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Asymmetric Twin Representation: the Transfer Matrix Symmetry*

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Abstract. The symmetry of the Hamiltonian describing the asymmetric twin model was partially studied in earlier works, and our aim here is to generalize these results for the open transfer matrix. In this spirit we first prove, that the so called boundary quantum algebra provides a symmetry for any generic – independent of the choice of model – open transfer matrix with a trivial left boundary. In addition it is shown that the boundary quantum algebra is the centralizer of the B type Hecke algebra. We then focus on the asymmetric twin representation of the boundary Temperley–Lieb algebra. More precisely, by exploiting exchange relations dictated by the reflection equation we show that the transfer matrix with trivial boundary conditions enjoys the recognized $\mathcal{U}_q(sl_2) \otimes \mathcal{U}_{\mathrm{i}}(sl_2)$ symmetry. When a non-diagonal boundary is implemented the symmetry as expected is reduced, however again certain familiar boundary non-local charges turn out to commute with the corresponding transfer matrix.

Key words: quantum integrability; boundary symmetries; quantum algebras; Hecke algebras

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1 Introduction

As is well known Yang–Baxter [1, 3] and reflection equations [5, 6] provide sets of algebraic constraints exactly defining integrable models with periodic and open boundary conditions respectively (see e.g. [6, 7]). On the other hand it was argued [8, 9] that the algebraic structures underlying Yang–Baxter equation may be seen as deformations of the usual Lie algebras, called quantum algebras. In a similar fashion the reflection algebras arise naturally in the frame of the reflection equation [6], and they have been the subject of ongoing interest (see e.g. [10, 11, 12, 13, 14]). The first step to comprehend such algebras is to systematically classify the solutions of the Yang–Baxter and reflection equations. This is a major problem, the study of which will significantly contribute into the deep understanding of the mathematical mechanisms that rule integrable models with boundaries. A consistent scheme to derive solutions of the reflection equation is to exploit the structural similarity between the Yang–Baxter and reflection equations with the cylinder braid group [15, 16, 17]. Then, using elements of representation theory one may construct physical systems based on the aforementioned solutions.

One of the main challenges within this context is to determine exact symmetries associated to the physical systems under consideration. In the present article we use already known representations of the boundary Temberley–Lieb (blob) algebra [15, 16, 17] as solutions of the reflection equation, and we derive explicitly conserved non-local charges belonging to the underlying reflection algebra, for various choices of boundary conditions. In particular, the symmetry

*This paper is a contribution to the Vadim Kuznetsov Memorial Issue "Integrable Systems and Related Topics". The full collection is available at http://www.emis.de/journals/SIGMA/kuznetsov.html of the transfer matrix of the so called asymmetric twin model is investigated. The main motivation for studying this model is the fact that it is a novel system altogether, and as such it offers a rather unconventional perspective on boundary effects. Historically the asymmetric twin representation was introduced in [18], whereas the corresponding physical model was constructed and studied in [17, 19]. It was shown in [19] that certain quantities were the centralizers of the asymmetric twin representation of the boundary Temperley–Lieb algebra. The corresponding Hamiltonian was then expressed in terms of the generators of the blob algebra, hence it turned out to commute with the aforementioned centralizers. In addition the form of the spectrum and the Bethe ansatz equations for the asymmetric twin transfer matrix were determined via the analytical Bethe ansatz formulation exploiting also the spectral equivalence between the open XXZ and asymmetric twin Hamiltonians. In the present work with the help of exchange relations satisfied by solutions of the reflection equation we are able to identify quantities commuting with the transfer matrix of the model. The transfer matrix as known gives rise to a whole hierarchy of charges in involution rendering the system integrable, with the Hamiltonian being the first one of the hierarchy.

The organization of this article is as follows: In the next section we introduce quotients of the cylinder braid group, i.e. the B type Hecke algebra and the boundary Temperley–Lieb algebra. We also review representations of the boundary Temperley–Lieb algebra, that is the XXZ and asymmetric twin representations. In Section 3 the transfer matrix of an open spin chain is reviewed. In Section 4 the underlying reflection algebra is introduced. It is also shown that the boundary quantum algebra – emerging from the reflection algebra with no spectral parameter – provides in fact a symmetry for any generic open transfer matrix with a trivial left boundary, and it is the centralizer of the B type Hecke algebra as well. The implementation of a non trivial left boundary breaks the symmetry down to a consistent subset. Note that these findings are generic that is they are independent of the choice of model. In the next section basic definitions regarding the quantum algebra $\mathcal{U}_q(\widehat{sl_2})$ are reviewed, and generalized intertwining relations are introduced. The symmetry for both the open XXZ and asymmetric twin models is also studied with the help of the aforementioned intertwining relations. The XXZ chain is considered mostly as a warm up exercise, however the main aim in this section is the derivation of familiar conserved quantities for the open asymmetric twin model for diagonal and non-diagonal right boundary. It will become clear later that the intricate asymptotic behavior of the asymmetric twin model [19] obliges us to invoke exchange relations involving solutions of the reflection equation in order to extract recognized conserved quantities. Finally in the last section we discuss the main results of the present work, and we also give some ideas for possible future directions.

2 The B type Hecke and blob algebras

It will be useful to first introduce a quotient of the cylinder braid group algebra called B type Hecke algebra $\mathcal{B}_N(q, \delta_e, \kappa)$, defined by generators $\mathbb{U}_1, \mathbb{U}_2, \dots, \mathbb{U}_{N-1}$ and e, and relations:

$$\begin{aligned}
&\mathbb{U}_i\mathbb{U}_i = \delta\mathbb{U}_i,\\
&\mathbb{U}_i\mathbb{U}_{i+1}\mathbb{U}_i - \mathbb{U}_i = \mathbb{U}_{i+1}\mathbb{U}_i\mathbb{U}_{i+1} - \mathbb{U}_{i+1},\\
&\big[\mathbb{U}_i, \mathbb{U}_j\big] = 0, \qquad |i-j| \neq 1
\end{aligned} \tag{2.1}$$

(so far we have the ordinary Hecke algebra $\mathcal{H}_N(q)$) with $-\delta = q + q^{-1}$, $q = e^{\mathrm{i}\mu}$

$$\begin{aligned}
&ee = \delta_e e,\\
&\mathbb{U}_1 e\mathbb{U}_1 e - \kappa\mathbb{U}_1 e = e\mathbb{U}_1 e\mathbb{U}_1 - \kappa e\mathbb{U}_1,\\
&\big[\mathbb{U}_i, e\big] = 0, \qquad i \neq 1.
\end{aligned} \tag{2.2}$$

We are free to renormalize e, changing only δ_e and κ (by the same factor). The quantities δ, δ_e, κ are expressed in terms of only two relevant parameters q and Q associated to the bulk and boundary parameters of the anticipated physical systems. It is convenient to parametrize δ_e and κ as:

$$\delta_e = -\mathrm{Q} - \mathrm{Q}^{-1}, \qquad \kappa = q\mathrm{Q}^{-1} + q^{-1}\mathrm{Q}. \tag{2.3}$$

In fact such a parametrization is quite natural from the point of view of the cylinder braid group.

It is known [8] that any tensor space representation $\pi : \mathcal{H}_N(q) \to \mathrm{End}(\mathbb{V}^{\otimes N})$ gives a solution to the Yang–Baxter equation [1, 2, 3, 4]

$$R_{12}(\lambda_1 - \lambda_2)R_{13}(\lambda_1)R_{23}(\lambda_2) = R_{23}(\lambda_2)R_{13}(\lambda_1)R_{12}(\lambda_1 - \lambda_2), \tag{2.4}$$

written as

$$R_{12}(\lambda) = \mathcal{P}_{12}\big(\sinh\mu(\lambda + \mathrm{i})\mathbb{I} + \sinh(\mu\lambda)\pi(\mathbb{U}_1)\big), \tag{2.5}$$

where $\mathcal{P}$ is the permutation operator on $\mathbb{V} \otimes \mathbb{V}$. Suppose π extends to a representation of $\mathcal{B}_N$. Then a solution of the reflection equation [5, 6]

$$\begin{aligned} &R_{12}(\lambda_1 - \lambda_2)\ K_1(\lambda_1)R_{21}(\lambda_1 + \lambda_2)\ K_2(\lambda_2) \\ &\qquad = K_2(\lambda_2)R_{12}(\lambda_1 + \lambda_2)K_1(\lambda_1)R_{21}(\lambda_1 - \lambda_2) \end{aligned} \tag{2.6}$$

can be written with the help of (2.1), (2.2) (see also [14, 17]) as

$$K_1(\lambda) = x(\lambda)\mathbb{I} + y(\lambda)\pi(e), \tag{2.7}$$

where

$$\begin{aligned} x(\lambda) &= -\delta_e \cosh\mu(2\lambda + \mathrm{i}) - \kappa\cosh(2\mu\lambda) - \cosh(2\mathrm{i}\mu\zeta), \\ y(\lambda) &= 2\sinh(2\mu\lambda)\sinh(\mathrm{i}\mu), \end{aligned} \tag{2.8}$$

here ζ is an arbitrary constant. By imposing further constraints on the B type Hecke algebra one obtains another quotient called boundary Temperley–Lieb (blob) algebra $b_N(q, \mathrm{Q})$ – an extension of the Temperley–Lieb algebra $T_N(q)$ – defined by (2.1), (2.2) and also

$$\mathbb{U}_i\mathbb{U}_{i+1}\mathbb{U}_i = \mathbb{U}_i, \qquad \mathbb{U}_1 e \mathbb{U}_1 = \kappa\mathbb{U}_1.$$

The expressions (2.5), (2.7) are apparently valid for any representation of the blob algebra as well. In what follows we shall briefly review two basic representations of the blob algebra, i.e. the XXZ and the asymmetric twin.

