8) of [12], which ensures this, is that β should not be a negative integer.

[2] Since the leading order term is killed, any non-zero result would be a polynomial of lower degree, which is not possible. This can also be seen by setting $k = n$ in (11).

with

$$
\begin{aligned}
\tilde{A}_n &= \frac{(n+1)(\gamma-n)}{\beta+2n} - \frac{n(\gamma-n+1)}{\beta+2n-2}, \\
\tilde{B}_n &= -\frac{n(\gamma-n+1)}{2(\beta+2n-2)}\left((n+1)\frac{\gamma-n}{\beta+2n-1} - 2n\frac{\gamma-n+1}{\beta+2n-2} + (n-1)\frac{\gamma-n+2}{\beta+2n-3}\right).
\end{aligned}
$$

The space of polynomials spanned by this basis is also invariant under the action of I_1:

$$
I_1 P_{n-k,k} = k\alpha P_{n-k,k} + k(k-\gamma-1)P_{n-k+1,k-1}, \qquad k = 1,\ldots,n.
$$

This array of polynomials, together with the action of I_1, I_2 and the 3-point recursion relations, are depicted in Fig. 1.

$$
\begin{tikzcd}
 & & & & P_{0,0} \arrow[dl, "\mathbf{r}_x"'] \arrow[dr, "\mathbf{r}_y"] & & & & \\
 & & & P_{1,0} \arrow[dl] & \rightleftarrows & P_{0,1} \arrow[dr] & & & \\
 & & P_{2,0} \arrow[dl] & \rightleftarrows & P_{1,1} & \rightleftarrows & P_{0,2} \arrow[dr] & & \\
 & P_{3,0} \arrow[dl] & \rightleftarrows & P_{2,1} & \rightleftarrows & P_{1,2} & \rightleftarrows & P_{0,3} \arrow[dr] & \\
P_{4,0} & \rightleftarrows & P_{3,1} & \rightleftarrows & P_{2,2} & \rightleftarrows & P_{1,3} & \rightleftarrows & P_{0,4}
\end{tikzcd}
$$

Figure 1. The triangular lattice of polynomials $P_{n-k,k}$, $k = 0,\ldots,n$ with $P_{0,0} = 1$, for system (9). Horizontal arrows denote the action of I_1 (left) and I_2 (right).

For each $n \geq 1$, the sequence of polynomials $P_{n-k,k}$, $k = 0,\ldots,n$, defines an $n+1$ dimensional representation of the algebra (10). These can be explicitly calculated. For instance, apart from $P_{1,0}$ and $P_{2,0}$, given above, we have

$$
\begin{aligned}
&P_{0,1} = y + \frac{\gamma}{\beta}, \qquad P_{1,1} = xy + \frac{1}{\beta+2}(\gamma x + \alpha y) + \frac{\alpha\gamma}{(\beta+2)(\beta+1)}, \\
&P_{0,2} = y^2 + 2\,\frac{(\gamma-1)y}{\beta+2} + \frac{\gamma(\gamma-1)}{(\beta+2)(\beta+1)}.
\end{aligned}
$$

It is easy to see that, for $n \geq 2$, the 4 representative matrices generate a larger Lie algebra (commutators lie outside the linear span), but *do* satisfy the *quadratic* relations (10).

4 The system with Killing tensors $\mathbf{H}^2$ and $\mathbf{F}^2$

Requiring *both* $I_1 = \mathbf{H}^2 + v_1$ and $I_2 = \mathbf{F}^2 + v_2$ to commute with L leads to the specific forms of $u(x,y)$, $v_i(x,y)$ in the operators below

$$
\begin{aligned}
L &= L_b + c_0\frac{x}{(y-1)^2} + \frac{c_1}{y} + c_2\,\frac{x^2}{(y-1)^3}, \\
I_1 &= \mathbf{F}^2 - 4\left(c_1\frac{(y-1)^2}{xy} + c_2\frac{xy}{(y-1)^2}\right), \\
I_2 &= \mathbf{H}^2 - 16\left(c_0\left(\frac{x}{y-1}\right) + c_2\left(\frac{x}{y-1}\right)^2\right).
\end{aligned}
$$

