Solvable Nonlinear Evolution PDEs in Multidimensional Space*

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Abstract. A class of *solvable* (systems of) nonlinear evolution PDEs in *multidimensional* space is discussed. We focus on a rotation-invariant system of PDEs of Schrödinger type and on a relativistically-invariant system of PDEs of Klein–Gordon type. *Isochronous* variants of these evolution PDEs are also considered.

Key words: nonlinear evolution PDEs in multidimensions; solvable PDEs; NLS-like equations; nonlinear Klein–Gordon-like equations; isochronicity

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This article is dedicated to the memory of Vadim Kuznetsov, with whom we spent several happy days during a Gathering of Scientists held at the Centro Internacional de Ciencias in Cuernavaca, and as well when he visited us in Rome, and when we met at several other meetings.

1 Introduction

Over a decade ago a class of *C-integrable* – i.e., *solvable* via a *Change of variables* – systems of PDEs in *multidimensional* space were identified [2]. (A problem involving nonlinear PDEs is considered *solvable* if its solution can be obtained by performing algebraic operations – such as finding the zeros of a given polynomial – and by solving *linear* PDEs; of course only seldom these operations can be performed *explicitly*.) In the present paper – motivated by the scarcity of *solvable* models of nonlinear evolution PDEs in *multidimensions* hence by the interest of *any* such model – we study (a subclass of) these *solvable* PDEs in more detail than it was done hitherto. We focus mainly on the system of PDEs of Schrödinger type

$$i\psi_{n,t} - \Delta\psi_n + W(\vec{r})\psi_n = 2\sum_{m=1,\, m\neq n}^{N} \frac{a + b\psi_n\psi_m - (\vec{\nabla}\psi_n)\cdot(\vec{\nabla}\psi_m)}{\psi_n - \psi_m}, \tag{1.1}$$

*This paper is a contribution to the Vadim Kuznetsov Memorial Issue "Integrable Systems and Related Topics". The full collection is available at http://www.emis.de/journals/SIGMA/kuznetsov.html which is *rotation-invariant* if $W(\vec{r}) = W(r)$, and on the *relativistically-invariant* system of PDEs of Klein–Gordon type

$$\psi_{n,tt} - \Delta\psi_n + M^2\psi_n = 2 \sum_{m=1,\, m\neq n}^{N} \frac{a + b\psi_n\psi_m + \psi_{n,t}\psi_{m,t} - \left(\vec{\nabla}\psi_n\right)\cdot\left(\vec{\nabla}\psi_m\right)}{\psi_n - \psi_m}. \tag{1.2}$$

Notation. Here and throughout N is an arbitrary positive integer ($N \geq 2$); the index n, as well as other analogous indices (see below), range generally from 1 to N; the dependent variables $\psi_n \equiv \psi_n(\vec{r}, t)$ are generally considered *complex* (although this is only mandatory for the first, (1.1), of these two systems of PDEs); the space variable $\vec{r}$ is a (*real*) S-vector (with S *arbitrary*: for $S = 3$, $\vec{r} \equiv (x, y, z)$), and r indicates its modulus, $r^2 = \vec{r}\cdot\vec{r}$; a dot sandwiched among two S-vectors denotes the standard (*rotation-invariant*) scalar product (for instance, for $S = 3$, $\vec{r}\cdot\vec{r} = x^2 + y^2 + z^2$); the "potential" $W(\vec{r})$ is an arbitrary function of the spatial coordinate $\vec{r}$ (we will often assume that it only depends on the modulus r of the S-vector $\vec{r}$); the constants a and b, as well as the "mass" parameter M, are arbitrary (they might also vanish); t is the (*real*) time variable; subscripted independent variables always denote partial differentiations; $\vec{\nabla}$ respectively $\Delta \equiv \vec{\nabla}\cdot\vec{\nabla}$ are the gradient respectively the Laplace operator in S-dimensional space. *Isochronous* variants of these evolution PDEs are also considered.

In the following Section 2 we review tersely the general class of PDEs treatable in this manner and the technique to solve them. In Sections 3 respectively 4 we treat in some detail the PDEs (1.1) respectively (1.2), describing various properties of their solutions and reporting some representative examples, and we exhibit their *isochronous* versions. In Section 5 we take advantage of the electronic format of this article to present a few animations displaying visually the time evolution of a few solutions of some of these solvable (systems of) nonlinear evolution PDEs: in this article we restrict these presentations to very few cases, all with space dimensionality less than three (we are of course aware that the three-dimensional case is probably the most interesting one – since we seem to live in a three-dimensional world – but the presentation of animations in a three-dimensional context is somewhat more tricky and we therefore postpone the display of such examples to future articles we hope to issue soon). In Section 6 we outline future directions of research.

2 A class of solvable (systems of) PDEs in multidimensional space

In this section we review tersely the basic idea allowing to identify a class of *solvable* nonlinear evolution PDEs. The interested reader will find a more detailed treatment in the paper where this approach was introduced [1], and especially in [3] where this method is treated in considerable detail: see Section 2.3 of this book, and other references quoted there in Section 2.N. These treatments focussed however on ODEs rather than PDEs: the extension to PDEs is rather straightforward, although it took some time to realize its feasibility [2] (see also Exercise 2.3.4.2-5 in [3]).

Let $\Psi(\psi; \vec{r}, t)$ be a (time- and space-dependent) *monic* polynomial of degree N in the variable ψ, and denote by $\psi_n(\vec{r}, t)$ its N zeros and by $\varphi_m(\vec{r}, t)$ its N coefficients:

$$\Psi(\psi; \vec{r}, t) = \prod_{n=1}^{N} \left[\psi - \psi_n(\vec{r}, t)\right], \tag{2.1a}$$

$$\Psi(\psi; \vec{r}, t) = \psi^N + \sum_{m=1}^{N} \varphi_m(\vec{r}, t)\, \psi^{N-m}. \tag{2.1b}$$

Assume then that the time evolution of the dependent variable Ψ – hence its dependence on the time and space independent variables t and $\vec{r}$, as well as its dependence on the independent variable ψ – is characterized by a *linear* evolution PDE, which must of course be consistent with the fact that Ψ is, for *all* time, a *monic* polynomial of degree N in ψ, see (2.1). The *linear* character of this PDE entails that the time-evolution of the N coefficients $\varphi_m(\vec{r},t)$, see (2.1b), is as well *linear*, possibly *explicitly* solvable (see below). On the other hand the corresponding time evolution of the N zeros $\psi_n(\vec{r},t)$ will be *nonlinear*, due to the *nonlinear* character of the mapping relating the N zeros $\psi_n(\vec{r},t)$ to the N coefficients $\varphi_m(\vec{r},t)$ of the polynomial $\Psi(\psi;\vec{r},t)$, see (2.1). These (systems of) *nonlinear* evolution PDEs satisfied by the dependent variables $\psi_n(\vec{r},t)$ are those referred to in the title of this paper: they are indeed generally *solvable* by taking advantage of the mapping, see (2.1), relating the N dependent variables $\psi_n(\vec{r},t)$ to the N functions $\varphi_m(\vec{r},t)$.

In particular it is easily seen (using, if need be, the formulas provided in Section 2.3 of [3]), that to the evolution PDE

$$i\Psi_t - \Delta\Psi - V(\vec{r})\left(\psi\Psi_\psi - N\Psi\right) + a\Psi_{\psi\psi} + b\left[\psi^2\Psi_{\psi\psi} - N(N-1)\Psi\right] = 0 \tag{2.2}$$

satisfied by $\Psi(\psi;\vec{r},t)$, there corresponds for the N zeros $\psi_n(\vec{r},t)$, see (2.1a), just the system of nonlinear evolution PDEs of Schrödinger type (1.1) with

$$W(\vec{r}) = V(\vec{r}) - 2(N-1)\,b, \tag{2.3}$$

while the corresponding evolution of the N coefficients $\varphi_m(\vec{r},t)$, see (2.1b), is clearly given by the system of *linear* evolution PDEs

$$\begin{aligned} &i\varphi_{m,t} - \Delta\varphi_m + \left[W(\vec{r}) - b(m+3)\right]m\varphi_m \\ &\qquad = -a(N-m+2)(N-m+1)\,\varphi_{m-2}, \qquad m = 1,\ldots,N \end{aligned} \tag{2.4}$$

with $\varphi_{-1} = 0$ and $\varphi_0 = 1$ (see (2.1b)). Note that this system of *linear* PDEs is *decoupled* if the constant a vanishes; as we shall see in the following section, it can also be replaced by a *decoupled* system if the constant a does *not* vanish, $a \neq 0$, but the potential $W(\vec{r})$ is constant, $W(\vec{r}) = C$, see Section 3.

