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u-Deformed WZW Model and Its Gauging*

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Abstract. We review the description of a particular deformation of the WZW model. The resulting theory exhibits a Poisson–Lie symmetry with a non-Abelian cosymmetry group and can be vectorially gauged.

Key words: gauged WZW model; Poisson–Lie symmetry

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1 Introduction

The theory of Poisson–Lie symmetric deformations of the standard WZW models [6] was developed in [2, 3, 4] and it is based on the concept of the twisted Heisenberg double [5]. This contribution is a review of a part of our work [4]. It is intended to the attention of those readers who are interested just in the direct description and gauging of one particular example of the Poisson–Lie WZW deformation and do not wish to go through the general theory of the twisted Heisenberg doubles exposed in [4].

2 u-deformed WZW model

K be a connected simple compact Lie group and denote by $(\cdot,\cdot)_{\mathcal{K}}$ the negative-definite Ad-invariant Killing form on its Lie algebra $\mathcal{K}$. Let LK be the group of smooth maps from a circle S^1 into K (the group law is given by pointwise multiplication) and define a non-degenerate Ad-invariant bilinear form $(\cdot|\cdot)$ on $L\mathcal{K} \equiv Lie(LK)$ by the following formula

$$(\alpha|\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma(\alpha(\sigma),\beta(\sigma))_{\mathcal{K}}. \tag{1}$$

Let $P_{\mathcal{H}} : L\mathcal{K} \to \mathcal{H}$ be the orthogonal projector to the Cartan subalgebra $\mathcal{H}$ of $\mathcal{K}$ and let $U : \mathcal{H} \to \mathcal{H}$ be a skew-symmetric linear operator with respect to the inner product $(\cdot,\cdot)_{\mathcal{K}}$. We denote by u the composition $U \circ P_{\mathcal{H}}$.

The u-deformed WZW model is a dynamical system whose phase space P is the direct product $P = L\mathcal{K} \times LK$, its symplectic form ω_u reads

$$\omega_u = \tfrac{1}{2}(dJ_L \wedge |r_{LK}) - \tfrac{1}{2}(dJ_R \wedge |l_{LK}) + \tfrac{1}{2}(u(dJ_L) \wedge |dJ_L) + \tfrac{1}{2}(u(dJ_R) \wedge |dJ_R) \tag{2}$$

and its Hamiltonian H is given by

$$H = -\tfrac{1}{2k}(J_L|J_L) - \tfrac{1}{2k}(J_R|J_R).$$

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Here k is a positive integer, $r_{LK} = dgg^{-1}$ and $l_{LK} = g^{-1}dg$ stand for the right and the left-invariant Maurer–Cartan forms on the group manifold LK and the $L\mathcal{K}$-valued functions J_L, J_R on P are defined as

$$J_L(\chi, g) = \chi, \qquad J_R(\chi, g) = -\mathrm{Ad}_{g^{-1}}\chi + kg^{-1}\partial_\sigma g, \qquad g \in LK, \quad \chi \in L\mathcal{K}. \tag{3}$$

If $U = 0$, the u-deformed WZW model becomes just the standard WZW model in the formulation [1, 2].

Consider the standard actions of the loop group LK on the phase space P:

$$h \triangleright_L (\chi, g) = (k\partial_\sigma hh^{-1} + h\chi h^{-1}, hg), \qquad h, g \in LK, \quad \chi \in L\mathcal{K},$$
$$h \triangleright_R (\chi, g) = (\chi, gh^{-1}), \qquad h, g \in LK, \quad \chi \in L\mathcal{K}.$$

It was established in [4] that these actions can infinitesimally be expressed via the Poisson bivector Π_u, corresponding to the symplectic form ω_u:

$$\xi_L f = (\Pi_u(df, J_L^* r_B)|\xi), \qquad \xi_R f = (\Pi_u(df, J_R^* r_B)|\xi). \tag{4}$$

Here ξ_L, ξ_R are, respectively, the vector fields on P, corresponding to an element $\xi \in L\mathcal{K}$, f is a function on P, $J_{L,R}^* r_B$ stand for pull-backs of the Maurer–Cartan form r_B on a Lie group B. As a set, B is just $L\mathcal{K}$, however, the group law is as follows:

$$\chi \bullet \tilde{\chi} = \chi + e^{u(\chi)}\tilde{\chi}e^{-u(\chi)}, \qquad \chi, \tilde{\chi} \in L\mathcal{K}, \qquad \chi^{-1} = -e^{-u(\chi)}\chi e^{u(\chi)}. \tag{5}$$

The reader can recognize in (4) the defining relations of the LK-Poisson–Lie symmetries with the cosymmetry group equal to B (cf. [2, eq. (5.30)]).

