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Multi-Hamiltonian Structures on Beauville's Integrable System and Its Variant*

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Abstract. We study Beauville's completely integrable system and its variant from a viewpoint of multi-Hamiltonian structures. We also relate our result to the previously known Poisson structures on the Mumford system and the even Mumford system.

Key words: completely integrable system; Mumford system; multi-Hamiltonian structure

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1 Introduction

Let $r \geq 2$ and $d \geq 1$ be integers. In [1], Beauville constructed an algebraically completely integrable Hamiltonian system on the space of the gauge equivalence classes of $r \times r$ polynomial matrices whose degree is less than or equal to d. This system is a generalization of the Mumford system [6], and the general level set is isomorphic to the complement of the theta divisor in the Jacobian variety of the spectral curve. Employing Beauville's approach, a variant is constructed in [5], which gives a generalization of the even Mumford system introduced by Vanhaecke [11, 12]. The remarkable difference with Beauville's system is that the general level set is isomorphic to the complement of the intersection of r translates of the theta divisor. We call these systems the Beauville system and the Beauville–Vanhaecke (BV) system in this paper.

The Poisson structures of these systems depend on a monic polynomial of degree $d+2$ having only simple roots. Our first aim in this article is to understand them in terms of the multi-Hamiltonian structure (see [9, § 4 and § 12]). This is achieved by extending them to those parameterized by a polynomial $\phi(x)$ of degree at most $d+2$; $\phi(x)$ need not be monic nor have simple roots, and $\deg \phi(x)$ can be less than $d+2$ (Propositions 1 and 3). It then turns out that the Poisson structures corresponding to $\phi(x) = 1, x, \dots, x^{d+2}$ give the integrable multi-Hamiltonian system (Theorems 1 and 2).

Since the Beauville system and the BV system are given by the quotient construction, it is an important problem to construct their spaces of representatives and to describe the vector fields and the Poisson structures on them. For the Beauville system, a space of representatives was constructed by Donagi and Markman for a certain subspace [2]. The vector fields on it was computed by Fu [4] and the multi-Hamiltonian structure was constructed in [8] using the group-theoretic approach. For the BV system, spaces of representatives were given in [5] and the vector fields for $r = 2$ and 3 were also given.

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Our second aim is to study the family of Poisson structures on the space of representatives. For the Beauville system, we introduce a new space of representatives for a certain subspace of codimension one which includes Donagi–Markman's result (Proposition 5). In the simplest case of $r = 2$ we write down the Poisson structures on this space and on the spaces of representatives for the BV system (Propositions 2 and 4) and compare them with previously known Poisson structures on the Mumford system and the even Mumford system [3, 7, 8].

This article is organized as follows. In Sections 2 and 3, we study Poisson structures and multi-Hamiltonian structures for the Beauville system and the BV system. We also give expressions of the Poisson structures on the spaces of representatives in the case $r = 2$ and compare them with those of the Mumford system and the even Mumford system. Section 4 is devoted to the construction of the new space of the representatives for the subset of the Beauville system.

2 Multi-Hamiltonian structure on the Beauville system

We fix numbers $r \in \mathbb{Z}_{\geq 2}$ and $d \in \mathbb{Z}_{\geq 1}$. Throughout this paper, we use the following notations: $S_d \subset \mathbb{C}[x]$ denotes the set of polynomials of degree at most d. $E_{ij} \in M_r(\mathbb{C})$ is the matrix whose (i, j)-th entry is one and other entries are zero. For a matrix $A(x) \in M_r(\mathbb{C}[x])$ with polynomial entries,

$$\begin{aligned} A_{ij}(x) \in \mathbb{C}[x] :& \quad \text{the } (i,j)\text{-th entry of } A(x), \\ A_{ij;k} \in \mathbb{C} :& \quad \text{the coefficient of } x^k \text{ in } A_{ij}(x), \\ A_k \in M_r(\mathbb{C}) :& \quad \text{the coefficient of } x^k \text{ in } A(x). \end{aligned}$$

Let W be a nonsingular algebraic variety.

Definition 1. A Poisson algebra structure on a sheaf of rings $\mathcal{F}$ on W is a morphism $\{\cdot,\cdot\} : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ satisfying skew-symmetry, the Leibniz rule, and the Jacobi identity. A Poisson structure on W is a Poisson algebra structure on the structure sheaf $\mathcal{O}_W$.

2.1 The Beauville system

Let $V(r, d)$ be the set

$$V(r,d) = \{P(x,y) = y^r + s_1(x)y^{r-1} + \cdots + s_r(x) \in \mathbb{C}[x,y] \mid s_i(x) \in S_{di}\}.$$

For $P(x, y) \in V(r, d)$, let C_P be the spectral curve obtained by taking the closure of the affine curve $P(x, y) = 0$ in the Hirzebruch surface $\mathbb{F}_d = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$ of degree d. Consider the set $M_r(S_d)$ of $r \times r$ matrices with entries in S_d and let ψ be the map:

$$\psi : M_r(S_d) \to V(r,d), \qquad A(x) \mapsto \det(y\mathbb{I}_r - A(x)).$$

The group $PGL_r(\mathbb{C})$ acts on $M_r(S_d)$ by conjugation:

$$PGL_r(\mathbb{C}) \ni g : A(x) \mapsto g(A(x)) = g^{-1}A(x)g.$$

Define a subset $M_r(S_d)_{ir}$ of $M_r(S_d)$ as

$$M_r(S_d)_{ir} = \{A(x) \in M_r(S_d) \mid C_{\psi(A(x))} \text{ is irreducible}\}.$$

Note that the $PGL_r(\mathbb{C})$-action is free on $M_r(S_d)_{ir}$. Let $\eta : M_r(S_d)_{ir} \to M_r(S_d)_{ir}/PGL_r(\mathbb{C})$ be the quotient map. The phase space of the Beauville system is $\mathcal{M}(r, d) = M_r(S_d)_{ir}/PGL_r(\mathbb{C})$. It was shown in [1] that if $P \in V(r, d)$ defines a smooth spectral curve C_P, then $\psi^{-1}(P)/PGL_r(\mathbb{C})$

is isomorphic to the complement of the theta divisor in $\mathrm{Pic}^{g-1}(C_P)$, where $g = \frac{1}{2}(r-1)(rd-2)$ is the genus of C_P.

Define the vector fields $Y_i^{(k)}$ on $M_r(S_d)$ by

$$\sum_{i=0}^{dk-1} a^i Y_i^{(k)}(A(x)) = \frac{1}{x-a}[A(x), A(a)^k], \qquad \text{for} \quad k = 1, \dots, r-1. \tag{1}$$

Here we have identified the tangent space at each $A(x) \in M_r(S_d)$ with $M_r(S_d)$. In [1], it was shown that $\eta_* Y_i^{(k)}$ generate the g-dimensional space of translation invariant vector fields on $\mathrm{Pic}^{g-1}(C_P)$.

2.2 Poisson structure

We use the following shorthand notations:

$$M^\bullet = M_r(S_{d+1}), \quad M = M_r(S_d), \quad M_{ir} = M_r(S_d)_{ir}, \quad \mathcal{M} = \mathcal{M}(r,d),, \quad G = PGL_r(\mathbb{C}).$$

Extending the result of [1, § 5], we are to equip $\mathcal{M}$ with a family of compatible Poisson structures depending on a polynomial $\phi(x) \in S_{d+2}$:

$$\phi(x) = \sum_{i=0}^{d+2} \sigma_i x^i.$$

Note that a Poisson structure on $\mathcal{M}$ is equivalent to a Poisson algebra structure on the sheaf of G-invariant functions $\mathcal{O}_{M_{ir}}^G$ on M_{ir}. Moreover, a Poisson algebra structure on $\mathcal{O}_M^G$ induces that on $\mathcal{O}_{M_{ir}}^G$ since M_{ir} is an open subset of M.