Likewise to the evolution PDE

$$\Psi_{tt} - \Delta\Psi - \mu^2\left[\psi\Psi_\psi - N\Psi\right] + a\Psi_{\psi\psi} + b\left[\psi^2\Psi_{\psi\psi} - N(N-1)\Psi\right] = 0, \tag{2.5}$$

there corresponds for the N zeros $\psi_n(\vec{r},t)$ just the system of *nonlinear* evolution PDEs of Klein–Gordon type (1.2) with

$$M^2 = \mu^2 - 2(N-1)\,b,$$

while the corresponding evolution of the N coefficients $\varphi_m(\vec{r},t)$ is clearly given by the system of *linear* evolution PDEs

$$\begin{aligned} &\varphi_{m,tt} - \Delta\varphi_m + \left[M^2 - b(m+3)\right]m\varphi_m \\ &\qquad = -a(N-m+2)(N-m+1)\,\varphi_{m-2}, \qquad m = 1,\ldots,N, \end{aligned} \tag{2.6}$$

again with $\varphi_{-1} = 0$ and $\varphi_0 = 1$ (see (2.1b)): this system is *decoupled* if the constant a vanishes, and can be replaced by a *decoupled* system even if a does *not* vanish, see Section 4.

Having conveyed, tersely but hopefully clearly, the main idea of this approach to identify *solvable* systems of *nonlinear* evolution PDEs, we turn, in the next two sections, to the study of the two systems of *nonlinear* evolution PDEs (1.1) and (1.2).

3 Solvable system of nonlinear PDEs of Schrödinger type

In this section we investigate the system of *nonlinear* evolution equations (1.1), firstly by analytic techniques and subsequently by reporting some of its solutions in a representative set of cases.

Remark 3.1. The "mean field" $\bar{\psi}\left(\vec{r},t\right)$

$$\bar{\psi}\left(\vec{r},t\right)=\frac{1}{N}\sum_{n=1}^{N}\psi_n\left(\vec{r},t\right) \tag{3.1}$$

satisfies the *linear* Schrödinger equation

$$i\bar{\psi}_t-\Delta\bar{\psi}+W\left(\vec{r}\right)\bar{\psi}=0.$$

Proof. Sum the nonlinear evolution PDEs (1.1) over n from 1 to N, use (3.1) in the left-hand side, and notice that the double sum in the right-hand side vanishes due to the antisymmetry of the summand under the exchange of the two dummy indices n and m. ■

Remark 3.2. If *all* the N coefficients φ_m vanish, $\varphi_m=0$, then $\Psi=\psi^N$ (see (2.1b)), hence *all* its N zeros ψ_n correspondingly vanish, $\psi_m=0$ (and more generally: if the first M coefficients φ_m vanish, $\varphi_m=0$ for $m=1,\ldots,M$, then M of the N zeros of Ψ vanish). Hence to a set of *localized* solutions $\psi_n\left(\vec{r},t\right)$ of the system of nonlinear evolution equations (1.1), characterized by the asymptotic conditions

$$\lim_{r\to\infty}\left[\psi_n\left(\vec{r},t\right)\right]=0,\qquad n=1,\ldots,N, \tag{3.2a}$$

there correspond a set of *localized* solutions of the system of *linear* Schrödinger PDEs (2.4) (with (2.3)), characterized by the analogous asymptotic conditions

$$\lim_{r\to\infty}\left[\varphi_n\left(\vec{r},t\right)\right]=0,\qquad n=1,\ldots,N; \tag{3.2b}$$

and, of course, viceversa, namely clearly (3.2b) entails (3.2a). However when two *different* components, $\psi_n\left(\vec{r},t\right)$ and $\psi_m\left(\vec{r},t\right)$ with $n\neq m$, of the system of nonlinear evolution equations (1.1) are equal, $\psi_n\left(\vec{r},t\right)=\psi_m\left(\vec{r},t\right)$, clearly this system runs into a singularity due to the vanishing of one of the denominators in the right-hand side of (1.1); and this situation gets even worst if the two *different* components, $\psi_n\left(\vec{r},t\right)$ and $\psi_m\left(\vec{r},t\right)$ with $n\neq m$, both *vanish*, $\psi_n\left(\vec{r},t\right)=\psi_m\left(\vec{r},t\right)=0$. Hence in all the examples considered below we shall try and avoid this problem, and in particular we shall focus, rather than on *localized* solutions $\psi_n\left(\vec{r},t\right)$ that vanish asymptotically (see (3.2a)), either on solutions that oscillate asymptotically, or on solutions that are asymptotically constant,

$$\lim_{r\to\infty}\left[\psi_n\left(\vec{r},t\right)\right]=a_n\qquad\text{with } a_n\neq a_m\quad\text{if}\quad n\neq m. \tag{3.3}$$

Note that these asymptotic values a_n might depend on the direction along which the space coordinate $\vec{r}$ diverges.

Remark 3.3. If the potential $V\left(r\right)$ is constant,

$$V\left(r\right)=B, \tag{3.4}$$

(entailing that $W\left(r\right)$ is as well *constant*,

$$W\left(r\right)=B-2\left(N-1\right)b,$$

see (2.3)), it may be convenient to replace the expression (2.1b) of the monic polynomial $\Psi(\psi;\vec{r},t)$ in terms of its coefficients $\varphi_m(\vec{r},t)$ by the following representation:

$$\Psi(\psi;\vec{r},t) = \frac{C_N^{\gamma}(c\psi)}{k_N c^N} + \sum_{m=1}^{N} \chi_m(\vec{r},t)\,\frac{C_{N-m}^{\gamma}(c\psi)}{k_{N-m}c^{N-m}}, \tag{3.5a}$$

$$c = \left(-\frac{b}{a}\right)^{1/2}, \tag{3.5b}$$

$$\gamma = -\frac{B+b}{2b}, \tag{3.5c}$$

where the polynomial $C_\ell^{\gamma}(z)$, of degree ℓ, is the standard Gegenbauer polynomial [5], satisfying the ODE

$$\left(1-z^2\right) C_l^{\gamma\prime\prime}(z) - (2\gamma+1)\, z C_l^{\gamma\prime}(z) + \ell(\ell+2\gamma)\, C_l^{\gamma}(z) = 0$$

(where appended primes denote derivatives with respect to the argument of the function they are appended to, in this case with respect to z), and being characterized by the asymptotic behavior

$$\lim_{z\to\infty}\left[\frac{C_l^{\gamma}(z)}{k_\ell}\right] = 1, \qquad \text{where} \quad k_\ell = \frac{2^\ell\,\Gamma(\gamma+\ell)}{\ell!\,\Gamma(\gamma)}.$$

Then the requirement that $\Psi(\psi;\vec{r},t)$ satisfy the PDE (2.2) with (3.4), and therefore that its N zeros $\psi_n(\vec{r},t)$ satisfy the system of nonlinear PDEs (1.1) with (3.4), entails that the N coefficients $\chi_m(\vec{r},t)$ satisfy the (system of *decoupled*) *linear* Schrödinger PDEs (with constant potentials)

$$i\chi_{m,t} - \Delta\chi_m + \left[B - b(2N-m-1)\right] m\chi_m = 0, \qquad m = 1,\ldots,N. \tag{3.6}$$

The investigation of the limiting cases in which some coefficients vanish, and therefore a different set of classical polynomials come into play in place of the Gegenbauer polynomials, is left as an exercise for the diligent reader.

