Non-Commutative Mechanics in Mathematical & in Condensed Matter Physics*

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Abstract. Non-commutative structures were introduced, independently and around the same time, in mathematical and in condensed matter physics (see Table 1). Souriau's construction applied to the two-parameter central extension of the planar Galilei group leads to the "exotic" particle, which has non-commuting position coordinates. A Berry-phase argument applied to the Bloch electron yields in turn a semiclassical model that has been used to explain the anomalous/spin/optical Hall effects. The non-commutative parameter is momentum-dependent in this case, and can take the form of a monopole in momentum space.

Key words: non-commutative mechanics; semiclassical models; Hall effect

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1 "Exotic" Galilean symmetry and mechanics in the plane

Central extensions first entered physics when Heisenberg realized that, in the quantum mechanics of a massive non-relativistic particle, the position and momentum operators did not commute. As a consequence, phase-space translations act only up-to-phase on the quantum Hilbert space. In more mathematical terms, it is not the [commutative] translation group itself, only its [non-commutative] 1-parameter central extension, the Heisenberg group, which is represented unitarily. Similarly, Galilean boosts act, for a massive non-relativistic system, only up-to phase. In other words, it is the 1-parameter central extension of the Galilei group, called the Bargmann group, that acts unitarily. True representations only arise for massless particles.

Are there further extension parameters? In $d \geq 3$ space dimensions, the Galilei group admits a 1-parameter central extension only [1]. The extension parameter, m, is identified with the physical mass. However, Lévy-Leblond [2] recognized that, in the plane, the Galilei group admits a second extension, highlighted by the non-commutativity of the Galilean boost generators,

$$[K_1, K_2] = i\kappa,$$

where κ is the new extension parameter. This has long been considered, however, a mere mathematical curiosity, as planar physics has itself been viewed a toy. The situation started to change around 1995, though, with the construction of physical models which realize this "exotic" symmetry [3, 4, 5, 6, 7, 8]. These models have the strange feature that the Poisson bracket of

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$$\{x_1, x_2\} = \frac{\kappa}{m^2} \equiv \theta.$$

Table 1. Exotic Galilean symmetry vs. semiclassical models with Berry term.

HIGH-ENERGY/MATH. PHYS.	CONDENSED MATTER PHYS.
1970 Lévy-Leblond: "exotic" planar Galilei group	1983 Laughlin: theory of FQHE
1995–97 Duval, Grigore, Brihaye, Lukierski mechanical models with exotic symmetry	1995–2000 Niu et al.: Berry term for semiclassical Bloch electron
2000–2001 Duval et al.: exotic particle in e.m. field & Hall effect; non-commutative mechanics	2002–2003 Jungwirth–Niu–MacDonald: Anomalous Hall effect
2004 Bérard, Mohrbach: momentum dependent monopole-type non-commutativity	2003 Fang et al.: monopole in momentum space in Anomalous Hall effect 2003 Murakami–Nagaosa–Zhang: spin-Hall effect
2000 Jackiw–Nair exotic structure from relativistic spin	2005 Sinova et al: observation of spin-Hall effect
2005 Duval et al: "SpinOptics"	2004 Onoda–Murakami–Nagaosa, Bliokh: Optical Magnus/Hall effect

2 The exotic model

In [3, 4, 5, 6, 9, 10, 11, 12] Souriau's "orbit method" [13] was used to construct a classical planar system associated with Lévy-Leblond's two-fold extended Galilean symmetry. It has an "exotic" symplectic form and a free Hamiltonian,

$$\Omega_0 = dp_i \wedge dx_i + \frac{1}{2}\theta\,\epsilon_{ij}\,dp_i \wedge dp_j, \qquad H_0 = \frac{\vec{p}^{\,2}}{2m}.$$

The associated (free) motions follow the usual straight lines; the "exotic" structure behaves, roughly, as spin: it enters the (conserved) boost and the angular momentum,

$$j = \epsilon_{ij}x_ip_j + \frac{\theta}{2}\,\vec{p}^{\,2}, \qquad K_i = mx_i - p_it + m\theta\,\epsilon_{ij}p_j.$$

The new terms are separately conserved, though. The new structure does not seem, hence, to lead to any new physics.