Consider the following Poisson structure on $M^\bullet$:

$$\{A_{ij}(x), A_{kl}(y)\} = \delta_{i,l}\frac{A_{kj}(x)\phi(y) - \phi(x)A_{kj}(y)}{x-y} - \delta_{k,j}\frac{A_{il}(x)\phi(y) - \phi(x)A_{il}(y)}{x-y}. \tag{2}$$

Let $\iota : M \hookrightarrow M^\bullet$ be the closed immersion. Let $\mathcal{I}_M$ be the ideal sheaf of ι. Writing α for the natural projection $\mathcal{O}_{M^\bullet} \to \mathcal{O}_{M^\bullet}/\mathcal{I}_M = \iota_*\mathcal{O}_M$, we set $\mathcal{N} := \alpha^{-1}(\iota_*\mathcal{O}_M^G) \subset \mathcal{O}_{M^\bullet}$.

Proposition 1. *$\mathcal{N}$ is a Poisson subalgebra of $\mathcal{O}_{M^\bullet}$. This Poisson algebra structure induces that on $\mathcal{N}/\mathcal{N} \cap \mathcal{I}_M$, hence on $\mathcal{O}_M^G$.*

Proof. Let U be an open subset of $M^\bullet$. Let us write $F, H \in \mathcal{N}(U)$ as

$$F = f + \sum_{1 \le i,j \le r} A_{ij;d+1} f_{ij}, \qquad H = h + \sum_{1 \le i,j \le r} A_{ij;d+1} h_{ij},$$

with $f, h \in \mathcal{O}_{M^\bullet}(U)^{PGL_r(\mathbb{C})}$ and $f_{ij}, h_{ij} \in \mathcal{O}_{M^\bullet}(U)$. Note that the Hamiltonian vector fields of $A_{ij;d+1}$ $(1 \le i,j \le r)$ are proportional to the vector fields $X_{[E_{ji}]}$ $(1 \le i,j \le r)$ which generate the infinitesimal actions corresponding to $[E_{ji}] \in \mathrm{Lie}\, PGL_r(\mathbb{C})$:

$$\{A_{ij;d+1}, *\} = -\sigma_{d+2} X_{[E_{ji}]} \qquad (1 \le i,j \le r).$$

Note also that $PGL_r(\mathbb{C})$-invariant functions f, h vanish when one applies $X_{[E_{ji}]}$ as derivations. Combining these facts, we obtain

$$\{F, H\} = \{f, h\} + \sum_{1 \le i,j \le r} A_{ij;d+1} l_{ij},$$

where $l_{ij} \in \mathcal{O}_{M^\bullet}(U)$. Since (2) is G-invariant, $\{f,h\} \in \mathcal{O}_{M^\bullet}^G$. Thus we have

$$\{F,H\} \in \mathcal{N}(U), \text{ and } \alpha\big(\{F,H\}\big) = \alpha\big(\{f,h\}\big).$$

Therefore $\mathcal{N}$ is a subalgebra of $\mathcal{O}_{M^\bullet}$ with respect to (2) and this Poisson algebra structure induces a Poisson algebra structure on $\mathcal{N}/\mathcal{N} \cap \mathcal{I}_M$. ■

Remark 1. The Poisson structure constructed in [1] corresponds to the case when $\phi(x)$ is monic of degree $d+2$ and has only simple roots. With such $\phi(x)$, the Poisson structure (2) on $M_r(S_{d+1})$ is equivalent to the canonical Poisson structure on $M_r(\mathbb{C})^{d+2}$, on which the discussion in [1] is based. See Appendix A for the explicit correspondence.

2.3 Multi-Hamiltonian structure

We define a family of Poisson structures on $\mathcal{M}$:

Definition 2. For $\phi(x) \in S_{d+2}$, $\{\cdot,\cdot\}_\phi : \mathcal{O}_\mathcal{M} \times \mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M}$ denotes the Poisson structure on $\mathcal{M}$ obtained in Proposition 1. For $0 \le i \le d+2$, we write $\{\cdot,\cdot\}_i := \{\cdot,\cdot\}_\phi$ with $\phi(x) = x^i$.

By construction, the Poisson structures are compatible:

$$\{\cdot,\cdot\}_{c_1\phi_1+c_2\phi_2} = c_1\{\cdot,\cdot\}_{\phi_1} + c_2\{\cdot,\cdot\}_{\phi_2} \qquad (c_1, c_2 \in \mathbb{C}, \phi_1(x), \phi_2(x) \in S_{d+2}).$$

Define the $PGL_r(\mathbb{C})$-invariant functions $H_i^{(k)}$ $(1 \le k \le r,\ 0 \le i \le kd)$ on $M_r(S_d)$ by

$$\frac{1}{k}\operatorname{Tr} A(x)^k = \sum_{i=0}^{kd} H_i^{(k)} x^i \qquad \text{for} \quad A(x) \in M_r(S_d).$$

Lemma 1. (Cf. [1, proposition in (5.2)].) *The Hamiltonian vector field of* $H_j^{(k)}$ $(1 \le k \le r,$ $0 \le j \le dk)$ *with respect to the Poisson structure* $\{\cdot,\cdot\}_\phi$ *is related to the vector fields* (1) *as follows:*

$$\{H_j^{(k)}, *\}_\phi = \sum_{i=0}^{\min(j,d+2)} \sigma_i\, \eta_* Y_{j-i}^{(k-1)}.$$

In particular, $H_j^{(1)}$ $(0 \le j \le d)$ *are Casimir functions.*

Proof. By direct calculation, we can show that for each $\phi(x) \in S_{d+2}$ and $k \ge 1$, the Hamiltonian vector field of $\frac{1}{k}\operatorname{Tr}A(a)^k$ $(a \in \mathbb{P}^1)$ on $M_r(S_{d+1})$ with respect to the Poisson structure (2) is

$$\frac{\phi(a)}{x-a}[A(x), A(a)^{k-1}].$$

It is easy to show that this is tangent to $M_r(S_d)$ and that its restriction to $M_r(S_d)$ is

$$\frac{\phi(a)}{x-a}[A(x), A(a)^{k-1}] = \sum_{i=0}^{d+2} \sum_{j=0}^{d(k-1)-1} \sigma_i a^{i+j} Y_j^{(k-1)}(A(x)). \tag{3}$$

By Proposition 1, the corresponding Hamiltonian vector field is given by a push forward of (3) by η. Comparing the coefficients of powers of a, we obtain the lemma. ■

Theorem 1. *(i) Each $\eta_* Y_j^{(k)}$ is a multi-Hamiltonian vector field with respect to the Poisson structures $\{\cdot,\cdot\}_i$ $(i = 0,\ldots,d+2)$:*

$$\eta_* Y_j^{(k)} = \{H_j^{(k+1)}, *\}_0 = \{H_{j+i}^{(k+1)}, *\}_i$$

for $1 \le k \le r-1$ and $0 \le j \le kd-2$.

(ii) With respect to $\{\cdot,\cdot\}_i$ $(0 \le i \le d+2)$, $H_0^{(k)},\ldots,H_{i-1}^{(k)}$ and $H_{d(k-1)+i-1}^{(k)},\ldots,H_{dk}^{(k)}$ $(1 \le k \le r)$ are Casimir functions.

Proof. By Lemma 1, we obtain

$$\begin{aligned}
&\{H_j^{(k+1)}, *\}_i = \eta_* Y_{j-i}^{(k)} && \text{for} \quad i \le j \le dk+i-1,\\
&\{H_j^{(k+1)}, *\}_i = 0 && \text{for} \quad 0 \le j \le i-1 \quad \text{and} \quad dk+i \le j \le d(k+1).
\end{aligned}$$

Moreover, $\eta_* Y_{dk-1}^{(k)} = 0$ since $Y_{dk-1}^{(k)}$ is tangent to $PGL_r(\mathbb{C})$-orbits by the definition (1). This proves the theorem. ■

2.4 Poisson structure for representatives of $\mathcal{M}(2,d)$

In this subsection, we explicitly write down the Poisson structure $\{\cdot,\cdot\}_\phi$ in the case of $r = 2$. We also discuss how this is related to the known Poisson structures on the Mumford system.

