Symmetry, Integrability and Geometry: Methods and Applications Vol. 2 (2006), Paper 084, 20 pages

$\mathcal{R}$-Matrix and Baxter $\mathcal{Q}$-Operators for the Noncompact $SL(N,\mathbb{C})$ Invariant Spin Chain*

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Received October 30, 2006; Published online December 02, 2006
Original article is available at http://www.emis.de/journals/SIGMA/2006/Paper084/

Abstract. The problem of constructing the $SL(N,\mathbb{C})$ invariant solutions to the Yang–Baxter equation is considered. The solutions ($\mathcal{R}$-operators) for arbitrarily principal series representations of $SL(N,\mathbb{C})$ are obtained in an explicit form. We construct the commutative family of the operators $\mathcal{Q}_k(u)$ which can be identified with the Baxter operators for the noncompact $SL(N,\mathbb{C})$ spin magnet.

Key words: Yang–Baxter equation; Baxter operator

2000 Mathematics Subject Classification: 82B23; 82B20

To the memory of Vadim Kuznetsov

1 Introduction

The Yang–Baxter equation (YBE) plays an important role in the theory of completely integrable systems. Its solutions, the so-called $\mathcal{R}$-matrices (operators), are basic ingredients of Quantum Inverse Scattering Method (QISM) [1, 2]. The problem of constructing solutions to the YBE was thoroughly analyzed in the works of Drinfeld [3, 4], Jimbo [5] and many others. In the most studied case of the quantum affine Lie (super)algebras the universal $\mathcal{R}$-matrix was constructed in works of Rosso [6], Kirillov and Reshetikhin [7] and Khoroshkin and Tolstoy [8, 9]. The interrelation of YBE with the representation theory was elucidated by Kulish, Reshetikhin and Sklyanin [10, 11], who studied solutions of YBE for finite dimensional representations of the $GL(N,\mathbb{C})$ group. The $\mathcal{R}$-matrix on infinite dimensional spaces were not considered till recently.

In the present paper we construct the $SL(N,\mathbb{C})$ invariant $\mathcal{R}$-operator which acts on the tensor product of two principal series representations of the $SL(N,\mathbb{C})$ group. The spin chains with an infinite dimensional Hilbert space, so called noncompact magnets, are interesting in connection with the problem of constructing the Baxter $\mathcal{Q}$-operators [12] and the representation of Separated Variables (SoV) [13, 14]. The Baxter $\mathcal{Q}$-operators are known now for a number of models. Beside few exceptions these are the spin chains with a symmetry group of rank one. No regular method of constructing Baxter operators for models with symmetry groups

*This paper is a contribution to the Vadim Kuznetsov Memorial Issue "Integrable Systems and Related Topics". The full collection is available at http://www.emis.de/journals/SIGMA/kuznetsov.html of higher rank exists so far. However, in the studies of the noncompact $sl(2)$ magnets it was noticed that the transfer matrices with generic (infinite dimensional) auxiliary space are factorized into the product of Baxter operators. This property seems to be quite a general feature of the noncompact magnets[1] and can be related to the factorization of $\mathcal{R}$-matrix suggested in [15]. Thus the problem of constructing the Baxter operators is reduced, at least for the noncompact spin chains, to the problem of factorization of the $\mathcal{R}$-matrix. The factorizing operators for the $sl(N)$ invariant $\mathcal{R}$-matrix acting on the tensor product of two generic lowest weight $sl(N)$ modules for $N = 2, 3$ were constructed in [15]. Unfortunately, for a higher N the defining equations for the factorizing operators become too complicated to be solved directly.

To get some insight into a possible structure of solutions for a general N it is instructive to consider the problem of constructing an $\mathcal{R}$-operator for principal series representations of $SL(N, \mathbb{C})$. We remark here that contrary to naive expectations the principal series noncompact magnets appear to be in some respects simpler than their (in)finite dimensional cousins. Different spin chain models with $sl(N)$ symmetry differ by a functional realization of a quantum space. The less restrictions are imposed on the functions from a quantum space, the simpler becomes the analysis of algebraic properties of a model and the harder its exact solution. For instance, the solution of the noncompact $SL(2, \mathbb{C})$ spin magnet [16, 17] presents a quite nontrivial problem already for the spin chain of the length $L = 3$, while the analysis of this model (constructing the Baxter operators, SoV representation, etc) becomes considerably easier in comparison with its compact analogs. In particularly, the $\mathcal{R}$-operator in this model ($SL(2, \mathbb{C})$ spin magnet) has a rather simple form and, as one can easily verify, admit the factorized representation. All this suggests to consider the problem of constructing the $\mathcal{R}$-operator for the principal series representations of $SL(N, \mathbb{C})$ in first instance. In the paper we give the complete solution to this problem. We will obtain the explicit expression for the $\mathcal{R}$-operator, prove that the latter satisfies Yang–Baxter equation and, as was expected, enjoys the factorization property. It allows us to construct the Baxter $\mathcal{Q}$-operators as traces of a product of the factorizing operators.

We hope that the obtained results will also be useful for the construction of the representation of Separated Variables. So far the latter was constructed in an explicit form only for a very limited number of models (see [20, 21, 22, 16, 23, 24, 25, 26, 27]). It was noticed by Kuznetsov and Sklyanin [28, 29] that the kernel of a separating operator and Baxter operator have to be related to each other. So the knowledge of the Baxter operator for the $SL(N, \mathbb{C})$ magnets can shed light on the form of the separating operator.

The paper is organized as follows. In Section 2 we recall the basic facts about the principal series representations of the complex unimodular group. We construct the operators which intertwine the equivalent representations and describe their properties. This material is well known [32, 33] but we represent intertwinning operators in the form which is appropriate for our purposes. Section 3 starts with the description of our approach for constructing the $\mathcal{R}$-operator. We derive the defining equations on the $\mathcal{R}$-operator and solve them. Finally, we show that the obtained operator satisfies the Yang–Baxter equation. In Section 4 we study the properties of the factorizing operators in more detail and discuss their relation with the Baxter $\mathcal{Q}$-operators. Finally we construct the explicit realization of the operators in question as integral operators.

2 Principal series representations of the group $SL(N, \mathbb{C})$

The unitary principal series representations of the group $SL(N, \mathbb{C})$ can be constructed as follows [30, 31]. Let Z and H be the groups of the lower triangular matrices with unit elements

[1] This property holds also for the integrable models with $U_q(\widehat{sl}_n)$ symmetry, at least for $n = 2, 3$ [18, 19].

on a diagonal and the upper triangular matrices with unit determinant, respectively,

$$z = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ z_{21} & 1 & 0 & \dots & 0 \\ z_{31} & z_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{N1} & z_{N2} & \dots & z_{N,N-1} & 1 \end{pmatrix} \in Z, \qquad h = \begin{pmatrix} h_{11} & h_{12} & h_{13} & \dots & h_{1N} \\ 0 & h_{22} & h_{23} & \dots & h_{2N} \\ 0 & 0 & h_{33} & \dots & h_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & h_{N,N} \end{pmatrix} \in H.$$

Almost any matrix $g \in G = SL(N, \mathbb{C})$ admits the Gauss decomposition $g = zh$ (the group Z is the right coset G/H). The element $z_1 \in Z$ satisfying the condition

$$g^{-1} \cdot z = z_1 \cdot h \tag{2.1}$$

will be denoted by $z\bar{g}$, so that $g^{-1}z = z\bar{g} \cdot h$. Thus we can speak about a local right action of G on Z. Later in (2.4), also h, dependent on g and z as in (2.1), will be used again.

Let $\boldsymbol{\alpha}$ be a character of the group H defined by the formula

$$\boldsymbol{\alpha}(h) = \prod_{k=1}^{N} h_{kk}^{-\sigma_k - k}\, \bar{h}_{kk}^{-\bar{\sigma}_k - k}, \tag{2.2}$$

where $\bar{h}_{kk} \equiv (h_{kk})^*$ is the complex conjugate of h_{kk}, whereas in general $\sigma_k^* \neq \bar{\sigma}_k$. Since $\det h = 1$ the function $\boldsymbol{\alpha}(h)$ depends only on the differences $\sigma_{k,k+1} \equiv \sigma_k - \sigma_{k+1}$ and can be rewritten in the form

$$\boldsymbol{\alpha}(h) = \prod_{k=1}^{N-1} \Delta_k^{1-\sigma_{k,k+1}}\, \bar{\Delta}_k^{1-\bar{\sigma}_{k,k+1}} = \prod_{k=1}^{N-1} \Delta_k^{n_k}\, |\Delta_k|^{2(1-\bar{\sigma}_{k,k+1})},$$

where $\Delta_k = \prod_{i=1}^{k} h_{ii}$ and $n_k = \bar{\sigma}_{k,k+1} - \sigma_{k,k+1}$, $k = 0, \dots, N-1$, are integer numbers[2]. One can always assume that parameters σ_k satisfy the restriction

$$\sigma_1 + \sigma_2 + \dots + \sigma_N = N(N-1)/2. \tag{2.3}$$

The map $g \to T^{\alpha}(g)$, where

$$[T^{\boldsymbol{\alpha}}(g)\Phi](z) = \boldsymbol{\alpha}(h^{-1})\Phi(z\bar{g}), \tag{2.4}$$

defines a principal series representation of the group $SL(N, \mathbb{C})$ on a suitable space of functions on the group Z, $\Phi(z) = \Phi(z_{21}, z_{31}, \dots, z_{NN-1})$ [30, 31]. The operator $T^{\alpha}(g)$ is a unitary operator on the Hilbert space $L^2(Z)$,

$$\langle \Phi_1 | \Phi_2 \rangle = \int \prod_{1 \leq i < k \leq N} d^2 z_{ki}\, (\Phi_1(z))^*\, \Phi_2(z)\,,$$

if the character $\boldsymbol{\alpha}'(h) = \boldsymbol{\alpha}(h) \prod_{k=1}^{N} |h_{kk}|^{2k}$ is a unitary one, i.e. $|\boldsymbol{\alpha}'| = 1$. This condition holds if $\sigma_{k,k+1}^* + \bar{\sigma}_{k,k+1} = 0$ for $k = 1, \dots, N-1$, i.e.

$$\sigma_{k,k+1} = -\frac{n_k}{2} + i\lambda_k, \qquad \bar{\sigma}_{k,k+1} = \frac{n_k}{2} + i\lambda_k, \qquad k = 1, 2, \dots, N-1, \tag{2.5}$$

n_k is integer and λ_k is real.

