The Painlevé Test and Reducibility to the Canonical Forms for Higher-Dimensional Soliton Equations with Variable-Coefficients

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Abstract. The general KdV equation (gKdV) derived by T. Chou is one of the famous $(1 + 1)$ dimensional soliton equations with variable coefficients. It is well-known that the gKdV equation is integrable. In this paper a higher-dimensional gKdV equation, which is integrable in the sense of the Painlevé test, is presented. A transformation that links this equation to the canonical form of the Calogero–Bogoyavlenskii–Schiff equation is found. Furthermore, the form and similar transformation for the higher-dimensional modified gKdV equation are also obtained.

Key words: KdV equation with variable-coefficients; Painlevé test; higher-dimensional integrable systems

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1 Introduction

Modern theories of nonlinear science have been widely developed over the last half-century. In particular, nonlinear integrable systems have attracted much interest among mathematicians and physicists. One of the reasons for this is the algebraic solvability of the integrable systems. Apart from their theoretical importance, they have remarkable applications to many physical systems such as hydrodynamics, nonlinear optics, plasma and field theories and so on [1, 2, 3]. Generally the notion of the nonlinear integrable systems [4] is not defined precisely, but rather is characterized by a number of common features: space-localized solutions (or *solitons*) [5, 6, 7, 8, 9], Lax pairs [10, 11, 12], bi-Hamiltonians [12, 13], Bäcklund transformations [9, 14] and Painlevé tests [15, 16, 17, 18, 19, 20, 21]. Moreover, solitons and nonlinear evolution equations are a major subject in mechanical and engineering sciences as well as mathematical and physical ones. Among the well-known soliton equations, the celebrated Korteweg–de Vries (KdV) [22] and Kadomtsev-Petviashvili (KP) [23] equations possess remarkable properties. For example, a real ocean is inhomogeneous and the dynamics of nonlinear waves is very influenced by refraction, geometric divergence and so on. The problem of evolution of transverse perturbations of a wave front is of theoretical and practical interest. However, finding new integrable systems is an important but difficult task because of their somewhat ambiguous definition and undeveloped mathematical background.

The physical phenomena in which many nonlinear integrable equations with constant coefficients arise tend to be very highly idealized. Therefore, equations with variable coefficients may provide various models for real physical phenomena, for example, in the propagation of small-amplitude surface waves, which runs on straits or large channels of slowly varying depth and width. On one hand, there has been much interest in the study of generalizations with variable coefficients of nonlinear integrable equations [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45].

For discovery of new nonlinear integrable systems, many researchers have mainly investigated $(1+1)$-dimensional nonlinear systems with constant coefficients. On the other hand, there are few research studies to find nonlinear integrable systems with variable coefficients, since they are essentially complicated. And their results are still in its early stages. Analysis of higher-dimensional systems is also an active topic in nonlinear integrable systems. Since then the study of integrable nonlinear equations in higher dimensions with variable coefficients has attracted much more attention. So the purpose of this paper is to construct a (2+1) dimensional integrable version of the KdV and modified KdV [46] equations with variable coefficients.

It is widely known that the Painlevé test in the sense of the Weiss–Tabor–Carnevale (WTC) method [16] is a powerful tool for investigating integrable equations with variable coefficients. We have discussed the following higher-dimensional 3rd order nonlinear evolution equation with variable coefficients for $u = u(x, z, t)$ [47, 48]:

$$\begin{aligned} &u_t + a(x,z,t)u + b(x,z,t)u_x + c(x,z,t)u_z + d(x,z,t)uu_z + e(x,z,t)u_x\partial_x^{-1}u_z \\ &\quad + f(x,z,t)u_{xxz} + g(x,z,t) = 0, \end{aligned} \tag{1}$$

where $d(x,z,t) + e(x,z,t) \neq 0$, $f(x,z,t) \neq 0$ and subscripts with respect to independent variables denote their partial derivatives, for example, $u_x = \partial u/\partial x$, $u_{xz} = \partial^2 u/\partial x\partial z$ etc, and $\partial_x^{-1}u := \int^x u(X)dX$. Here $a(x,z,t), b(x,z,t), \ldots, g(x,z,t)$ are coefficient functions of two spatial variables x, z and one temporal variable t. We have carried out the WTC method for equation (1), and have presented two sets of the coefficient function so that it is shown in next section. Equations of the form (1) include one of the integrable higher-dimensional KdV equations:

$$u_t + uu_z + \frac{1}{2}u_x\partial_x^{-1}u_z + \frac{1}{4}u_{xxz} = 0, \tag{2}$$

which is called the Calogero–Bogoyavlenskii–Schiff (CBS) equation [49, 50, 51, 52, 53]. Equation (2) will be the standard KdV equation for $u = u(x, t)$:

$$u_t + \frac{3}{2}uu_x + \frac{1}{4}u_{xxx} = 0$$

by a dimensional reduction $\partial_z = \partial_x$, and the Ablowitz–Kaup–Newell–Segur (AKNS) equation for $u = u(x, t)$ [54, 55]:

$$u_t + uu_t + \frac{1}{2}u_x\partial_x^{-1}u_t + \frac{1}{4}u_{xxt} = 0$$

by another dimensional reduction $\partial_z = \partial_t$. Here (and hereafter) $\partial_x \equiv \partial/\partial x$, $\partial_t \equiv \partial/\partial t$ and so on.