Consider the subspace $\mathcal{S}_\infty \subset M_2(S_d)$ defined by

$$\mathcal{S}_\infty = \left\{ S(x) = \begin{pmatrix} v_d & 0 \\ 1 & 0 \end{pmatrix} x^d + \begin{pmatrix} v_{d-1} & u_{d-1} \\ w_{d-1} & 0 \end{pmatrix} x^{d-1} + \begin{pmatrix} v_{d-2} & u_{d-2} \\ w_{d-2} & t_{d-2} \end{pmatrix} x^{d-2} + \cdots \;\middle|\; u_{d-1} \neq 0 \right\}.$$

In Section 4, we will see that $\mathcal{S}_\infty$ is a space of representatives for M_∞ which is an open subset of

$$M_{2d} = \left\{ A(x) \in M_2(S_d) \mid H_{2d}^{(2)} = 0 \right\}.$$

Lemma 2. *If $\phi(x) \in S_{d+1}$, (2) induces a Poisson structure on $\mathcal{S}_\infty$.*

Proof. In this proof we write M for $M_2(S_d)$. By Proposition 1, we have the Poisson algebra structure $\{\cdot,\cdot\}_\phi$ on the sheaf $\mathcal{O}_M^G$. Moreover, $H_{2d}^{(2)}$ is its Casimir function since $\deg \phi \le d+1$ (Theorem 1). Therefore the Poisson algebra structure induces that on $\mathcal{O}_M^G / \mathcal{O}_M^G \cap \mathcal{I}_{M_{2d}}$, where $\mathcal{I}_{M_{2d}}$ is the ideal sheaf of M_{2d} in $M = M_2(S_d)$. Thus (2) induces the Poisson structure on $M_\infty / G \cong \mathcal{S}_\infty$. ■

By a direct calculation (cf. proof of Proposition 4), we obtain the next result.

Proposition 2. *For $\phi(x) = \sigma_{d+1} x^{d+1} + \cdots + \sigma_0 \in S_{d+1}$, the Poisson structure $\{\cdot,\cdot\}_\phi$ is written as follows*

$$\begin{aligned}
\{S(x) \overset{\otimes}{,} S(y)\}_\phi = {}& \phi(y)[r(x,y), S(x) \otimes \mathbb{I}_2] - \phi(x)[\bar{r}(x,y), \mathbb{I}_2 \otimes S(y)] \\
& + [K(x,y), S(x) \otimes \mathbb{I}_2] - [\bar{K}(x,y), \mathbb{I}_2 \otimes S(y)],
\end{aligned} \tag{4}$$

where

$$\{S(x) \overset{\otimes}{,} S(y)\}_\phi = \sum_{1 \le i,j,k,l \le r} E_{ij} \otimes E_{kl} \, \{S_{ij}(x), S_{kl}(y)\}_\phi$$

and

$$\begin{aligned}
&r(x,y) = \frac{1}{x-y}\mathbb{P}_2 + \frac{1}{u_{d-1}} \begin{pmatrix} v_d & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
&\bar{r}(x,y) = \mathbb{P}_2 \cdot r(y,x) \cdot \mathbb{P}_2, \\
&K(x,y) = \frac{1}{u_{d-1}} \begin{pmatrix} y v_d \sigma_{d+1} & -u_{d-1}\sigma_{d+1} \\ -(w_{d-1}-y)\sigma_{d+1} + \sigma_d & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & S_{12}(y) \\ -S_{21}(y) & 0 \end{pmatrix}, \\
&\bar{K}(x,y) = \mathbb{P}_2 \cdot K(y,x) \cdot \mathbb{P}_2, \qquad \mathbb{P}_2 = \sum_{1 \le i,j \le 2} E_{ij} \otimes E_{ji}.
\end{aligned}$$

We write $F_j^{(1)}$ ($j = 0, \dots, d-2$) for the vector field on $\mathcal{S}_\infty$ induced by $\eta_* Y_j^{(1)}$. As a consequence of Theorem 1 and Proposition 2 we obtain

Corollary 1. *Each $F_j^{(1)}$ ($j = 0, \dots, d-2$) is the multi-Hamiltonian vector field with respect to the Poisson structure* (4)*. They are written as the Lax form:*

$$\sum_{j=0}^{d-2} y^j F_j^{(1)}(S(x)) = \frac{1}{y^i}\{H^{(2)}(y), S(x)\}_i = \left[S(x), \frac{1}{x-y}S(y) + \frac{S_{12}(y)}{u_{d-1}} \begin{pmatrix} v_d & 0 \\ 1 & 0 \end{pmatrix}\right], \tag{5}$$

for $i = 0, \dots, d+1$.

Now we derive a Poisson structure of the Mumford system from (4). The phase space $\mathcal{S}_{\mathrm{Mum}}$ of the Mumford system is the subspace of $\mathcal{S}_\infty$ defined as

$$\mathcal{S}_{\mathrm{Mum}} = \{S(x) \in \mathcal{S}_\infty \mid \operatorname{Tr} S(x) = 0, u_{d-1} = 1\}.$$

Lemma 3. (4) *induces a Poisson structure on $\mathcal{S}_{\mathrm{Mum}}$ if $\sigma_{d+1} = 0$.*

Proof. When $\deg \phi(x) \le d$, $H_{2d-1}^{(2)} = u_{d-1}$ is a Casimir of $\{\cdot,\cdot\}_\phi$ by Theorem 1. Therefore (4) induces a Poisson structure on $\mathcal{S}_{\mathrm{Mum}}$. ■

This is the same as the Poisson structure in [8, § 5.1]. The Poisson structures in [3, (4)] and [7] correspond to the case $\sigma_{d+1} = \sigma_d = 0$ and the case $\phi(x) = x$ respectively. The formula (5) reduces to the Lax form for the Mumford system [3, (7)].

3 Multi-Hamiltonian structure on the Beauville–Vanhaecke system

3.1 The Beauville–Vanhaecke system

Following [5], we define the set $M'(r,d)$ and the group G_r as

$$\begin{aligned}
&M'(r,d) = \left\{ A(x) \in M_r(\mathbb{C}[x]) \;\middle|\; \begin{matrix} A(x)_{11} \in S_d, & A(x)_{1j} \in S_{d+1}, \\ A(x)_{i1} \in S_{d-1}, & A(x)_{ij} \in S_d, \end{matrix} \quad (2 \le i,j \le r) \right\}, \\
&G_r = \left\{ g(x) = \begin{pmatrix} 1 & {}^t\vec{b}_1 x + {}^t\vec{b}_0 \\ 0 & B \end{pmatrix} \;\middle|\; B \in GL_{r-1}(\mathbb{C}), \quad \vec{b}_1, \vec{b}_0 \in \mathbb{C}^{r-1} \right\}.
\end{aligned}$$

Here we use the notation such as $\vec{b}$ for a column vector and ${}^t\vec{b}$ for a row vector. The group G_r acts on $M'(r,d)$ by conjugation. Let $\psi : M'(r,d) \to V(r,d)$ be the map $\psi(A(x)) = \det(y\mathbb{I} - A(x))$ and define

$$M'(r,d)_{ir} = \{A(x) \in M'(r,d) \mid \text{the spectral curve } C_{\psi(A(x))} \text{ is irreducible}\}.$$

The G_r-action is free on $M_r(S_d)_{ir}$ [5, Lemma 2.6]. Let $\eta' : M'(r,d)_{ir} \to M'(r,d)_{ir}/G_r$ be the quotient map. The phase space of the Beauville–Vanhaecke system is $\mathcal{M}'(r,d) = M'(r,d)_{ir}/G_r$. It was shown that if $P \in V(r,d)$ defines a smooth spectral curve C_P, then $\psi^{-1}(P)/G_r$ is isomorphic to the complement of the intersection of r-translates of the theta divisor in $\mathrm{Pic}^g(C_P)$ [5, Theorem 2.8].