The situation changes dramatically, though, if the particle is coupled to a gauge field. Souriau's minimal coupling prescription [13] yields indeed

$$\Omega = \Omega_0 + eB\,dx_1 \wedge dx_2, \qquad H = H_0 + eV.$$

The associated Poisson bracket automatically satisfies the Jacobi identity; equations of motion read

$$m^*\dot{x}_i = p_i - em\theta\,\epsilon_{ij}E_j, \qquad \dot{p}_i = eE_i + eB\,\epsilon_{ij}\dot{x}_j,$$

where $\theta = k/m^2$ is the non-commutative parameter and

$$m^* = m(1 - e\theta B). \tag{1}$$

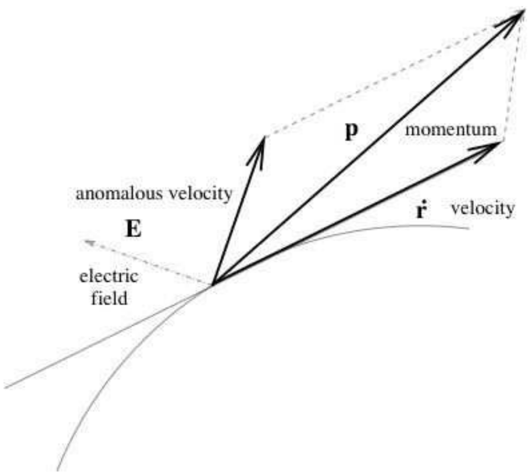

Figure 1. Due to the anomalous velocity term perpendicular to the electric field, the velocity and the momentum may be non-parallel.

The novel features, crucial for physical applications, are two-fold. They both concern the first equation in (2).

Firstly, the interplay between the exotic structure and the magnetic field yields the *effective mass* m^* in (1).

Secondly, the *anomalous velocity term*, perpendicular to the direction of the electric field, makes that velocity and momentum, $\dot{x}_i$ and p_i, are not parallel in general, cf. Fig. 1.

Such a possibility has been discarded by some high-energy physicists. However, it has been argued a long time ago [14, 15, 16], that *no first principle requires that velocity and momentum be proportional*, and that relaxing this restriction allows for perfectly consistent theories. This has been well-known in condensed matter physics, where the velocity is $\partial\epsilon/\partial\boldsymbol{p}$ where the band energy, ϵ, may differ from the simple quadratic expression $\boldsymbol{p}^2/2m$. The novelty is the additional "anomalous" velocity term, see Section 3 below.

Equations (2) derive from the first-order "phase-space" Lagrangian

$$\int(\boldsymbol{p}-\boldsymbol{A}\,)\cdot d\boldsymbol{x}-\frac{p^2}{2}\,dt+\frac{\theta}{2}\,\boldsymbol{p}\times d\boldsymbol{p}. \tag{2}$$

When $m^*\neq 0$, (2) is also a Hamiltonian system, $\dot{\xi}=\{h,\xi_\alpha\}$, with $\xi=(p_i,x_j)$ and Poisson brackets

$$\{x_1,x_2\}=\frac{m}{m^*}\,\theta,\qquad \{x_i,p_j\}=\frac{m}{m^*}\,\delta_{ij},\qquad \{p_1,p_2\}=\frac{m}{m^*}\,eB.$$

A remarkable property is that for *vanishing effective mass* $m^*=0$, i.e., when the magnetic field takes the critical value $B=1/(e\theta)$, the system becomes singular. Then "Faddeev–Jackiw" (alias symplectic) reduction yields an essentially two-dimensional, simple system, reminiscent of "Chern–Simons mechanics" [17, 18][1]. The symplectic plane plays, simultaneously, the role of both configuration and phase space.

The clue is to introduce the "twisted" coordinate

$$\boldsymbol{Q}=\boldsymbol{r}-\boldsymbol{q},\qquad \boldsymbol{r}=(x_i),\qquad q_i=\epsilon_{ij}\frac{p_j}{eB}.$$

In the critical case $eB\theta=1$ the momentum stops to be dynamical: it is determined by the position according to

$$p_i=m\epsilon_{ij}\frac{E_j}{B}.$$

[1]The model goes back to [19], see [20].