The *nonlinear* mapping among the N dependent variables $\psi_n(\vec{r},t)$ satisfying the system of *nonlinear* PDEs (1.1) and the N functions $\varphi_m(\vec{r},t)$ respectively $\chi_m(\vec{r},t)$ satisfying the system of *linear* PDEs (2.4) respectively (3.6), entailed by the simultaneous validity of (2.1a) and (2.1b) respectively (3.5), is the key to the *solvability* of the system of *nonlinear* PDEs (1.1). This can be taken advantage of in two ways: to solve the *initial-value* problem for the system of *nonlinear* PDEs (1.1), or to manufacture special, possibly quite *explicit*, solutions of this system of *nonlinear* PDEs.

3.1 Solution of the initial-value problem for the system of nonlinear Schrödinger-like PDEs (1.1)

The *initial-value* problem consists in the determination of the solution $\psi_n(\vec{r},t)$, $n = 1,\ldots,N$, of the system of Schrödinger-like *nonlinear* PDEs (1.1) corresponding to *assigned* initial data $\psi_m(\vec{r},0)$, $m = 1,\ldots,N$.

The *first* step is to determine the corresponding initial data $\varphi_m(\vec{r},0)$ of the system of Schrödinger-like *linear* PDEs (2.4). This is achieved by solving for the N functions $\varphi_m(\vec{r},0)$ the system

$$\psi^N + \sum_{m=1}^{N} \varphi_m(\vec{r},0)\,\psi^{N-m} = \prod_{n=1}^{N}\left[\psi - \psi_n(\vec{r},0)\right], \tag{3.7}$$

entailed by the simultaneous validity of (2.1a) and (2.1b) at $t = 0$. This amounts to the determination of the N coefficients φ_m of a monic polynomial given its N zeros ψ_n; the relevant, *explicit* formulas are of course well-known:

$$\varphi_1(\vec{r},0) = -\sum_{n=1}^{N} \psi_n(\vec{r},0), \qquad \varphi_2(\vec{r},0) = \sum_{n,m=1,\, m\neq n}^{N} \psi_n(\vec{r},0)\,\psi_m(\vec{r},0),$$

and so on, up to

$$\varphi_N(\vec{r},0) = (-)^N \prod_{n=1}^{N} \psi_n(\vec{r},0).$$

The *second* step is to solve the initial-value problem for the system of evolution PDEs (2.4), obtaining thereby its solution $\varphi_m(\vec{r},t)$ at time t. The *linear* character of this (coupled) system of evolution PDEs, (2.4), provides the main simplification; of course an *explicit* solution is only possible for special choices of the potential $W(\vec{r})$.

The *third* step is to obtain the solution $\psi_n(\vec{r},t)$ from the, now assumedly known, functions $\varphi_m(\vec{r},t)$, via the relation

$$\prod_{n}^{N} [\psi - \psi_n(\vec{r},t)] = \psi^N + \sum_{m=1}^{N} \varphi_m(\vec{r},t)\,\psi^{N-m}, \tag{3.8}$$

again entailed by the simultaneous validity of (2.1a) and (2.1b), but now at time t. This amounts of course just to the purely algebraic task of finding the zeros of a given monic polynomial of degree N: an *explicit* solution is of course only possible for $N \leq 4$.

In the special case of a constant potential $V(r)$, see (3.4), an alternative procedure of solution can be based on the representation (3.5) rather than (2.1b): this eases the *second* of the steps outlined above, but makes a bit less simple the *first* step. The diligent reader will easily figure out the relevant details.

3.2 How to manufacture explicit solutions of the system of nonlinear Schrödinger-like PDEs (1.1)

Clearly the appropriate strategy – underlining all the examples exhibited below – is to identify an *explicit* solution $\varphi_m(\vec{r},t)$ of the system of *linear* evolution PDEs (2.4), and then to obtain the corresponding solution $\psi_n(\vec{r},t)$ of the system of *nonlinear* evolution PDEs (1.1) via (3.8), namely by identifying the N zeros $\psi_n(\vec{r},t)$ of the monic polynomial of degree N in ψ having the coefficients $\varphi_m(\vec{r},t)$, see (2.1b). This can of course be done *explicitly* only for $N \leq 4$: not a significant restriction when it comes to the *explicit* exhibition of examples, which would indeed be impractical for larger values of N (in Section 5 we indeed limit our exhibition of animations to the 3-body case, $N = 3$).

An alternative route – applicable when the potential is *constant*, see (3.4) – takes as starting point an *explicit* solution $\chi_m(\vec{r},t)$ of the (*decoupled*) system of *linear* evolution PDEs (3.6), and then obtains the corresponding solution $\psi_n(\vec{r},t)$ of the system of *nonlinear* evolution PDEs (1.1) via (3.5), namely by identifying the N zeros $\psi_n(\vec{r},t)$ of the monic polynomial of degree N in ψ given by this expression (3.5).

3.3 Examples

Example 3.3.1. The simplest example is characterized by the assignment

$$a = b = 0, \qquad W(\vec{r}) = V(\vec{r}) = 0,$$

namely by the system of *nonlinear* evolution PDEs in one-dimensional space (see (1.1))

$$i\psi_{n,t} - \Delta\psi_n = -2 \sum_{m=1,\, m\neq n}^{N} \frac{\vec{\nabla}\psi_n \cdot \vec{\nabla}\psi_m}{\psi_n - \psi_m}. \tag{3.9}$$

The corresponding system of *linear* PDEs satisfied by the coefficients $\varphi_m(\vec{r},t)$ of the polynomial $\Psi(\psi;\vec{r},t)$ in the variable ψ of which the solutions $\psi_n(\vec{r},t)$ are the N zeros,

$$\Psi(\psi;\vec{r},t) = \psi^N + \sum_{m=1}^{N} \varphi_m(\vec{r},t)\,\psi^{N-m} = \prod_{n}^{N} \left[\psi - \psi_n(\vec{r},t)\right], \tag{3.10}$$

reads as follows (see (2.4)):

$$i\varphi_{m,t} - \Delta\varphi_m = 0, \qquad m = 1,\ldots,N. \tag{3.11}$$

A special class of "traveling wave" solutions of these (decoupled) *linear* PDEs reads

$$\varphi_m(\vec{r},t) = \varphi_m(\vec{r}-\vec{v}t) = A_m + B_m \exp\left[-i\vec{v}\,(\vec{r}-\vec{v}t)\right], \tag{3.12}$$

with $\vec{v}$ an arbitrary *real* constant S-vector and the $2N$ scalar constants A_m, B_m also arbitrary (possibly *complex*). These solutions (which are clearly the most general ones of "traveling wave" character, namely depending on the single S-vector $\vec{r}-\vec{v}t$ rather than separately on the S-vector space variable $\vec{r}$ and the scalar time variable t) are *not* localized: they are *constant* along the $S-1$ space directions orthogonal to $\vec{v}$, *periodic* with period $L = |2\pi/v|$ along the space direction parallel to $\vec{v}$, and *periodic* in t with period $T = 2\pi v^{-2}$. The corresponding *traveling wave* solutions $\psi_n(x,t)$ of the system of *nonlinear* PDEs (3.9) are the N zeros of the following polynomial of degree N in the variable ψ:

$$P_N(\psi) + Q_{N-1}(\psi)\exp\left[-i\vec{v}\,(\vec{r}-\vec{v}t)\right] = \prod_{n}^{N}\left[\psi - \psi_n(\vec{r},t)\right], \tag{3.13a}$$

where $P_N(\psi)$ is an arbitrary *monic* polynomial of degree N and $Q_{N-1}(\psi)$ is an arbitrary polynomial of degree $N-1$,

$$P_N(\psi) = \psi^N + \sum_{m=1}^{N} A_m \psi^{N-m}, \qquad Q_{N-1}(\psi) = \sum_{m=1}^{N} B_m \psi^{N-m}. \tag{3.13b}$$

This entails of course that these solutions are as well of the same traveling wave type, $\psi_n(\vec{r},t) = \psi_n(\vec{r}-\vec{v}t)$, with the same periodicity properties described above for $\varphi_m(\vec{r},t) = \varphi_m(\vec{r}-\vec{v}t)$ – or possibly with time periods which are *integer multiples* of that of the coefficients $\varphi_m(\vec{r},t)$ due to the possibility that through the time evolution the zeros of the polynomial (3.13b) exchange their roles; for each component $\psi_n(\vec{r},t)$ this integer multiple cannot of course exceed N, while for the entire solution $\{\psi_n(\vec{r},t)\,; n = 1,\ldots,N\}$ the period can be somewhat larger – but generally not too much [6].