Define the vector fields $Y_i^{(k)}$ on $M'(r,d)$ by

$$\sum_{i=0}^{kd} a^i Y_i^{(k)}(A(x)) = \frac{1}{x-a}[A(x), A(a)^k], \qquad \text{for} \quad k = 1, \dots, r-1. \tag{6}$$

It was shown that $\eta'_* Y_i^{(k)}$ generate the g-dimensional space of translation invariant vector fields on $\mathrm{Pic}^g(C_P)$[1].

3.2 Poisson structure

We equip $\mathcal{M}'(r,d)$ with a family of Poisson structures, extending the results in [5, § 3]. The key idea is that (2) induces the Poisson structure on $\mathcal{M}'(r,d)$ as in the case of the Beauville system. However, due to the technical difficulties arising from the G_r-action, we need a modification of the argument.

We use the following shorthand notations:

$$M' = M'(r,d), \qquad M'_{ir} = M'(r,d)_{ir}, \qquad \mathcal{M}' = \mathcal{M}'(r,d).$$

Let us write $A(x) \in M^\bullet$ and $A(x) \in M'$ as

$$A(x) = \begin{pmatrix} v(x) & {}^t\vec{w}(x) \\ \vec{u}(x) & t(x) \end{pmatrix},$$

where

$$v(x) = A_{11}(x), \qquad {}^t\vec{w}(x) = (A_{12}(x), \dots, A_{1r}(x)),$$
$$\vec{u}(x) = {}^t(A_{21}(x), \dots, A_{r1}(x)), \qquad t(x) = \big(A_{ij}(x)\big)_{2 \le i,j \le r}.$$

Let $\iota' : M' \hookrightarrow M^\bullet$ be the closed immersion and $\pi' : M^\bullet \to M'$ be the surjection:

$$A(x) = \begin{pmatrix} \sum_{k=0}^{d+1} v_k x^k & \sum_{k=0}^{d+1} {}^t\vec{w}_k x^k \\ \sum_{k=0}^{d+1} \vec{u}_k x^k & \sum_{k=0}^{d+1} t_k x^k \end{pmatrix} \mapsto \begin{pmatrix} \sum_{k=0}^{d} v_k x^k & \sum_{k=0}^{d+1} {}^t\vec{w}_k x^k \\ \sum_{k=0}^{d-1} \vec{u}_k x^k & \sum_{k=0}^{d} t_k x^k \end{pmatrix}.$$

Note that $\pi' \circ \iota' = id_{M'}$. Let γ be the composition of the morphisms:

$$\mathrm{Hom}((\mathcal{O}_{M^\bullet})^2, \mathcal{O}_{M^\bullet}) \xrightarrow{\pi'_*} \mathrm{Hom}((\pi'_*\mathcal{O}_{M^\bullet})^2, \pi'_*\mathcal{O}_{M^\bullet}) \to \mathrm{Hom}((\mathcal{O}_{M'})^2, \mathcal{O}_{M'}),$$

where the second morphism is given by

$$\Phi \mapsto \Big[(\mathcal{O}_{M'})^2 \xrightarrow{(\pi'^\#)^2} (\pi'_*\mathcal{O}_{M^\bullet})^2 \xrightarrow{\Phi} \pi'_*\mathcal{O}_{M^\bullet} \xrightarrow{\pi'_*(\iota'^\#)} \pi'_*\iota'_*\mathcal{O}_{M'} = O_{M'}\Big].$$

[1]Although $Y_i^{(k)}$ is not G_r-invariant, $\eta'_* Y_i^{(k)}$ is well-defined because the difference between $g(x)_* Y_i^{(k)}$ and $Y_i^{(k)}$ is tangent to G_r-orbits [5, Lemma 3.2].

We define $\{\cdot,\cdot\}^{BV} \in \mathrm{Hom}((\mathcal{O}_{M'})^2, \mathcal{O}_{M'})$ to be the image of (2) by γ. For the coordinate functions $A_{ij;k}$ of M', it is written explicitly as

$$\{A_{ij}(x), A_{kl}(y)\}^{BV} = \left[\delta_{i,l}\frac{A_{kj}(x)\phi(y)-\phi(x)A_{kj}(y)}{x-y} - \delta_{k,j}\frac{A_{il}(x)\phi(y)-\phi(x)A_{il}(y)}{x-y}\right]_{\le d_{ij},\le d_{kl}} \tag{7}$$

where $[\cdot]_{\le d_{ij},\le d_{kl}}$ means taking the terms whose degree in x is smaller or equal to d_{ij} and whose degree in y is smaller or equal to d_{kl}. Here $d_{ij}=d$, $d_{1j}=d+1$, $d_{i1}=d-1$ for $2\le i,j\le r$ and $d_{11}=d$.

Proposition 3. *The sheaf $\mathcal{O}_{M'}^{G_r}$ of G_r-invariant regular functions on M' is closed with respect to* (7)*. Moreover,* (7) *is a Poisson algebra structure on $\mathcal{O}_{M'}^{G_r}$.*

The proof is delegated to Subsection 3.3. As an immediate consequence of this proposition, we obtain the Poisson algebra structure on $\mathcal{O}_{M'_{ir}}^{G_r}$, which is equivalent to the Poisson structure on $\mathcal{M}'$.

Remark 2. The Poisson structure constructed in [5, § 3] corresponds to the case where $\phi(x)$ is monic of degree $d+2$ and has only simple roots.

3.3 Proof of Proposition 3

We prove Proposition 3 in the cases of $\deg\phi(x)=d+2$, $\deg\phi(x)=d+1$ and $\deg\phi(x)\le d$ separately.

The case of $\deg\phi(x)=d+2$**:** We equip $M^\bullet$ with the Poisson structure (2). We extend the G_r-action on M' to $M^\bullet$ as follows[2]:

$$G_r \ni g(x) : A(x) \mapsto \tilde{A}(x), \tag{8}$$

where $\tilde{A}(x) \in M_r(S_{d+1})$ is the matrix uniquely determined by

$$g(x)^{-1}A(x)g(x) = \tilde{A}(x) + \phi(x)\hat{A}(x), \qquad \hat{A}(x)\in M_r(S_1). \tag{9}$$

By direct calculation, we can show that the Poisson structure (2) is invariant with respect to this G_r-action.

Let $\mathcal{I}_{M'}$ be the ideal sheaf of ι', and set $\mathcal{N}' := {\alpha'}^{-1}(\iota'_*\mathcal{O}_{M'}^{G_r}) \subset \mathcal{O}_{M^\bullet}$ by writing α' for the natural projection $\mathcal{O}_{M^\bullet} \to \mathcal{O}_{M^\bullet}/\mathcal{I}_{M'} = \iota'_*\mathcal{O}_{M'}$.

Lemma 4. *(1) $\mathcal{N}'$ is a Poisson subalgebra of $\mathcal{O}_{M^\bullet}$.*

(2) The Poisson algebra structure of (1) induces that on $\mathcal{N}'/\mathcal{N}'\cap\mathcal{I}_{M'}$, hence on $\mathcal{O}_{M'}^{G_r}$. Moreover, it is given by (7)*.*

Proof. Note that for $f\in\mathcal{O}_{M^\bullet}(U)$ where U is any open subset,

$$\begin{aligned}
&\sum_{i=1}^{r}\{A_{ii;d+1}, f\} = 0,\\
&\{A_{ij;d+1}, f\} = -\sigma_{d+2}X_{E_{ji}}f \qquad (2\le i,j\le r),\\
&\{A_{i1;d+1}, f\} = -\sigma_{d+2}X_{E_{1i}}f, \qquad \{A_{i1;d}, f\} = -\sigma_{d+2}X_{E'_{1i}}f - \sigma_{d+1}X_{E_{1i}}f \qquad (2\le i\le r).
\end{aligned}$$

[2] In the case $\phi(x)$ has only simple roots, this action is the same as the one used in [5]. See Appendix B for a proof.