The unitary principal series representation $T^{\boldsymbol{\alpha}}$ is irreducible. Two representations $T^{\boldsymbol{\alpha}}$ and $T^{\boldsymbol{\alpha}'}$ are unitary equivalent if and only if the corresponding parameters $(\sigma_1, \dots, \sigma_N)$ and $(\sigma'_1, \dots, \sigma'_N)$ are related to each other by a permutation [30, 31].

[2]From now in, since each variable a comes along with its antiholomorphic twin $\bar{a}$ we will write down only holomorphic variant of equations.

2.1 Generators and right shifts

We will need the explicit expression for the generators of infinitesimal $SL(N,\mathbb{C})$ transformations. The latter are defined in the standard way

$$\left[T^{\boldsymbol{\alpha}}\left(\mathbb{1}+\sum_{ik}\epsilon^{ik}\mathcal{E}_{ki}\right)\Phi\right](z)=\Phi(z)+\sum_{ik}\epsilon^{ik}E_{ki}\Phi(z)+O(\epsilon^2),$$

where $\mathcal{E}_{ik}$, $(1\le i,k\le N)$, are the generators in the fundamental representation of the $SL(N,\mathbb{C})$ group,

$$(\mathcal{E}_{ik})_{nm}=\delta_{in}\delta_{km}-\frac{1}{N}\delta_{ik}\delta_{nm}.$$

The generators E_{ik} are linear differential operators in the variables z_{mn}, $(1\le n<m\le N)$ which satisfy the commutation relation

$$[E_{ki},E_{nm}]=\delta_{in}E_{km}-\delta_{km}E_{ni}. \tag{2.6}$$

It follows from definition (2.4) that the lowering generators E_{ki}, $k>i$, are the generators of left shifts, $\Phi(z)\to L(z_0)\Phi(z)=\Phi(z_0^{-1}z)$. In a similar way we define the generators of right shifts, $\Phi(z)\to R(z_0)\Phi(z)=\Phi(zz_0)$,

$$\Phi\left(z\left(\mathbb{1}+\sum_{k>i}\epsilon^{ik}\mathcal{E}_{ki}\right)\right)=\left(1+\sum_{k>i}\epsilon^{ik}D_{ki}+O\big(\epsilon^2\big)\right)\Phi(z). \tag{2.7}$$

Since right and left shifts commute one concludes that $[E_{ki},D_{nm}]=0$ $(k>i,n>m)$. Clearly, the generators D_{ki} satisfy the same commutation relation as E_{ki}, equation (2.6)

$$[D_{ki},D_{nm}]=\delta_{in}D_{km}-\delta_{km}D_{ni}.$$

The explicit expression for the generators of left and right shifts reads

$$E_{ki}=-\sum_{m=1}^{i}z_{im}\frac{\partial}{\partial z_{km}},\qquad D_{ki}=\sum_{m=k}^{N}z_{mk}\frac{\partial}{\partial z_{mi}}=-\sum_{m=1}^{i}\tilde z_{im}\frac{\partial}{\partial\tilde z_{km}}, \tag{2.8}$$

where $\tilde z_{ki}=(z^{-1})_{ki}$ and we recall that $z_{ii}=1$. Let us notice here that the operator D_{ki} depends on the variables in the k-th and i-th columns of the matrix z, or on the variables in the k-th and i-th rows of the inverse matrix z^{-1}.

The generators E_{ki} can be expressed in terms of the generators D_{ki} as follows

$$E_{ki}=-\sum_{mn}z_{im}\big(D_{nm}+\delta_{nm}\sigma_m\big)\big(z^{-1}\big)_{nk}, \tag{2.9}$$

where D_{nm} is nonzero only for $n>m$. To derive equation (2.9) it is sufficient to notice that for the infinitesimal transformation $g=\mathbb{1}+\epsilon\cdot\mathcal{E}=\mathbb{1}+\sum_{ik}\epsilon^{ik}\mathcal{E}_{ki}$ the elements of the Gauss decomposition of the matrix $g^{-1}z$ can be represented in the following form: $z\bar g=z(\mathbb{1}-\epsilon_-(z))$ and $h=\mathbb{1}-\epsilon_+(z)$. The matrices $\epsilon_-(z)$ and $\epsilon_+(z)$ are the lower and upper diagonal parts of the matrix $z^{-1}(\epsilon\cdot\mathcal{E})z$. Namely, the matrix $\epsilon_-(z)_{ki}$ is nonzero only for $k>i$, $\epsilon_-(z)_{ki}=(z^{-1}(\epsilon\cdot\mathcal{E})z)_{ki}$, while $\epsilon_+(z)_{ki}=(z^{-1}(\epsilon\cdot\mathcal{E})z)_{ki}$, for $k\le i$ and zero otherwise. Thus one finds that the infinitesimal transformation (2.4) is essentially the right shift generated by $z_0=\mathbb{1}-\epsilon_-(z)$. Finally, taking into account that $\sum\sigma_k=N(N-1)/2$ one obtains the representation (2.9) for the symmetry generators.

It is convenient to rewrite equation (2.9) in the matrix form

$$E = -z\,(D+\sigma)\,z^{-1}, \tag{2.10}$$

where $E = \sum_{mn} e_{mn}E_{nm}$, $D = \sum_{n>m} e_{mn}D_{nm}$ and $\sigma = \sum_n e_{nn}\sigma_n$ and the matrices e_{nm} form the standard basis in $\mathrm{Mat}(N\times N)$, $(e_{nm})_{ik} = \delta_{in}\delta_{mk}$. Let us recall that z and z^{-1} are lower triangular matrices while $D+\sigma$ is an upper triangular matrix

$$D+\sigma = \begin{pmatrix} \sigma_1 & D_{21} & D_{31} & \dots & D_{N1} \\ 0 & \sigma_2 & D_{32} & \dots & D_{N2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma_{N-1} & D_{N,N-1} \\ 0 & 0 & \dots & 0 & \sigma_N \end{pmatrix} \tag{2.11}$$

The dependence of the generators E on the representation $T^{\boldsymbol{\alpha}}$ resides in the parameters σ_n entering the matrix σ. The same formulae hold for the antiholomorphic generators $\bar{E}$.

Closing this subsection we give some identities that will be useful in the further analysis:

$$D_{ki}z = z\,e_{ki}, \tag{2.12a}$$

$$D_{ki}\,z^{-1} = -e_{ki}z^{-1}, \tag{2.12b}$$

$$[D_{ki}, D] = \sum_{n<i} D_{kn}e_{ni} - \sum_{n>k} e_{kn}D_{ni}. \tag{2.12c}$$

The first two formulae follow directly from definition (2.7), while the last one is a consequence of the commutation relations for the generators D_{ki}. We remind also that the operators D_{ki} are defined (nonzero) only for $k>i$.

2.2 Intertwining operators

It is known that two principal series representations $T^{\boldsymbol{\alpha}}$ and $T^{\boldsymbol{\alpha}'}$ are equivalent if the parameters which specify the representations, $\vec{\sigma} = (\sigma_1,\dots,\sigma_N)$ and $\vec{\sigma}' = (\sigma_1',\dots,\sigma_N')$, are related to each other by a permutation, $\vec{\sigma}' = P\,\vec{\sigma}$ (of course, it is assumed that the parameters $\{\bar{\sigma}_k\}$, $\{\bar{\sigma}_k'\}$ in the antiholomorphic sector are related by the same permutation, $\vec{\bar{\sigma}}' = P\vec{\bar{\sigma}}$) [30]. An arbitrary permutation can be represented as a composition of the elementary permutations, P_k, which interchange the k-th and $k+1$-st component of the vectors, $\vec{\sigma}\,(\vec{\bar{\sigma}})$,

$$P_k, (\dots,\sigma_k,\sigma_{k+1},\dots) = (\dots,\sigma_{k+1},\sigma_k,\dots).$$

The operator U_k intertwining the representations, $T^{\boldsymbol{\alpha}}$ and $T^{\boldsymbol{\alpha}'}$ which differ by the elementary permutation of spins P_k, $\boldsymbol{\alpha}' = P_k\boldsymbol{\alpha}$, has the form

$$U_k = D_{k+1,k}^{\sigma_{k,k+1}}\,\bar{D}_{k+1,k}^{\bar{\sigma}_{k,k+1}}, \qquad U_kT^{\boldsymbol{\alpha}} = T^{\boldsymbol{\alpha}'}U_k, \tag{2.13}$$

where $\sigma_{k,k+1} = \sigma_k - \sigma_{k+1}$ ($\bar{\sigma}_{k,k+1} = \bar{\sigma}_k - \bar{\sigma}_{k+1}$), and $D_{k+1,k}(\bar{D}_{k+1,k})$ is the generator of the right shift. Let us note that the operator $D_{k+1,k}$ depends on the variables z_{nm} in the k-th and $k+1$-st columns of the matrix z. After the change of variables

$$z_{k+1,k} = x_{k+1,k}, \quad z_{m,k+1} = x_{m,k+1}, \quad z_{m,k} = x_{m,k} + x_{k+1,k}x_{m,k+1}, \quad k<m\le N \tag{2.14}$$

the operator $D_{k+1,k}$ turns into a derivative with respect to $x_{k+1,k}$, $D_{k+1,k} = \partial_{x_{k+1,k}}$, which means that the construction given in equation (2.13) results in well-defined operator. Moreover, if the powers in equation (2.13) satisfy the condition $\sigma^*_{k,k+1} + \bar{\sigma}_{k,k+1} = 0$, the operator U_k is a unitary operator on $L^2(Z)$.

To prove that operator U_k is an intertwining operator it is sufficient to check that it intertwines the generators E and E' in the representations $T^{\boldsymbol{\alpha}}$ and $T^{\boldsymbol{\alpha}'}$, $E'U_k = U_kE$. To this end it is convenient to use the representation (2.10) for the generators. Using commutation relations (2.12) one finds

$$U_k z = z\left(1+\alpha D_{k+1,k}^{-1} e_{k+1,k}\right) U_k, \qquad U_k z^{-1} = \left(1-\alpha D_{k+1,k}^{-1} e_{k+1,k}\right) z^{-1} U_k \tag{2.15}$$

and

$$U_k D = \left(D + \alpha D_{k+1,k}^{-1}\left(\sum_{n<k} D_{k+1,n} e_{nk} - \sum_{n>k+1} e_{k+1,n} D_{n,k}\right)\right) U_k, \tag{2.16}$$

where $\alpha = \sigma_{k,k+1}$. (Let us note that the operator $D_{k+1,k}^{-1}$ commutes with $e_{k+1,k}z^{-1}$ and with the operators in the sum in (2.16), so its position can be changed). Starting from U_kE and moving the operator U_k to the right with the help of equations (2.15) and (2.16) one gets after some simplifications

$$\begin{aligned} U_k E &= -z\left(D + \sigma + \alpha(e_{k+1,k+1} - e_{k,k}) - \alpha(\alpha - \sigma_{k,k+1}) e_{k+1,k} D_{k+1,k}^{-1}\right) z^{-1} U_k \\ &= -z(D+\sigma')z^{-1}U_k = E'U_k. \end{aligned}$$

(One has to take into account that $\alpha = \sigma_{k,k+1}$ and the matrix $\sigma' = \sigma + \sigma_{k,k+1}(e_{k+1,k+1} - e_{k,k})$ differs from the matrix σ by the transposition $\sigma_k \leftrightarrow \sigma_{k+1}$.)