(I) The XXZ representation

For any given N let the map $\mathcal{R}_q$ on the generators of $T_N(q)$ into $\mathrm{End}(\mathbb{V}^{\otimes N})$ be given by

$$\mathcal{R}_q(\mathbb{U}_l) = \mathbb{I} \otimes \mathbb{I} \otimes \cdots \otimes U(q) \otimes \cdots \otimes \mathbb{I} \otimes \mathbb{I}, \tag{2.9}$$

where

$$U(q) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(acting non-trivially on $\mathbb{V}_l \otimes \mathbb{V}_{l+1}$). This is the usual XXZ representation of the Temperley–Lieb algebra. This may be extended to a representation of $b_N(q,Q)$ by also introducing the matrix [20]

$$\mathcal{M} = -\frac{\delta_e}{Q+Q^{-1}} \begin{pmatrix} -Q^{-1} & 1 \\ 1 & -Q \end{pmatrix} \otimes \cdots \otimes \mathbb{I} \otimes \mathbb{I}$$

acting non trivially on $\mathbb{V}_1$. There exists a representation $\mathcal{R}_q : b_N(q,Q) \to \mathrm{End}(\mathbb{V}^{\otimes N})$ provided by (2.9) and $\mathcal{R}_q(e) = \mathcal{M}$ [20]. For this we have:

$$\delta_e = -\frac{Q+Q^{-1}}{2\mathrm{i}\sinh(\mathrm{i}\mu)}, \qquad \kappa = \frac{qQ^{-1}+q^{-1}Q}{2\mathrm{i}\sinh(\mathrm{i}\mu)}.$$

(II) The asymmetric twin representation

By combining two XXZ representations with different quantum parameters one may construct a novel representation known as asymmetric twin representation [18, 17, 19]. Indeed set

$$r = \mathrm{i}\sqrt{\mathrm{i}q}, \qquad \hat{r} = \sqrt{\mathrm{i}q},$$

so that $r\hat{r} = -q$ ($\mathrm{i} = \sqrt{-1}$). The representation $\Theta : T_N(q) \to \mathrm{End}(\mathbb{V}^{\otimes 2N})$ (introduced in [18]) is constructed by combining parts of the representations $\mathcal{R}_r$ of $T_{2N}(r)$ and $\mathcal{R}_{\hat{r}}$ of $T_{2N}(\hat{r})$ as follows:

$$\Theta(\mathbb{U}_l) = \mathcal{R}_r(\mathbb{U}_{N-l})\mathcal{R}_{\hat{r}}(\mathbb{U}_{N+l}).$$

A striking feature of this is that extends to a representation of b_N in a variety of ways [17, 19]:

Fixing q, define matrices in $\mathrm{End}(\mathbb{V}^{\otimes 2N})$ as follows:

$$\mathcal{M}^i(Q) = -\frac{\delta_e}{Q+Q^{-1}}\, \mathbb{I} \otimes \mathbb{I} \otimes \cdots \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -Q & 1 & 0 \\ 0 & 1 & -Q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \cdots \otimes \mathbb{I} \otimes \mathbb{I}, \tag{2.10}$$

where the 4×4 matrix acts on $\mathbb{V}_N \otimes \mathbb{V}_{N+1}$ and Q is some scalar;

$$\mathcal{M}^{ii}(Q) = -\frac{\delta_e}{Q+Q^{-1}} \mathbb{I} \otimes \mathbb{I} \otimes \cdots \otimes \begin{pmatrix} -Q & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -Q^{-1} \end{pmatrix} \otimes \cdots \otimes \mathbb{I} \otimes \mathbb{I}, \tag{2.11}$$

$$\mathcal{M}^+(Q) = \mathcal{M}^i(Q) + \mathcal{M}^{ii}(Q), \tag{2.12}$$

$$\begin{aligned} \mathcal{M}^{iii}(Q_1,Q_2) = {} & \frac{\delta_e}{(Q_1+Q_1^{-1})(Q_2+Q_2^{-1})} \mathbb{I} \otimes \cdots \otimes \begin{pmatrix} -Q_1 & 1 \\ 1 & -Q_1^{-1} \end{pmatrix} \\ & \otimes \begin{pmatrix} -Q_2^{-1} & 1 \\ 1 & -Q_2 \end{pmatrix} \otimes \cdots \otimes \mathbb{I}, \end{aligned} \tag{2.13}$$

where the 2×2 matrices act separately on $\mathbb{V}_N$, $\mathbb{V}_{N+1}$ respectively; and

$$\mathcal{M}^{iii}(Q) = \mathcal{M}^{iii}(\mathrm{i}\sqrt{\mathrm{i}Q}, \sqrt{\mathrm{i}Q}).$$

For each $I \in \{i, ii, +, iii\}$ there is a representation $\Theta^I : b_N \to \mathrm{End}(\mathbb{V}^{\otimes 2N})$ given by $\Theta^I(\mathbb{U}_i) = \Theta(\mathbb{U}_i)$, $\Theta^I(e) = \mathcal{M}^I(Q)$, provided

$$\delta_e = -\frac{Q+Q^{-1}}{2\mathrm{i}\sinh(\mathrm{i}\mu)}, \qquad \kappa^i = \frac{q^{-1}Q+qQ^{-1}}{2\mathrm{i}\sinh(\mathrm{i}\mu)}, \qquad \kappa^{ii} = \frac{\mathrm{i}Q-\mathrm{i}Q^{-1}}{2\mathrm{i}\sinh(\mathrm{i}\mu)},$$

$$\kappa^{+} = \kappa^{i} + \kappa^{ii} \qquad \kappa^{iii} = \frac{q^{-1}Q + qQ^{-1} + 2}{2\mathrm{i}\sinh(\mathrm{i}\mu)}. \tag{2.14}$$

Notice that the algebraic parameter Q and the parameter Q of the representation do not necessarily coincide. For the XXZ case and the representation (ii) they do, up to appropriate normalization, but in general the one parameter may be expressed in terms of the other by simply comparing (2.3), (2.14).

Let us finally note that in [19] a novel class of representations of the (boundary) Temperley–Lieb algebra, called crossing representation, was introduced. In fact it was shown that both the XXZ and the asymmetric twin representation belong to this class. For a more detailed analysis of this type of representations we refer the reader to [19].

3 The open spin chain

Our objective is to examine the symmetry of the transfer matrix of the asymmetric twin model. To achieve that we need to introduce the R and K matrices associated to the asymmetric twin representation of $T_N(q)$, and then build the corresponding open transfer matrix [6]. We shall also examine the open XXZ chain, mainly to familiarize ourselves with the approach, therefore we shall also recall the XXZ R and K matrices.

(I) The XXZ representation

Let us first briefly recall the form of the R and K matrices for the XXZ representation. The XXZ R matrix, acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is given by

$$R_{l\,l+1}(\lambda) = \sinh\mu(\lambda + \mathrm{i})\mathcal{P}_{l\,l+1} + \sinh(\mu\lambda)\mathcal{P}_{l\,l+1}\mathcal{R}_q(\mathbb{U}_l),$$

where $\mathcal{P}$ is the permutation operator acting on $(\mathbb{C}^2)^{\otimes 2}$, and $\mathcal{R}_q$ is the XXZ representation of the Temperley–Lieb algebra given by (2.9). The XXZ K matrix is given in a 2×2 form as

$$K_1(\lambda) = x(\lambda)\mathbb{I} + y(\lambda)\mathcal{M}. \tag{3.1}$$

The latter matrix coincides with the solution found in [21, 22] subject to certain identifications among the various boundary parameters (for more details see [13]).

(II) The asymmetric twin representation

To derive the R and K matrices of the twin representation it is convenient to introduce the following relabelling (see also [19]),

$$N - l + 1 \to l^{-}, \qquad N + l \to l^{+}, \tag{3.2}$$

then we can write

$$\Theta(\mathbb{U}_l) = U_{l^{-}(l+1)^{-}}(r^{-1})U_{l^{+}(l+1)^{+}}(\hat{r}),$$

acting on $\mathbb{V}_{\tilde{l}} \otimes \mathbb{V}_{\widetilde{l+1}}$, where the index $\tilde{l} = (l^{-}, l^{+})$ in the space/mirror space notation [19]. The 16×16 explicit expression of $\Theta(\mathbb{U}_l)$ is given in the Appendix A. The asymmetric twin R matrix is given by the following expression [17, 19] (the tilted indices are suppressed from R, K from now on for the sake of simplicity)

$$R_{l\,l+1}(\lambda) = \sinh\mu(\lambda + \mathrm{i})\mathcal{P}_{l\,l+1} + \sinh(\mu\lambda)\mathcal{P}_{l\,l+1}\Theta(\mathbb{U}_l),$$

where now the permutation operator $\mathcal{P}$ acts on $(\mathbb{C}^4)^{\otimes 2}$. The asymmetric twin K matrix can be written in 4×4 form with the help of the representations (i), (ii), (iii) as

$$K_1^I(\lambda) = x(\lambda)\mathbb{I} + y(\lambda)\mathcal{M}^I(Q), \tag{3.3}$$

$x(\lambda)$, $y(\lambda)$ are given by (2.8), and M^I given by (2.10)–(2.13) (for explicit 4×4 expressions see [19]).

The transfer matrix

Given any $R(\lambda) \in \mathrm{End}(\mathbb{V} \otimes \mathbb{V})$ one may define the more general object $L(\lambda) \in \mathrm{End}(\mathbb{V}) \otimes \mathcal{A}$ where $\mathcal{A}$ is the algebra associated to the R matrix and defined by the following fundamental relation

$$R_{12}(\lambda_1 - \lambda_2)L_1(\lambda_1)L_2(\lambda_2) = L_2(\lambda_2)L_1(\lambda_1)R_{12}(\lambda_1 - \lambda_2). \tag{3.4}$$

It is clear that when the second space, which is a copy of $\mathcal{A}$, is mapped to $\mathbb{V}$ then $L(\lambda) \mapsto R(\lambda)$. Note that for the XXZ case the L matrix is known and $\mathcal{A} = \mathcal{U}_q(\widehat{sl_2})$, whereas for the asymmetric twin model a generic L matrix is not available for the moment. Hence whenever we discuss about the asymmetric twin case we restrict ourselves to the R matrix only.