Here $X_{E_{ji}}$, $X_{E_{1i}}$, $X_{E'_{1i}}$ are the vector fields generating the infinitesimal actions corresponding to E_{ji}, E_{1i}, $E'_{1i} = xE_{1i} \in \operatorname{Lie} G_r$:

$$\begin{aligned} &X_{E_{ij}}(A(x)) = [A(x), E_{ij}] && (2 \le i, j \le r),\\ &X_{E_{1j}}(A(x)) = [A(x), E_{1j}] && (2 \le j \le r),\\ &X_{E'_{1j}}(A(x)) = \left[xA(x) - \frac{\sigma_{d+1}}{\sigma_{d+2}}A^{d+1}, E_{1j}\right] && (2 \le j \le r). \end{aligned}$$

By the same argument as that of Proposition 1, we can show that $\mathcal{N}'$ is a Poisson subalgebra of $\mathcal{O}_{M'}$ and that this induces a Poisson algebra structure on $\mathcal{N}'/\mathcal{N}' \cap I_{M'}$, hence on $\mathcal{O}_{M'}^{G_r}$. By construction, the Poisson algebra structure on $\mathcal{O}_{M'}^{G_r}$ coincides with the restriction of (7) to $\mathcal{O}_{M'}^{G_r}$. ■

The Case of $\deg \phi(x) = d + 1$**:** For an open subset U of M', a function $F \in \mathcal{O}_{M'}^{G_r}(U)$ is characterized by the condition $X_{E_{ij}}F = X_{E_{1j}}F = X_{E'_{1j}}F = 0$ $(2 \le i, j \le r)$ where $X_{E_{ij}}$, $X_{E_{1j}}$, $X_{E'_{1j}}$ are the infinitesimal action on M'corresponding to $E_{ij}, E_{1j}, E'_{1j} \in \operatorname{Lie} G_r$:

$$\begin{aligned} &X_{E_{ij}}(A(x)) = [A(x), E_{ij}] && (2 \le i, j \le r),\\ &X_{E_{1j}}(A(x)) = [A(x), E_{1j}] && (2 \le j \le r),\\ &X_{E'_{1j}}(A(x)) = [xA(x), E_{1j}] && (2 \le j \le r). \end{aligned} \tag{10}$$

Using this fact, we can show that $\mathcal{O}_{M'}^{G_r}$ is closed with respect to $\{\cdot,\cdot\}^{BV}$. We can also show that the Jacobi identity holds on $\mathcal{O}_{M'}^{G_r}$ although it does not on $\mathcal{O}_{M'}$. Thus $\{\cdot,\cdot\}^{BV}$ is a Poisson algebra structure on $\mathcal{O}_{M'}^{G_r}$.

The Case of $\deg \phi(x) \le d$**:** When $\deg \phi(x) \le d$, $\{\cdot,\cdot\}^{BV}$ satisfies the Jacobi identity on $\mathcal{O}_{M'}$ since it gives $\mathcal{O}_{M'}$ a Poisson subalgebra structure. By using the action (10) of $\operatorname{Lie} G_r$ on M', we can show that $\mathcal{O}_{M'}^{G_r}$ is closed with respect to $\{\cdot,\cdot\}^{BV}$ as in the case of $\deg \phi(x) = d+1$. Therefore $\{\cdot,\cdot\}^{BV}$ is a Poisson algebra structure on $\mathcal{O}_{M'}^{G_r}$.

3.4 Multi-Hamiltonian structure

Definition 3. For $\phi(x) \in S_{d+2}$, $\{\cdot,\cdot\}'_\phi : \mathcal{O}_{\mathcal{M}'} \times \mathcal{O}_{\mathcal{M}'} \to \mathcal{O}_{\mathcal{M}'}$ denotes the Poisson structure defined in Proposition 3. For $0 \le i \le d+2$, we write $\{\cdot,\cdot\}'_i := \{\cdot,\cdot\}'_\phi$ with $\phi(x) = x^i$.

By construction, these Poisson structures are compatible in the sense of Subsection 2.3.

Define the G_r-invariant functions $H_i^{(k)}$ $(1 \le k \le r,\ 0 \le i \le kd)$ on $M'(r,d)$ by

$$\frac{1}{k}\operatorname{Tr} A(x)^k = \sum_{i=0}^{kd} H_i^{(k)} x^i \qquad \text{for} \quad A(x) \in M'(r,d).$$

Lemma 5. (Cf. [5, Lemma 3.10].) *The Hamiltonian vector field of* $H_j^{(k)}$ $(1 \le k \le r, 0 \le i \le dk)$ *is related to the vector fields* (6) *as*

$$\{H_j^{(k)}, *\}'_\phi = \sum_{i=0}^{\min(j,d+2)} \sigma_i \eta_* Y_{j-i}^{(k-1)}.$$

In particular, $H_j^{(1)}$ $(0 \le j \le d)$ *are Casimir functions of* $\{\cdot,\cdot\}'_\phi$.

This lemma can be proved in the same way as Lemma 1 using Proposition 3. From Lemma 5, we obtain the following theorem similar to Theorem 1.

Theorem 2. *(i) Each $\eta_* Y_j^{(k)}$ is a multi-Hamiltonian vector field with respect to the Poisson structures $\{\cdot,\cdot\}'_i$ $(i = 0,\dots,d+2)$:*

$$\eta_* Y_j^{(k)} = \{H_j^{k+1}, *\}'_0 = \{H_{j+i}^{k+1}, *\}'_i$$

for $1 \le k \le r-1$ and $0 \le j \le kd-2$.

(ii) With respect to $\{\cdot,\cdot\}'_i$ $(0 \le i \le d+2)$, $H_0^{(k)},\dots,H_{i-1}^{(k)}$ and $H_{d(k-1)+i-1}^{(k)},\dots,H_{dk}^{(k)}$ $(1 \le k \le r)$ are Casimir functions.

3.5 Poisson structure for representatives of $\mathcal{M}'(2,d)$

We present the Poisson structure $\{\cdot,\cdot\}'_\phi$ in the case $r = 2$ on the space of representatives $\mathcal{S}'_\infty$ of $\mathcal{M}'(2,d)$ constructed in [5, § 4].

The space $\mathcal{S}'_\infty$ is written as

$$\mathcal{S}'_\infty = \left\{ S(x) = \begin{pmatrix} 0 & w_{d+1} \\ 0 & 0 \end{pmatrix} x^{d+1} + \begin{pmatrix} v_d & w_d \\ 0 & 0 \end{pmatrix} x^d + \begin{pmatrix} v_{d-1} & w_{d-1} \\ 1 & 0 \end{pmatrix} x^{d-1} + \text{lower terms in } x \right\}.$$

We obtain the following result by a direct calculation.