It is insightful to detail the fundamental relations (4) in the standard Cartan basis $H^{\mu,n} = H^\mu e^{in\sigma} \in L\mathcal{K}^{\mathbf{C}}$, $E^{\alpha,n} = E^\alpha e^{in\sigma} \in L\mathcal{K}^{\mathbf{C}}$, $n \in \mathbb{Z}$. We have

$$H_L^{\mu,m} f = \{f, J_L^{\mu,m}\}_u, \qquad E_L^{\alpha,n} f = \{f, J_L^{\alpha,n}\}_u + \langle \alpha, U(H^\mu)\rangle J_L^{\alpha,n}\{f, J_L^{\mu,0}\}_u, \tag{6}$$
$$H_R^{\mu,m} f = \{f, J_R^{\mu,m}\}_u, \qquad E_R^{\alpha,n} f = \{f, J_R^{\alpha,n}\}_u + \langle \alpha, U(H^\mu)\rangle J_R^{\alpha,n}\{f, J_R^{\mu,0}\}_u, \tag{7}$$

where

$$J_{L,R}^{\alpha,n} \equiv (J_{L,R}|E^\alpha e^{in\sigma}), \qquad J_{L,R}^{\mu,n} \equiv (J_{L,R}|H^\mu e^{in\sigma}).$$

For completeness, note that E^α are the step generators of the complexified Lie algebra $\mathcal{K}^{\mathbf{C}}$ and H^μ are the orthonormalized generators of the Cartan subalgebra $\mathcal{H}^{\mathbf{C}}$:

$$[H^\mu, E^\alpha] = \langle \alpha, H^\mu\rangle E^\alpha, \qquad [E^\alpha, E^{-\alpha}] = \alpha^\vee, \qquad [E^\alpha, E^\beta] = c^{\alpha\beta}E^{\alpha+\beta},$$
$$(H^\mu, H^\nu)_{\mathcal{K}} = \delta^{\mu\nu}, \qquad (E^\alpha, E^{-\alpha})_{\mathcal{K}^{\mathbf{C}}} = \tfrac{2}{|\alpha|^2}, \qquad (E^\alpha)^\dagger = E^{-\alpha}, \qquad (H^\mu)^\dagger = H^\mu.$$

The coroot $\alpha^\vee$ is defined as

$$\alpha^\vee = \tfrac{2}{|\alpha|^2}\langle \alpha, H^\mu\rangle H^\mu.$$

We observe, that the actions $\triangleright_{L,R}$ are not Hamiltonian, unless $u = 0$. This suggests that the current algebra brackets cannot be the same as they are in the non-deformed WZW model. Indeed, u-corrections are present and we underline them for the better orientation of the reader:

$$\{J_L^{\mu,m}, J_L^{\nu,n}\}_u = k\delta^{\mu\nu}in\delta_{m+n,0}, \qquad \{J_L^{\mu,m}, J_L^{\alpha,n}\}_u = \langle \alpha, H^\mu\rangle J_L^{\alpha,n+m},$$
$$\{J_L^{\alpha,m}, J_L^{-\alpha,n}\}_u = \tfrac{2}{|\alpha|^2}\big(\langle \alpha, H^\mu\rangle J_L^{\mu,n+m} + ikn\delta_{m+n,0}\big),$$