Making use of the change of variables (2.14) one can represent the operator U_k in the form of an integral operator

$$[U_k\Phi](z) = A(\sigma_{k,k+1}) \int d^2\zeta [z_{k+1,k} - \zeta]^{-1-\sigma_{k,k+1}} \Phi(z_\zeta), \tag{2.17}$$

where $[z]^\sigma = z^\sigma \bar{z}^{\bar\sigma}$,

$$A(\sigma) \stackrel{\text{def}}{=} A(\sigma,\bar\sigma) = \frac{1}{\pi} i^{\bar\sigma - \sigma}\Gamma(1+\sigma)/\Gamma(-\bar\sigma) \tag{2.18}$$

and

$$z_\zeta = z\big(1 + (\zeta - z_{k+1,k}) e_{k+1,k}\big).$$

Let us note that the matrices z and z_ζ differ from each other by the elements in the k-th column only.

The operator U_k depends on the difference of $\sigma_{k,k+1}$, $U_k = U(\sigma_{k,k+1})$, which is determined by the representation which the operator acts on. We do not display the dependence on this parameter explicitly, but want to note here that in the product $U_{k+1}U_k$, the first operator is $U_k = U_k(\sigma_{k,k+1})$, while the second one is $U_{k+1} = U_{k+1}(\sigma_{k,k+2})$, since the operator U_k interchanges the parameter σ_k with σ_{k+1}. Similarly, one finds that $U_k\,U_k = U_k(\sigma_{k+1,k})U_k(\sigma_{k,k+1}) = \mathbb{1}$.

The intertwining operators U_k satisfy the same commutation relations as the operators of elementary permutations P_k

$$U_kU_k = \mathbb{1}, \tag{2.19a}$$

$$U_kU_n = U_k\,U_n, \qquad \text{for} \quad |k-n|>1 \tag{2.19b}$$

$$U_kU_{k+1}U_k = U_{k+1}U_kU_{k+1}. \tag{2.19c}$$

The first relation had been already explained. The second one is a trivial consequence of the commutativity of the generators of right shifts, $[D_{k+1,k}, D_{n+1,n}] = 0$ for $|k-n|>1$. The last relation can be checked by the direct calculation with the help of the integral representation (2.17).

Obviously, the operator intertwining the representations $T^{\boldsymbol{\alpha}}$ and $T^{\boldsymbol{\alpha}'}$ such that the characters $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ are related to each other by some permutation can be constructed as a certain combination of the operators U_k, $k=1,\dots,N-1$. The intertwining operators, U_k, play an important role in the construction of an $\mathcal{R}$-operator which we will discuss in the next section.

3 $\mathcal{R}$-operator

Let us recall that an $\mathcal{R}$-operator is a linear operator which acts on the tensor product of two spaces $\mathbb{V}_1 \otimes \mathbb{V}_2$, depends on the spectral parameter u and satisfies the Yang–Baxter relation

$$\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u-w)\mathcal{R}_{23}(v-w) = \mathcal{R}_{23}(v-w)\mathcal{R}_{13}(u-w)\mathcal{R}_{12}(u-w). \tag{3.1}$$

As usual, it is implied that the operator $\mathcal{R}_{ik}(u)$ acts nontrivially on the tensor product $\mathbb{V}_i \otimes \mathbb{V}_k$. In the case under consideration each space $\mathbb{V}_i$ ($\mathbb{V}_k$) is assumed to be a vector space of some representation of the group $SL(N,\mathbb{C})$. We are interested in constructing an $\mathcal{R}$-operator which acts on the space $L^2(Z) \otimes L^2(Z)$ and is invariant with respect to $SL(N,\mathbb{C})$ transformations,

$$[T^{\boldsymbol{\alpha}_1}(g) \otimes T^{\boldsymbol{\alpha}_2}(g), \mathcal{R}_{12}(u)] = 0.$$

The form of the $\mathcal{R}$-operator strongly depends on the spaces $\mathbb{V}_1$ and $\mathbb{V}_2$ which it acts on. The $\mathcal{R}$-operator has the simplest form when both space $\mathbb{V}_1$ and $\mathbb{V}_2$ is the vector space of the fundamental (N-dimensional) representation of the $SL(N,\mathbb{C})$ group, $\mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_f$. Namely, in the case that $\mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_3 = \mathbb{V}_f$ the solution of the Yang–Baxter equation (3.1) is given by the operator $\mathcal{R}_{ik}(u) = u + P_{ik}$, where P_{ik} is the permutation operator on $\mathbb{V}_i \otimes \mathbb{V}_k$. This solution can also be represented in the form

$$\mathcal{R}_{ik}(u) = u + \sum_{mn} e_i^{nm} e_k^{mn},$$

Substituting the generators e_i^{nm} by the generators E^{nm} in some generic representation of the $SL(N,\mathbb{C})$ group in the above formula one gets the $\mathcal{R}$-operator on the space $\mathbb{V}_1 \otimes \mathbb{V}_f$. Such an operator is called Lax operator. Let us recall here that in a generic representation of the $SL(N,\mathbb{C})$ group there are two sets of generators, holomorphic E_{ik} and the antiholomorphic $\bar{E}_{ik}$, so that we define two Lax operators, the holomorphic one, $L(u)$, and the antiholomorphic one, $\bar{L}(\bar{u})$,

$$L(u) = u + \sum_{mn} e_{mn}\, E_{nm}, \qquad \bar{L}(\bar{u}) = \bar{u} + \sum_{mn} e_{mn} \bar{E}_{nm}. \tag{3.2}$$

The Yang–Baxter equation (3.1) on the spaces $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_f$ takes the form

$$\mathcal{R}_{12}(u-v, \bar{u}-\bar{v}) L_1(u) L_2(v) = L_2(v) L_1(u) \mathcal{R}_{12}(u-v, \bar{u}-\bar{v}), \tag{3.3a}$$
$$\mathcal{R}_{12}(u-v, \bar{u}-\bar{v}) \bar{L}_1(\bar{u}) \bar{L}_2(\bar{v}) = \bar{L}_2(\bar{v}) \bar{L}_1(\bar{u}) \mathcal{R}_{12}(u-v, \bar{u}-\bar{v}). \tag{3.3b}$$

These equations can be considered as defining equations for the operator $\mathcal{R}_{12}(u,\bar{u})$. Let us notice that the $\mathcal{R}$-operator depends on two spectral parameters u and $\bar{u}$, which are not supposed to be related each other. We will show later that the spectral parameters are subject to the restriction, $u - \bar{u} = n$, where n is an integer number[3]. In the next subsection we discuss the approach for solving equations (3.3) suggested in [15].

3.1 Factorized ansatz for $\mathcal{R}$-matrix

Let us remark that the Lax operator (3.2) depends on the spectral parameter u and N parameters σ_k. Since the character $\boldsymbol{\alpha}$ (equation (2.2)) and, hence, the generators of the $SL(N,\mathbb{C})$ group depend only on the differences $\sigma_{k,k+1} = \sigma_k - \sigma_{k+1}$, one concludes that the Lax operators depend on N-independent parameters which can be chosen as $u_k = u - \sigma_k$, $k = 1, \ldots, N$, $L(u) =$

[3]The quantization of the spectral parameters for the $SL(2,\mathbb{C})$ spin magnet was observed in [16].

$L(u_1,\dots,u_N)$. These parameters appear quite naturally when one uses the representation (2.10) for the generators. Indeed, the Lax operator can be represented as

$$L(u) = z\big(u-\sigma-D\big)z^{-1}, \tag{3.4}$$

and the parameters u_k are nothing else as diagonal elements of the matrix $(u-\sigma-D)$.

Let us represent the operator $\mathcal{R}_{12}$ in the form $\mathcal{R}_{12} = P_{12}\mathbb{R}_{12}$. Note, that since $\mathbb{V}_1 = \mathbb{V}_2 = L^2(Z)$ the permutation operator P_{12} is unambiguously defined on $\mathbb{V}_1\otimes\mathbb{V}_2$. We will associate the matrix variables z and w with the spaces $\mathbb{V}_1$ and $\mathbb{V}_2$, respectively, so that the vectors in $\mathbb{V}_1\otimes\mathbb{V}_2$ are functions $\Phi(z,w)$. The action of the permutation operator P_{12} is defined in a conventional way, $P_{12}\Phi(z,w) = \Phi(w,z)$. The Yang–Baxter relation (3.3) can be rewritten as follows[4]

$$\begin{aligned}&\mathbb{R}_{12}(u-v)L_1(u_1,\dots,u_N)L_2(v_1,\dots,v_N)\\&\qquad= L_1(v_1,\dots,v_N)L_2(u_1,\dots,u_N)\mathbb{R}_{12}(u-v).\end{aligned} \tag{3.5}$$

The parameters v_k are defined as $v_k = v-\rho_k$ where the parameters $\{\rho\}$ specify the representation of $SL(N,\mathbb{C})$ group on the space $\mathbb{V}_2$. It is seen from equation (3.5) that action of the operator $\mathbb{R}_{12}(u-v)$ results in permutation of the arguments $\{u\}$ and $\{v\}$ of the Lax operators L_1 and L_2. Since we have already constructed the operators $\{U_k\}$ which interchange the components of the string $\{u\}$ ($\{v\}$) it seems reasonable to try to find operators that carry out permutations of the strings $\{u\}$ and $\{v\}$.