A solution of this kind (with $S = 2$ and $N = 3$, and a specific choice of the remaining free parameters) is displayed as an animation in Section 5.

Another special set of solutions of (3.11) – written below, for simplicity, for the one-dimensional case ($S = 1$) – reads

$$\varphi_m(x,t) = A_m + \frac{B_m}{(t + t_m + i\eta_m)^{1/2}} \exp\left[\frac{-i\,(x - x_m + i\eta_m v_m)^2}{4\,(t + t_m + i\eta_m)}\right], \tag{3.14}$$

where the constants A_m and B_m are *arbitrary* (possibly *complex*; but see below for some conditions on the constants A_m), the constants x_m, t_m and v_m are also arbitrary but *real*, and the constants η_m are also arbitrary but *positive*, $\eta_m > 0$: it is indeed easily seen that these conditions are sufficient to guarantee that $\varphi_m(x,t)$ is *nonsingular* for all *real* values of the independent variables x and t and tend asymptotically to the constants A_m:

$$\lim_{x\to\pm\infty}\left[\varphi_m(x,t)\right] = A_m.$$

The corresponding solutions $\psi_n(x,t)$ of the system of *nonlinear* PDEs (3.9) are the N zeros of the following polynomial of degree N in the variable ψ:

$$\psi^N + \sum_{m=1}^{N}\left\{A_m + \frac{B_m}{(t+t_m+i\eta_m)^{1/2}}\exp\left[\frac{-i\,(x-x_m+i\eta_m v_m)^2}{4\,(t+t_m+i\eta_m)}\right]\right\}\psi^{N-m}$$
$$= \prod_{n=1}^{N}\left[\psi-\psi_n(x,t)\right]. \tag{3.15}$$

Note that this entails that these solutions satisfy the asymptotic property (3.3), with the constants a_n being the N zeros of the polynomial, of degree N in the variable a,

$$a^N + \sum_{m=1}^{N} A_m a^{N-m} = \prod_{n=1}^{N}(a-a_n).$$

Of course the constants A_m should be assigned so that these N zeros a_n are *all different*, see (3.3).

A solution of this kind (with $N=3$, and a specific choice of all the remaining free parameters) is displayed as an animation in Section 5.

3.4 Isochronous version of the class of nonlinear PDEs of Schrödinger type

In this subsection we report, without much commentary, an "ω-modified" version of (a subclass of) the system of *nonlinear* PDEs of Schrödinger type (1.1), which is characterized by the property to possess an ample class of solutions *completely periodic* in time with period

$$T = \frac{2\pi}{\omega} \tag{3.16}$$

(the reason why this is so will be rather obvious from what follows; for more details on, and other examples of, *isochronous* PDEs obtained in an analogous manner see [7] and Chapter 6 of [4]).

We start from the subcase of the system of *nonlinear* PDEs of Schrödinger type (1.1) with

$$b=0, \qquad W(\vec r)=0,$$

and we set

$$\tilde\psi_n(\vec r,t) = e^{i\lambda\omega t}\psi_n(\vec\rho,\tau), \qquad \vec\rho\equiv\vec\rho(t) = e^{\frac{i\omega t}{2}}\vec r, \qquad \tau\equiv\tau(t) = \frac{e^{i\omega t}-1}{i\omega},$$

with ω an arbitrary *positive* constant and λ an *arbitrary real rational* number if a vanishes ($a=0$, see (1.1) and below), otherwise $\lambda=-1/2$. It is then easily seen that the new dependent variables $\tilde\psi_n(\vec r,t)$ satisfy the following system of *nonlinear* PDEs of Schrödinger type:

$$i\tilde\psi_{n,t} - \Delta\tilde\psi_n + \lambda\omega\tilde\psi_n + \frac{\omega}{2}\vec r\cdot\vec\nabla\tilde\psi_n = 2\sum_{m=1,\,m\neq n}^{N}\frac{a-\left(\vec\nabla\tilde\psi_n\right)\cdot\left(\vec\nabla\tilde\psi_m\right)}{\tilde\psi_n-\tilde\psi_m}.$$

Note that this system is *autonomous* with respect to the time variable, but it features an *explicit* dependence on the space variable $\vec{r}$ (see the last term in the left-hand side); and it is clearly *rotation invariant*.

Example 3.4.1. The simplest example is again characterized by the assignment

$$S = 1, \qquad a = 0,$$

namely by the system of *nonlinear* evolution PDEs in one-dimensional space (see (1.1))

$$i\tilde{\psi}_{n,t} - \tilde{\psi}_{n,xx} + \lambda\omega\tilde{\psi}_n + \frac{\omega}{2}x\tilde{\psi}_{n,x} = -2\sum_{m=1,\, m\neq n}^{N} \frac{\tilde{\psi}_{n,x}\tilde{\psi}_{m,x}}{\tilde{\psi}_n - \tilde{\psi}_m}.$$

A special class of solutions of this system of *nonlinear* PDEs obtains, via the formula

$$\tilde{\psi}_n(x,t) = \exp(i\lambda\omega t)\,\psi_n(\xi,\tau),$$
$$\xi \equiv \xi(t) = \exp\left(\frac{i\omega t}{2}\right)x, \qquad \tau \equiv \tau(t) = \frac{\exp(i\omega t) - 1}{i\omega}, \tag{3.17}$$

from the solutions $\psi_n(\xi,\tau)$ of (3.13a), now reading

$$P_N(\psi) + Q_{N-1}(\psi)\exp[-iv(\xi - v\tau)] = \prod_{n}^{N}[\psi - \psi_n(\xi,\tau)],$$

where $P_N(\psi)$ is an arbitrary *monic* polynomial of degree N and $Q_{N-1}(\psi)$ is an arbitrary polynomial of degree $N-1$, see (3.13b).

Another class of solutions is provided via (3.17) from the solutions ψ_n of (3.15), now reading

$$\psi^N + \sum_{m=1}^{N}\left\{A_m + \frac{B_m}{(\tau + \tau_m + i\eta_m)^{1/2}}\exp\left[\frac{-i(\xi - \xi_m + i\eta_m v_m)^2}{4(\tau + \tau_m + i\eta_m)}\right]\right\}\psi^{N-m}$$
$$= \prod_{n=1}^{N}[\psi - \psi_n(\xi,\tau)].$$

4 Solvable systems of nonlinear PDEs of Klein-Gordon type

In this section we investigate, firstly by analytic techniques and subsequently via the explicit display of a few of its solutions, the system of *nonlinear* evolution equations of Klein–Gordon type (1.2). The first remarks are analogous to those given in the first part of the preceding section and are therefore reported below without much commentary (their proofs are analogous to those given in the preceding section; we also use occasionally the same notation, confiding that this will cause no misunderstandings).

Remark 4.1. The "mean field" $\bar{\psi}(\vec{r},t)$ defined by (3.1) satisfies now the *linear* Klein–Gordon equation

$$\bar{\psi}_{tt} - \Delta\bar{\psi} + M^2\bar{\psi} = 0.$$

Remark 4.2. To a set of *localized* solutions $\psi_n(\vec{r},t)$ of the system of *nonlinear* evolution equations of Klein–Gordon type (1.2) characterized by the asymptotic conditions (3.2a) there correspond a set of *localized* solutions of the system of *linear* Klein–Gordon PDEs (2.6) characterized by the analogous asymptotic conditions (3.2b); and, of course, viceversa. But such localized solutions cause the same kind of problem discussed in Remark 3.1. Hence in all the examples considered below we shall try and avoid this problem, just as indicated in Remark 3.2.