$$\{J_L^{\alpha,m},J_L^{\beta,n}\}_u=c^{\alpha\beta}J_L^{\alpha+\beta,m+n}\underline{-\langle\alpha,U(H^\mu)\rangle\langle\beta,H^\mu\rangle J_L^{\alpha,m}J_L^{\beta,n}}, \tag{8}$$
$$\{J_R^{\mu,m},J_R^{\nu,n}\}_u=-k\delta^{\mu\nu}in\delta_{m+n,0},\qquad \{J_R^{\mu,m},J_R^{\alpha,n}\}_u=\langle\alpha,H^\mu\rangle J_R^{\alpha,n+m},$$
$$\{J_R^{\alpha,m},J_R^{-\alpha,n}\}_u=\tfrac{2}{|\alpha|^2}\big(\langle\alpha,H^\mu\rangle J_R^{\mu,n+m}-ikn\delta_{m+n,0}\big),$$
$$\{J_R^{\alpha,m},J_R^{\beta,n}\}_u=c^{\alpha\beta}J_R^{\alpha+\beta,m+n}\underline{-\langle\alpha,U(H^\mu)\rangle\langle\beta,H^\mu\rangle J_R^{\alpha,m}J_R^{\beta,n}}, \tag{9}$$
$$\{J_L,J_R\}_u=0. \tag{10}$$

Note that the brackets of the left currents differ from those of the right currents just by the sign in front of the parameter k.

The relations (6), (7) and (8)–(10) almost determine the Poisson bracket $\{\cdot,\cdot\}_u$, corresponding to the symplectic form ω_u. The remaining relation, which completes the description of $\{\cdot,\cdot\}_u$, is as follows:

$$\{\phi,\psi\}_u=U(H^{\mu,0})_L\phi\ H_L^{\mu,0}\psi-H_R^{\mu,0}\phi\ U(H^{\mu,0})_R\psi. \tag{11}$$

Here ϕ, ψ are functions on P which depend only on LK but not on $L\mathcal{K}$.

3 Symplectic reduction

The symplectic reduction of a dynamical system (P,ω) consists in singling out a particular set of observables $\phi_i\in\mathrm{Fun}(P)$ called first class constraints. One just requires from ϕ_i that on the common locus L, where all ϕ_i vanish, also all Poisson brackets $\{\phi_i,\phi_j\}$ vanish. This requirement and the Frobenius theorem guarantee that the kernels of the restriction of the symplectic form ω to L form an integrable distribution on L. Under certain conditions, the set of integrated surfaces of this distribution is itself a manifold P_r which is called the reduced symplectic manifold. The reduced symplectic form ω_r on P_r is uniquely fixed by a condition that the pull-back of ω_r to L coincides with the restriction of the symplectic form ω to L.

In many interesting situations, the integrated surfaces of the integrable distribution can be naturally identified with orbits of a Lie group acting on L. This is the reason why the symplectic reduction is sometimes called the gauging of that Lie group action. As an warm-up example, let us first perform the (vectorial) gauging of the standard WZW model corresponding to the choice $u=0$ in the formula (2).

Let Υ be a subset of the set of all positive roots of the Lie algebra $\mathcal{K}^{\mathbf{C}}$ and suppose that the complex vector space $\mathcal{S}^{\mathbf{C}}$

$$\mathcal{S}^{\mathbf{C}}=\mathrm{Span}\{E^\gamma,E^{-\gamma},[E^\gamma,E^{-\gamma}]\},\qquad \gamma\in\Upsilon$$

is the Lie subalgebra of $\mathcal{K}^{\mathbf{C}}$ (as an example take the block diagonal embedding of sl_3 in sl_4). The complex Lie algebra $\mathcal{S}^{\mathbf{C}}$ has a natural compact real form $\mathcal{S}$ consisting of the anti-Hermitean elements of $\mathcal{S}^{\mathbf{C}}$. Consider the corresponding compact semi-simple group S and view it as the subgroup of K.

For the first class constraints, we take

$$\phi^{\gamma,n}\equiv J_L^{\gamma,n}+J_R^{\gamma,n},\qquad \phi^{\nu,n}\equiv J_L^{\nu,n}+J_L^{\nu,n}, \tag{12}$$

where $\gamma\in\pm\Upsilon$ and ν is such that H^ν is in the Cartan subalgebra $\mathcal{H}_S$ of $\mathcal{S}$. For $u=0$, we obtain

$$\{\phi^{\mu,m},\phi^{\nu,n}\}_{u=0}=0,\qquad \{\phi^{\mu,m},\phi^{\alpha,n}\}_{u=0}=\langle\alpha,H^\mu\rangle\phi^{\alpha,n+m},$$
$$\{\phi^{\alpha,m},\phi^{-\alpha,n}\}_{u=0}=\tfrac{2}{|\alpha|^2}\langle\alpha,H^\mu\rangle\phi^{\mu,n+m},\qquad \{\phi^{\alpha,m},\phi^{\beta,n}\}_{u=0}=c^{\alpha\beta}\phi^{\alpha+\beta,m+n}.$$