It was suggested in [15] to look for the operator $\mathbb{R}_{12}$ in the factorized form

$$\mathbb{R}_{12}(u-v) = \mathbb{R}^{(1)}\,\mathbb{R}^{(2)}\cdots\mathbb{R}^{(N)}, \tag{3.6}$$

where each operator $\mathbb{R}^{(k)}$ interchanges the arguments u_k and v_k of the Lax operators,

$$\begin{aligned}&\mathbb{R}^{(k)}L_1(u_1,\dots,u_k,\dots u_N)L_2(v_1,\dots,v_k,\dots,v_N)\\&\qquad= L_1(u_1,\dots,v_k,\dots,u_N)L_2(v_1,\dots,u_k,\dots,v_N)\mathbb{R}^{(k)}.\end{aligned} \tag{3.7}$$

Evidently, if such operators can be constructed then equation (3.5) will follow immediately from equations (3.6) and (3.7). We note also that since any permutation inside the string $\{u\}$ can be carried out with the use of the operators U_k (and similarly for the string $\{v\}$), it is sufficient to find an operator which interchanges some elements u_k and v_i of strings $\{u\}$ and $\{v\}$.

3.2 Exchange operator

In this subsection we will construct operator S which exchanges the arguments u_1 and v_N of the Lax operators L_1 and L_2,

$$\begin{aligned}&S\,L_1(u_1,u_2,\dots,u_N)L_2(v_1,\dots,v_{N-1},v_N)\\&\qquad= L_1(v_N,u_2,\dots,u_N)L_2(v_1,\dots,v_{N-1},u_1)S.\end{aligned} \tag{3.8}$$

It turns out that the operator S has a surprisingly simple form. Taking into account that the character $\boldsymbol{\alpha}(h)$ (equation (2.2)) can be written in the form

$$\boldsymbol{\alpha}(h) = \prod_{k=1}^{N} h_{kk}^{u_k-k}\bar{h}_{kk}^{\bar{u}_k-k}, \tag{3.9}$$

[4]From now on we will write down the equations in the holomorphic sector only and suppress the dependence on "barred" variables, $\bar{u},\bar{v}$, etc for brevity.

it is easy to figure out that operator S has to intertwine the representations $T^{\boldsymbol{\alpha}} \otimes T^{\boldsymbol{\beta}}$ and $T^{\boldsymbol{\alpha}'} \otimes T^{\boldsymbol{\beta}'}$,

$$S\big(T^{\boldsymbol{\alpha}} \otimes T^{\boldsymbol{\beta}}\big) = \big(T^{\boldsymbol{\alpha}'} \otimes T^{\boldsymbol{\beta}'}\big)S, \tag{3.10}$$

where the characters $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$, ($\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$) are related to each other as follows (see equation (2.2))

$$\boldsymbol{\alpha}'(h) = h_{11}^{v_N-u_1}\bar{h}_{11}^{\bar{v}_N-\bar{u}_1}\boldsymbol{\alpha}(h), \qquad \boldsymbol{\beta}'(h) = h_{NN}^{u_1-v_N}\bar{h}_{NN}^{\bar{u}_1-\bar{v}_N}\boldsymbol{\beta}(h).$$

It turns out that the simplest operator intertwining the representations in question gives a solution to equation (3.8). To construct an operator with the required transformation properties let us consider the matrix $w^{-1}z$. Under the transformation $z \to g^{-1}z = (z\bar{g})h(z,g)$, $w \to g^{-1}w = (w\bar{g})h(w,g)$ it transforms as follows

$$w^{-1}z = h^{-1}(w,g)(w\bar{g})^{-1}(z\bar{g})h(z,g).$$

Taking into account that the matrices z and w are lower triangular matrices while $h(z,g)$, $h(w,g)$ are upper triangular ones, one finds that the matrix element $(w^{-1}z)_{N1}$ transforms in a simple way

$$\big(w^{-1}z\big)_{N1} = h_{NN}^{-1}(w,g)\big((w\bar{g})^{-1}(z\bar{g})\big)_{N1}h_{11}(z,g). \tag{3.11}$$

It suggests to define the operator S as follows

$$[S(\gamma,\bar{\gamma})\Phi]\,(z,w) = \big(\big(w^{-1}z\big)_{N1}\big)^{\gamma}\big(\big(w^{-1}z\big)_{N1}^{*}\big)^{\bar{\gamma}}\Phi(z,w), \tag{3.12}$$

where $\gamma = u_1 - v_N$, $\bar{\gamma} = \bar{u}_1 - \bar{v}_N$. Notice, that the difference $\gamma - \bar{\gamma}$ has to be an integer number.

It follows from equations (3.11) and (3.12) that the operator S has necessary transformation properties (3.10). Thus it remains to show that the operator S satisfies equation (3.8). We start with two useful identities for the Lax operator $L(u_1,\dots,u_N)$

$$\sum_m z_{Nm}^{-1}L_{mk} = u_N\, z_{Nk}^{-1}, \qquad \sum_m L_{km}z_{m1} = u_1 z_{k1}, \tag{3.13}$$

which follow immediately from the representation (3.4) and a triangularity of the matrix D, equation (2.11). Next, using the definitions of the operator of right shifts (2.7), (2.8) one finds

$$\left(\sum_{k>i} e_{ik}D_{ki}^{(z)}\right)\big(w^{-1}z\big)_{N1} = \sum_{k>1} e_{1k}\big(w^{-1}z\big)_{Nk} = -\big(w^{-1}z\big)_{N1}e_{11} + \sum_{k\geq 1} e_{1k}\big(w^{-1}z\big)_{Nk}.$$

Then it is straightforward to derive

$$S(\gamma,\bar{\gamma})\,(L_1)_{nm}\,S^{-1}(\gamma,\bar{\gamma}) = \big(z(u-\gamma e_{11}-D)z^{-1}\big)_{nm} + \frac{\gamma}{(w^{-1}z)_{N1}}z_{n1}w_{Nm}^{-1}. \tag{3.14}$$

where u is a diagonal matrix, $u = \sum_k u_k e_{kk}$. Taking into account that $\gamma = u_1 - v_N$ one finds that the first term in the rhs is the Lax operator $L(v_N, u_2, \dots, u_N)$. Quite similarly, one derives for the Lax operator L_2

$$\begin{aligned} S(\gamma,\bar{\gamma})\,(L_2)_{nm}\,S^{-1}(\gamma,\bar{\gamma}) &= \big(w(v+\gamma e_{NN}-D)w^{-1}\big)_{nm} - \frac{\gamma}{(w^{-1}z)_{N1}}z_{n1}w_{Nm}^{-1} \\ &= \big(L_2(v_1,\dots,v_{N-1},u_1)\big)_{nm} - \frac{\gamma}{(w^{-1}z)_{N1}}z_{n1}w_{Nm}^{-1}. \end{aligned} \tag{3.15}$$

where $v = \sum_k v_k e_{kk}$. The equation (3.8) then follows immediately from equations (3.14), (3.15) and (3.13).

3.3 Permutation group and the star-triangle relation

Let us consider the operators we have constructed in more details. As was already noted, a character $\boldsymbol{\alpha}$ can be written in the form (3.9) so that the representation $T^{\boldsymbol{\alpha}}$ is completely determined by the numbers $u_1,\ldots,u_N$ (and $\bar u_1,\ldots,\bar u_N$ which are always implied). Respectively, the tensor product $T^{\boldsymbol{\alpha}}\otimes T^{\boldsymbol{\beta}}$ is determined by the numbers $u_1,\ldots,u_N$ and $v_1,\ldots,v_N$ where the latter refer to the representation $T^{\boldsymbol{\beta}}$. Let us join these parameters into the string

$$\boldsymbol{v}=(v_1,\ldots,v_N,u_1,\ldots,u_N)\equiv(v_1,\ldots,v_N,v_{N+1},\ldots,v_{2N})$$

(notice the inverse order of the parameters $\{u\}$ and $\{v\}$) and accept the notation $\mathbb{T}^{\boldsymbol{v}}$ for the tensor product $T^{\boldsymbol{\alpha}}\otimes T^{\boldsymbol{\beta}}$. The string $\boldsymbol{v}$ fixes not only the characters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ but also the spectral parameters, u and v of the Lax operators L_1 and L_2.

For a definiteness we will assume that the representations $T^{\boldsymbol{\alpha}}$ and $T^{\boldsymbol{\beta}}$ are unitary. We recall that this results in the restriction (2.5) which can be represented as

$$u^*_{k,k+1}+\bar u_{k,k+1}=0,\qquad v^*_{k,k+1}+\bar v_{k,k+1}=0,\qquad k=1,\ldots,N-1.$$

We will assume also that $(v_N-u_1)^*+(\bar v_N-u_1)=v^*_{N,N+1}-\bar v_{N,N+1}=0$, so that $v^*_{k,k+1}+\bar v_{k,k+1}=0$ for $k=1,\ldots,2N-1$. Under this condition the representation $\mathbb{T}^{\boldsymbol{v}'}$ where $\boldsymbol{v}'$ is an arbitrary permutation of the string $\boldsymbol{v}$ is unitary. The vector space of the representation $\mathbb{T}^{\boldsymbol{v}}$ will be denoted as $\mathbb{V}_{\boldsymbol{v}}=L^2(Z\times Z)$.

In Section 2.2 we have constructed the intertwining operators U_k, equation (2.13). Since one has now two sets of such operators and also the exchange operator S, it is convenient to introduce the notation

$$\mathbb{U}_k=\begin{cases}\mathbb{1}\otimes U_k, & k=1,\ldots,N-1,\\ S, & k=N,\\ U_{k-N}\otimes\mathbb{1}, & k=N+1,\ldots,2N-1.\end{cases}\tag{3.16}$$

The operators $\mathbb{U}_k$ depend on the "quantum numbers" of the space they act on, namely $\mathbb{U}_k=\mathbb{U}_k(v_{k+1,k})$, $v_{k+1,k}=v_{k+1}-v_k$, and

$$\mathbb{U}_k:\mathbb{V}_{\boldsymbol{v}}\mapsto\mathbb{V}_{\boldsymbol{v}_k},\qquad\text{where}\qquad \boldsymbol{v}_k=P_k\boldsymbol{v}.$$