Remark 4.3. It may be convenient to replace the expression (2.1b) of the monic polynomial $\Psi(\psi;\vec{r},t)$ in terms of its coefficients $\varphi_m(\vec{r},t)$ by the representation (3.5a), again with (3.5b) but now with

$$\gamma = -\frac{M^2 + b(2N-m)}{2b}, \tag{4.1}$$

instead of (3.5c). Then the requirement that $\Psi(\psi;\vec{r},t)$ satisfy the PDE (2.5), and therefore that its N zeros $\psi_n(\vec{r},t)$ satisfy the system of nonlinear PDEs (1.2), entails that the N coefficients $\chi_m(\vec{r},t)$ satisfy the (system of) *decoupled linear* Klein–Gordon PDEs

$$\chi_{m,tt} - \Delta\chi_m + \left[M^2 - b(m+3)\right] m\chi_m = 0, \qquad m = 1,\ldots,N. \tag{4.2}$$

Analogous developments to those reported in the preceding Section 3 in the context of the system of Schrödinger-like *nonlinear* evolution PDEs (1.1) can now be elaborated in the present context of the system of Klein–Gordon-like *nonlinear* evolution PDEs (1.2). Our presentation below is more terse than in the preceding section, to avoid repetitions.

4.1 Solution of the initial-value problem for the system of nonlinear Klein–Gordon-like PDEs (1.2)

The *initial-value* problem consists now in the determination of the solution $\psi_n(\vec{r},t)$ of the system of Klein–Gordon-like *nonlinear* PDEs (1.2) corresponding to *assigned* initial data $\psi_m(\vec{r},0)$ and $\psi_{m,t}(\vec{r},0)$, $m = 1,\ldots,N$.

The *first* step is to determine the corresponding initial data $\varphi_m(\vec{r},0)$ and $\varphi_{m,t}(\vec{r},0)$ of the system of Klein–Gordon-like *linear* PDEs (2.6). This is achieved by firstly solving, as above, for the N functions $\varphi_m(\vec{r},0)$ the system (3.7), and then by solving for the N functions $\varphi_{m,t}(\vec{r},0)$ the system

$$\sum_{m=1}^{N} \varphi_{m,t}(\vec{r},0)\,\psi^{N-m} = -\sum_{n=1}^{N}\left\{\psi_{n,t}(\vec{r},0)\prod_{m=1,\,m\neq n}^{N}\left[\psi - \psi_m(\vec{r},0)\right]\right\}.$$

The *second* step is to obtain the solution $\varphi_m(\vec{r},t)$ at time t of the system of evolution PDEs (2.6). Again, the *linear* character of this system of evolution PDEs provides the main simplification.

The *third* step is, as above, to obtain the solution $\psi_n(\vec{r},t)$ from the, now assumedly known, functions $\varphi_m(\vec{r},t)$ via the relation (3.8), amounting again just to the purely algebraic task of finding the N zeros of a given monic polynomial of degree N.

An alternative procedure of solution can be based on the representation (3.5a) (with (3.5b) and (4.1) rather than (3.5c)): this eases the *second* of the steps outlined above, but makes a bit less simple the *first* step. The diligent reader will easily figure out the relevant details.

4.2 How to manufacture explicit solutions of the system of nonlinear Klein–Gordon-like PDEs (1.2)

As above the strategy – that underlies all the examples discussed below – is to identify an *explicit* solution $\varphi_m(\vec{r},t)$ of the system of *linear* evolution PDEs (2.6), and then to obtain the corresponding solution $\psi_n(\vec{r},t)$ of the system of *nonlinear* evolution PDEs (1.2) via (3.8), namely by identifying the N zeros $\psi_n(\vec{r},t)$ of the monic polynomial of degree N in ψ with coefficients $\varphi_m(\vec{r},t)$, see (2.1b).

An alternative route takes as starting point an *explicit* solution $\chi_m(\vec{r},t)$ of the (*decoupled*) system of *linear* evolution PDEs (4.2), and then obtains the corresponding solution $\psi_n(\vec{r},t)$

of the system of *nonlinear* evolution PDEs (1.2) by identifying the quantities $\psi_n(\vec{r},t)$ as the N zeros of the monic polynomial of degree N in ψ given by the expression (3.5a) with (3.5b) and (4.1).

4.3 Examples

Example 4.3.1. The simplest example is characterized by the assignment

$$a = b = 0, \qquad M = 0,$$

namely by the system of *nonlinear* evolution PDEs (see (1.2))

$$\psi_{n,tt} - \Delta\psi_n = 2 \sum_{m=1,\, m\neq n}^{N} \frac{\psi_{n,t}\psi_{m,t} - \vec{\nabla}\psi_n \cdot \vec{\nabla}\psi_m}{\psi_n - \psi_m}. \tag{4.3}$$

The corresponding system of *linear* PDEs satisfied by the coefficients $\varphi_m(\vec{r},t)$ of the polynomial $\Psi(\psi;\vec{r},t)$ in the variable ψ of which the solutions $\psi_n(\vec{r},t)$ are the N zeros, see (3.10), reads as follows (see (2.6)):

$$\varphi_{m,tt} - \Delta\varphi_m = 0, \qquad m = 1,\ldots,N.$$

The general solution of these *linear* PDEs reads

$$\varphi_m(\vec{r},t) = \sum_{k=1}^{K} f_{mk}(\vec{r} - \vec{u}_k t), \tag{4.4}$$

with the KN functions $f_{mk}(\vec{r})$ arbitrary and the K constant S-vectors $\vec{u}_k$ having unit length, $u_k = 1$, but being otherwise arbitrary. Of course these solutions $\varphi_m(\vec{r},t)$ are *localized* if the arbitrary functions $f_{mk}(\vec{r})$ are themselves *localized*, but (motivated by Remark 4.2) we shall rather consider solutions that tend asymptotically to *nonvanishing* asymptotic values; and $\varphi_m(\vec{r},t)$ has the character of a *traveling wave* if $K = 1$.

The corresponding solutions $\psi_n(\vec{r},t)$ of the system of *nonlinear* PDEs (4.3) are the N zeros of the following polynomial of degree N in ψ:

$$\Psi(\psi;\vec{r},t) = \psi^N + \sum_{m=1}^{N} \psi^{N-m} \sum_{k=1}^{K} f_{mk}(\vec{r} - \vec{u}_k t) = \prod_{n}^{N} [\psi - \psi_n(\vec{r},t)].$$

Two solutions of this kind (with $S = 2$ respectively $S = 1$, $N = 3$, and a specific choice of the remaining free parameters) are displayed as animations in Section 5.

Example 4.3.2. An analogous example – but reported here for simplicity in the two-dimensional case ($S = 2$) – is characterized by the analogous assignment

$$S = 2, \qquad a = b = 0, \qquad M = 0,$$

namely by the system of *nonlinear* evolution PDEs in two-dimensional space (see (1.1))

$$\psi_{n,tt} - \psi_{n,xx} - \psi_{n,yy} = 2 \sum_{m=1,\, m\neq n}^{N} \frac{\psi_{n,t}\psi_{m,t} - \psi_{n,x}\psi_{m,x} - \psi_{n,y}\psi_{m,y}}{\psi_n - \psi_m}.$$

The corresponding system of (decoupled) PDEs satisfied by the coefficients $\varphi_m(x,y,t)$ reads

$$\varphi_{m,tt} - \varphi_{m,xx} - \varphi_{m,yy} = 0.$$

A class of regular solutions of this system of PDEs reads

$$\varphi_m\left(x,y,t\right)=J_0\left(\sqrt{(x-x_m)^2+(y-y_m)^2}\right)\left[A_m\cos(t)+B_m\sin(t)\right]+C_m, \tag{4.5}$$

where A_m, B_m and C_m are $3N$ arbitrary constants (possibly *complex*), and $J_0(r)$ is the zeroth-order Bessel function of the first kind. A solution of this kind (with $N=3$, and a specific choice of the remaining free parameters) is displayed as an animation in Section 5.

4.4 Isochronous version of the class of nonlinear PDEs of Klein–Gordon type

In this subsection we report, with even less commentary than in the (analogous) Subsection 3.4, an "ω-modified" version of (a subclass of) the system of *nonlinear* PDEs of Klein–Gordon type (1.2), which is again characterized by the property to possess an ample class of solutions *completely periodic* in time with period T, see (3.16).