We immediately observe that the Poisson brackets of the first class constraints vanish on the common locus $L=\{p\in P;\phi^{\gamma,n}(p)=0,\phi^{\nu,n}(p)=0\}$, therefore the symplectic reduction can be performed. As the result of analysis, it turns out that the integrated surfaces of the integrable distribution are given as the orbits of the following action of the loop group LS on L:

$$s \triangleright (\chi, g) = \left(k\partial_\sigma s s^{-1} + s\chi s^{-1}, sgs^{-1}\right), \qquad s \in LS, \quad (\chi, g) \in L.$$

If $u \neq 0$, the Poisson brackets of the constraints (12) do not vanish on the common locus L and, therefore, they cannot serve as the base for a symplectic reduction. It is not difficult to find a way out from the trouble, however. For that, we take an inspiration from the case $u = 0$ where the sum of the left and right currents can be interpreted as the product in the Abelian cosymmetry group $L\mathcal{K}$ (the group multiplication is the addition in the vector space $L\mathcal{K}$). Thus, for $u \neq 0$, it looks plausible to use the product (5) in the non-Abelian cosymmetry group B. This gives the following constraints:

$$\phi_u^{\gamma,n} \equiv (J_L \bullet J_R | E^{\gamma,n}) = J_L^{\gamma,n} + e^{-\langle \gamma, U(H^\nu)\rangle J_L^{\nu,0}} J_R^{\gamma,n},$$
$$\phi_u^{\nu,n} \equiv (J_L \bullet J_R | H^{\nu,n}) = J_L^{\nu,n} + J_L^{\nu,n},$$

where, again, $\gamma \in \pm\Upsilon$ and ν is such that H^ν is in $\mathcal{H}_S$. Suppose, moreover, that it holds for all $\gamma \in \Upsilon$:

$$(\gamma \circ U)(\mathcal{H}_S^\perp) = 0,$$

where the subscript $\perp$ stands for the orthogonal complement with respect to the restriction of the Killing–Cartan form $(\cdot,\cdot)_\mathcal{K}$ to $\mathcal{H}$. Then the Poisson brackets of the constraints ϕ_u vanish on the common locus $L_u = \{p \in P; \phi_u^{\gamma,n}(p) = 0, \phi_u^{\nu,n}(p) = 0\}$, as it is obvious from the following explicit formulas:

$$\{\phi_u^{\mu,m}, \phi_u^{\nu,n}\}_u = 0, \qquad \{\phi_u^{\mu,m}, \phi_u^{\alpha,n}\}_u = \langle \alpha, H^\mu\rangle \phi_u^{\alpha,n+m},$$
$$\{\phi_u^{\alpha,m}, \phi_u^{-\alpha,n}\}_u = \tfrac{2}{|\alpha|^2}\langle \alpha, H^\mu\rangle \phi_u^{\mu,n+m},$$
$$\{\phi_u^{\alpha,m}, \phi_u^{\beta,n}\}_u = c^{\alpha\beta}\phi_u^{\alpha+\beta,m+n} - \langle \alpha, U(H^\mu)\rangle\langle \beta, H^\mu\rangle \phi_u^{\alpha,m}\phi_u^{\beta,n}.$$

The symplectic reduction now can be performed and the question arises whether we can identify the orbits of a LS action on L_u which would coincide with the integrated surfaces of the integrable distribution. The answer is affirmative [4] and it reads:

$$s \triangleright_u (\chi, g) = \left(s\chi s^{-1} + k\partial_\sigma s s^{-1}, sgs_L^{-1}\right), \qquad s_L = e^{-u(sJ_Ls^{-1} + \kappa\partial s s^{-1})} s e^{u(J_L)}.$$

We conclude that the reduced symplectic manifold P_{ru} can be identified with the coset space L_u/LS.

4 Outlook

We believe that the u-deformed WZW model may become a useful laboratory for the study of possible generalizations of the standard axioms of the conformal field theory in two dimensions.

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