In general given any L and K matrices one can derive the algebraic open transfer matrix, which provides the conserved quantities of the model. Define first the monodromy matrix $T(\lambda)$, ($\hat{T}(\lambda) = T^{-1}(-\lambda)$) [23]

$$T_0(\lambda) = L_{0N}(\lambda) \cdots L_{02}(\lambda)L_{01}(\lambda), \qquad \hat{T}_0(\lambda) = \hat{L}_{01}(\lambda)\hat{L}_{02}(\lambda) \cdots \hat{L}_{0N}(\lambda),$$

where $\hat{L}(\lambda) = L^{-1}(-\lambda)$. The monodromy matrix satisfies fundamental algebraic relation (3.4) as well. The transfer matrix of the open chain [6] is defined as

$$t(\lambda) = \mathrm{tr}_0\big\{M_0\ K_0^{(L)}(\lambda)\mathcal{T}_0(\lambda)\big\}, \qquad \mathcal{T}_0(\lambda) = T_0(\lambda)K_0^{(R)}(\lambda)\hat{T}_0(\lambda). \tag{3.5}$$

Note that as customary the 'quantum' indices $1, \dots, N$ are suppressed from T, $\hat{T}$ and $\mathcal{T}$. Recall that for the moment the quantum spaces are not represented, but they are simply copies of the algebra $\mathcal{A}$, hence the transfer matrix at this stage is a purely algebraic object $t(\lambda) \in \mathcal{A}^{\otimes N}$. The operator $\mathcal{T}$ is also a solution of the reflection equation, $K^{(L)}$ is associated to the left boundary of the spin chain, and in what follows it will be unit, and $K^{(R)}$ denotes the right boundary of the chain. It will be either unit, or given by (3.1) for XXZ, and by (3.3) for the asymmetric twin model. The matrix M is given by

$$\begin{aligned} &\text{XXZ:} \quad M = \mathrm{diag}\,(q, q^{-1}), \\ &\text{Twin:} \quad M = \mathrm{diag}\,(\mathrm{i}, q^{-1}, q, -\mathrm{i}), \end{aligned} \tag{3.6}$$

where recall $q = -r\hat{r}$.

As mentioned our main aim is to study the symmetry of the twin transfer matrix. Usually this is carried out by investigating the asymptotics of $\mathcal{T}$ as $\lambda \to \infty$ (see e.g. [24, 25, 26]). However, as discussed in [19] the study of the asymptotics for the twin R matrix and consequently for $\mathcal{T}$ is a rather intricate task. Hence one can not easily obtain recognized conserved non-local charges with a simple coproduct structure, as is the case in e.g. the XXZ model. To circumvent this complication we adopt the approach developed in [13, 14], and will be reviewed in the Section 5. More precisely using the methodology of [13, 14] we shall be able to identify familiar non-local charges, which however do not provide the full symmetry of the asymmetric twin model, as already pointed out in [19]. Nevertheless in the subsequent section we shall give a generic description of the symmetry algebra emerging in any lattice model with particular integrable boundaries.

4 The underlying algebra and the symmetry

As argued in [6] given any L, K matrices, solutions of the fundamental relation (3.4) and reflection equation (2.6) respectively, one may build the more general solution of (2.6)

$$\mathbb{K}(\lambda' \mp \lambda) = L(\lambda' \mp \lambda)(K(\lambda') \otimes \mathbb{I})\hat{L}(\lambda' \pm \lambda), \tag{4.1}$$

where K is a c-number solution of the reflection equation. The entries of the matrix $\mathbb{K}$ are elements of the so called reflection algebra $\mathbb{R}$ (2.6) (see also [6]).

One may easily show that all the elements of the reflection algebra 'commute' with the solutions of the reflection equation (see also [12]). Indeed recalling (2.6) and the above expressions (4.1) it is straightforward to show that

$$\mathbb{K}_{ab}(\lambda' - \lambda)\ K(\lambda) = K(\lambda)\ \mathbb{K}_{ab}(\lambda' + \lambda), \tag{4.2}$$

which implies that any solution of the reflection equation commutes with the elements of the reflection algebra and the opposite.

The reflection algebra is also endowed with a coproduct. More precisely, the L matrix is equipped with a coproduct derived from the Yang–Baxter equation $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, where recall $\mathcal{A}$ is the algebra defined by (3.4) (see also [8])

$$\Delta(L_{ab}) = \sum_c L_{cb} \otimes L_{ac}, \qquad \hat{L}_{ab} = \sum_c \hat{L}_{ac} \otimes \hat{L}_{cb}. \tag{4.3}$$

L_{ab} are the entries of the L matrix. It is also convenient to introduce $\Delta' = \sigma \circ \Delta$ where $\sigma : a \otimes b \to b \otimes a$. The n coproduct is obtained by iteration: $\Delta^{(n)} = (\mathrm{id} \otimes \Delta^{(n-1)})\Delta$, $\Delta'^{(n)} = (\mathrm{id} \otimes \Delta^{(n-1)})\Delta'$.

Taking into account (4.3) we conclude that the reflection algebra is also endowed with a coproduct, $\Delta : \mathbb{R} \to \mathbb{R} \otimes \mathcal{A}$ (see also [12])

$$\Delta(\mathbb{K}_{ab}(\lambda)) = \sum_{k,l=1}^{n} \mathbb{K}_{kl}(\lambda) \otimes L_{ak}(\lambda)\hat{L}_{lb}(\lambda).$$

In fact, the entries $\mathcal{T}_{ab}$ are simply N coproducts of the elements of the reflection algebra (2.6), i.e. $\mathcal{T}_{ab}(\lambda) = \Delta^{(N)}(\mathbb{K}_{ab}(\lambda))$. It may be shown, bearing in mind the fact that $\mathcal{T}$ is also a solution of the (2.6), that (see also [13, 14])

$$\Delta'^{(N+1)}(\mathbb{K}_{ab}(\lambda' - \lambda))\ \mathcal{T}(\lambda) = \mathcal{T}(\lambda)\Delta'^{(N+1)}(\mathbb{K}_{ab}(\lambda' + \lambda)). \tag{4.4}$$

The generic solution (4.1) allows the asymptotic expansion as $\lambda' \to \pm\infty$ providing the reflection algebra generators. The first order of such expansion yields the generators of the boundary quantum algebra $\mathbb{B}$, which obey exchange relations dictated by the defining relations (2.6) as $\lambda' \to \infty$. More precisely, as $\lambda' \to \pm\infty$ $R(\lambda') \propto R^{\pm}$, $K(\lambda') \to K^{\pm}$ and $\mathbb{K} \to \mathbb{K}^{\pm}$, where $R^{\pm}$, $K^{\pm}$, $\mathbb{K}^{\pm}$ have no spectral dependance anymore,

$$R_{12}^{\pm(\mp)}\ \mathbb{K}_1^{\pm}\hat{R}_{12}^{\pm}\mathbb{K}_2^{\pm} = \mathbb{K}_2^{\pm}\ R_{12}^{\pm}\mathbb{K}_1^{\pm}\hat{R}_{12}^{\pm(\mp)} \tag{4.5}$$

and the entries $\mathbb{K}_{ab}^{\pm}$ form the boundary quantum algebra $\mathbb{B}$. The later formula (4.5) is in fact an immediate consequence of the quadratic relation of the B type Hecke algebraic (cylinder braid group) (2.2) (for further comments on this point see also [14]).

Let us stress once more that depending on the choice of the R matrix one obtains distinct (boundary) quantum algebras. Moreover, the explicit form of the boundary quantum algebra generators depends on the choice of R matrix as well as the choice of K matrix. For instance for the XXZ case with $K = \mathbb{I}$ the corresponding boundary quantum algebra coincides with $\mathcal{U}_q(sl_2)$ [28], whereas if K is given by (3.1) the boundary quantum algebra consists of only one element (abelian), which may be expressed in terms of the $\mathcal{U}_q(sl_2)$ generators (see later in the text in Section 5, and also in [13]). On the other hand, for models associated to higher rank algebras for trivial boundary conditions the symmetry of the transfer matrix coincides with $\mathcal{U}_q(gl_n)$ ($K^{(L,R)} = \mathbb{I}$) or $\mathcal{U}_q(gl_l) \otimes \mathcal{U}_q(gl_{n-l})$ ($K^{(L)} = \mathbb{I}$, $K^{(R)} = \mathrm{diag}$) [26]. For a particular non diagonal right boundary on the other hand the boundary non-local charges form a non-Abelian boundary quantum algebra identified in [14]. Exploiting (4.4) for $\lambda' \to \pm\infty$, i.e.

$$\sum_{b,c} R^{\pm}_{ab} \hat{R}^{\pm}_{cd} \otimes \mathcal{T}^{\pm}_{bc} \mathcal{T}(\lambda) = \mathcal{T}(\lambda) \sum_{b,c} R^{\pm}_{ab} \hat{R}^{\pm}_{cd} \otimes \mathcal{T}^{\pm}_{bc} \tag{4.6}$$

one could show that the 'boundary non local charges' $\mathcal{T}^{\pm}_{ab} = \Delta^{(N)}(\mathbb{K}^{\pm}_{ab})$ commute with the asymmetric twin transfer matrix [13, 14]. We do not attempt this rather technical proof here, although it is a routine exercise based on (4.6) and the explicit expression of the $R^{\pm}$ matrices (see e.g. [13, 14] for the XXZ and $\mathcal{U}_q(gl_n)$ cases).

• *Symmetry:* Instead we shall show that the boundary quantum algebra $\mathbb{B}$ is a symmetry for any open transfer matrix with trivial left boundary following a different approach (see also [26]). Consider any open transfer matrix (3.5) with trivial left boundary $K^{(L)} = \mathbb{I}$, $K^{(R)}(\lambda) \in \mathrm{End}(\mathbb{C}^n)$ is any solution of the reflection equation of the type (2.7), and $R(\lambda) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ is a solution of the Yang–Baxter equation (2.5). Assume also that the R matrix satisfies the following:

$$R_{12}(\lambda) R_{21}(-\lambda) \propto \mathbb{I}, \qquad M_1^{-1} R_{12}^{t_1}(\lambda) M_1 R_{21}^{t_1}(-\lambda - 2i\rho) \propto \mathbb{I}, \tag{4.7}$$

ρ is a constant and for both XXZ and the asymmetric twin representations is unit (for the $\mathcal{U}_q(gl_n)$ case $\rho = \frac{n}{2}$). Suppose also that the matrix M satisfies:

$$M = M^t, \qquad \big[M_1 M_2, R_{12}(\lambda)\big] = 0, \tag{4.8}$$

and for the XXZ and asymmetric twin models is given by (3.6). We also introduce the following quantity, which structurally resembles the open transfer matrix [26], i.e.

$$\tau^{\pm} = tr_0\big\{\mathbb{P}_0 \mathcal{T}_0^{\pm}\big\}, \qquad \mathcal{T}^{\pm} \propto \mathcal{T}(\lambda \to \pm\infty),$$

with $\mathbb{P}$ being a priori an arbitrary $n \times n$ matrix. Using (4.7), (4.8) and the fact that $\mathcal{T}$ satisfies the reflection equation (2.6), in particular we employ (2.6) for $\lambda_1 \to \pm\infty$, we may show following the steps of the proof of integrability in [6] (for more details see e.g. [26]) that

$$\big[\tau^{\pm}, t(\lambda)\big] = 0.$$

By choosing the matrix $\mathbb{P}$ appropriately, i.e. $\mathbb{P} = E_{ab}$ where we define $(E_{ab})_{cd} = \delta_{ac}\delta_{bd}$, it is clear that

$$\tau^{\pm} = \mathcal{T}^{\pm}_{ab} \quad \Rightarrow \quad \big[\mathcal{T}^{\pm}_{ab},\ t(\lambda)\big] = 0, \tag{4.9}$$

and this concludes our proof.