Now we start from the subcase of the system of *nonlinear* PDEs of Klein–Gordon type (1.2) with

$$b=0, \qquad M=0,$$

and we set

$$\tilde{\psi}_n\left(\vec{r},t\right)=e^{i\lambda\omega t}\psi_n\left(\vec{\rho},\tau\right), \qquad \vec{\rho}\equiv\vec{\rho}\left(t\right)=e^{i\omega t}\vec{r}, \qquad \tau\equiv\tau\left(t\right)=\frac{e^{i\omega t}-1}{i\omega},$$

with ω an arbitrary *positive* constant and λ an *arbitrary real rational* number if a vanishes ($a=0$, see (1.2) and below), otherwise $\lambda=-1$. It is then easily seen that the new dependent variables $\tilde{\psi}_n\left(\vec{r},t\right)$ satisfy the following system of *nonlinear* PDEs of Klein–Gordon type:

$$\begin{aligned}
&\tilde{\psi}_{n,tt}-\Delta\tilde{\psi}_n-2i\omega(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_{n,t}-i\left(2\lambda+1\right)\omega\tilde{\psi}_{n,t}\\
&\quad-\lambda\left(\lambda+1\right)\omega^2\tilde{\psi}_n-\left(2\lambda+1\right)\omega^2(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_n-\omega^2(\vec{r}\cdot\vec{\nabla})^2\tilde{\psi}_n\\
&\quad=2\sum_{m=1,\,m\neq n}^{N}\left\{\frac{a-\left(\vec{\nabla}\tilde{\psi}_n\right)\cdot\left(\vec{\nabla}\tilde{\psi}_m\right)}{\tilde{\psi}_n-\tilde{\psi}_m}\right.\\
&\quad\left.+\frac{\left[\tilde{\psi}_{n,t}-i\lambda\omega\tilde{\psi}_n-i\omega(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_n\right]\left[\tilde{\psi}_{m,t}-i\lambda\omega\tilde{\psi}_m-i\omega(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_m\right]}{\tilde{\psi}_n-\tilde{\psi}_m}\right\}.
\end{aligned}$$

Note that this system is *autonomous* with respect to the time variable, but it features an *explicit* dependence on the space variable $\vec{r}$ (incidentally, in the last term in the left-hand side, $(\vec{r}\cdot\vec{\nabla})^2\equiv(\vec{r}\cdot\vec{\nabla})(\vec{r}\cdot\vec{\nabla})=S(\vec{r}\cdot\vec{\nabla})+\sum_{k,j=1}^{S}r_kr_j\nabla_k\nabla_j$, where of course r_j denotes the j-th component of the S-vector $\vec{r}$); and it is clearly *rotation invariant.*

5 Animations

In this section we show a few solutions of the Schrödinger and Klein–Gordon type problems treated in the previous sections, displayed as animations over time.

Let us begin with a brief description of the methodology – implemented via a software written using *Mathematica* – employed to obtain the numerical results presented below. After assigning the space dimension S, the coefficients $\varphi_n\left(\vec{r},t\right)$ (see (2.4) and (2.6)) and the monic polynomial $\Psi\left(\psi;\vec{r},t\right)$ of degree N in ψ (see (2.1)), we compute the N zeros $\psi_n\left(\vec{r},t\right)$ of this polynomial

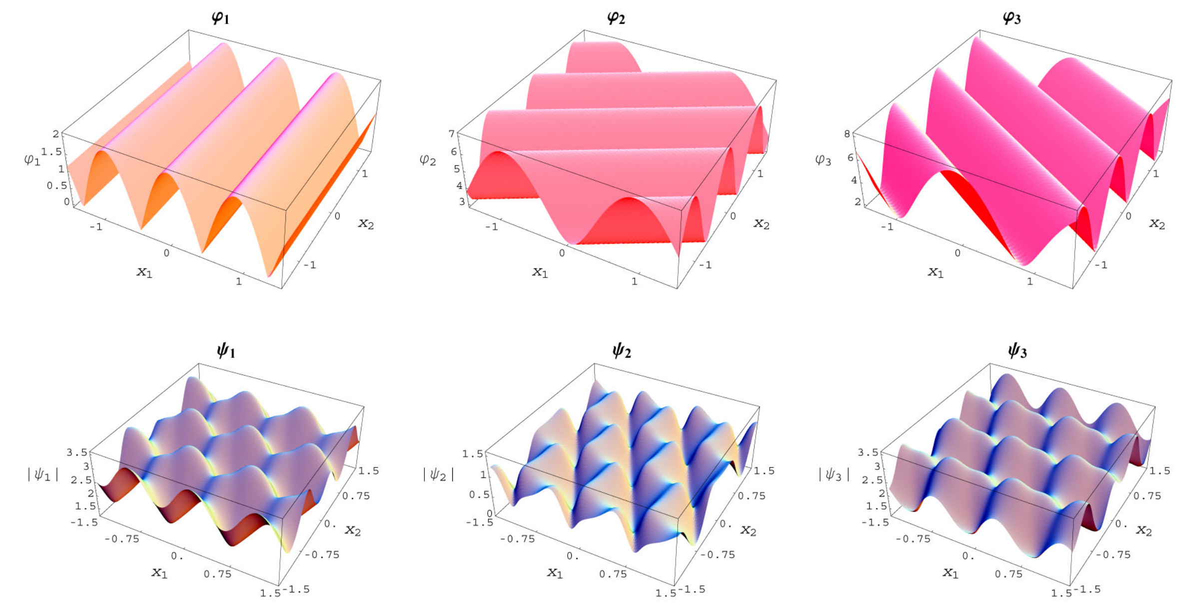

Figure 5.1.1. The $\varphi_m(\vec{r},t)$ functions and the absolute values of the $\psi_m(\vec{r},t)$ solutions for $0 \le t \le 1/8$.

as follows: firstly we create in the $(S+1)$-dimensional space of the independent variables $\vec{r}, t$ a lattice; next, we use a root-finding routine to calculate, at an appropriately chosen point of the lattice, the (generally *complex*) values of the N zeros $\psi_n(\vec{r},t)$ of the polynomial $\Psi(\psi;\vec{r},t)$, taking them in a generic order; and then we use an iterative root-finding procedure to calculate the N zeros of the polynomial $\Psi(\psi;\vec{r},t)$ at any new point of the lattice, making use of the zeros previously calculated at the nearest points of the lattice so as to preserve the same initial ordering of the zeros.

In the following we consider only examples with $N=3$. The animations are organized as arrays of synchronized subanimations, showing above the time evolution of the three $\varphi_m(\vec{r},t)$ functions, and below the corresponding time evolution of the three $\psi_n(\vec{r},t)$ solutions. In each frame of the animation (namely, for a fixed value of the time variable), we display the values of the functions $\varphi_m(\vec{r},t)$ and $\psi_n(\vec{r},t)$ (or their *absolute* values $|\varphi_m(\vec{r},t)|$ and $|\psi_n(\vec{r},t)|$, when these functions are *complex*), with respect to the x variable in two-dimensional plots if $S=1$, or with respect to the x and y variables in three-dimensional plots if $S=2$. In this paper we restrict attention only to cases with space dimension $S=1$ and $S=2$.

5.1 Solutions of (1.1)

In this subsection we present two numerical solutions of (1.1) displayed as animations, the first one with $S=2$ and the other two with $S=1$.