These are, in fact, directly transformed from (7) under the action of the involution (6). Either by composing the gauge transformation (8) with this involution or by directly transforming (9) by (6) (and dropping 'bars'), we obtain

$$
\begin{aligned}
L &= x^2\partial_x^2 + 2xy\partial_x\partial_y + (y^2-y)\partial_y^2 + \left(\beta x + \frac{2(\beta+\gamma-1)x}{y-1} - \frac{\alpha x^2}{(y-1)^2}\right)\partial_x + (\beta y+\gamma)\partial_y,\\
I_1 &= 4xy\partial_x^2 + 4y(y-1)\partial_x\partial_y + \frac{y(y-1)^2}{x}\partial_y^2 + 2\left((1-\gamma)y + \frac{\alpha xy}{y-1} + \gamma\right)\partial_x\\
&\quad + \left(\alpha y - \frac{\gamma(y-1)^2}{x}\right)\partial_y,\\
I_2 &= x^2\partial_x^2 + \left((2-\beta-\gamma)x - \frac{\alpha x^2}{1-y}\right)\partial_x,
\end{aligned}
$$

where $c_0 = \alpha(2-\beta-\gamma)/2$, $c_1 = (2\gamma+1)(2\gamma+3)/16$, $c_2 = \alpha^2/4$. These satisfy the same quadratic relations (10). Starting from these operators, it would not be so easy to find a simple function in the kernel of I_1, and then to build the sequence of eigenfunctions, but there is no need to do this, since the functions

$$
Q_{n-k,k}(x,y) = P_{n-k,k}\left(\frac{(y-1)^2}{x}, y\right)
$$

are obtained directly through the involution. The operator L is no longer 'admissible' in the Krall–Sheffer sense, and the eigenfunctions are no longer polynomial, but the same set of relations hold, leading to the same diagram of Fig. 1.

5 The system with Killing tensors $\mathbf{E}^2$ and $\mathbf{F}^2$

Requiring *both* $I_1 = \mathbf{E}^2 + v_1$ and $I_2 = \mathbf{F}^2 + v_2$ to commute with L leads to the specific forms of $u(x,y)$, $v_i(x,y)$ in the operators below

$$
L = L_b + \frac{c_1}{y}, \qquad I_1 = \mathbf{E}^2 - 4c_1\,\frac{x}{y}, \qquad I_2 = \mathbf{F}^2 - 4c_1\,\frac{(y-1)^2}{xy}. \tag{12}
$$

However, it can be seen that the potential of L is a subcase of (7), with $c_0 = c_2 = 0$, so there should also be an integral of the form $I_3 = \mathbf{H}^2 + v_3$. Indeed, it should be *exactly* as in the case of (7). However, in this reduction, we just have $I_3 = \mathbf{H}^2$, so may consider the algebra generated by L, I_1, I_2 and $\mathbf{H}$, which satisfy the commutation relations

$$
[\mathbf{H}, I_1] = 4I_1, \qquad [\mathbf{H}, I_2] = -4I_2, \qquad [I_1, I_2] = 2(8c_1-1)\mathbf{H} - \mathbf{H}^3 + 16\mathbf{H}\,L, \tag{13}
$$

as well as $[L, I_1] = [L, I_2] = [L, \mathbf{H}] = 0$. Under the action of the involution (6), $L \mapsto L$, $I_1 \mapsto I_2$, $I_2 \mapsto I_1$, $\mathbf{H} \mapsto -\mathbf{H}$, which preserves the commutation relations (13). The Casimir operator of the algebra (13) is

$$
\mathcal{C} = I_1 I_2 + I_2 I_1 + \frac{1}{2}(8c_1-5)\mathbf{H}^2 - \frac{1}{8}\mathbf{H}^4 + 4\mathbf{H}^2 L.
$$

This is not, of course, independent of L and can be written $\mathcal{C} = 32L^2 - 16(4c_1+1)L + 16c_1(2c_1-1)$.