Consider now a non trivial left boundary $K^{(L)} \neq \mathbb{I}$. The proof of the corresponding symmetry goes along the lines described above although it is technically more involved and it will be reported elsewhere. The intricate point in this case is that one has to impose a series of requirements satisfied by R, $K^{(L,R)}$ and $\mathbb{P}$ matrices. In any case the corresponding conserved quantities are linear combinations of $\mathcal{T}^{\pm}_{ab}$, hence the remaining symmetry is a consistent subset of the boundary quantum algebra $\mathbb{B}$.

• *Centralizer:* We shall now restrict our attention to the case where both quantum and auxiliary spaces correspond to $\mathbb{V}$, thus $L \mapsto R$ and the open transfer matrix is not an algebraic object any more as in (3.5), but is mapped to $\mathrm{End}(\mathbb{V}^{\otimes N})$,

$$t(\lambda) = \mathrm{tr}_0\big\{M_0\ K_0^{(L)}(\lambda)\mathcal{T}_0(\lambda)\big\},$$
$$\mathcal{T}_0(\lambda) = R_{0N}(\lambda)\cdots R_{01}(\lambda)K_0^{(R)}(\lambda)\hat{R}_{01}(\lambda)\cdots\hat{R}_{0N}(\lambda). \tag{4.10}$$

Due to the fact that the R matrix reduces to the permutation operator $\mathcal{P}$ for $\lambda = 0$, a local Hamiltonian may be deduced from (4.10) ($\mathcal{H} \propto \frac{dt(\lambda)}{d\lambda}|_{\lambda=0}$), which can be expressed in terms of representations of the B-type Hecke algebra (for more details see e.g. [14, 19]).

The boundary quantum algebra is also represented, and we shall show that apart from providing a symmetry of the open transfer matrix (4.10) with trivial left boundary, is also the centralizer of the B type Hecke algebra. Indeed consider solutions of the Yang–Baxter and reflection equations expressed as in (2.5), (2.7). Set $\check{R} = \mathcal{P}R$, also consider the parametrization (2.3), also recall that $\pi : \mathcal{B}_N \to \mathrm{End}(\mathbb{V}^{\otimes N})$ then from (2.5), (2.7) as $\lambda \to \pm\infty$ it follows that

$$\check{R}_{i\ i+1}(\lambda \to \pm\infty) \propto \check{R}^{\pm}_{i\ i+1} = \pi(\mathbb{U}_i) + q^{\pm 1}, \qquad K_1(\lambda \to \pm\infty) \propto K_1^{\pm} = \pi(e) + \mathrm{Q}^{\pm 1}. \tag{4.11}$$

From the Yang–Baxter equation (2.4) for $\lambda_{1,2} \to \pm\infty$ one obtains

$$\check{R}^{\pm}_{i\,i+1}R^{\pm}_{0\,i+1}\ R^{\pm}_{0i} = R^{\pm}_{0\,i+1}\ R^{\pm}_{0i}\check{R}^{\pm}_{i\ i+1}, \qquad \check{R}^{\pm}_{i\,i+1}\hat{R}^{\pm}_{0i}\hat{R}^{\pm}_{0\,i+1} = \hat{R}^{\pm}_{0i}\hat{R}^{\pm}_{0\,i+1}\check{R}^{\pm}_{i\,i+1}. \tag{4.12}$$

Finally recalling the structure of $\mathcal{T}$ (4.10), and using relations (4.11), (4.12) and the reflection equation we obtain

$$\big[\pi(\mathbb{U}_i), \mathcal{T}^{\pm}_{ab}\big] = 0, \qquad \big[\pi(e), \mathcal{T}^{\pm}_{ab}\big] = 0, \qquad i \in \{1, \dots, N-1\}, \tag{4.13}$$

where now apparently $\mathcal{T}^{\pm}_{ab} \in \mathrm{End}(\mathbb{V}^{\otimes N})$. The latter commutation relations imply the duality between the boundary quantum algebra and the B type Hecke algebra (for a relevant discussion see also [27]). When dealing with representations of the blob algebra such as the XXZ and the asymmetric twin, the boundary quantum algebra becomes evidently the centralizer of the blob algebra [13].

5 Familiar conserved quantities

The presentation of the previous section relies primarily on abstract algebraic considerations, and as such it does not offer explicit expressions of the algebra generators, and consequently of conserved quantities that determine the symmetry of the open spin chain. Usually the study of the symmetry of an open transfer matrix (see e.g. [24, 25, 26, 28]) rests on the fact that the monodromy matrix T reduces to upper (lower) triangular matrix as $\lambda \to \pm\infty$, which facilitates enormously the algebraic manipulations. There exist however cases such as the asymmetric twin model, where the monodromy matrix does not reduce to such a convenient form as $\lambda \to \pm\infty$. In these cases the most effective way to investigate the corresponding symmetry is to derive in some way (e.g. by direct computation) linear intertwining relations of the type (4.2) and (4.4), by means of which exchange relations between the entries of $\mathcal{T}$ and the corresponding non-local charges can be deduced. This is carried out in what follows, where we provide explicit expressions of conserved quantities associated to familiar quantum algebras.

Before we proceed with the derivation of recognized conserved quantities for the asymmetric twin model it is useful to recall basic definitions regarding the quantum algebra $\mathcal{U}_{\mathrm{q}}(sl_2)$. Let

$\mathcal{E}$, $\mathcal{F}$ and $\mathcal{H}$ be the generators of the quantum algebra $\mathcal{U}_{\mathrm{q}}(sl_2)$ [8, 9], satisfying the defining relations,

$$\big[\mathcal{E},\mathcal{F}\big]=\frac{\mathcal{H}^2-\mathcal{H}^{-2}}{\mathrm{q}-\mathrm{q}^{-1}},\qquad \mathcal{H}\mathcal{E}=\mathrm{q}\;\mathcal{E}\mathcal{H},\qquad \mathcal{H}\;\mathcal{F}=\mathrm{q}^{-1}\mathcal{F}\mathcal{H}.$$

There is a coproduct Δ: $\mathcal{U}_{\mathrm{q}}(sl_2)\to\mathcal{U}_{\mathrm{q}}(sl_2)\otimes\mathcal{U}_{\mathrm{q}}(sl_2)$ given by

$$\Delta(\chi)=\mathcal{H}^{-1}\otimes\chi+\chi\otimes\mathcal{H},\qquad \chi\in\{\mathcal{E},\mathcal{F}\},\qquad \Delta(\mathcal{H}^{\pm1})=\mathcal{H}^{\pm1}\otimes\mathcal{H}^{\pm1}.$$

The n coproducts are obtained by iteration as defined in the previous section.

Henceforth we shall focus on the case where both spaces quantum and auxiliary are associated to $\mathbb{V}$, we shall deal basically with the transfer matrix (4.10). Let us make a general statement, which shall be used for both the XXZ and the asymmetric twin models. Consider a generic R matrix, solution of the Yang–Baxter equation, satisfying the following intertwining relations with a representation $\mathrm{h}:\mathcal{U}_q(sl_2)\to\mathrm{End}(\mathbb{V})$

$$\mathrm{h}^{\otimes2}(\Delta'(x))R_{12}(\lambda)=R_{12}(\lambda)\mathrm{h}^{\otimes2}(\Delta(x)),\qquad x\in\{\mathcal{E},\mathcal{F},\mathcal{H}\}. \tag{5.1}$$

Then one can show in a straightforward fashion that generalized intertwining relations hold also for the operator $\mathcal{T}$ (4.10) for $K^{(L,R)}=\mathbb{I}$ (see also [13, 14] for a detailed proof)

$$(\mathrm{h}\otimes\mathrm{h}^{\otimes N})\Delta'^{(N+1)}(x)\mathcal{T}(\lambda)=\mathcal{T}(\lambda)(\mathrm{h}\otimes\mathrm{h}^{\otimes N})\Delta'^{(N+1)}(x). \tag{5.2}$$

The latter relations are exactly of the type (4.4) with no spectral parameter, i.e. for $\lambda'\to\pm\infty$. Although similar relations hold for the generators of the affine $\mathcal{U}_{\mathrm{q}}(\widehat{sl_2})$ algebra, here we restrict our attention to the non affine case $\mathcal{U}_{\mathrm{q}}(sl_2)$. The reason for such a restriction is the fact that the asymmetric twin R matrix, which is our main concern, does not satisfy any obvious intertwining relations with the elements of $\mathcal{U}_{\mathrm{q}}(\widehat{sl_2})$. As a consequence no generalized intertwining relations can be derived, and that is why we remain focused on the non-affine case.

It will be useful for the remaining part to present the operator $\mathcal{T}$ (4.10) in a matrix form for both the XXZ and the asymmetric twin representations respectively

$$\mathcal{T}(\lambda)=\begin{pmatrix}\mathcal{A}&\mathcal{B}\\ \mathcal{C}&\mathcal{D}\end{pmatrix}\qquad\text{and}\qquad t(\lambda)=q\mathcal{A}+q^{-1}\mathcal{D}, \tag{5.3}$$

$$\mathcal{T}(\lambda)=\begin{pmatrix}\mathcal{A}&\mathcal{B}_1&\mathcal{B}_2&\mathcal{B}\\ \mathcal{C}_1&\mathcal{A}_1&\mathcal{B}_5&\mathcal{B}_3\\ \mathcal{C}_2&\mathcal{C}_5&\mathcal{A}_2&\mathcal{B}_4\\ \mathcal{C}&\mathcal{C}_3&\mathcal{C}_4&\mathcal{D}\end{pmatrix}\qquad\text{and}\qquad t(\lambda)=\mathrm{i}\mathcal{A}+q^{-1}\mathcal{A}_1+q\mathcal{A}_2-\mathrm{i}\mathcal{D}. \tag{5.4}$$

5.1 The XXZ open transfer matrix ($K^{(L,R)}=\mathbb{I}$)

Let us first consider, mostly as a warm up exercise, the XXZ representation. The case with trivial boundaries ($K^{(L,R)}=\mathbb{I}$) will be considered here. The symmetry of this model has been already studied in [28] extending the results of [29], however here we rederive the result using the method described in [13].