Example 5.1.1. The first animation corresponds to Example 3.3.1, with $S=2$, $a=b=0$, $W(r)=V(r)=0$ and the $\varphi_m(\vec{r},t)$ functions as in (3.12), namely (but with 3 different 2-vector parameters $\vec{v}_m$)

$$\varphi_m(\vec{r},t) = A_m + B_m \exp\left[-i\vec{v}_m(\vec{r}-\vec{v}_m t)\right],$$

with

$$A_1 = 1, \qquad A_2 = 5, \qquad A_3 = -5,$$

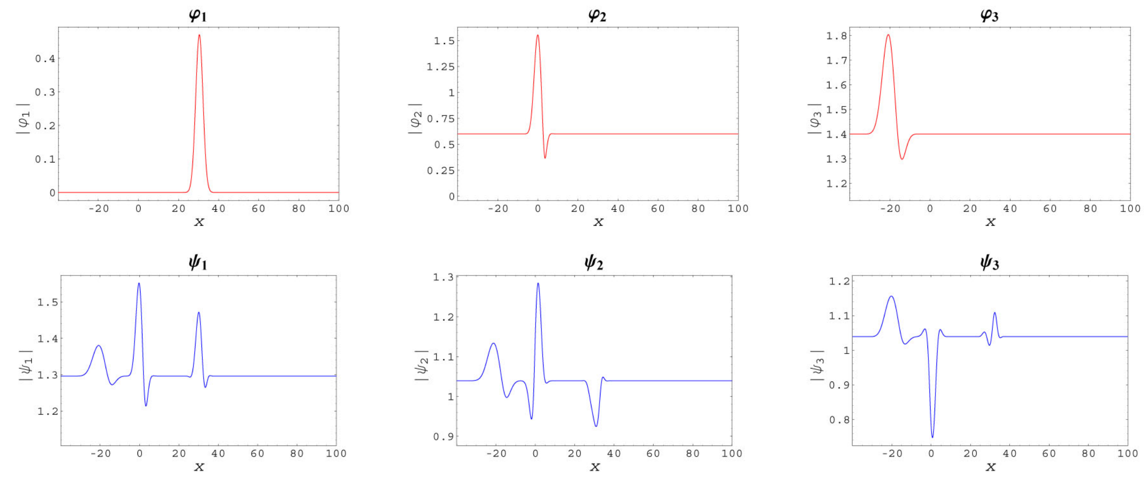

Figure 5.1.2. The absolute values of the $\varphi_m(x,t)$ functions and the absolute values of the $\psi_m(x,t)$ solutions for $0 \le t \le 20$.

$$B_1 = 1, \qquad B_2 = 2, \qquad B_3 = 3,$$
$$\vec{v}_1 = (4\sqrt{\pi}, 0), \qquad \vec{v}_2 = (-2\sqrt{\pi}, 2\sqrt{3\pi}), \qquad \vec{v}_3 = (-2\sqrt{\pi}, -2\sqrt{3\pi}).$$

Here the periods in time of the three $\varphi_m(\vec{r},t)$ functions are chosen to be the same, $T = 1/8$, and the animation is performed on the time interval $0 \le t \le T$, then closed in loop.

Fig. 5.1.1 is the first frame. To see the whole animation, please click on the following (external) link: http://www.emis.de/journals/SIGMA/2006/Paper088/Animation5.1.1.gif.

Example 5.1.2. The second animation corresponds to the Example 3.3.1, with $S = 1$, $a = b = 0$, $W(r) = V(r) = 0$ and the $\varphi_m(x,t)$ functions as in (3.14), namely

$$\varphi_m(x,t) = A_m + \frac{B_m}{(t + t_m + i\eta_m)^{1/2}} \exp\left[\frac{-i\,(x - x_m + i\eta_m v_m)^2}{4\,(t + t_m + i\eta_m)}\right],$$

with

$$\begin{aligned}
&A_1 = 0, && B_1 = 0.5, && x_1 = 30, && v_1 = 0.5, && \eta_1 = 1, && t_1 = 0.8,\\
&A_2 = -0.6, && B_2 = -0.9, && x_2 = 0, && v_2 = 1, && \eta_2 = 2, && t_2 = 0.4,\\
&A_3 = 1.4, && B_3 = 0.8, && x_3 = -20, && v_3 = 0.5, && \eta_3 = 7, && t_3 = 1.
\end{aligned}$$

Here the animation is performed on the time interval $0 \le t \le 20$.

Fig. 5.1.2 is the first frame. To see the whole animation, please click on the following (external) link: http://www.emis.de/journals/SIGMA/2006/Paper088/Animation5.1.2.gif.

5.2 Solutions of (1.2)

In this subsection we present three numerical solutions of (1.2) displayed as animations, two with $S = 2$ and the last with $S = 1$.

Example 5.2.1. The first animation corresponds to the Example 4.3.1, with $S = 2$, $a = b = 0$, $M = 0$ and the $\varphi_m(x,t)$ functions as in (4.4), with $K = 4$ and a very particular choice of the functions $\varphi_m(x,y,t)$:

$$\varphi_m(x,y,t) = A_m \exp\left[-\frac{\left(x - x_m^{(1)} - t\right)^2}{a_m}\right] + B_m \exp\left[-\frac{\left(x - x_m^{(2)} + t\right)^2}{b_m}\right]$$

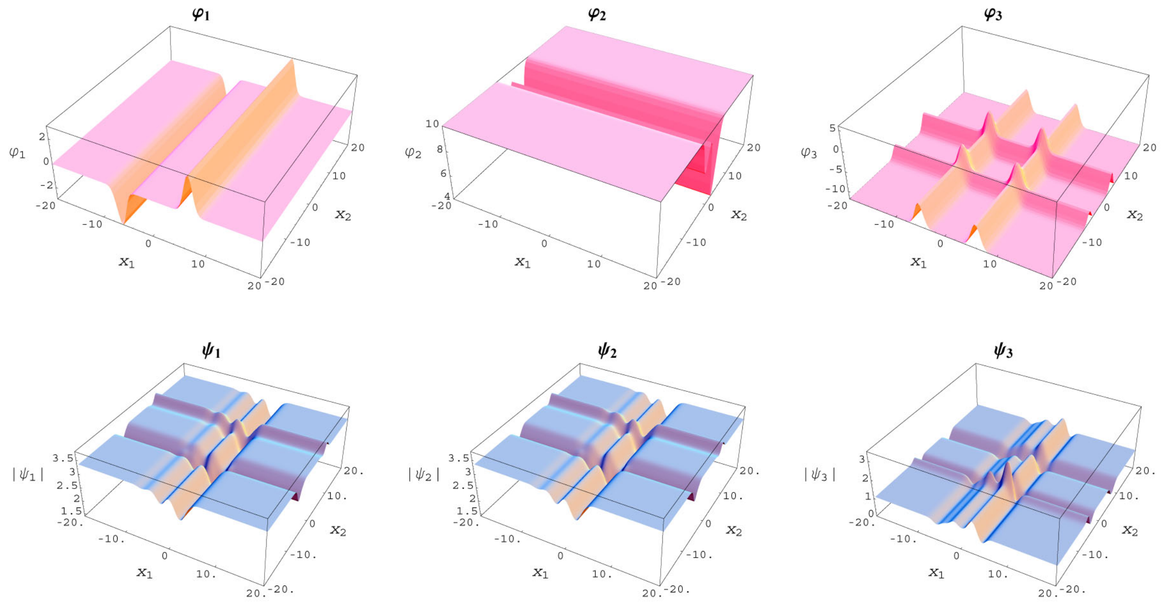

Figure 5.2.1. The $\varphi_m(\vec{r},t)$ functions and the absolute values of the $\psi_m(\vec{r},t)$ solutions for $0 \le t \le 20$.

$$+ C_m \exp\left[-\frac{\left(y - y_m^{(1)} - t\right)^2}{c_m}\right] + D_m \exp\left[-\frac{\left(y - y_m^{(2)} + t\right)^2}{d_m}\right] + E_m,$$

where

$$\begin{array}{lllll}
A_1 = -3, & B_1 = 3, & C_1 = 0, & D_1 = 0, & E_1 = 0, \\
A_2 = 0, & B_2 = 0, & C_2 = -6, & D_2 = -3, & E_2 = 10, \\
A_3 = 6, & B_3 = 6, & C_3 = 3, & D_3 = 4.5, & E_3 = -12, \\
a_1 = 1.5, & b_1 = 1.3, & c_1 = 1.3, & d_1 = 1.6, & \\
a_2 = 1.8, & b_2 = 1.2, & c_2 = 2, & d_2 = 1.6, & \\
a_3 = 1.4, & b_3 = 1.5, & c_3 = 1.4, & d_3 = 1.2, & \\
x_1^{(1)} = -6, & x_1^{(2)} = 7, & y_1^{(1)} = 0, & y_1^{(2)} = 0, & \\
x_2^{(1)} = 0, & x_2^{(2)} = 0, & y_2^{(1)} = 2.5, & y_2^{(2)} = -1.5, & \\
x_3^{(1)} = -5, & x_3^{(2)} = 6, & y_3^{(1)} = -5.5, & y_3^{(2)} = 4.5. &
\end{array}$$

Here the animation is performed on the time interval $0 \le t \le 20$.