It will always be implied that the argument of the operator $\mathbb{U}_k$ is determined by the representation $\mathbb{T}^{\boldsymbol{v}}$ it acts on. It can be formulated in the following way: Let us consider the lattice $\mathcal{L}_{\boldsymbol{v}}$ in $\mathbb{C}^{2N}$ formed by vectors $\{\boldsymbol{v}_i\}$ obtained from the vector $\boldsymbol{v}$ by all possible permutation of its components. For each point $\boldsymbol{v}_i$ we identify the corresponding space $\mathbb{V}_{\boldsymbol{v}_i}$. The operators $\mathbb{U}_k$ map the spaces attached to the lattice points related by elementary permutations to one another. So one can omit the index k of the operator but show the lattice points $\boldsymbol{v}_i$ and $\boldsymbol{v}_{i'}$ as the argument of the operator $\mathbb{U}_k(v_{k+1,k})\to\mathbb{U}(\boldsymbol{v}_{i'},\boldsymbol{v}_i)$. So far we consider operators which connect the points related by elementary permutation. But it is obvious that one can construct the operator which maps $\mathbb{V}_{\boldsymbol{v}_i}$ to $\mathbb{V}_{\boldsymbol{v}_j}$ where $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are two arbitrary points of the lattice. We will show that such operator depends only on the points $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$, and does not depend on the path connecting these points. To this end one has to show that the operators $\mathbb{U}_k$ satisfy the same commutation relations as the operators of the elementary permutation P_k: $P_k^2=\mathbb{1}$, $[P_k,P_n]=0$ for $|n-k|>1$ and $P_{k+1}P_kP_{k+1}=P_kP_{k+1}P_k$. Thus we have to show that

$$\mathbb{U}_k\mathbb{U}_k=\mathbb{1},\tag{3.17a}$$

$$\mathbb{U}_k\mathbb{U}_n=\mathbb{U}_n\mathbb{U}_k,\qquad |n-k|>1\tag{3.17b}$$

$$\mathbb{U}_k\mathbb{U}_{k+1}\mathbb{U}_k=\mathbb{U}_{k+1}\mathbb{U}_k\mathbb{U}_{k+1}.\tag{3.17c}$$

Taking into account equations (2.19) which hold for the operators U_k one easily figures out that it is sufficient to check only those of equations (3.17) which involve the operator $\mathbb{U}_N$. The first equation, $\mathbb{U}_N\mathbb{U}_N = \mathbb{1}$, is obvious. The second one, $\mathbb{U}_N\mathbb{U}_k = \mathbb{U}_k\mathbb{U}_N$, $|N-k|>1$, follows immediately from equations (2.13) and (3.12) if one takes into account that $D^z_{k+1,k}(w^{-1}z)_{N1} = D^w_{i+1,i}(w^{-1}z)_{N1} = 0$ for $k>1$ and $i<N-1$.

The last identity, equation (3.17c), is nothing else as a slightly camouflaged version of the integral identity known as the star-triangle relation[5]. If we restore the dependence on the spectral parameters it takes the form

$$\mathbb{U}_N(b)\mathbb{U}_{N+1}(a+b)\mathbb{U}_N(a) = \mathbb{U}_{N+1}(a)\mathbb{U}_N(a+b)\mathbb{U}_{N+1}(b), \tag{3.18a}$$

$$\mathbb{U}_N(b)\mathbb{U}_{N-1}(a+b)\mathbb{U}_N(a) = \mathbb{U}_{N-1}(a)\mathbb{U}_N(a+b)\mathbb{U}_{N-1}(b). \tag{3.18b}$$

Let us consider equation (3.18a). We recall that $\mathbb{U}_{N+1}(a) = (D^z_{21})^a(\bar{D}^{\bar{z}}_{21})^{\bar{a}}$. After the change of variables (2.14) this operator takes the form $\mathbb{U}_{N+1}(a) = \partial^a\,\bar{\partial}^{\bar{a}}$, where $\partial \equiv \partial_x$ and $x = x_{21}$. It turn, the matrix element $(w^{-1}z)_{N1}$ is a linear function of x,

$$(w^{-1}z)_{N1} = Cx + D = C(x+x_0), \qquad x_0 = D/C,$$

where C and D depend on other variables and can be treated as constants. It is easy to see that equation (3.18a) is equivalent to the statement about commutativity of the operators

$$G_a = (x^a\,\partial^a)(\bar{x}^{\bar{a}}\bar{\partial}^{\bar{a}}) \qquad \text{and} \qquad H_b = (\partial^b x^b)(\bar{\partial}^{\bar{b}}\bar{x}^{\bar{b}}).$$

The latter follows from the observation that both operators are some functions of the operators $x\partial$ and $\bar{x}\bar{\partial}$, $G_a = g_a(x\partial,\bar{x}\bar{\partial})$ and $H_b = h_a(x\partial,\bar{x}\bar{\partial})$. This line of reasoning is due to Isaev [36].

The property of commutativity of the operators H_b and G_a can be expressed as the integral identity which presents a more conventional form of the star-triangle relation

$$A(a)A(b)\int d^2\xi \frac{1}{[x-\xi]^{a+1}[\xi-\zeta]^{-a-b}[\xi-x']^{b+1}} \\ = \frac{A(a+b)}{[x-\zeta]^{-b}[x-x']^{1+a+b}[x'-\zeta]^{-a}}. \tag{3.19}$$

Here the function $A(a)$ is defined in equation (2.17) and $[x]^a = x^a\bar{x}^{\bar{a}}$. Using equation (3.19) and the integral representation (2.17) for the operators U_k it is straightforward to verify equations (3.18).

Thus we have proved that the operators $\mathbb{U}_k$ (3.16) satisfy the commutation relations (3.17). Now one can construct the operator $\mathbb{U}(\boldsymbol{v}_j,\boldsymbol{v}_i)$ which maps $\mathbb{V}_{\boldsymbol{v}_i} \mapsto \mathbb{V}_{\boldsymbol{v}_j}$ where vectors $\boldsymbol{v}_i$, $\boldsymbol{v}_j$ belong to the lattice $\mathcal{L}_{\boldsymbol{v}}$. By definition, the vectors $\boldsymbol{v}_i$, $\boldsymbol{v}_j$ are related by some permutation $P(ij)$, $\boldsymbol{v}_j = P(ij)\boldsymbol{v}_i$. The permutation $P(ij)$ can be always represented as the product of the elementary permutations, $P(ij) = P_{i_m}\cdots P_{i_1}$. Then we define the operator $\mathbb{U}(\boldsymbol{v}_j,\boldsymbol{v}_i)$ as

$$\mathbb{U}(\boldsymbol{v}_j,\boldsymbol{v}_i) = \mathbb{U}_{i_m}\cdots\mathbb{U}_{i_1}.$$

Due to equations (3.17) the operator $\mathbb{U}(\boldsymbol{v}_j,\boldsymbol{v}_i)$ does not depend on the way the decomposition of the permutation $P(ij)$ onto elementary permutations is done. The operator $\mathbb{U}(\boldsymbol{v}',\boldsymbol{v})$ intertwines the representations $\mathbb{T}^{\boldsymbol{v}}$ and $\mathbb{T}^{\boldsymbol{v}'}$

$$\mathbb{U}(\boldsymbol{v}',\boldsymbol{v})\mathbb{T}^{\boldsymbol{v}} = \mathbb{T}^{\boldsymbol{v}'}\mathbb{U}(\boldsymbol{v}',\boldsymbol{v})$$

[5]The star-triangle relation is, in some sense, a key feature of the Yang–Baxter equation (3.1), see [35] for a nice review.

and interchanges the parameters in the product of Lax operators as follows

$$\mathbb{U}(\boldsymbol{v}',\boldsymbol{v})L_1(v_{N+1},\ldots,v_{2N})L_2(v_1,\ldots,v_N)=L_1(v'_{N+1},\ldots,v'_{2N})L_2(v'_1,\ldots,v'_N)\mathbb{U}(\boldsymbol{v}',\boldsymbol{v}).$$

It follows from equation (3.5) that the operator $\mathbb{R}_{12}(u-v)$ can be identified with $\mathbb{U}(\boldsymbol{v}',\boldsymbol{v})$ for the special $\boldsymbol{v}'$. Namely, one gets

$$\mathbb{R}_{12}(u-v)=\mathbb{U}(\boldsymbol{v}',\boldsymbol{v}), \tag{3.20}$$

where

$$\boldsymbol{v}=(v_1,\ldots,v_N,u_1,\ldots,u_N),\qquad \boldsymbol{v}'=(u_1,\ldots,u_N,v_1,\ldots,v_N).$$

As a consequence the $\mathcal{R}$-operator takes the form

$$\mathcal{R}_{12}(u-v)=P_{12}\mathbb{U}(\boldsymbol{v}',\boldsymbol{v}).$$

Note, that since each operator $\mathbb{U}_k$ depends on the difference $v_{k,k+1}$, the operator $\mathbb{U}(\boldsymbol{v}',\boldsymbol{v})$ depends on the differences $\sigma_{k,k+1}$ and $\rho_{k,k+1}$, which specify the representations $T^{\boldsymbol{\alpha}}$ and $T^{\boldsymbol{\beta}}$ in the tensor product $T^{\boldsymbol{\alpha}}\otimes T^{\boldsymbol{\beta}}$, and the spectral parameter $u-v$.

It is quite easy to show that the constructed $\mathcal{R}$-operator satisfies the Yang–Baxter relation (3.1). The latter can be rewritten in the form

$$\begin{aligned}&(P_{23}P_{12}P_{23})\,\mathbb{R}_{23}(\vec{u},\vec{v})\mathbb{R}_{12}(\vec{u},\vec{w})\mathbb{R}_{23}(\vec{v},\vec{w})\\&\qquad=(P_{12}P_{23}P_{12})\,\mathbb{R}_{12}(\vec{v},\vec{w})\mathbb{R}_{23}(\vec{u},\vec{w})\mathbb{R}_{12}(\vec{u},\vec{v}),\end{aligned} \tag{3.21}$$

where we have shown all arguments of $\mathbb{R}$-operators explicitly, that is $\mathbb{R}_{12}(u-v)=\mathbb{R}_{12}(\vec{u},\vec{v})$, $\vec{u}=(u_1,\ldots,u_N)$ and so on. Since $P_{23}P_{12}P_{23}=P_{12}P_{23}P_{12}$ one has to check that the product of $\mathbb{R}$-operators in the l.h.s and r.h.s are equal. One can easily find that the operators on the both sides result in the same permutation of the parameters $(\vec{u},\vec{v},\vec{w})\to(\vec{w},\vec{v},\vec{u})$. Since the operators are constructed from the operators $\mathbb{U}_k$ which obey the relations (3.17), these operators, as was explained earlier, are equal.