Consider the representation $\rho:\mathcal{U}_q(sl_2)\to\mathrm{End}(\mathbb{C}^2)$ defined by

$$\rho(\mathcal{H})=q^{\frac{1}{2}\sigma^z},\qquad \rho(\mathcal{E})=\sigma^+,\qquad \rho(\mathcal{F})=\sigma^-,$$

where the parameter $q=e^{\mathrm{i}\mu}$ coincides with the parameter of the Temperley–Lieb algebra. It is convenient to introduce some notations. Let

$$E^{(N)}=\rho^{\otimes N}(\Delta^{(N)}(\mathcal{E})),\qquad F^{(N)}=\rho^{\otimes N}(\Delta^{(N)}(\mathcal{F})),\qquad H^{(N)}=\rho^{\otimes N}(\Delta^{(N)}(\mathcal{H})). \tag{5.5}$$

It is clear that $E^{(N)}$, $F^{(N)}$, $H^{(N)}$ form a tensor representation of $\mathcal{U}_q(sl_2)$ acting on $(\mathbb{C}^2)^{\otimes N}$.

One can show that the $T_N(q)$ generators in the XXZ representation (2.9) commute with the action of the quantum group (5.5) (see also [19, 29])

$$\big[E^{(N)}, \mathcal{R}_q(\mathbb{U}_l)\big] = \big[F^{(N)}, \mathcal{R}_q(\mathbb{U}_l)\big] = \big[H^{(N)}, \mathcal{R}_q(\mathbb{U}_l)\big] = 0, \qquad l \in \{1, \ldots, N-1\}. \tag{5.6}$$

The commutation relations (5.6) were exploited in [29] for proving that the Hamiltonian of the open XXZ spin chain, with trivial boundaries $K^{(L,R)} = \mathbb{I}$, is $\mathcal{U}_q(sl_2)$ invariant.

The XXZ R matrix [30] satisfies linear intertwining relations (5.1) with the representation $\rho : \mathcal{U}_q(sl_2) \to \mathrm{End}(\mathbb{C}^2)$, and as a consequence the operator $\mathcal{T}$ for the XXZ model, with $K^{(L,R)} = \mathbb{I}$, satisfies (5.2). From the generalized relations (5.2) and the form of the $\mathcal{T}$ matrix for the XXZ model (5.3) algebraic Bethe ansatz type relations are entailed [13] (see Appendix B for explicit expressions (B.1)–(B.3)). With the help of relations (B.1)–(B.3) it is possible to study the symmetry of the open transfer matrix. Indeed recall (5.3) then by virtue of the aforementioned relations it can be easily shown that

$$\big[E^{(N)}, t(\lambda)\big] = \big[F^{(N)}, t(\lambda)\big] = \big[H^{(N)}, t(\lambda)\big] = 0. \tag{5.7}$$

The commutation relations between the transfer matrix and the representations of the $\mathcal{U}_q(sl_2)$ generators (5.7) imply that the full transfer matrix enjoys $\mathcal{U}_q(sl_2)$ symmetry as already proved in [28]. Note that for the finite XXZ chain with periodic (or twisted) boundary condition no symmetry has been identified for generic values of q. However in the case where q is root of unity it was shown in [31] that the periodic XXZ chain enjoys the sl_2 loop symmetry. The symmetry of the transfer matrix for $K^{(R)}$ non-diagonal given by (3.1) was studied in [13, 14], where it was shown that the transfer matrix of the XXZ chain with non-diagonal right boundary commutes with the generator of the boundary quantum algebra $\mathbb{B}(\mathcal{U}_q(sl_2))$ [10, 12, 13].

Let us mention that for the XXZ case the associated L matrix is known, and it satisfies intertwining relations of the type (5.1) (see e.g. [30])

$$\begin{aligned}
&(\rho \otimes \mathrm{id})\Delta'(x) L_{12}(\lambda) = L_{12}(\lambda)(\rho \otimes \mathrm{id})\Delta(x) \quad \Rightarrow \\
&(\rho \otimes \mathrm{id}^{\otimes N}){\Delta'}^{(N+1)}(x)\mathcal{T}(\lambda) = \mathcal{T}(\lambda)\ (\rho \otimes \mathrm{id}^{\otimes N}){\Delta'}^{(N+1)}(x), \qquad x \in \{\mathcal{E}, \mathcal{F}, \mathcal{H}\}.
\end{aligned}$$

Hence one can show the symmetry of the algebraic transfer matrix (3.5). In fact commutation relations (5.7) hold for the algebraic transfer matrix (3.5), but now $E^{(N)}$, $F^{(N)}$, $H^{(N)}$ are not represented, namely

$$E^{(N)} = \Delta^{(N)}(\mathcal{E}), \qquad F^{(N)} = \Delta^{(N)}(\mathcal{F}), \qquad H^{(N)} = \Delta^{(N)}(\mathcal{H}).$$

5.2 The asymmetric twin open transfer matrix, $K^{(L,R)} = \mathbb{I}$

We come now to our main objective, which is the derivation of conserved quantities, associated to familiar quantum algebras, for the asymmetric twin transfer matrix. Unfortunately an L matrix associated to the asymmetric twin R matrix is not available at this stage, hence we are compelled to restrict ourselves to the symmetry of the represented transfer matrix (4.10). We shall first examine the case where both boundaries are trivial, i.e. $K^{(L,\ R)} = \mathbb{I}$.

• *The $\mathcal{U}_q(sl_2)$, $\mathcal{U}_{\mathrm{i}}(sl_2)$ symmetries:* We introduce representations of $\mathcal{U}_{\mathrm{i}}(sl_2)$, $\mathcal{U}_q(sl_2)$ respectively (recall, $q = e^{i\mu}$ is a the parameter of the blob algebra and $\mathrm{i} = \sqrt{-1}$ fixed):

$$\sigma_1 : \mathcal{U}_{\mathrm{i}}(sl_2) \to \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \qquad \text{and} \qquad \sigma_2 : \mathcal{U}_q(sl_2) \to \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

such that

$$\sigma_1(\mathcal{H}) = \mathrm{i}^{\frac{1}{2}(e_{11} - e_{44})}, \qquad \sigma_1(\mathcal{E}) = e_{14}, \qquad \sigma_1(\mathcal{F}) = e_{41},$$

$$\sigma_2(\mathcal{H}) = q^{-\frac{1}{2}(e_{22}-e_{33})}, \qquad \sigma_2(\mathcal{E}) = e_{32}, \qquad \sigma_2(\mathcal{F}) = e_{23}, \tag{5.8}$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Let also

$$E_j^{(N)} = \sigma_j^{\otimes N}(\Delta^{(N)}(\mathcal{E})), \qquad F_j^{(N)} = \sigma_j^{\otimes N}(\Delta^{(N)}(\mathcal{F})),$$
$$H_j^{(N)} = \sigma_j^{\otimes n}(\Delta^{(N)}(\mathcal{H})), \qquad j \in \{1,2\}.$$

It was shown in [19] that

$$\big[E_j^{(N)}, \Theta(\mathbb{U}_l)\big] = \big[F_j^{(N)}, \Theta(\mathbb{U}_l)\big] = \big[H_j^{(N)}, \Theta(\mathbb{U}_l)\big] = 0, \qquad l \in \{1,\dots,N-1\}. \tag{5.9}$$

$E_1^{(N)}$, $F_1^{(N)}$, $H_1^{(N)}$ provide a tensor representations of $\mathcal{U}_{\mathrm{i}}(sl_2)$ acting on $(\mathbb{C}^4)^{\otimes N}$, while $E_2^{(N)}$, $F_2^{(N)}$, $H_2^{(N)}$ form a representation of $\mathcal{U}_q(sl_2)$ acting on $(\mathbb{C}^4)^{\otimes N}$. The commutation relations (5.9) were exploited in [19] to show that the Hamiltonian of the model with $K^{(L,R)} = \mathbb{I}$ is $\mathcal{U}_q(sl_2) \otimes \mathcal{U}_{\mathrm{i}}(sl_2)$ symmetric.

We shall now show that the full transfer matrix (4.10) enjoys also the $\mathcal{U}_q(sl_2) \otimes \mathcal{U}_{\mathrm{i}}(sl_2)$ symmetry for $K^{(L,R)} = \mathbb{I}$. To achieve that we use the generalized intertwining relations between the operator $\mathcal{T}$ and the co-products of the quantum algebras $\mathcal{U}_q(sl_2)$, $\mathcal{U}_{\mathrm{i}}(sl_2)$. As shown in [19] the asymmetric twin R matrix satisfies intertwining relations for both σ_1 and σ_2, and consequently generalized 'commutation' relations are valid for the operator $\mathcal{T}$ of the asymmetric twin model (4.10). By exploiting these relations (5.2) for σ_j, and recalling the form of $\mathcal{T}$ (5.4) for the asymmetric twin model exchange relations involving the entries of the $\mathcal{T}$ operator (5.4) are obtained. In Appendix B we report only the necessary ones for the study of the transfer matrix symmetry, although they are already quite involved. It is then easy to show using (B.4), (B.5) and (5.4) that

$$\big[t(\lambda), E_j^{(N)}\big] = \big[t(\lambda), F_j^{(N)}\big] = \big[t(\lambda), H_j^{(N)}\big] = 0,$$

which proves that the transfer matrix for $K^{(L,R)} = \mathbb{I}$ is indeed $\mathcal{U}_q(sl_2) \otimes \mathcal{U}_{\mathrm{i}}(sl_2)$ symmetric.