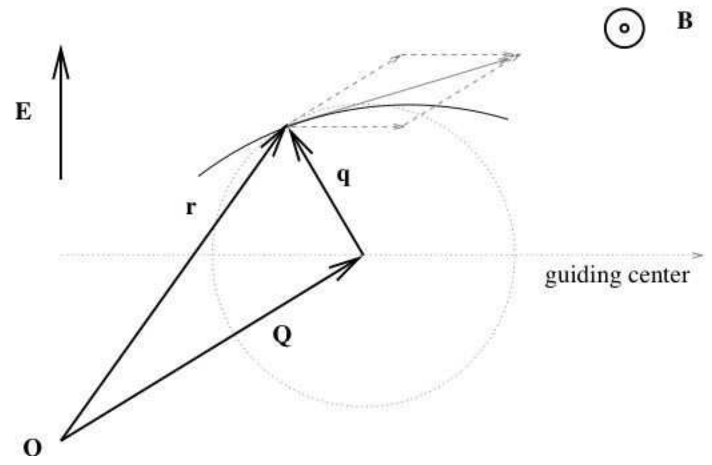

Figure 2. Composition of the uniform guiding center motion and of uniform rotation around it yields the usual cyclotronic motion.

Then $\boldsymbol{Q}$ becomes the guiding center, $\boldsymbol{Q} = \boldsymbol{r} + m\boldsymbol{E}/(eB^2)$. The reduced system has Poisson bracket & energy

$$\{Q_1, Q_2\}_{\mathrm{red}} = \frac{1}{eB}, \qquad H_{\mathrm{red}} = eV(Q_1, Q_2) + \frac{\theta^2 e^2 m}{2}\boldsymbol{E}^2.$$

$\boldsymbol{Q}$ follows therefore a generalized Hall law. The "rotating coordinate" $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{Q}$ becomes in turn "frozen" to the $\boldsymbol{q} = -m\boldsymbol{E}/eB^2$, determined (via the electric field) by the position alone. The evolution of $\boldsymbol{r}$ is hence rigidly determined by that of $\boldsymbol{Q}$.

Quantization of the reduced system yields the wave functions Laughlin starts with [21][2].

Let us now illustrate our general theory on examples.

Constant fields: $\boldsymbol{E} = \text{const}$, $B = \text{const}$. Generically, a particle follows the usual cyclotronic motion around the guiding center, as shown on Fig. 3. $\boldsymbol{q}$ is now a constant. The velocity, $\dot{\boldsymbol{r}}$, is tangent to the trajectory. It is the sum of the velocity of the guiding center (perpendicular to the electric field), $\dot{\boldsymbol{Q}}$, and that, coming from the rotation of $\boldsymbol{q}$ Fig. 2.

In the *critical case* $e\theta B = 1$, however (see Fig. 3), the electric force is canceled by the Lorentz force:

$$e\dot{\boldsymbol{r}} \times \boldsymbol{B} = e\boldsymbol{E} \quad \Rightarrow \quad \dot{x}_i = \epsilon_{ij}\frac{E_j}{B}.$$

Exotic oscillator: $\boldsymbol{E} = -\omega^2\boldsymbol{r}$. The general motions follow elliptical orbits. In the *critical case* $\theta B = 1$, however (see Fig. 4), the guiding center and the real-space position become proportional, $\boldsymbol{q} = (1 + \theta^2\omega^2)\boldsymbol{r}$. The only consistent motions are circular, with "Hall" angular velocity

$$\Omega = \frac{\omega^2 B}{B^2 + \omega^2}.$$

The electric force is *not* compensated by Lorentz force in this case. The dynamics is in fact *non-newtonian:* $m\ddot{\boldsymbol{r}} = (\text{force}) + (\text{terms})$!