4 Factorizing operators and Baxter $\mathcal{Q}$-operators

In the previous section we have obtained the expression for the operator $\mathbb{R}_{12}$, equation (3.20). For the construction of the Baxter $\mathcal{Q}$-operators it is quite useful to represent the $\mathbb{R}$-operator in the factorized form (3.6). Each operator $\mathbb{R}^{(k)}$ interchanges the components u_k and v_k of the Lax operators, equation (3.7) and can be expressed in terms of the elementary permutation operators $\mathbb{U}_k$ as follows

$$\mathbb{R}^{(k)}=(\mathbb{U}_{N+k-1}\cdots\mathbb{U}_{N+1})(\mathbb{U}_k\cdots\mathbb{U}_{N-1})\mathbb{U}_N(\mathbb{U}_{N-1}\cdots\mathbb{U}_k)\,(\mathbb{U}_{N+1}\cdots\mathbb{U}_{N+k-1}) \tag{4.1}$$

Taking into account the definition of the operators $\mathbb{U}_k$, equation (3.16), it is straightforward to check that the operator $\mathbb{R}^{(k)}$ satisfies equation (3.7). Indeed, sequence of the operators in (4.1) results in the following permutation of the parameters $\vec{u}$ and $\vec{v}$ in the product of Lax operators

$$\begin{aligned}&(\cdots v_k,v_{k+1}\cdots v_N|u_1\cdots u_{k-1},u_k\cdots)\xrightarrow{\mathbb{U}_{N+1}\cdots\mathbb{U}_{N+k-1}}(\cdots v_k,v_{k+1}\cdots v_N|u_k,u_1\cdots u_{k-1}\cdots)\\&\xrightarrow{\mathbb{U}_{N-1}\cdots\mathbb{U}_k}(\cdots v_{k+1}\cdots v_N,v_k|u_k,u_1\cdots u_{k-1}\cdots)\xrightarrow{\mathbb{U}_N}(\cdots v_{k+1}\cdots v_N,u_k|v_k,u_1\cdots u_{k-1}\cdots)\\&\xrightarrow{\mathbb{U}_k\cdots\mathbb{U}_{N-1}}(\cdots u_k,v_{k+1}\cdots v_N|v_k,u_1\cdots u_{k-1}\cdots)\\&\xrightarrow{\mathbb{U}_{N+k-1}\cdots\mathbb{U}_{N+1}}(\cdots u_k,v_{k+1}\cdots v_N|u_1\cdots u_{k-1},v_k\cdots),\end{aligned}$$

where we have displayed the relevant arguments only.

It is easy to show that the operator (4.1) intertwines the representations $T^{\boldsymbol{\alpha}} \otimes T^{\boldsymbol{\beta}}$ and $T^{\boldsymbol{\alpha}_{k,\lambda}} \otimes T^{\boldsymbol{\beta}_{k,-\lambda}}$, where

$$\boldsymbol{\alpha}_{k,\lambda}(h) = h_{kk}^{-\lambda}\, \bar{h}_{kk}^{-\bar{\lambda}}\, \boldsymbol{\alpha}(h), \qquad \boldsymbol{\beta}_{k,-\lambda}(h) = h_{kk}^{\lambda}\, \bar{h}_{kk}^{\bar{\lambda}}\, \boldsymbol{\beta}(h), \tag{4.2}$$

with $\lambda = u_k - v_k$, $\bar{\lambda} = \bar{u}_k - \bar{v}_k$. The operator $\mathbb{R}^{(k)}$ is completely determined by the characters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and the parameter $\lambda, (\bar{\lambda})$, which we will refer to as the spectral parameter, i.e. $\mathbb{R}^{(k)} = \mathbb{R}^{(k)}(\lambda|\boldsymbol{\alpha}, \boldsymbol{\beta})$. Henceforth we accept the shorthand notation, $\mathbb{R}^{(k)}(\lambda|\boldsymbol{\alpha}, \boldsymbol{\beta}) \to \mathbb{R}^{(k)}(\lambda)$, omitting the dependence on the characters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$. To avoid misunderstanding we stress that the product of operators $\mathbb{R}^{(i)}(\mu)\mathbb{R}^{(k)}(\lambda)$ reads in explicit form as

$$\mathbb{R}^{(i)}(\mu)\mathbb{R}^{(k)}(\lambda) \equiv \mathbb{R}^{(i)}(\mu|\boldsymbol{\alpha}_{k,\lambda}, \boldsymbol{\beta}_{k,-\lambda})\mathbb{R}^{(k)}(\lambda|\boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{4.3}$$

In a full analogy with $\mathcal{R}$-operator we accept the notation $\mathbb{R}^{(k)}_{ab}$ for the operator which acts nontrivially on the tensor product of spaces $\mathbb{V}_a$ and $\mathbb{V}_b$. The expression (3.6) for the $\mathcal{R}$-operator can be written in the form

$$\mathcal{R}_{12}(u-v) = P_{12}\mathbb{R}_{12}(u-v) = P_{12}\mathbb{R}^{(1)}_{12}(u_1 - v_1)\mathbb{R}^{(2)}_{12}(u_2 - v_2)\cdots\mathbb{R}^{(N)}_{12}(u_N - v_N). \tag{4.4}$$

We recall that $u_k = u - \sigma_k$ and $v_k = v - \rho_k$, where the parameters σ and ρ define the characters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, see equations (2.2) and (2.3).

The operators $\mathbb{R}^{(k)}_{ab}(\lambda)$ possess a number of remarkable properties

$$\mathbb{R}^{(k)}_{12}(0) = \mathbb{1}, \tag{4.5a}$$
$$\mathbb{R}^{(k)}_{12}(\lambda)\mathbb{R}^{(k)}_{12}(\mu) = \mathbb{R}^{(k)}_{12}(\lambda+\mu), \tag{4.5b}$$
$$\mathbb{R}^{(k)}_{12}(\lambda)\mathbb{R}^{(j)}_{23}(\mu) = \mathbb{R}^{(j)}_{23}(\mu)\mathbb{R}^{(k)}_{12}(\lambda), \qquad j \neq k, \tag{4.5c}$$
$$\mathbb{R}^{(k)}_{12}(\lambda)\mathbb{R}^{(k)}_{23}(\lambda+\mu)\mathbb{R}^{(k)}_{12}(\mu) = \mathbb{R}^{(k)}_{23}(\mu)\mathbb{R}^{(k)}_{12}(\lambda+\mu)\mathbb{R}^{(k)}_{23}(\lambda), \tag{4.5d}$$
$$\mathbb{R}^{(k)}_{12}(\lambda - \sigma_k + \rho_k)\mathbb{R}^{(i)}_{12}(\lambda - \sigma_i + \rho_i) = \mathbb{R}^{(i)}_{12}(\lambda - \sigma_i + \rho_i)\mathbb{R}^{(k)}_{12}(\lambda - \sigma_k + \rho_k). \tag{4.5e}$$

Equations (4.5a) and (4.5b) follow from equations (4.1) and (3.12), while to prove the last three equations it is sufficient to check that the operators on both sides result in the same permutation of the parameters $\vec{u}$, $\vec{v}$, $\vec{w}$, (see equation (3.21)). Namely, the equations (4.5c)–(4.5e) are the deciphered form of the equations

$$\mathbb{R}^{(k)}_{12}\mathbb{R}^{(j)}_{23} = \mathbb{R}^{(j)}_{23}\mathbb{R}^{(k)}_{12}, \qquad \mathbb{R}^{(k)}_{12}\mathbb{R}^{(k)}_{23}\mathbb{R}^{(k)}_{12} = \mathbb{R}^{(k)}_{23}\mathbb{R}^{(k)}_{12}\mathbb{R}^{(k)}_{23} \qquad \text{and} \qquad \mathbb{R}^{(k)}_{12}\mathbb{R}^{(i)}_{12} = \mathbb{R}^{(i)}_{12}\mathbb{R}^{(k)}_{12}.$$

Let us notice that equation (4.5e) indicates that the operators $\mathbb{R}^k(u_k - v_k)$ in the expression for the $\mathcal{R}$-matrix, equation (4.4), can stand in arbitrary order.

We remark here that YBE for the $\mathcal{R}$-operator (3.6) is the corollary of the properties (4.5c), (4.5d) and (4.5e) of the operators $\mathbb{R}^{(k)}$. One can expect that these equations will hold for the $\mathcal{R}$ matrix for the generic representation of $sl(N)$ algebra since they express the property of consistency of equations (3.7). At the same time the possibility to represent the operators $\mathbb{R}^{(k)}$ as the product of elementary operators $\mathbb{U}_k$ is a specific feature of the principal series representations. Taking in mind this possibility we give here the alternative proof of YBE. We recall that YBE is equivalent to the following identity for the operators $\mathbb{R}_{ik}$ (see discussion after equation (3.21))

$$\mathbb{R}_{23}(u-v)\mathbb{R}_{12}(u-w)\mathbb{R}_{23}(v-w) = \mathbb{R}_{12}(v-w)\mathbb{R}_{23}(u-w)\mathbb{R}_{12}(u-v). \tag{4.6}$$

Using the factorized form (4.4) for the operator $\mathbb{R}_{ik}$ and using equations (4.5c) and (4.5e) one can bring the l.h.s and r.h.s of equation (4.6) into the form

$$\begin{aligned}&\mathbb{R}^{(1)}_{23}(u_1 - v_1)\mathbb{R}^{(1)}_{12}(u_1 - w_1)\mathbb{R}^{(1)}_{23}(v_1 - w_1)\cdots\mathbb{R}^{(N)}_{23}(u_N - v_N)\\ &\qquad\times \mathbb{R}^{(N)}_{12}(u_N - w_N)\mathbb{R}^{(N)}_{23}(v_N - w_N)\end{aligned}$$

and

$$\mathbb{R}^{(1)}_{12}(v_1-w_1)\mathbb{R}^{(1)}_{23}(u_1-w_1)\mathbb{R}^{(1)}_{12}(u_1-v_1)\cdots\mathbb{R}^{(N)}_{12}(v_N-w_N)\\ \times\mathbb{R}^{(N)}_{23}(u_N-w_N)\mathbb{R}^{(N)}_{12}(u_N-v_N),$$

respectively. By virtue of equation (4.5d) one finds that they are equal.