Proposition 4. *For $\phi(x) = \sigma_{d+2}x^{d+2} + \sigma_{d+1}x^{d+1} + \dots + \sigma_0$, the Poisson bracket $\{\cdot,\cdot\}'_\phi$ is written as follows:*

$$\begin{aligned} \{S(x) \overset{\otimes}{,} S(y)\}'_\phi &= \phi(y)[r(x,y), S(x) \otimes \mathbb{I}_2] - \phi(x)[\bar r(x,y), \mathbb{I}_2 \otimes S(y)] \\ &\quad + B(x,y)[K(x,y), S(x) \otimes \mathbb{I}_2] - B(x,y)[\bar K(x,y), \mathbb{I}_2 \otimes S(y)], \end{aligned} \tag{11}$$

where

$$\begin{aligned} r(x,y) &= \frac{1}{x-y}\mathbb{P}_2 + \begin{pmatrix} v_d & A(x+y) \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \bar r(x,y) = \mathbb{P}_2 \cdot r(y,x) \cdot \mathbb{P}_2, \\ K(x,y) &= \begin{pmatrix} 0 & B(x,y) \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -S_{12}(y) \\ S_{21}(y) & 0 \end{pmatrix}, \qquad \bar K(x,y) = \mathbb{P}_2 \cdot K(y,x) \cdot \mathbb{P}_2, \\ A(x) &= w_{d+1}(x - u_{d-2}) + w_d, \\ B(x,y) &= \sigma_{d+2}(x^2 + y^2 + xy + (u_{d-2} - x - y)u_{d-2} - u_{d-3}) + \sigma_{d+1}(u_{d-2} - x - y) + \sigma_d. \end{aligned}$$

Proof. In this proof, we write $A(x) \in M'$ as

$$A(x) = \begin{pmatrix} \sum_{i=0}^{d} \tilde v_i x^i & \sum_{i=0}^{d+1} \tilde w_i x^i \\ \sum_{i=0}^{d-1} \tilde u_i x^i & \sum_{i=0}^{d} \tilde t_i x^i \end{pmatrix}$$

and regard $\tilde v_i$ $(0 \le i \le d)$, $\tilde w_i$ $(0 \le i \le d+1)$, $\tilde u_i$ $(0 \le i \le d-1)$, $\tilde t_i$ $(0 \le i \le d)$ as coordinate functions of M'. (Here we use $\tilde{}$ to distinguish from v_i, w_i, u_i, t_i which we use as coordinates of $\mathcal{S}'_\infty$.) The bracket $\{\cdot,\cdot\}^{BV}$ (7) among $\tilde v_i,\dots,\tilde t_i$ are explicitly written as follows:

$$\begin{aligned} &\{\tilde v(x), \tilde v(y)\} = \{\tilde u(x), \tilde u(y)\} = \{\tilde w(x), \tilde w(y)\} = \{\tilde t(x), \tilde t(y)\} = \{\tilde v(x), \tilde t(y)\} = 0, \\ &\{\tilde v(x), \tilde u(y)\} = -\{\tilde t(x), \tilde u(y)\} = \frac{\tilde u(x)\phi(y) - \phi(x)\tilde u(y)}{x-y} \\ &\qquad + \sigma_{d+2}\tilde u(y)x^{d+1} + \sigma_{d+2}\tilde u(x)y^{d+1} - (\sigma_{d+1} - \sigma_{d+2}x)\tilde u(x)y^d, \end{aligned}$$

$$\begin{aligned}
\{\tilde v(x), \tilde w(y)\} &= -\{\tilde t(x), \tilde w(y)\} = -\frac{\tilde w(x)\phi(y) - \phi(x)\tilde w(y)}{x-y} - \sigma_{d+2}\tilde w(y)x^{d+1},\\
\{\tilde u(x), \tilde w(y)\} &= \frac{(\tilde v(x) - \tilde t(x))\phi(y) - \phi(x)(\tilde v(y) - \tilde t(y))}{x-y}\\
&\quad + \sigma_{d+2}(\tilde v(y) - \tilde t(y))x^{d+1} - (\sigma_{d+1} - \sigma_{d+2}y)(\tilde v(y) - \tilde t(y))x^d,
\end{aligned} \tag{12}$$

where $\tilde v(x) := \sum_{i=0}^{d} \tilde v_i x^i$ and so on.

Similarly let v_i $(0 \le i \le d)$, w_i $(0 \le i \le d+1)$, u_i $(0 \le i \le d-2)$, t_i $(0 \le i \le d-2)$ denote coordinate functions of $\mathcal{S}'_\infty$. Let $(b_1, b_0, c) \in \mathbb{C}^2 \times \mathbb{C}^*$ be the following coordinate functions of G_r:

$$G_r \ni \begin{pmatrix} 1 & b_1 x + b_0 \\ 0 & c \end{pmatrix}.$$

In the neighborhood of $\mathcal{S}'_\infty$, (v_i $(0 \le i \le d)$, w_i $(0 \le i \le d+1)$, u_i $(0 \le i \le d-2)$, t_i $(0 \le i \le d-2)$, b_0, b_1, c) forms a local coordinate system of M'. The transformation between the two coordinate systems is given by

$$\begin{pmatrix} \tilde v(x) & \tilde w(x) \\ \tilde u(x) & \tilde t(x) \end{pmatrix} = \begin{pmatrix} 1 & b_1 x + b_0 \\ 0 & c \end{pmatrix}^{-1} \begin{pmatrix} v(x) & w(x) \\ u(x) & t(x) \end{pmatrix} \begin{pmatrix} 1 & b_1 x + b_1 \\ 0 & c \end{pmatrix}. \tag{13}$$

Substituting the RHS of (13) into (12) and using the Leibniz rule, we obtain the system of equations for brackets among ($v(x)$, $w(x)$, $u(x)$, $t(x)$, b_0, b_1, c). Solving this and restricting to $\mathcal{S}'_\infty$ (i.e. setting $b_0 = b_1 = 0$, $c = 1$), we arrive at the result of Proposition 4. ■

As in the case of the Beauville system, we write $F_j^{(1)}$ $(j = 0, \dots, d-2)$ for the vector field on $\mathcal{S}'_\infty$ induced by $\eta_* Y_j^{(1)}$. From Theorem 2 and Proposition 4 we obtain

Corollary 2. *Each $F_j^{(1)}$ $(j = 0, \dots, d-2)$ is the multi-Hamiltonian vector field with respect to the Poisson structure* (11). *They are written as the Lax form:*

$$\begin{aligned}
\sum_{j=0}^{d-2} y^j F_j^{(1)}\big(S(x)\big) &= \frac{1}{y^i}\{H^{(2)}(y), S(x)\}_i\\
&= \left[S(x), \frac{1}{x-y}S(y) + S_{21}(y)\begin{pmatrix} 0 & A(x+y) \\ 0 & -v_d \end{pmatrix}\right],
\end{aligned}$$

for $i = 0, \dots, d+2$.

We remark that this Lax form already appeared in [5, (4.9)] for general r.

In closing this subsection, we discuss the Poisson structure on the even Mumford system. The phase space of the even Mumford system is given by

$$\{S(x) \in \mathcal{S}'_\infty \mid \operatorname{Tr} S(x) = 0, w_{d+1} = 1\}.$$

Lemma 6. *If $\sigma_{d+2} = 0$,* (11) *gives a Poisson structure on the phase space of the even Mumford system.*

Proof. By Theorem 2, $w_{d+1} = H_{2d}^{(2)}$ is Casimir of $\{\cdot,\cdot\}'_\phi$ if $\deg \phi(x) \le d+1$. Therefore in such a case, (11) induces a Poisson algebra structure on $\mathcal{O}(\mathcal{S}'_\infty)/H_{2d}^{(2)}\mathcal{O}(\mathcal{S}'_\infty)$. ■

The Poisson structure in [3] corresponds to the case $\sigma_{d+2} = \sigma_{d+1} = \sigma_d = 0$.