The reduced energy is proportional to the reduced angular momentum,

$$H_{\mathrm{red}} = \frac{\omega^2}{2}\big(1 + \omega^2\theta^2\big)Q^2 \ \propto \ \mathrm{I}_{\mathrm{red}} = \frac{B}{2}Q^2.$$

The spectrum is, therefore,

$$E_n = \frac{\omega^2\theta}{1 + \theta^2\omega^2(1/2 + n)}, \qquad n = 0, 1, \ldots.$$

[2]On the quantum Hall effect see, e.g., [22].

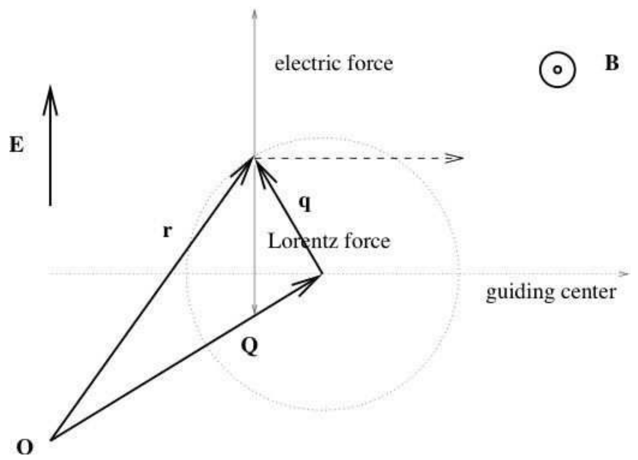

Figure 3. In a constant electromagnetic field, the electric and the Lorentz forces cancel in the critical case. The motion around the guiding center is "frozen" and the particle moves according to the Hall law.

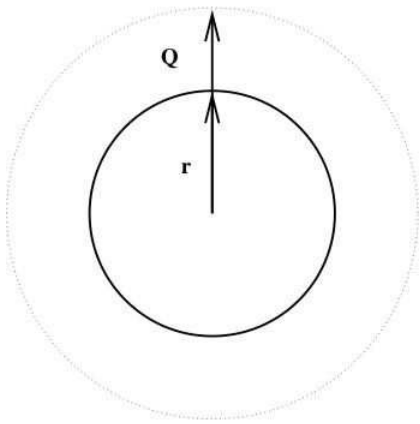

Figure 4. If the electric force is harmonic, then, for critical the magnetic field, all trajectories are circular, and are determined by the motion of the guiding center.

3 The semiclassical Bloch electron

With no relation to the above developments, a similar theory has arisen, around the same time, in solid state physics [23, 24]. One starts with the Bloch wave functions

$$\partial si_{n,\boldsymbol{p}}(\boldsymbol{r}) = e^{i\boldsymbol{p}\cdot\boldsymbol{r}} u_{n,\boldsymbol{p}}(\boldsymbol{r}),$$

where $u_{n,\boldsymbol{p}}(\boldsymbol{r})$ is periodic. The vector $\boldsymbol{p}$ here is the crystal (quasi)momentum. The *Berry connection* is

$$\mathcal{A}_j = i\left\langle u_{n,\boldsymbol{p}} \,\middle|\, \frac{\partial u_{n,\boldsymbol{p}}}{\partial p_j}\right\rangle.$$

Its curvature,

$$\boldsymbol{\Theta}(\boldsymbol{p}) = \nabla_{\boldsymbol{p}} \times \mathcal{A}_l(\boldsymbol{p})$$

is hence purely momentum-dependent.

Then the authors of [23, 24] argue that the semiclassical equations of motion in n^{th} band should be modified by including the Berry term, according to

$$\dot{\boldsymbol{r}} = \frac{\partial \epsilon_n(\boldsymbol{p})}{\partial \boldsymbol{p}} - \dot{\boldsymbol{p}} \times \boldsymbol{\Theta}(\boldsymbol{p}), \tag{3}$$

$$\dot{\boldsymbol{p}} = -e\boldsymbol{E} - e\dot{\boldsymbol{r}} \times \boldsymbol{B}(\boldsymbol{r}), \tag{4}$$

where $\boldsymbol{r} = (x_i)$ denotes the electron's three-dimensional intracell position and $\epsilon_n(\boldsymbol{p})$ is the band energy. Equations (3)–(4) derive from the Lagrangian