So we have shown that if the operators $\mathbb{R}^{(k)}$ obey equations (4.5c)–(4.5e) then the $\mathcal{R}$-matrix (4.4) satisfies YBE (3.1). Moreover, the operators $\mathbb{R}^{(k)}$ can be considered as a special case of the $\mathcal{R}$-operator. Namely, let us take the character $\boldsymbol{\beta}(h)$ to be $\boldsymbol{\beta}(h)=\boldsymbol{\alpha}_{k,\lambda}(h)$. Then it follows from equations (4.2) that the operator

$$\mathcal{R}^{(k)}_{12}(\lambda)=P_{12}\mathbb{R}^{(k)}_{12}(\lambda) \tag{4.7}$$

is a $SL(N,\mathbb{C})$ invariant operator,

$$\mathcal{R}^{(k)}_{12}(\lambda)\left(T^{\boldsymbol{\alpha}}\otimes T^{\boldsymbol{\alpha}_{k,\lambda}}\right)(g)=\left(T^{\boldsymbol{\alpha}}\otimes T^{\boldsymbol{\alpha}_{k,\lambda}}\right)(g)\mathcal{R}^{(k)}_{12}(\lambda).$$

Let us note that the operators $\mathbb{R}^{(k)}(\lambda)$ depend only on the part of parameters σ_i and ρ_i which specify the characters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. For instance, an inspection of equation (4.1) shows that the operator $\mathbb{R}^{(N)}(\lambda)$ depends only on the parameters $\sigma_{k,k+1}=\sigma_k-\sigma_{k+1}$ (the character $\boldsymbol{\alpha}$), $(\mathbb{R}^{(N)}(\lambda)\equiv\mathbb{R}^{(k)}(\lambda|\sigma_{12},\ldots,\sigma_{N-1,N}))$, while the operator $\mathbb{R}^{(1)}(\lambda)$ depends on $\rho_{k,k+1}$ (the character $\boldsymbol{\beta}$), and similarly for others. It means that the condition $\boldsymbol{\beta}(h)=\boldsymbol{\alpha}_{k,\lambda}(h)$ does not reduce the number of the independent parameters the operator $\mathbb{R}^{(k)}$ depends on.

Further, let us return to equation (4.4) and put $u_k-v_k=\lambda$. Then, provided that $\boldsymbol{\beta}(h)=\alpha_{k,\lambda}(h)$ one finds $u_i-v_i=0$ unless $i=k$. Taking into account equation (4.5a) and $u-v=\sum_k(u_k-v_k)/N=\lambda/N$ one derives

$$\mathcal{R}^{(k)}_{12}(\lambda)=P_{12}\mathbb{R}^{(k)}_{12}(\lambda)=\mathcal{R}_{12}\left(\frac{\lambda}{N}\right)\Big|_{\boldsymbol{\beta}(h)=\boldsymbol{\alpha}_{k,\lambda}(h)}.$$

It means that the operator $\mathcal{R}^{(k)}$ coincides with the $\mathcal{R}$-operator at the "special point". In case $\lambda+\bar\lambda^*=0$, the operator $\mathcal{R}^{(k)}_{12}(\lambda)$ is a unitary operator on the Hilbert space $\mathbb{V}_1\otimes\mathbb{V}_2=L^2(Z\times Z)$ which carries the representation $T^{\boldsymbol{\alpha}}\otimes T^{\boldsymbol{\alpha}_{k,\lambda}}$,

$$\left(\mathcal{R}^{(k)}_{12}(\lambda)\right)^\dagger\mathcal{R}^{(k)}_{12}(\lambda)=\mathbb{1}.$$

Moreover, these operators for different k are unitary equivalent, for example

$$\mathcal{R}^{(k)}_{12}(\lambda)=W_k^\dagger\mathcal{R}^{(N)}_{12}(\lambda)W_k. \tag{4.8}$$

The operator W_k can be easily read off from equation (4.1). It factorizes into the product of two operators, $W_k=W_k^{(1)}\widetilde{W}_k^{(2)}$, where the unitary operators $W_k^{(1)}$ and $\widetilde{W}_k^{(2)}$ act in the spaces $\mathbb{V}_1$ and $\mathbb{V}_2$, respectively,

$$W_k^{(1)}=\mathbb{U}_{2N-1}(\sigma_{k-1,N})\cdots\mathbb{U}_{N+k-1}(\sigma_{k-1,k}),\\ \widetilde{W}_k^{(2)}=\mathbb{U}_{N-1}(\sigma_{k,N}+\lambda)\cdots\mathbb{U}_k(\sigma_{k,k+1}+\lambda).$$

Here we display explicitly the arguments of the intertwining operators.

Let us consider the homogeneous spin chain of length L and define the operators $\mathcal{Q}_k(\lambda)$ by[6]

$$\mathcal{Q}_k(\lambda+\sigma_k)=\mathrm{Tr}_0\left\{\mathcal{R}^{(k)}_{10}(\lambda)\cdots\mathcal{R}^{(k)}_{L0}(\lambda)\right\}. \tag{4.9}$$

[6]We recall our convention is to display only the holomorphic parameter λ, i.e. the operator $\mathcal{Q}_k(\lambda)\equiv\mathcal{Q}_k(\lambda,\bar\lambda)$, where $\lambda-\bar\lambda$ is integer.

The operators $\mathcal{Q}_k$ act on the Hilbert space $\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_L = L^2(Z \times \cdots \times Z)$. It is assumed that the representation $T^{\boldsymbol{\alpha}}$ of the group $SL(N,\mathbb{C})$ is defined at each site. The trace is taken over the auxiliary space $\mathbb{V}_0 = L^2(Z)$ and exists (we give below the explicit realization of $\mathcal{Q}_k(\lambda)$ as an integral operator).

The operators (4.9) can be identified as the Baxter operators for the noncompact $SL(N,\mathbb{C})$ invariant spin magnet. Since the operators $\mathcal{R}^{(k)}_{i0}(\lambda)$ coincide with the $\mathcal{R}_{i0}$-operator for the special choice of auxiliary space one concludes that the Baxter operators $\mathcal{Q}_k(\lambda)$ commute with each other for different values of a spectral parameter

$$[\mathcal{Q}_k(\lambda), \mathcal{Q}_i(\mu)] = 0.$$

As follows from equation (4.8) all Baxter operators are unitary equivalent, namely

$$\mathcal{Q}_k(\lambda + \sigma_k) = \left(\prod_{i=i}^{L} W_k^{(i)}\right)^{\dagger} \mathcal{Q}_N(\lambda + \sigma_N) \left(\prod_{i=i}^{L} W_k^{(i)}\right). \tag{4.10}$$

This property is a special feature of the principal series $SL(N,\mathbb{C})$ magnets (see also [16] where the $SL(2,\mathbb{C})$ magnet was studied in detail). For conventional (infinite-dimensional) $sl(N)$ magnets the property (4.10) does not hold (see [34, 37, 38]).

Another property of the Baxter operators which we want to mention is related to the factorization property of the transfer matrix. The latter is defined as

$$\mathsf{T}_{\boldsymbol{\beta}}(\lambda) = \mathrm{Tr}_0 \left\{\mathcal{R}_{10}(\lambda) \cdots \mathcal{R}_{L0}(\lambda)\right\}, \tag{4.11}$$

where the trace is taken over the auxiliary space and the index $\boldsymbol{\beta}$ refers to the representation $T^{\boldsymbol{\beta}}$ which is defined on the space $\mathbb{V}_0$. We will suppose that the trace exists (this is true at least for $N = 2$ [16]). Under this assumption one can prove that the transfer matrix factorizes in the product of $\mathcal{Q}_k$ operators as follows

$$\mathsf{T}_{\boldsymbol{\beta}}(\lambda) = \mathcal{Q}_1(\lambda + \rho_1)\mathcal{P}^{-1}\mathcal{Q}_2(\lambda + \rho_2)\mathcal{P}^{-1} \cdots \mathcal{P}^{-1}\mathcal{Q}_N(\lambda + \rho_N),$$

where $\mathcal{P}$ is the operator of cyclic permutation

$$\mathcal{P}\Phi(z_1, \ldots, z_L) = \Phi(z_L, z_1, \ldots, z_{L-1})$$

and the parameters ρ_k determine the character $\boldsymbol{\beta}$, equations (2.2), (2.3). The proof repeats that given in [38] and is based on disentangling of the trace equation (4.11) with the help of the relation (4.5c) for $\mathbb{R}^{(k)}$ operators. The analysis of analytical properties of the Baxter $\mathcal{Q}$-operators and a derivation of the T-$\mathcal{Q}$ relation will be given elsewhere.

4.1 Integral representation

In this subsection we give the integral representation for the operators in question. Due to equations (4.8) and (4.10) it suffices to consider the operators $\mathcal{Q}_N$ and $\mathbb{R}^{(N)}$ only. The latter can be represented as the product of three operators, $\mathbb{R}^{(N)}(\lambda) = \widetilde{\mathcal{X}_1}(\lambda)S(\lambda)\mathcal{X}_1$. The operator $S(\lambda)$ defined in equation (3.12) is an multiplication operator

$$[S(\lambda)\Phi](z, w) = [w, z]^{\lambda}\, \Phi(z, w), \tag{4.12}$$

where $[w, z]^{\lambda} \equiv (w^{-1}z)^{\lambda}_{N1}((w^{-1}z)^{*}_{N1})^{\bar{\lambda}}$. The operator $\mathcal{X}_1$ is given by a product of the operators $(U_1 \otimes \mathbb{1})(U_2 \otimes \mathbb{1}) \cdots (U_{N-1} \otimes \mathbb{1})$, see equations (4.1) and (3.16). Making use of equation (2.17) one gets

$$[\mathcal{X}_1\Phi](z, w) = \prod_{k=1}^{N-1} A(\sigma_{kN}) \int d^2\zeta_k \, [z_{k+1,k} - \zeta_k]^{-1-\sigma_{kN}}\, \Phi(z_{\boldsymbol{\zeta}}, w). \tag{4.13}$$

The factor A is defined in equation (2.18), $\sigma_{kN} = \sigma_k - \sigma_N$,

$$z_{\boldsymbol{\zeta}} = z\big(1 + (\zeta_1 - z_{21})e_{21}\big)\cdots\big(1 + (\zeta_{N-1} - z_{NN-1})e_{NN-1}\big)$$
$$= z\left(1 + \sum_{k=1}^{N-1}(\zeta_k - z_{k+1,k})e_{k+1,k}\right).$$

and we recall that $[z]^a \equiv z^a\bar{z}^{\bar{a}}$. Similarly, for the operator $\widetilde{\mathcal{X}_1}(\lambda)$ one derives

$$[\widetilde{\mathcal{X}_1}(\lambda)\Phi](z,w) = \prod_{k=1}^{N-1} A(\sigma_{Nk}+\lambda)\int d^2\zeta_k\,[z_{k+1,k} - \zeta_k]^{-(1+\lambda+\sigma_{Nk})}\Phi(\tilde{z}_{\boldsymbol{\zeta}}, w), \tag{4.14}$$

with

$$\tilde{z}_{\boldsymbol{\zeta}} = z\,(1 + (\zeta_{N-1} - z_{NN-1})e_{NN-1})\cdots(1 + (\zeta_1 - z_{21})e_{21})\,.$$