We mimic the highest weight representation of $sl(2,\mathbb{C})$ and seek function $\varphi_1(x,y)$, satisfying

$$
\mathbf{H}\varphi_1 = s\varphi_1, \qquad L\varphi_1 = \lambda\varphi_1, \qquad I_1\varphi_1 = 0.
$$

The first of these gives $\varphi_1(x,y) = x^{s/4}\psi(y)$, so the remaining equations lead to

$$16y^2(y-1)\psi' + 8y((s+3)y-1)\psi' + (16c_1 + (s(s+2) - 16\lambda)y)\psi = 0,$$
$$2y^2\psi' + y\psi' - 2c_1\psi = 0,$$

which can be solved for both λ and ψ to give

$$\lambda_{m,s} = \frac{1}{16}(s+m+1)(s+m+3), \qquad \psi = y^{(m+1)/4}, \qquad \text{where} \qquad c_1 = \frac{m^2-1}{16}.$$

We now define

$$\varphi_k = I_2^{k-1}\varphi_1, \qquad \text{satisfying} \qquad \mathbf{H}\varphi_k = (s - 4(k-1))\varphi_k.$$

Under the involution, we have $\varphi_1 \mapsto \bar{\varphi}_1$ and

$$\mathbf{H}\bar{\varphi}_1 = -s\bar{\varphi}_1, \qquad I_2\bar{\varphi}_1 = 0.$$

For this to be in our sequence φ_k, we must have $s - 4(k-1) = -s$ for some k. For some integer n, we therefore have

$$s = 2n \qquad \text{and} \qquad \bar{\varphi}_1 = \varphi_{n+1}, \qquad I_2\varphi_{n+1} = 0.$$

In this case, $\varphi_k,\ k = 1,\dots,n+1$, define a *finite dimensional* representation of the algebra. For other values of s, the representation is infinite.

The representations obtained in this way are *deformations* of representations of $sl(2,\mathbb{C})$ (given in the Appendix), which correspond to the reduction $c_1 = 0$ (which is $m = -1$). In this reduction, the eigenvalue reduces to the value $\lambda_{-1,2n} = \frac{1}{4}n(n+1)$. When $c_1 = 0$, I_2 reduces to $\mathbf{F}^2$, so our sequence $\varphi_1, \varphi_2, \dots$ reduces to $\psi_1^n, \psi_3^n, \dots$, jumping in steps of 2. We thus retrieve only $n+1$ of the $2n+1$ vectors of the highest weight representation.

Example 1 (The case $n = 2$). *Here we have*

$$\varphi_1 = xy^p, \qquad \varphi_2 = 4(3y - 1 + 4p(y-1))y^p, \qquad \varphi_3 = 8(y-1)^2(3 + 16p + 16p^2)y^p,$$

where $m = -1 + 4p$. When $p = 0$, these reduce to ψ_1^2, ψ_3^2 and ψ_5^2, given in the Appendix.

6 Conclusions

The main message of this paper is that *super-integrability* in a quantum system leads *directly and explicitly* to the construction of eigenfunctions. This leads to *exact solvability*, which is much stronger than just *complete integrability*.

The examples presented here are fairly simple, but the general approach can be used for *any* super-integrable system. The examples presented here are associated with the Laplace–Beltrami operator of a 2-dimensional space of constant curvature. The main feature used was the existence of a large number of symmetries of the underlying space. Using these it is possible to deform the Laplace–Beltrami operator, by adding a potential, in such a way that it possesses (generally) higher order (non-geometric) symmetries in the form of higher order commuting operators. This can be thought of as a 'ghost' of the previous geometric symmetry. These higher symmetries generate an algebra with *nonlinear* commutation relations, whose representations are built in a way analogous to the highest weight representations of Lie algebras. In 2 dimensions the symmetry algebra is very small, being just $sl(2,\mathbb{C})$ in our case (the 2-dimensional Euclidean algebra in the flat case), so the representations are very simple. In higher dimensions, the usual highest weight representations apply for the symmetry algebra of the Laplace–Beltrami operator. In this case there is no longer a single chain of eigenfunctions, so we can expect similar behaviour for the algebra of higher symmetries.

Appendix. Eigenfunctions of the Laplace–Beltrami operator

We use the *highest weight representations* of the symmetry algebra $sl(2,\mathbb{C})$ to construct eigenfunctions of L_b. Since $\mathbf{H}$ and L_b commute, they share eigenfunctions and for $\mathbf{H}$ these are built by the highest weight construction. Furthermore, starting with *any* eigenfunction of L_b, we may use the symmetry algebra to construct further eigenfunctions (with the same eigenvalue). Since this eigenspace is *invariant* under the action of $sl(2,\mathbb{C})$ (by construction), it can be decomposed into irreducible components, which are just weight spaces. Therefore *all* eigenfunctions of L_b can be written as linear combinations of those we construct below.