$$L^{\mathrm{Bloch}} = (p_i - eA_i(\boldsymbol{r},t))\dot{x}^i - (\epsilon_n(\boldsymbol{p}) + eV(\boldsymbol{r},t)) + a^i(\boldsymbol{p})\dot{p}_i, \tag{5}$$

and are also consistent with the Hamiltonian structure

$$\{x_i, x_j\}^{\mathrm{Bloch}} = \frac{\epsilon^{ijk}\Theta_k}{1 + e\boldsymbol{B}\cdot\boldsymbol{\Theta}}, \qquad \{x_i, p_j\}^{\mathrm{Bloch}} = \frac{\delta^i_j + eB_i\Theta_j}{1 + e\boldsymbol{B}\cdot\boldsymbol{\Theta}},$$

$$\{p_i, p_j\}^{\mathrm{Bloch}} = -\frac{\epsilon_{ijk}eB^k}{1 + e\boldsymbol{B}\cdot\boldsymbol{\Theta}}$$

and Hamiltonian $h = \epsilon_n + eV$ [25, 26][3]. Restricted to the plane, these equations reduce to the exotic equations, (2), when $\Theta_i = \theta\delta_{i3}$, $\epsilon_n(\boldsymbol{p}) = \boldsymbol{p}^2/2m$, $A_i = -(\theta/2)\epsilon_{ij}p_j$. Then the semiclassical Bloch Lagrangian (5) becomes (2). The exotic Galilean symmetry is lost if θ is not constant, though.

Recent applications of the semiclassical model include the Anomalous [33, 34, 35] and the spin Hall effects [36, 37, 38]. All these developments are based on the *anomalous velocity term* in the equations of motion, $\dot{\boldsymbol{p}} \times \boldsymbol{\Theta}(\boldsymbol{p})$.

4 The anomalous Hall effect

The anomalous Hall effect (AHE), observed in some ferromagnetic crystals, is characterized by the absence of a magnetic field. While it has been well established experimentally, its explanation is still controversial. One of them, put forward by Karplus and Luttinger [39] fifty years ago, suggests that the effect is due to an anomalous current. Here we propose to study the AHE in the semiclassical framework.

A remarkable discovery concerns the AHE in $SrRuO_3$. Fang et al. [35] have shown, by some first-principle calculation, that the experimental data are consistent with $\boldsymbol{\Theta}$ which behaves near $\boldsymbol{p} \approx 0$ as *monopole in momentum space*[4]. Let us consider instead the toy model given by

$$\boldsymbol{\Theta} = \theta \frac{\boldsymbol{p}}{p^3}, \tag{6}$$

$p \neq 0$. This is indeed the only possibility consistent with rotational symmetry [41, 42].

For $\boldsymbol{B} = 0$ and a constant electric field, $\boldsymbol{E} = \mathrm{const}$ and assuming a parabolic profile $\epsilon_n(\boldsymbol{p}) = \boldsymbol{p}^2/2$, equation (4) with non-commutative parameter (6), $\dot{\boldsymbol{p}} = e\boldsymbol{E}$, is integrated as $\boldsymbol{p}(t) = e\boldsymbol{E}\,t + \boldsymbol{p}_0$. The velocity relation (3) becomes in turn

$$\dot{\boldsymbol{r}} = \boldsymbol{p}_0 + e\boldsymbol{E}t + \frac{e\theta E k_0}{p^3}\,\widehat{\boldsymbol{n}}, \tag{7}$$

where $\widehat{\boldsymbol{n}} = \widehat{\boldsymbol{p}}_0 \times \widehat{\boldsymbol{E}}$ ["hats" denote vectors normalized to unit length]. The component of $\boldsymbol{p}_0$ parallel to $\boldsymbol{E}$ has no interest; we can assume therefore that $\boldsymbol{p}_0$ is perpendicular to the electric field. Writing $\boldsymbol{r}(t) = x(t)\widehat{\boldsymbol{p}}_0 + y(t)\widehat{\boldsymbol{E}} + z(t)\widehat{\boldsymbol{n}}$, equation (7) yields that the component parallel to $\boldsymbol{p}_0$ moves uniformly, $x(t) = p_0 t$, and its component parallel to the electric field is uniformly accelerating, $y(t) = \frac{1}{2}eEt^2$. (Our choices correspond to choosing time so that the turning point is at $t = 0$.) However, due to the anomalous term in (3), the particle is also deviated perpendicularly to $\boldsymbol{p}_0$ and $\boldsymbol{E}$, namely by