• *The $\mathcal{U}_r(sl_2)$, $\mathcal{U}_{\hat{r}}(sl_2)$ symmetries:* As discussed in [19] in addition to the $\mathcal{U}_q(sl_2)$, $\mathcal{U}_{\mathrm{i}}(sl_2)$ symmetry the open twin Hamiltonian also enjoys the $\mathcal{U}_r(sl_2) \otimes \mathcal{U}_{\hat{r}}(sl_2)$ symmetry. We shall show that the full transfer matric enjoys these symmetries as well. In particular, the following actions of $\mathcal{U}_{\hat{r}}(sl_2)$, $\mathcal{U}_r(sl_2)$ on $\mathbb{C}^4$ were introduced, i.e.

$$\rho_1(\mathcal{H}) = \mathbb{I} \otimes \hat{r}^{\frac{1}{2}\sigma^z}, \qquad \rho_1(\mathcal{E}) = \mathbb{I} \otimes \sigma^+, \qquad \rho_1(\mathcal{F}) = \mathbb{I} \otimes \sigma^-,$$
$$\rho_2(\mathcal{H}) = r^{-\frac{1}{2}\sigma^z} \otimes \mathbb{I}, \qquad \rho_2(\mathcal{F}) = \sigma^+ \otimes \mathbb{I}, \qquad \rho_2(\mathcal{E}) = \sigma^- \otimes \mathbb{I}.$$

Setting

$$\tilde{E}_j^{(N)} = \rho_j^{\otimes N}(\Delta^{(N)}(\mathcal{E})), \qquad \tilde{F}_j^{(N)} = \rho_j^{\otimes N}(\Delta^{(N)}(\mathcal{F})),$$
$$\tilde{H}_j^{(N)} = \rho_j^{\otimes N}(\Delta^{(N)}(\mathcal{H})), \qquad j \in \{1,2\}$$

it was shown in [19] that

$$\big[\tilde{E}_j^{(N)}, \Theta(\mathbb{U}_l)\big] = \big[\tilde{F}_j^{(N)}, \Theta(\mathbb{U}_l)\big] = \big[\tilde{H}_j^{(N)}, \Theta(\mathbb{U}_l)\big] = 0, \qquad l \in \{1,\dots,N-1\}.$$

As in the $\mathcal{U}_q(sl_2)$, $\mathcal{U}_{\mathrm{i}}(sl_2)$ cases generalized intertwining relations between $\mathcal{T}$ (4.10) and the representations ρ_j introduced above are valid. Exploiting such relations we may immediately obtain exchange relations given in Appendix B (B.6), (B.7). In a straightforward manner, with the use of (B.6), (B.7) and (5.4) it may be shown that

$$\big[t(\lambda), \tilde{E}_j^{(N)}\big] = \big[t(\lambda), \tilde{F}_j^{(N)}\big] = \big[t(\lambda), \tilde{H}_j^{(N)}\big] = 0$$

which proves that the transfer matrix is also $\mathcal{U}_r(sl_2) \otimes \mathcal{U}_{\hat{r}}(sl_2)$ symmetric.

It is finally worth noting that the maps σ_i can be in fact expressed in terms of the representations ρ_i in the following manner

$$\sigma_1(\mathcal{E}) = \rho_1(\mathcal{E})\rho_2(\mathcal{F}), \qquad \sigma_1(\mathcal{F}) = \rho_1(\mathcal{F})\rho_2(\mathcal{E}), \qquad \sigma_1(\mathcal{H})\sigma_2(\mathcal{H}) = (-1)^{\frac{1}{2}}\rho_1(\mathcal{H})\rho_2(\mathcal{H}),$$
$$\sigma_2(\mathcal{E}) = \rho_1(\mathcal{E})\rho_2(\mathcal{E}), \qquad \sigma_2(\mathcal{F}) = \rho_1(\mathcal{F})\rho_2(\mathcal{F}).$$

We have been able to show, for the moment, that the asymmetric twin open transfer matrix (4.10) with trivial boundary conditions is $\mathcal{U}_q(sl_2) \otimes \mathcal{U}_{\mathrm{i}}(sl_2)$ $(\mathcal{U}_r(sl_2) \otimes \mathcal{U}_{\hat{r}}(sl_2))$ symmetric. Whether there exist further recognized symmetries associated to the twin spin chain with trivial boundaries is still a question under investigation. It should be stressed however that up to date we have not been able to identify further charges, associated to some familiar quantum algebra, commuting with the open asymmetric twin transfer matrix (4.10).

5.3 Non-trivial boundary conditions, $K^{(R)} \neq \mathbb{I}$

It is also desirable to investigate the transfer matrix symmetry when a non-trivial right boundary (3.3), emerging from the representations Θ^I (2.10)–(2.13), is implemented. The left boundary is kept trivial i.e. $K^{(L)} = \mathbb{I}$. Inspired basically by the symmetry of the open XXZ spin chain with non-diagonal right boundary [10, 12, 13] we consider the combination of generators of the quantum algebra $\mathcal{U}_{\mathrm{q}}(sl_2)$, $\mathrm{q} \in \{q, \mathrm{i}, r, \hat{r}\}$

$$\mathcal{Q}_{\mathrm{q}} = \mathrm{q}^{-\frac{1}{2}\mathcal{H}}\, \mathcal{E} + \mathrm{q}^{\frac{1}{2}\mathcal{H}}\, \mathcal{F} + x_{\mathrm{q}}\mathcal{H}^2 - x_{\mathrm{q}}\mathbb{I}$$

the constants x_{q} will be identified later on in this section. The charge $\mathcal{Q}_{\mathrm{q}}$ is equipped with a co-product structure, i.e.,

$$\Delta(\mathcal{Q}_{\mathrm{q}}) = \mathbb{I} \otimes \mathcal{Q}_{\mathrm{q}} + \mathcal{Q}_{\mathrm{q}} \otimes \mathcal{H}^2.$$

As in the case of trivial boundary conditions we shall state the following general argument. Consider a solution of the Yang–Baxter equation satisfying (5.1), and a $K^{(R)}$ matrix, solution of the reflection equation, satisfying

$$\mathrm{h}(\mathcal{Q}_{\mathrm{q}})\; K^{(R)}(\lambda) = K^{(R)}(\lambda)\; \mathrm{h}(\mathcal{Q}_{\mathrm{q}}). \tag{5.10}$$

Then using (5.1) and (5.10) one can show in a straightforward fashion that generalized intertwining relations hold also for the corresponding $\mathcal{T}$ (4.10) (see also [13, 14] for a detailed proof)

$$(\mathrm{h} \otimes \mathrm{h}^{\otimes N})\Delta'^{(N+1)}(\mathcal{Q}_{\mathrm{q}})\; \mathcal{T}(\lambda) = \mathcal{T}(\lambda)\; (\mathrm{h} \otimes \mathrm{h}^{\otimes N})\Delta'^{(N+1)}(\mathcal{Q}_{\mathrm{q}}). \tag{5.11}$$

Again the latter relations are of the type (4.4) for $\lambda' \to \pm\infty$

We could have considered the combination of the quantum algebra $\mathcal{U}_{\mathrm{q}}(\widehat{sl_2})$ generators [13, 19] and then exploit intertwining relations between the co-products of the $\mathcal{U}_{\mathrm{q}}(\widehat{sl_2})$ generators and the operator $\mathcal{T}$, but as already discussed the twin R matrix does not satisfy any obvious intertwining relations with the elements of $\mathcal{U}_{\mathrm{q}}(\widehat{sl_2})$, therefore we focus on the non affine case. In what follows we shall treat separately each one of the boundaries associated to Θ^I.

Type (i) It was shown in [19] for the solution type (i) (3.3), that

$$\sigma_1(x)K^{(R)}(\lambda) = K^{(R)}(\lambda)\sigma_1(x), \qquad x \in \mathcal{U}_{\mathrm{i}}(sl_2),$$
$$\sigma_2(\mathcal{Q}_q)\; K^{(R)}(\lambda) = K^{(R)}(\lambda)\; \sigma_2(\mathcal{Q}_q), \tag{5.12}$$

provided that

$$x_q = \frac{Q - Q^{-1}}{q - q^{-1}}, \tag{5.13}$$

σ_j are given by (5.8). The first of the equations (5.12) implies that the presence of the non-trivial boundary (i) does not break the $\mathcal{U}_{\mathrm{i}}(sl_2)$ symmetry, namely

$$\big[t(\lambda), E_1^{(N)}\big] = \big[t(\lambda), F_1^{(N)}\big] = \big[t(\lambda), K_1^{(N)}\big] = 0. \tag{5.14}$$

Using the second equation in (5.12) it is clear that intertwining relations (5.11) hold for $\mathrm{h} \to \sigma_2$. Setting also

$$\mathbb{Q}_q^{(N)} = \sigma_2^{\otimes N}(\Delta^{(N)}(\mathcal{Q}_q)) \tag{5.15}$$

we obtain via (5.11) for σ_2 and (5.4) the following exchange relations

$$\begin{aligned} &\big[\mathbb{Q}_q^{(N)}, \mathcal{A}\big] = \big[\mathbb{Q}_q^{(N)}, \mathcal{D}\big] = 0, \qquad \big[\mathbb{Q}_q^{(N)}, \mathcal{A}_1\big] = q(\mathcal{B}_5 - \mathcal{C}_5),\\ &\big[\mathbb{Q}_q^{(N)}, \mathcal{A}_2\big] = -q^{-1}(\mathcal{B}_5 - \mathcal{C}_5). \end{aligned} \tag{5.16}$$

Using the latter relations and the form of the transfer matrix (5.4) we can show that

$$\big[t(\lambda), \mathbb{Q}_q^{(N)}\big] = 0. \tag{5.17}$$

The boundary associated to solution (i) preserves the $\mathcal{U}_{\mathrm{i}}(sl_2)$ symmetry (5.14), and also preserves part of $\mathcal{U}_q(sl_2)$, that is the charge $\mathbb{Q}_q^{(N)}$ (5.17).