A *highest weight vector* ψ_1^n, of weight $2n$, satisfies

$$\mathbf{E}\,\psi_1^n = 0, \qquad \mathbf{H}\,\psi_1^n = 2n\psi_1^n,$$

which constitute a pair of partial differential equations for the eigenfunction. These are compatible on the zeros of the differential operator $\mathbf{E}$, since

$$\mathbf{H}\,\mathbf{E}\,\psi_1^n - \mathbf{E}\,\mathbf{H}\,\psi_1^n = 2\mathbf{E}\,\psi_1^n = 0.$$

The specific form of ψ_1^n depends upon n and upon the choice of representation for $sl(2,\mathbb{C})$. However, the general structure of the representation is *independent* of this specific form, being a consequence only of the commutation relations (4) (see [7]).

Defining $\psi_r^n = \mathbf{F}^{r-1}\psi_1^n$, the commutation relations imply:

$$\mathbf{H}\,\psi_r^n = 2(n+1-r)\psi_r^n, \qquad \mathbf{E}\,\psi_r^n = (r-1)(2n+2-r)\,\psi_{r-1}^n. \tag{14}$$

Our definition of Laplace–Beltrami operator L_b as Casimir operator (5) implies that

$$L_b\,\psi_r^n = \frac{1}{4}\,n(n+1)\psi_r^n, \qquad \text{for all}\quad r,\ n.$$

We can also construct a 3-point recursion relation between these eigenfunctions, but this *is* explicitly dependent upon the form of the operators. Let

$$\mathbf{H} = h_1\,\partial_{z_1} + h_2\,\partial_{z_2}, \qquad \mathbf{E} = e_1\,\partial_{z_1} + e_2\,\partial_{z_2}, \qquad \mathbf{F} = f_1\,\partial_{z_1} + f_2\,\partial_{z_2},$$

where h_i, etc are functions of the coordinates z_i. We have (with μ_r and a_r defined by (14))

$$\left.\begin{aligned} \mathbf{H}\,\psi_r^n &= \mu_r\psi_r^n, \\ \mathbf{E}\,\psi_r^n &= a_r\psi_{r-1}^n \end{aligned}\right\} \quad\Rightarrow\quad \begin{pmatrix} \partial_{z_1}\psi_r^n \\ \partial_{z_2}\psi_r^n \end{pmatrix} = \frac{1}{h_1e_2 - e_1h_2}\begin{pmatrix} e_2 & -h_2 \\ -e_1 & h_1 \end{pmatrix}\begin{pmatrix} \mu_r\psi_r^n \\ a_r\psi_{r-1}^n \end{pmatrix}.$$

The relation $\psi_{r+1}^n = \mathbf{F}\,\psi_r^n$ then implies that

$$\psi_{r+1}^n = \frac{f_1e_2 - e_1f_2}{h_1e_2 - e_1h_2}\mu_r\psi_r^n + \frac{h_1f_2 - f_1h_2}{h_1e_2 - e_1h_2}a_r\psi_{r-1}^n.$$

When this is singular the representation reduces to 1 dimension.

In the case of our vector field representation (3), this takes explicit form

$$\psi_{r+1}^n = 2(n-r+1)\sqrt{\frac{y}{x}}\,\psi_r^n + (r-1)(2n+2-r)\left(\frac{y-1}{x}\right)\psi_{r-1}^n,$$

with

$$\psi_1^n = \sqrt{x^n}, \qquad \psi_2^n = 2n\sqrt{x^{n-1}y}.$$

When n is an integer, these representations are of finite dimension $2n+1$ and irreducible, but infinite dimensional otherwise. For instance, when $n=2$, we have

$$\psi_1^2 = x, \qquad \psi_2^2 = 4\sqrt{xy}, \qquad \psi_3^2 = 4(3y-1),$$
$$\psi_4^2 = 24(y-1)\sqrt{\frac{y}{x}}, \qquad \psi_5^2 = \frac{24(y-1)^2}{x}.$$

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