$$z(t) = \frac{\theta}{p_0}\frac{eEt}{\sqrt{p_0^2 + e^2E^2t^2}}.$$

It follows that the trajectory leaves its initial plane and suffers indeed, between $t = -\infty$ to $t = \infty$, a *finite transverse shift*, namely

$$\Delta z = \frac{2\theta}{p_0}. \tag{8}$$

Cf. Fig. 5, then continue with θ becomes a half-integer upon quantization, $\theta = N/2$, and hence (8) is indeed N/k_0. The constant $p_0 \neq 0$, the minimal possible value of momentum, plays the role of an impact parameter. Let us observe that while (8) does not depend on the field $\boldsymbol{E}$ or the electric charge e, the limit $eE \to 0$ is singular. For $eE = 0$, the motion is uniform along a straight line.

[3]This clarifies the controversy raised by Xiao et al. [27], see also [28, 29, 30, 31, 32].

[4]The relevance of the model to the AHE is still under discussion [40].

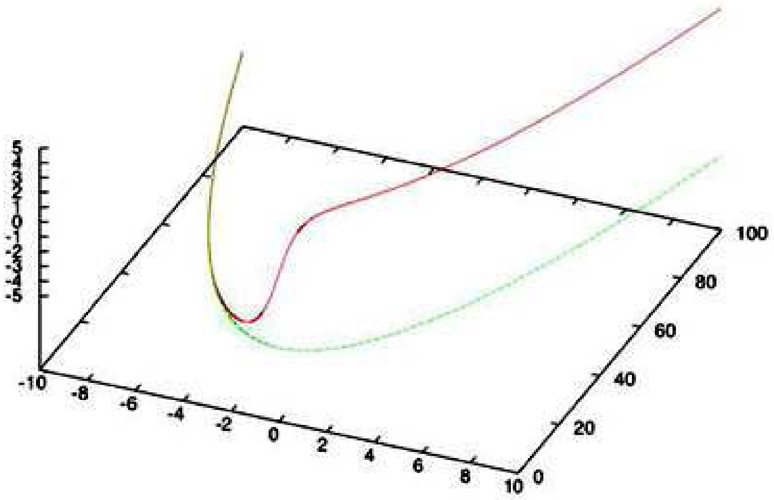

Figure 5. The anomalous velocity term deviates the trajectory from the plane. Most contribution to the shift comes when the momentum is small, i.e., when the particle passes close to the "$\boldsymbol{p}$-monopole".

The transverse shift, reminiscent of the recently discovered optical Hall effect [43], can also be derived using the conservation of angular momentum. The free expression[5] [41],

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} - \theta\,\widehat{\boldsymbol{p}}, \tag{9}$$

is plainly broken by the electric field to its component parallel to $\boldsymbol{E}$,

$$J = J_y = z(t)p_0 - \theta\,\frac{eEt}{\sqrt{k_0^2 + e^2E^2t^2}},$$

whose conservation yields once again the shift (8).

Our model is plainly not realistic: what we described is, rather, the deviation of a freely falling non-commutative particle from the classical parabola found by Galileo. Particles in a semiconductor are not free, though, and their uniform acceleration in the direction of $\boldsymbol{E}$ should be damped by some mechanism.

Interestingly, a similar calculation has been performed in the Spin-Hall context [36, 37, 38]. The similarity to optics [43, 44, 45, 46, 47, 48] is explained by that the three-dimensional system with Θ given in equation (6) studied here is indeed mathematically equivalent to "SpinOptics", described in [47, 48]. It follows that the NC system carries therefore a massless Poincaré dynamical symmetry. I am indebted to C. Duval for calling my attention to this point.

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