4 Representatives of the Beauville system

First we introduce some notations. Let us define a subset $M_{\rm reg}$ of $M_r(\mathbb{C})$:

$$M_{\rm reg} = \{A \in M_r(\mathbb{C}) \,|\, \deg(\text{the minimal polynomial of } A) = r\}.$$

For $A \in M_r(\mathbb{C})$, $A \in M_{\rm reg}$ is equivalent to the condition that only one Jordan block corresponds to each eigenvalue of A. For $A \in M_{\rm reg}$, let $\alpha_1, \dots, \alpha_k$ $(k < r)$ be the distinct eigenvalues and $\nu_1, \dots, \nu_k$ be the size of the corresponding Jordan blocks. Define the subspace of $\mathbb{C}^r$ as

$$W_{\alpha_i:j} = \{\vec{u} \in \mathbb{C}^r \mid (A - \alpha_i \mathbb{I}_r)^j \vec{u} = 0\}$$

for $0 \le i \le k$, $j \in \mathbb{Z}_{\ge 0}$. The spaces $W_{\alpha_i:1}$ and $W_{\alpha_i:\nu_i}$ are respectively the eigenspace and the generalized eigenspace of A. There is the filtration

$$W_{\alpha_i:\nu_i} \supset W_{\alpha_i:\nu_i-1} \supset \cdots \supset W_{\alpha_i:1} \supset W_{\alpha_i:0} = \{\vec{0}\}.$$

By the assumption of A, $\dim(W_{\alpha_i:j}/W_{\alpha_i:j-1}) = 1$ for all α_i and $j = 1, \dots, \nu_i$. We fix a base $\vec{v}_{\alpha_i}(A)$ of $W_{\alpha_i:1}$. Let Π_{α_i} be the projection map $\Pi_{\alpha_i} : \mathbb{C}^r \to W_{\alpha_i:\nu_i}$, and define

$$V(A) = \{\vec{u} \in \mathbb{C}^r \mid \Pi_{\alpha_i}(\vec{u}) \notin W_{\alpha_i:\nu_i-1} \text{ for } i = 0, \dots, k\}.$$

Now we introduce the subspaces M_∞ and $\mathcal{S}_\infty$ of $M_r(S_d)$:

$$\begin{aligned} M_\infty &= \Big\{A(x) = \sum_{k=0}^{d} A_k x^k \in M_r(S_d) \,\Big|\, A_d \in M_{\rm reg},\ \det A_d = 0,\ A_{d-1}\vec{v}_0(A_d) \in V(A_d)\Big\}, \\ \mathcal{S}_\infty &= \Big\{A(x) \in M_r(S_d) \,|\, A(x) = \omega x^d + \rho x^{d-1} + \text{lower terms in } x,\ \omega \in \Omega,\ \rho \in \mathcal{T}\Big\}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} \Omega &= \left\{ \begin{pmatrix} -\beta_1 & \cdots & -\beta_{r-1} & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \in M_r(\mathbb{C}) \,\middle|\, \beta_1, \cdots, \beta_{r-1} \in \mathbb{C} \right\} \\ \mathcal{T} &= \{\rho \in M_r(\mathbb{C}) \,|\, \rho_{1r} \neq 0,\ \rho_{jr} = 0 \text{ for } j = 2, \dots, r\}. \end{aligned}$$

The main result of this section is as follows:

Proposition 5. *(i)* $\mathcal{S}_\infty \subset M_\infty$.

(ii) The action of $PGL_r(\mathbb{C})$ *on* M_∞ *induces an isomorphism* $\mathcal{S}_\infty \times PGL_r(\mathbb{C}) \simeq M_\infty$. *Thus the space* $\mathcal{S}_\infty$ *is a set of representatives of* $M_\infty/PGL_r(\mathbb{C})$.

Remark 3. One can define $\mathcal{M}_c$ and $\mathcal{S}_c$ for $c \in \mathbb{C}$ as

$$\begin{aligned} M_c &= \Big\{A(x) \in M_r(S_d) \,\Big|\, A(c) \in M_{\rm reg},\ \det A(c) = 0,\ A'(c)\vec{v}_0(A(c)) \in V(A(c))\Big\}, \\ \mathcal{S}_c &= \{A(x) \in M_r(S_d) \,|\, A(x) = \omega + \rho(x-c) + \text{higher terms in } (x-c),\ \omega \in \Omega,\ \rho \in \mathcal{T}\Big\}. \end{aligned}$$

Then Proposition 5 also holds for $(M_c, \mathcal{S}_c)$.

Let us recall the following lemmas on linear algebra.

Lemma 7. *Let $A \in M_{\mathrm{reg}}$. For $\vec{u} \in \mathbb{C}^r$, the followings are equivalent*

(i) $\vec{u} \in V(A)$;

(ii) ${}^t\vec{v}_\alpha({}^tA) \cdot \vec{u} \neq 0$ for all eigenvalues α of A;

(iii) the vectors $\vec{u}, A\vec{u}, \ldots, A^{r-1}\vec{u}$ generate $\mathbb{C}^r$.

The proof is left for readers.

For $A \in M_{\mathrm{reg}}$, set

$$\xi_i(A) = A^i + \beta_1(A)A^{i-1} + \cdots + \beta_i(A)\,\mathbb{I}_r \in M_r(\mathbb{C}) \qquad (i = 1, \ldots, r-1),$$

where $\beta_1(A), \ldots, \beta_r(A)$ are coefficients of the characteristic polynomial of A, $\det(y\mathbb{I}_r - A) = y^r + \beta_1(A)y^{r-1} + \cdots + \beta_r(A)$.

Lemma 8. *Let $A \in M_{\mathrm{reg}}$ and $\vec{u} \in V(A)$.*

(i) The matrix $g(\vec{u}, A) = (\vec{u}, \xi_1(A)\vec{u}, \ldots, \xi_{r-1}(A)\vec{u}) \in M_r(\mathbb{C})$ is invertible. Moreover, it satisfies

$$g(\vec{u}, A)^{-1}Ag(\vec{u}, A) = \begin{pmatrix} -\beta_1(A) & \cdots & -\beta_{r-1}(A) & -\beta_r(A) \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

(ii) $g(\vec{u}, A)$ makes $B \in M_r(\mathbb{C})$ into the following form

$$g(\vec{u}, A)^{-1}Bg(\vec{u}, A) = \begin{pmatrix} * & \cdots & * & * \\ * & \cdots & * & 0 \\ \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & 0 \end{pmatrix},$$

if and only if $\vec{u}$ is an eigenvector of $B\xi_{r-1}(A)$. Moreover the $(1, r)$-th entry of the RHS is equal to the eigenvalue.

Proof. (i) The invertibility of g follows from Lemma 7. Another claim is checked by a direct computation.

(ii) Let $\tilde{B} = g(\vec{u}, A)^{-1}Bg(\vec{u}, A)$. If $\tilde{B}$ has the form of the RHS, we obtain $B\xi_{r-1}(A)\vec{u} = \tilde{B}_{1r}\vec{u}$ by comparing the r-th columns of Bg and $g\tilde{B}$. Conversely if $\vec{u}$ is an eigenvalue of $B\xi_{r-1}(A)$, then we see by direct calculation that $\tilde{B}_{1r}$ is equal to its eigenvalue and $\tilde{B}_{jr} = 0$ for $2 \leq j \leq r$. ■

Lemma 9. *Let $A \in M_{\mathrm{reg}}, B \in M_r(\mathbb{C})$ and assume that* $\det A = 0$.

(i) $\xi_{r-1}(A) = c\vec{v}_0(A) \otimes {}^t\vec{v}_0({}^tA)$ for some $c \in \mathbb{C}^\times$.

(ii) $B\vec{v}_0(A)$ is an eigenvector of $B\xi_{r-1}(A)$.

Proof. (i) By the assumption on A, the rank of $\xi_{r-1}(A)$ is one, and ξ_{r-1} satisfies $A\xi_{r-1}(A) = \xi_{r-1}(A)A = 0$. Thus $\xi_{r-1}(A)$ have to be written as $c\vec{v}_0(A) \otimes {}^t\vec{v}_0({}^tA)$ with some $c \in \mathbb{C}^\times$.