Fig. 5.2.1 is the first frame. To see the whole animation, please click on the following (external) link: http://www.emis.de/journals/SIGMA/2006/Paper088/Animation5.2.1.gif.

Example 5.2.2. The second animation corresponds to the Example 4.3.2, with $S = 2$, $a = b = 0$, $M = 0$ and the $\varphi_m(\vec{r},t)$ functions as in (4.5):

$$\varphi_m(x,y,t) = J_0\left(\sqrt{(x-x_m)^2 + (y-y_m)^2}\right)\left[A_m \cos(t) + B_m \sin(t)\right] + C_m,$$

where

$$\begin{array}{lllll}
A_1 = 1, & B_1 = 0.1, & C_1 = 0, & x_1 = 5, & y_1 = 0,
\end{array}$$

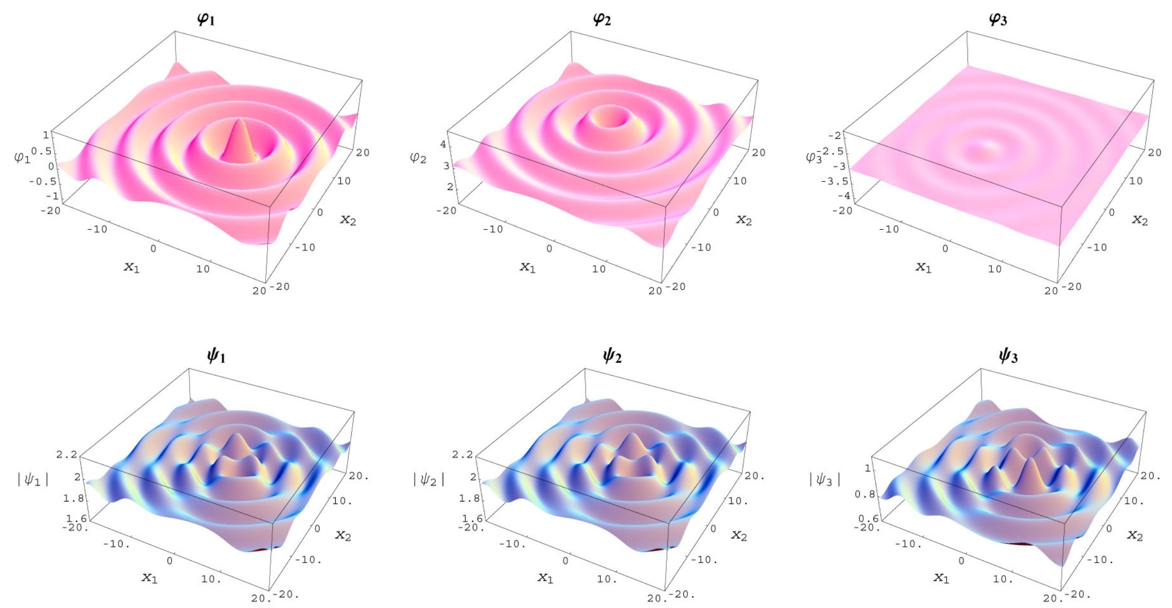

Figure 5.2.2. The $\varphi_m(\vec{r},t)$ functions and the absolute values of the $\psi_m(\vec{r},t)$ solutions for $0 \leq t \leq 2\pi$.

$$
\begin{aligned}
&A_2 = -1, \qquad B_2 = 1, \qquad C_2 = 3, \qquad x_2 = -5/2, \qquad y_2 = 5\sqrt{3}/2,\\
&A_3 = 0.1, \qquad B_3 = 1, \qquad C_3 = -3, \qquad x_3 = -5/2, \qquad y_3 = -5\sqrt{3}/2.
\end{aligned}
$$

The $\varphi_m(\vec{r},t)$ functions are periodic in time with period $T = 2\pi$. The animation is performed on the time interval $0 \leq t \leq T$, then closed in loop.

Fig. 5.2.2 is the first frame. To see the whole animation, please click on the following (external) link: http://www.emis.de/journals/SIGMA/2006/Paper088/Animation5.2.2.gif.

Example 5.2.3. The third animation corresponds again to the Example 4.3.1, with $S = 1$, $a = b = 0$, $M = 0$ and the $\varphi_m(x,t)$ functions as in (4.4) with $K = 2$, and with a very particular choice of the functions $\varphi_m(x,t)$:

$$
\varphi_m(x,t) = A_m \cos(x - t + B_m) + C_m \cos(x + t + D_m) + E_m,
$$

where

$$
\begin{aligned}
&A_1 = 1, \qquad B_1 = 0, \qquad C_1 = 3, \qquad D_1 = 0.1, \qquad E_1 = 0,\\
&A_2 = 2, \qquad B_2 = \pi, \qquad C_2 = -1, \qquad D_2 = \pi + 0.1, \qquad E_2 = 5,\\
&A_3 = -1, \qquad B_3 = \pi/2 + 0.1, \qquad C_3 = 2, \qquad D_3 = \pi/2, \qquad E_3 = -5.
\end{aligned}
$$

The $\varphi_m(x,t)$ functions are periodic in time with period $T = 2\pi$ and periodic with respect to the x variable with period $L = \pi$. The animation is performed on the time interval $0 \leq t \leq T$, then closed in loop.

Fig. 5.2.3 is the first frame. To see the whole animation, please click on the following (external) link: http://www.emis.de/journals/SIGMA/2006/Paper088/Animation5.2.3.gif.

6 Outlook

In future articles we plan to report additional investigations of these solvable (systems of) nonlinear evolution PDEs, and in particular to display visual animations of solutions of certain of these models in *three-dimensional* space.

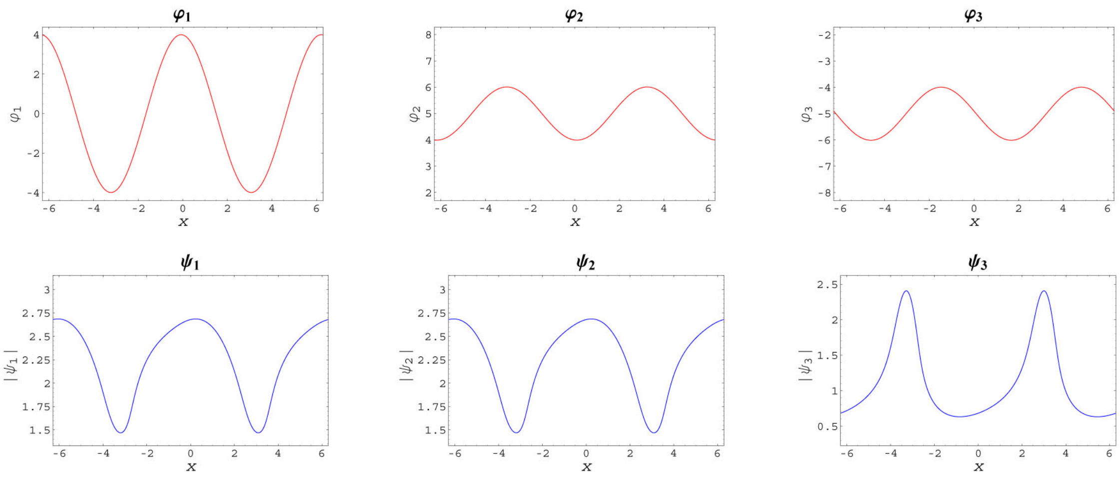

Figure 5.2.3. The $\varphi_m(x,t)$ functions and the absolute values of the $\psi_m(x,t)$ solutions for $0 \leq t \leq 2\pi$.

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