Thus the operator $\mathbb{R}^{(N)}(\lambda)$ can be represented in the following form

$$[\mathbb{R}_{12}^{(N)}(\lambda)\Phi](z,w) = \prod_{k=1}^{N-1} A(\sigma_{kN})A(\sigma_{Nk}+\lambda)\int d^2\zeta_k \int d^2\zeta_k'$$
$$\times\,[z_{k+1,k} - \zeta_k]^{-(1+\lambda+\sigma_{Nk})}[\zeta_k - \zeta_k']^{-(1+\sigma_{kN})}[w,\tilde{z}_\zeta]^\lambda\Phi(z_{\zeta\zeta'}, w), \tag{4.15}$$

where

$$z_{\zeta\zeta'} = \tilde{z}_{\boldsymbol{\zeta}}\,\big(1 + (\zeta_1' - \zeta_1)e_{21}\big)\cdots\big(1 + (\zeta_{N-1}' - \zeta_{N-1})e_{NN-1}\big)\,.$$

To obtain the integral representation for the Baxter operator $\mathcal{Q}_N$ we notice that the operator $\mathcal{R}_{12}^{(N)}(\lambda)$ leaves the second argument of the function $\Phi(z,w)$ intact. Therefore the integral kernel of the operator $\mathcal{R}_{12}^{(N)}(\lambda)$,

$$[\mathbb{R}_{12}^{(N)}(\lambda)\Phi](z,w) = \int Dz'Dw' R_\lambda(z,w|z',w')\Phi(z',w'),$$

where $Dz' = \prod_{k>i} d^2z_{ki}'$, $Dw' = \prod_{k>i} d^2w_{ki}'$, has the following form

$$R_\lambda(z,w|z',w') = R_\lambda(z,w|z')\delta(w-w'),$$

with $\delta(w-w') = \prod_{k>i}\delta^2(w_{ki} - w_{ki}')$. Making use of equations (4.7) and (4.9) one finds that the kernel $Q_\lambda^{(N)}$ of the operator $\mathcal{Q}_N$

$$[\mathcal{Q}_N(\lambda+\sigma_N)\Phi](z_1,\dots,z_L) = \int\prod_{k=1}^{L} Dz_k'\,Q_\lambda^{(N)}(z_1,\dots,z_L|z_1',\dots,z_L')\,\Phi(z_1',\dots,z_L')\,.$$

for a spin chain of a length L has the following form

$$Q_\lambda^{(N)}(z_1,\dots,z_L|z_1',\dots,z_L') = \prod_{k=1}^{L} R_\lambda(z_k, z_{k+1}|z_{k+1}'),$$

where the periodic boundary conditions are implied $z_{L+1} = z_1$. Further, taking into account equations (4.12), (4.13), (4.14) one can represent the operator $\mathcal{Q}^{(N)}$ in the form

$$\mathcal{Q}^{(N)}(\lambda+\sigma_N) = \overline{\mathcal{Q}}(\lambda+\sigma_N)\mathcal{X}_1\mathcal{X}_2\cdots\mathcal{X}_L. \tag{4.16}$$

The operator $\mathcal{X}_k$, which acts non-trivially only on the k-th component in the tensor product $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \cdots \otimes \mathbb{V}_L$, is given by equation (4.13). In turn, for the operator $\overline{\mathcal{Q}}(\lambda)$ one obtains

$$[\overline{\mathcal{Q}}(\lambda)\Phi](z_1, \ldots, z_L) = q^L(\lambda) \prod_{k=1}^{L} \left(\prod_{j=1}^{N-1} \int d^2\zeta_{k,j}[(z_k)_{j+1,j} - \zeta_{k,j}]^{-(1+\lambda-\sigma_j)} \right) \times [z_{k+1}, (\tilde{z}_k)_{\zeta_k}]^{\lambda-\sigma_N} \, \Phi((\tilde{z}_L)_{\zeta_L}, (\tilde{z}_1)_{\zeta_1}, \ldots, (\tilde{z}_{L-1})_{\zeta_{L-1}}),$$

where

$$q(\lambda) = \prod_{j=1}^{N-1} A(\lambda - \sigma_j).$$

4.2 $SL(2, \mathbb{C})$ magnet

Closing this section we give the explicit expression for the operator $\mathbb{R}^{(N)}$ and the Baxter $\mathcal{Q}$-operator for the $SL(2, \mathbb{C})$ spin magnet. For $N = 2$ equation (4.15) takes the form

$$\left[\mathbb{R}_{12}^{(N=2)}(\lambda)\Phi\right](z, w) = A(\sigma_{12})A(\lambda - \sigma_{12}) \int d^2\zeta \int d^2\zeta' \times [z - \zeta]^{-(1+\lambda-\sigma_{12})}[\zeta - \zeta']^{-(1+\sigma_{12})}[\zeta - w]^{\lambda}\Phi(\zeta', w),$$

where we put $z_{12} = z$, $w_{12} = w$. Integrating over ζ with help of equation (3.19) one derives

$$\begin{aligned}\left[\mathbb{R}_{12}^{(N=2)}(\lambda)\Phi\right](z, w) &= A(\lambda)[z - w]^{\sigma_{12}} \int d^2\zeta'[z - \zeta']^{-1-\lambda}[\zeta' - w]^{\lambda-\sigma_{12}} \, \Phi(\zeta', w) \\ &= A(\lambda) \int d^2\alpha[\alpha]^{-1-\lambda} \, [1 - \alpha]^{\lambda-\sigma_{12}}\Phi(z(1 - \alpha) + w\alpha, w).\end{aligned}$$

Similarly, for the operator $\mathbb{R}_{12}^{(1)}$ one finds

$$\begin{aligned}\left[\mathbb{R}_{12}^{(1)}(\lambda)\Phi\right](z, w) &= A(\lambda)[z - w]^{\rho_{12}} \int d^2\zeta' \, [w - \zeta']^{-1-\lambda}[z - \zeta']^{\lambda-\rho_{12}}\Phi(z, \zeta') \\ &= (-1)^{\lambda-\bar{\lambda}}A(\lambda) \int d^2\alpha[\alpha]^{-1-\lambda}[1 - \alpha]^{\lambda-\rho_{12}}\Phi(z, w(1 - \alpha) + z\alpha).\end{aligned}$$

With the help of equations (4.4) and (4.3) one can easily find that the integral kernel of the operator $\mathcal{R}_{12}(\lambda)$

$$[\mathcal{R}_{12}(\lambda)\Phi] \, (z, w) = \int d^2z'd^2w' \, R_\lambda(z, w|z', w')\Phi(z', w')$$

takes the form

$$R_\lambda(z, w|z', w') = \frac{(-1)^{\lambda-\bar{\lambda}-\sigma_1+\bar{\sigma}_1+\rho_1-\bar{\rho}_1}A(\lambda - \sigma_1 + \rho_1)A(\lambda - \sigma_2 + \rho_2)}{[z - w]^{-\lambda+\sigma_2-\rho_1} \, [z' - w']^{-\lambda+\sigma_1-\rho_2}[z - w']^{1+\lambda-\sigma_2+\rho_2}[z' - w]^{1+\lambda-\sigma_1+\rho_1}}.$$

The straightforward check shows that up to a prefactor this expression coincides with the kernel for the $\mathcal{R}$-operator obtained in [16].

The expression for the $\overline{\mathcal{Q}}$ operator for $N = 2$ case can be rewritten in the form

$$[\overline{\mathcal{Q}}(\lambda)\Phi](z_1, \ldots, z_L) = A^L(\lambda - \sigma_1) \prod_{k=1}^{L} \int d^2\zeta_k[z_k - \zeta_{k+1}]^{-(1+\lambda-\sigma_1)}[\zeta_k - z_k]^{\lambda-\sigma_2} \times \Phi(\zeta_1, \zeta_2, \ldots, \zeta_L). \tag{4.17}$$

Again it can be checked that the expression (4.17) together with equation (4.16) matches (up to a λ-dependent prefactor) the expression for the kernel of the Baxter $\mathcal{Q}$-operator obtained in [16].

Finally, we give one more representation for the Baxter operator $\mathcal{Q}^{(N=2)}(\lambda)$

$$\mathcal{Q}^{(N=2)}(\lambda+\sigma_2) = A^L(\lambda)\prod_{k=1}^{L}\int d^2\alpha_k[\alpha_k]^{-1-\lambda}\,[1-\alpha_k]^{\lambda-\sigma_{12}} \\ \times \Phi((1-\alpha_1)z_L+\alpha_1 z_1,(1-\alpha_2)z_1+\alpha_2 z_2,\dots,(1-\alpha_L)z_{L-1}+\alpha_L z_L),$$

which is instructive to compare with the expression for the Baxter $\mathcal{Q}$-operator for $su(1,1)$ spin chain [39].

5 Summary

In this paper we developed an approach of constructing the solutions of the Yang–Baxter equation for the principal series representations of $SL(N,\mathbb{C})$. We obtained the $\mathcal{R}$-operator as the product of elementary operators $\mathbb{U}_k$, $k=1,\dots,2N-1$. The latter, except for the operator $\mathbb{U}_N$, are the intertwining operators for the principal series representations of $SL(N,\mathbb{C})$. The operator $\mathbb{U}_N$ is a special one. It intertwines the tensor products of $SL(N,\mathbb{C})$ representation and a product of Lax operators, see equations (3.8), (3.10). The operators $\mathbb{U}_k$ satisfy the same commutation relations as the operators of the elementary permutation, P_k. In other words, they define the representation of the permutation group S_N. It means that any two operators constructed from the operator $\mathbb{U}_k$ corresponding to the same permutation are equal. As a result a proof of the Yang–Baxter relation becomes trivial.

We have represented the $\mathcal{R}$-operator in the factorized form and constructed the factorizing operators $\mathbb{R}^{(k)}$. Having in mind application of this approach to spin chains with generic representations of $sl(N)$ algebra we figured out which properties of the factorizing operators are vital to proving of the Yang–Baxter equation. The operators $\mathbb{R}_k$ play the fundamental role in constructing the Baxter operators. Namely, using them as building blocks, one can construct the commutative family of the operators $\mathcal{Q}_k$ which can be identified as the Baxter operators. We obtain the integral representation for the latter and show that for $N=2$ our results coincide with the results of [16].

Acknowledgments

We are grateful to M.A. Semenov-Tian-Shansky for valuable discussions and attracting our attention to [32, 33]. This work was supported by the RFFI grant 05-01-00922 and DFG grant 436 Rus 17/9/06 (S.D.) and by the Helmholtz Association, contract number VH-NG-004 (A.M.)

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