Type (ii) For the solution type (ii) (3.3) one has that [19]

$$\begin{aligned} &\sigma_2(x)K^{(R)}(\lambda) = K^{(R)}(\lambda)\sigma_2(x), \qquad x \in \mathcal{U}_q(sl_2),\\ &\sigma_1(\mathcal{Q}_{\mathrm{i}})K^{(R)}(\lambda) = K^{(R)}(\lambda)\sigma_1(\mathcal{Q}_{\mathrm{i}}), \end{aligned} \tag{5.18}$$

provided that

$$x_{\mathrm{i}} = -\frac{Q - Q^{-1}}{2\mathrm{i}}. \tag{5.19}$$

The first of the equations (5.18) implies that the presence of the non-trivial boundary (ii) does not break the $\mathcal{U}_q(sl_2)$ symmetry, i.e.

$$\big[t(\lambda), E_2^{(N)}\big] = \big[t(\lambda), F_2^{(N)}\big] = \big[t(\lambda), H_2^{(N)}\big] = 0. \tag{5.20}$$

It can be shown using the second equation in (5.18) that generalized intertwining relations (5.11) are valid for $\mathrm{h} \to \sigma_1$. Recalling (5.15) and setting

$$\mathbb{Q}_{\mathrm{i}}^{(N)} = \sigma_1^{\otimes N}(\Delta^{(N)}(\mathcal{Q}_{\mathrm{i}})) \tag{5.21}$$

we obtain the following exchange relations

$$\begin{aligned} &\big[\mathbb{Q}_{\mathrm{i}}^{(N)}, \mathcal{A}_1\big] = \big[\mathbb{Q}_{\mathrm{i}}^{(N)}, \mathcal{A}_2\big] = 0, \qquad \big[\mathbb{Q}_{\mathrm{i}}^{(N)}, \mathcal{A}\big] = \mathrm{i}^{-1}(\mathcal{B} - \mathcal{C}),\\ &\big[\mathbb{Q}_{\mathrm{i}}^{(N)}, \mathcal{D}\big] = -\mathrm{i}(\mathcal{B} - \mathcal{C}). \end{aligned} \tag{5.22}$$

From the latter relations and the form of the transfer matrix (5.4) we conclude that

$$\big[t(\lambda), \mathbb{Q}_{\mathrm{i}}^{(N)}\big] = 0. \tag{5.23}$$

The presence of boundary of type (ii) preserves the $\mathcal{U}_q(sl_2)$ symmetry (5.20), and also the boundary charge $\mathbb{Q}_{\mathrm{i}}^{(N)}$ commutes with the transfer matrix.

Type (+) It is clear from the form of the solution (+) (3.3), (2.12) and taking into account relations (5.11) for $\mathrm{h} \to \sigma_1, \sigma_2$ (which apparently also hold for M^i (2.10) and M^{ii} (2.11)), provided that relations (5.13), (5.19) hold simultaneously. Both sets of commutation relations (5.16), (5.22) are valid and therefore

$$\big[t(\lambda), \mathbb{Q}_{\mathrm{i}}^{(N)}\big] = \big[t(\lambda), \mathbb{Q}_q^{(N)}\big] = 0. \tag{5.24}$$

The presence of solution type (+) breaks both $\mathcal{U}_q(sl_2)$ and $\mathcal{U}_{\tilde{q}}(sl_2)$, and the remaining conserved quantities are the boundary non-local charges $\mathbb{Q}_q^{(N)}$, $\mathbb{Q}_{\mathrm{i}}^{(N)}$.

Type (iii) Finally, for the solution type (iii) (3.3) the following commutation relations are valid [19]

$$\rho_1(\mathcal{Q}_{\hat{r}})K^{(R)}(\lambda) = K^{(R)}(\lambda)\rho_1(\mathcal{Q}_{\hat{r}}),$$
$$\rho_2(\mathcal{Q}_r)\ K^{(R)}(\lambda) = K^{(R)}(\lambda)\rho_2(\mathcal{Q}_r)$$

provided that the constants x_r, $x_{\hat{r}}$ are given by

$$x_r = \mathrm{i}\frac{\sqrt{\mathrm{i}Q} + \sqrt{-\mathrm{i}Q^{-1}}}{r - r^{-1}}, \qquad x_{\hat{r}} = \frac{\sqrt{\mathrm{i}Q} - \sqrt{-\mathrm{i}Q^{-1}}}{\hat{r} - \hat{r}^{-1}}.$$

Again we can show that (5.11) are valid for both ρ_1, ρ_2. Setting

$$\mathbb{Q}_{\hat{r}}^{(N)} = \rho_1^{\otimes N}(\Delta^N(\mathcal{Q}_{\hat{r}})), \qquad \mathbb{Q}_r^{(N)} = \rho_2^{\otimes N}(\Delta^N(\mathcal{Q}_r)) \tag{5.25}$$

and with the help of (5.11) for $\mathrm{h} \to \rho_1, \rho_2$ and (5.4) we obtain exchange relations of the type:

$$r^{-1}\big[\mathbb{Q}_r^{(N)}, \mathcal{A}\big] = -r\big[\mathbb{Q}_r^{(N)}, \mathcal{A}_2\big] = (\mathcal{C}_2 - \mathcal{B}_2),$$
$$r^{-1}\big[\mathbb{Q}_r^{(N)}, \mathcal{A}_1\big] = -r\big[\mathbb{Q}_r^{(N)}, \mathcal{D}\big] = (\mathcal{C}_3 - \mathcal{B}_3),$$
$$-\hat{r}^{-1}\big[\mathbb{Q}_{\hat{r}}^{(N)}, \mathcal{A}\big] = \hat{r}\big[\mathbb{Q}_{\hat{r}}^{(N)}, \mathcal{A}_1\big] = (\mathcal{C}_1 - \mathcal{B}_1),$$
$$-\hat{r}\big[\mathbb{Q}_{\hat{r}}^{(N)}, \mathcal{A}_2\big] = \hat{r}^{-1}\big[\mathbb{Q}_{\hat{r}}^{(N)}, \mathcal{D}\big] = (\mathcal{C}_4 - \mathcal{B}_4).$$

The latter relations lead to the following,

$$\big[t(\lambda), \mathbb{Q}_{\hat{r}}^{(N)}\big] = \big[t(\lambda), \mathbb{Q}_r^{(N)}\big] = 0. \tag{5.26}$$

In this case both $\mathcal{U}_q(sl_2)$, $\mathcal{U}_{\tilde{q}}(sl_2)$ ($\mathcal{U}_r(sl_2)$, $\mathcal{U}_{\hat{r}}(sl_2)$) are broken, and the remaining conserved quantities are the charges $\mathbb{Q}_{\hat{r}}^{(N)}$, $\mathbb{Q}_r^{(N)}$.

As in the case with trivial boundary conditions, discussed in the previous section, the crucial point raised is whether there exist further well known symmetries associated to the model under consideration. Up to date we have not succeed to identify further commuting quantities, associated to some familiar quantum algebra. A pertinent question is whether the highly involved boundary non-local charges $\mathcal{T}_{ab}^{\pm}$ introduced in Section 4 may be written in terms of the more familiar charges (5.15), (5.21), (5.25). Presumably some of the non-local charges $\mathcal{T}_{ab}^{\pm}$ may be expressed in terms of those but not all of them. It is worth emphasizing that all the conserved quantities found for both trivial and non trivial right boundary satisfy algebraic relations of the type (4.4), with no spectral dependance. Finally in the case of a non trivial left boundary one has to exploit exchange relations involving all the entries of the $\mathcal{T}$ matrix and then extract the appropriate combination of non-local charges commuting with transfer matrix.

6 Discussion

Let us now briefly review the main findings of the present article. The main objective of this work was the study of the symmetry of the open asymmetric twin chain. In this spirit we were able to show that the boundary quantum algebra provides a symmetry for any open transfer matrix with trivial boundary (4.9). It was also shown that the boundary quantum algebra is in addition the centralizer of the B type Hecke algebra (4.13). Furthermore we derived sets of convenient exchange relations for the asymmetric twin model with both trivial and non trivial boundaries emerging from the generalized intertwining relations (5.2), (5.11). By exploiting such relations we proved the commutation of the transfer matrix with certain non-local charges. More precisely, in the case of trivial boundaries the derived conserved charges consist tensor representations of $\mathcal{U}_q(sl_2) \otimes \mathcal{U}_{\mathrm{i}}(sl_2)$ $(\mathcal{U}_r(sl_2) \otimes \mathcal{U}_{\hat{r}}(sl_2))$. When a non-trivial right boundary is implemented the symmetry of the transfer matrix as expected is reduced (5.14), (5.17), (5.20), (5.23), (5.24), (5.26). Depending on the choice of boundaries some of the symmetry is preserved, and the new conserved quantities are expressed as combinations of the generators of the aforementioned quantum algebras. Notice that we essentially extend the results of [19] in as much as the results of [29] are extended in [28]. As already pointed out the discovered 'familiar' symmetries do not seem to consist the full symmetry of the transfer matrix in [19]. The full symmetry (non-Abelian) of the model is presumably the boundary quantum algebra, but at this stage this is rather a conjecture, which needs to be further checked.

The relation between the boundary non-local charges and the spectrum and Bethe ansatz equations is also an intricate problem. Usually such relations emerge from the asymptotic behavior of the $\mathcal{T}$ matrix, which as already mentioned is not at all straightforward for the asymmetric twin model. More precisely, it is clear that the asymptotics of the transfer matrix may be expressed as (see also e.g. [26, 32])

$$t(\lambda \to \pm\infty) \propto \sum_{a=1}^{4} \mathcal{T}_{aa}^{\pm},$$

hence the spectrum of $\mathcal{T}_{aa}^{\pm}$ will provide consequential information regarding the asymptotic behaviour of the spectrum of the asymmetric twin chain. The key point is to derive the explicit form of the objects $\mathcal{T}_{aa}^{\pm}$, and express them, if possible, in terms of the familiar non-local charges of Sections 5.2, 5.3. For the moment there is no apparent link between the conserved quantities and the spectrum and this is the main obstacle in deriving the spectrum of the twin transfer matrix (see also [19]). However, in [19] the equivalence of the spectrum of the open twin and XXZ Hamiltonians was established, and consequently the form of the spectrum of the asymmetric twin chain was derived. It is worth pointing out that the diagonalization of the non-local charges is also a particularly challenging problem, and it has been already solved for the XXZ model for particular representations in [33, 34, 35, 36]. In fact the boundary non-local charges of Section 5.3 having exactly the same structure as the ones of the XXZ model may be diagonalized along the lines described in [33, 34, 35, 36]. Finally, an interesting point to pursue is the generalization of the asymmetric twin representation consisting of representations associated to higher rank algebras. We hope to report on these matters in a forthcoming work.