(ii) By (i), any $\vec{w} \in \mathbb{C}^r$ satisfy $\xi_{r-1}(A)\vec{w} = c\vec{v}_0(A)$ with some $c \in \mathbb{C}^\times$. By multiplying the both sides by B from the left and setting $\vec{w} = B\vec{v}_0(A)$, we see that $B\vec{v}_0(A)$ is an eigenvector of $B\xi_{r-1}(A)$. ■

Proofs of Proposition 5. (i) We write $S(x) = S_dx^d + S_{d-1}x^{d-1} + \cdots + S_0 \in \mathcal{S}_\infty$ as

$$S(x) = \begin{pmatrix} -\beta_1 & \cdots & -\beta_{r-1} & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} x^d + \begin{pmatrix} * & \cdots & * & \beta \\ * & \cdots & * & 0 \\ \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & 0 \end{pmatrix} x^{d-1} + \text{lower terms in } x, \qquad (15)$$

where $\beta \neq 0$. Then it is easy to see $\det S_d = 0$, and we can set $\vec{v}_0(S_d) = {}^t(0, \dots, 0, 1)$. A direct calculation shows that $\vec{u} := S_{d-1}\vec{v}_0(S_d) = {}^t(\beta, 0, \dots, 0)$ and S_d satisfy

$$\det(\vec{u}, S_d\vec{u}, \dots, S_d^{r-1}\vec{u}) = \det(\mathrm{diag}(\beta, \dots, \beta)) = \beta^r \neq 0.$$

Thus we see $\vec{u} \in V(S_d)$ due to Lemma 7, and the claim follows.

(ii) It is easy to see that M_∞ is invariant under the action of $PGL_r(\mathbb{C})$, thus the map

$$\mu : \mathcal{S}_\infty \times PGL_r(\mathbb{C}) \to M_\infty; (S(x), g) \mapsto gS(x)g^{-1}$$

is well-defined. In the following we show that μ is bijection.

First we show the surjectivity of μ. For $A(x) = A_d x^d + A_{d-1}x^{d-1} + \cdots + A_0 \in M_\infty$, set $g = g(A_{d-1}\vec{v}_0(A_d), A_d)$. By Lemma 7, g is invertible and $g^{-1}A_d g \in \Omega$ by Lemma 8(i). By Lemma 7 and 9, $A_{d-1}\vec{v}_0(A_d)$ is an eigenvector of $A_{d-1}\xi_{r-1}(A_d)$ belonging to a nonzero eigenvalue. Thus $g^{-1}A_{d-1}g \in \mathcal{T}$ by Lemma 8(ii). Consequently we obtain $S(x) = gA(x)g^{-1} \in \mathcal{S}_\infty$, i.e. $\mu(S(x), g) = A(x)$.

To check the injectivity of μ, we only have to check the following: for any $S(x) \in \mathcal{S}_\infty$, $g \in GL_r(\mathbb{C})$ satisfies $g^{-1}S(x)g \in \mathcal{S}_\infty$ only when g is a scalar matrix. When $S(x)$ is given by (15), we get

$$\xi_{r-1}(S_d) = {}^t(0, \dots, 0, 1) \cdot (1, \beta_1, \dots, \beta_{r-1}),$$
$$\big(\xi_k(S_d)\big)_{j1} = \delta_{k,j-1} \quad \text{for} \quad k = 1, \dots, r-2.$$

The first equation implies that $\big(S_{d-1}\xi_{r-1}(S_d)\big)_{ij} = \delta_{i,1}\delta_{j,1}\beta$. This matrix has only one nonzero eigenvalue β and the corresponding eigenvector is ${}^t(a, 0, \dots, 0)$ for some $a \in \mathbb{C}^\times$. By Lemma 8(ii), we only have to show that $g({}^t(a, 0, \dots, 0), S_d) = c\mathbb{I}_r$ for some $c \in \mathbb{C}^\times$. This follows from the second equation. ■

In closing, we give some remarks. The space M_∞ (14) is an affine subspace of

$$\{A(x) \in M_r(S_d) \,|\, H_{rd}^{(r)} = 0\}$$

which is the codimension one subspace of $\mathcal{M}(r, d)$. As a generalization of Lemma 2, we easily obtain the following:

Lemma 10. *If $\phi(x) \in S_{d+1}$, then* (2) *induces the Poisson structure on $M_\infty/PGL_r(\mathbb{C}) \simeq \mathcal{S}_\infty$.*

The space of representatives introduced by Donagi and Markman [2, Lemma 4.1] is a subspace of $\mathcal{S}_\infty$ (14) defined by

$$\{S(x) = S_d x^d + \cdots + S_0 \in \mathcal{S}_\infty \,|\, \mathrm{Tr}S(x) = 0,\ \beta_1(S_d) = \cdots = \beta_{r-1}(S_d) = 0\}.$$

The phase space discussed in [10, § 3.2] is obtained by removing the first condition in the above.

A On the Poisson structure (2)

Assume that $\phi(x)$ is a monic polynomial with only simple roots, $a_1, \dots, a_{d+2}$. Consider the following isomorphism φ [1, (5.6)]:

$$\varphi : M_r(S_{d+1}) \to M_r(\mathbb{C})^{\oplus d+2},\ A(x) \mapsto \big(c_1 A(a_1), \dots, c_{d+2}A(a_{d+2})\big), \tag{16}$$

where $c_\alpha = \prod_{\beta\neq\alpha} (a_\alpha - a_\beta)^{-1}$. The inverse of φ^{-1} is given by the Lagrange interpolation formula:

$$\varphi^{-1} : M_r(\mathbb{C})^{d+2} \to M_r(S_{d+1}), \qquad \big(A^{(1)}, \dots, A^{(d+2)}\big) \mapsto \sum_{\alpha=1}^{d+2} A^{(\alpha)} \prod_{\beta\neq\alpha} (x - a_\beta).$$

Then the pullback by φ of the canonical Poisson structure on $M_r(\mathbb{C})^{\oplus d+2}$:

$$\{A^{(\alpha)}_{ij}, A^{(\beta)}_{kl}\} = \delta_{\alpha,\beta}\big(\delta_{j,k}A^{(\alpha)}_{il} - \delta_{i,l}A^{(\alpha)}_{kj}\big) \qquad \text{for} \quad (A^{(1)}, \dots, A^{(d+2)}) \in M_r(\mathbb{C})^{\oplus d+2},$$

is equal to (2). This is easily checked if one uses the elementary identity

$$A(x)\phi(y) - \phi(x)A(y) = (y - x) \sum_{\alpha=1}^{d+2} A^{(\alpha)} \prod_{\mu\neq\alpha} (x - a_\mu)(y - a_\mu).$$

B On the G_r-action (8)

In the construction of Poisson structures in [5], the isomorphism φ given in (16) and the following G_r-action on $M_r(\mathbb{C})^{\oplus d+2}$ were used:

$$G_r \ni g(x) : \big(A^{(1)}, \dots, A^{(d+2)}\big) \mapsto \big(g(a_\alpha)^{-1} A^{(\alpha)}\, g(a_\alpha)\big)_{1\le\alpha\le d+2}, \tag{17}$$

where $a_\alpha \neq a_\beta$ if $\alpha \neq \beta$. We show that this action is compatible with the G_r-action (8) under the isomorphism φ when $\phi(x) = \prod_{\alpha=1}^{d+2} (x - a_\alpha)$.

From (17), we have

$$(\varphi^{-1} \circ g(x) \circ \varphi)A(x) = \sum_{\alpha=1}^{d+2} c_\alpha\, g(a_\alpha)^{-1} A(a_\alpha) g(a_\alpha) \prod_{\mu\neq\alpha} (x - a_\mu).$$

On the other hand, substituting $x = a_\alpha$ into (9), we have $\tilde{A}(a_\alpha) = g(a_\alpha)^{-1} A(a_\alpha) g(a_\alpha)$. Then expressing $\tilde{A}(x)$ by the Lagrange interpolation formula, we see that $\tilde{A}(x) = (\varphi^{-1} \circ g(x) \circ \varphi)A(x)$.

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