A Appendix

We present here $\Theta(\mathbb{U}_1)$ as a 16×16 matrix acting *not* on $\mathbb{V}_{2^-} \otimes \mathbb{V}_{1^-} \otimes \mathbb{V}_{1^+} \otimes \mathbb{V}_{2^+}$, but on $(\mathbb{V}_{1^-} \otimes \mathbb{V}_{1^+}) \otimes (\mathbb{V}_{2^-} \otimes \mathbb{V}_{2^+}) = \mathbb{V}_{\tilde{1}} \otimes \mathbb{V}_{\tilde{2}}$, according to the space/mirror space notation (see also (3.2))

$$\Theta(\mathbb{U}_1) = \left(\begin{array}{cccccccccccccccc}
0&&&&&&&&&&&&&&& \\
&0&&&&&&&&&&&&&& \\
&&0&&&&&&&&&&&&& \\
&&&-i&&&-r^{-1}&&&-\hat{r}&&&1&&& \\
&&&&0&&&&&&&&&&& \\
&&&&&0&&&&&&&&&& \\
&&&-r^{-1}&&&-q^{-1}&&&1&&&-\hat{r}^{-1}&&& \\
&&&&&&&0&&&&&&&& \\
&&&&&&&&0&&&&&&& \\
&&&-\hat{r}&&&1&&&-q&&&-r&&& \\
&&&&&&&&&&0&&&&& \\
&&&&&&&&&&&0&&&& \\
&&&1&&&-\hat{r}^{-1}&&&-r&&&i&&& \\
&&&&&&&&&&&&&0&& \\
&&&&&&&&&&&&&&0& \\
&&&&&&&&&&&&&&&0
\end{array}\right) \otimes \cdots \otimes \mathbb{I}.$$

B Appendix

In this Appendix exchange relations arising from the generalized intertwining relations are reported. First we present exchange relations involving the entries of $\mathcal{T}$ and representations of the quantum algebra $\mathcal{U}_q(sl_2)$. Let $[X,Y]_{\mathrm{q}} = XY - \mathrm{q}YX$, then

$$\begin{aligned}
&\big[\mathcal{A}, H^{(N)}\big] = 0, \qquad \big[\mathcal{D}, H^{(N)}\big] = 0, \\
&\big[\mathcal{C}, (H^{(N)})^{\pm 1}\big]_{q^{\mp 1}} = 0, \qquad \big[\mathcal{B}, (H^{(N)})^{\pm 1}\big]_{q^{\pm 1}} = 0,
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
&\big[E^{(N)}, \mathcal{A}\big] = -q^{-\frac{1}{2}}(H^{(N)})^{-1}\mathcal{C}, \qquad \big[E^{(N)}, \mathcal{D}\big] = q^{\frac{1}{2}}\mathcal{C}(H^{(N)})^{-1}, \\
&\big[E^{(N)}, \mathcal{C}\big]_q = 0, \qquad \big[E, \mathcal{B}\big]_{q^{-1}} = q^{-\frac{1}{2}}\big(\mathcal{A}(H^{(N)})^{-1} - (H^{(N)})^{-1}\mathcal{D}\big),
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
&\big[F^{(N)}, \mathcal{A}\big] = q^{-\frac{1}{2}}\mathcal{B}(H^{(N)})^{-1}, \qquad \big[F^{(N)}, \mathcal{D}\big] = -q^{\frac{1}{2}}(H^{(N)})^{-1}\mathcal{B}, \\
&\big[F^{(N)}, \mathcal{B}\big]_{q^{-1}} = 0, \qquad \big[F^{(N)}, \mathcal{C}\big]_q = q^{\frac{1}{2}}\big(\mathcal{D}(H^{(N)})^{-1} - (H^{(N)})^{-1}\mathcal{A}\big).
\end{aligned} \tag{B.3}$$

Exchange relations between the entries of $\mathcal{T}$ and representations of the quantum algebras $\mathcal{U}_q(sl_2)$, $\mathcal{U}_{\mathrm{i}}(sl_2)$ are given below

$$\begin{aligned}
&\big[H_j^{(N)}, \mathcal{A}\big] = \big[H_j^{(N)}, \mathcal{D}\big] = \big[H_j^{(N)}, \mathcal{A}_i\big] = 0, \qquad i, j \in \{1, 2\}, \\
&\big[\mathcal{C}, (H_1^{(N)})^{-1}\big]_{\mathrm{i}} = \big[\mathcal{B}, (H_1^{(N)})^{-1}\big]_{\mathrm{i}^{-1}} = \big[\mathcal{C}_5, (H_2^{(N)})^{-1}\big]_{q^{-1}} = \big[\mathcal{B}_5, (H_2^{(N)})^{-1}\big]_q = 0,
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
&\big[E_1^{(N)}, \mathcal{A}\big] = -\mathrm{i}^{-\frac{1}{2}}(H_1^{(N)})^{-1}\mathcal{C}, \qquad \big[E_1^{(N)}, \mathcal{D}\big] = \mathrm{i}^{\frac{1}{2}}\mathcal{C}\ (H_1^{(N)})^{-1}, \qquad \big[E_1^{(N)}, \mathcal{A}_j\big] = 0, \\
&\big[F_1^{(N)}, \mathcal{A}\big] = \mathrm{i}^{-\frac{1}{2}}\mathcal{B}(H_1^{(N)})^{-1}, \qquad \big[F_1^{(N)}, \mathcal{D}\big] = -\mathrm{i}^{\frac{1}{2}}(H_1^{(N)})^{-1}\mathcal{B}, \qquad \big[F_1^{(N)}, \mathcal{A}_j\big] = 0, \\
&\big[F_2^{(N)}, \mathcal{A}_1\big] = -q^{\frac{1}{2}}(H_2^{(N)})^{-1}\mathcal{C}_5, \qquad \big[F_2^{(N)}, \mathcal{A}_2\big] = q^{-\frac{1}{2}}\mathcal{C}_5(H_1^{(N)})^{-1}, \\
&\big[F_2^{(N)}, \mathcal{A}\big] = 0, \qquad \big[F_2^{(N)}, \mathcal{D}\big] = 0, \\
&\big[E_2^{(N)}, \mathcal{A}_1\big] = q^{\frac{1}{2}}\mathcal{B}_5(H_2^{(N)})^{-1}, \qquad \big[E_2^{(N)}, \mathcal{A}_2\big] = -q^{-\frac{1}{2}}(H_2^{(N)})^{-1}\mathcal{B}_5, \\
&\big[E_2^{(N)}, \mathcal{A}\big] = 0, \qquad \big[E_2^{(N)}, \mathcal{D}\big] = 0.
\end{aligned} \tag{B.5}$$

Finally exchange relations involving the entries of $\mathcal{T}$ and representations of the quantum algebras $\mathcal{U}_r(sl_2)$, $\mathcal{U}_{\hat r}(sl_2)$ are presented below

$$\big[\mathcal{B}_1,(\tilde H_1^{(N)})^{-1}\big]_{\hat r^{-1}}=\big[\mathcal{B}_4,(\tilde H_1^{(N)})^{-1}\big]_{\hat r^{-1}}=\big[\mathcal{C}_1,(\tilde H_1^{(N)})^{-1}\big]_{\hat r}=\big[\mathcal{C}_4,\tilde(H_1^{(N)})^{-1}\big]_{\hat r}=0,$$
$$\big[\mathcal{B}_2,(\tilde H_2^{(N)})^{-1}\big]_{r}=\big[\mathcal{B}_3,(\tilde H_2^{(N)})^{-1}\big]_{r}=\big[\mathcal{C}_2,(\tilde H_2^{(N)})^{-1}\big]_{r^{-1}}=\big[\mathcal{C}_3,(\tilde H_2^{(N)})^{-1}\big]_{r^{-1}}=0, \tag{B.6}$$
$$\big[\tilde E_1^{(N)},\mathcal{A}\big]=-\hat r^{-\frac12}(\tilde H_1^{(N)})^{-1}\mathcal{C}_1,\qquad \big[\tilde E_1^{(N)},\mathcal{A}_1\big]=\hat r^{\frac12}\mathcal{C}_1(\tilde H_1^{(N)})^{-1},$$
$$\big[\tilde E_1^{(N)},\mathcal{A}_2\big]=-\hat r^{-\frac12}(\tilde H_1^{(N)})^{-1}\mathcal{C}_4,\qquad \big[\tilde E_1^{(N)},\mathcal{D}\big]=\hat r^{\frac12}\mathcal{C}_4(\tilde H_1^{(N)})^{-1},$$
$$\big[\tilde F_1^{(N)},\mathcal{A}\big]=\hat r^{-\frac12}\mathcal{B}_1(\tilde H_1^{(N)})^{-1},\qquad \big[\tilde F_1^{(N)},\mathcal{A}_1\big]=-\hat r^{\frac12}(\tilde H_1^{(N)})^{-1}\mathcal{B}_1,$$
$$\big[\tilde F_1^{(N)},\mathcal{A}_2\big]=\hat r^{-\frac12}\mathcal{B}_4(\tilde H_1^{(N)})^{-1},\qquad \big[\tilde F_1^{(N)},\mathcal{D}\big]=r^{\frac12}\mathcal{B}_2(\tilde H_2^{(N)})^{-1},$$
$$\big[\tilde E_2^{(N)},\mathcal{A}\big]=r^{\frac12}\mathcal{B}_2(\tilde H_1^{(N)})^{-1},\qquad \big[\tilde E_2^{(N)},\ \mathcal{A}_1\big]=r^{\frac12}\mathcal{B}_3(\tilde H_1^{(N)})^{-1},$$
$$\big[\tilde E_2^{(N)},\mathcal{A}_2\big]=-r^{-\frac12}(\tilde H_2^{(N)})^{-1}\mathcal{B}_2,\qquad \big[\tilde E_2^{(N)},\mathcal{D}\big]=-r^{-\frac12}(\tilde H_1^{(N)})^{-1}\mathcal{B}_3$$
$$\big[\tilde F_2^{(N)},\mathcal{A}\big]=-r^{\frac12}(\tilde H_2^{(N)})\mathcal{C}_2,\qquad \big[\tilde F_2^{(N)},\mathcal{A}_1\big]=-r^{\frac12}(\tilde H_2^{(N)})^{-1}\mathcal{C}_3,$$
$$\big[\tilde F_2^{(N)},\mathcal{A}_2\big]=r^{-\frac12}\mathcal{C}_2(\tilde H_2^{(N)})^{-1},\qquad \big[\tilde E_2^{(N)},\mathcal{D}\big]=r^{-\frac12}\mathcal{C}_3(\tilde H_2^{(N)})^{-1}. \tag{B.7}$$

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