This paper is organized as follows. In Section 2, we will review the process of the WTC method of equation (1) in brief. In Section 3 and 4 we will consider a general KdV and a general modified KdV equations in $(2+1)$ dimensions corresponding a special but interesting case of the integrable higher-dimensional KdV equation with variable coefficients given in Section 2. Section 5 will be devoted to conclusions.

2 Painlevé test of equation (1)

Let us now briefly review the process of the Painlevé test in the sense of the WTC method for equation (1) put forward in [47] and then in [48].

Weiss et al. said in [16] that a partial differential equation (PDE) has the Painlevé property when the solutions of the PDE are single-valued around the movable singularity manifold. They have proposed a technique that determines whether or not a given PDE is integrable, which we call the WTC method:

When the singularity manifold is determined by

$$\phi(z_1, \ldots, z_n) = 0, \tag{3}$$

and $u = u(z_1, \ldots, z_n)$ is a solution of PDE given, then we assume that

$$u = \sum_{j=0}^{\infty} u_j \phi^{j-\alpha}, \tag{4}$$

where $\phi = \phi(z_1, \ldots, z_n)$, $u_j = u_j(z_1, \ldots, z_n)$, $u_0 \neq 0$ $(j = 0, 1, 2, \ldots)$ are analytic functions of z_j in a neighborhood of the manifold (3), and α is a positive integer called the leading order. Substitution of expansion (4) into the PDE determines the value of α and defines the recursion relations for u_j. When expansion (4) is correct, the PDE possesses the Painlevé property and is conjectured to be integrable.

Now we show the WTC method for equation (1). For that, a nonlocal term of equation (1) should be eliminated. We have a potential form of equation (1) in terms of $U = U(x, z, t)$:

$$\begin{aligned} &U_{xt} + a(x,z,t)U_x + b(x,z,t)U_{xx} + c(x,z,t)U_{xz} + d(x,z,t)U_xU_{xz} + e(x,z,t)U_{xx}U_z \\ &\qquad + f(x,z,t)U_{xxxz} + g(x,z,t) = 0, \end{aligned} \tag{5}$$

when defining $u = U_x$. We are now looking for a solution of equation (5) in the Laurent series expansion with $\phi = \phi(x, z, t)$:

$$U = \sum_{j=0}^{\infty} U_j \phi^{j-\alpha}, \tag{6}$$

where $U_j = U_j(x, z, t)$ are analytic functions in a neighborhood of $\phi = 0$. In this case, the leading order (α) is 1 and

$$U_0 = 12\frac{f(x,z,t)}{d(x,z,t) + e(x,z,t)}\phi_x$$

is given. Then, after substituting of the expansion (6) into equation (5), the recursion relations for the U_j are presented as follows:

$$(j+1)(j-1)(j-4)(j-6)f(x,z,t)\phi_x^3\phi_z U_j = F(U_{j-1}, \ldots, U_0, \phi_t, \phi_x, \phi_z, \ldots),$$

where the explicit dependence on x, z, t of the right-hand side comes from that of the coefficients. It is found that the resonances occur at

$$j = -1,\ 1,\ 4,\ 6. \tag{7}$$

Then one can check that the numbers of the resonances (7) correspond to arbitrary functions ϕ, U_1, U_4 and U_6, though we have omitted the details in this paper. We have succeeded in finding of two forms of the higher-dimensional KdV equation with variable coefficients given by

$$u_t + \left\{\frac{d'(t)}{d(t)} - \frac{f'(t)}{f(t)} + \frac{4}{3}\big(\alpha(z,t) - \beta(t) + c_z(z,t)\big)\right\} u + \frac{2}{3}x\big(\alpha(z,t) - \beta(t) + c_z(z,t)\big)u_x$$

$$+ c(z,t)u_z + d(t)uu_z + \frac{d(t)}{2}u_x\partial_x^{-1}u_z + f(t)u_{xxz} + g(z,t) = 0, \tag{8}$$

and

$$\begin{aligned} &u_t + \big(2A(z,t) - \eta'(t)\big)u + c(z,t)u_z + \big(A(z,t)x + B(z,t)\big)u_x + d(z,t)uu_z \\ &\quad + \frac{d(z,t)}{2}u_x\partial_x^{-1}u_z + \frac{3}{2}d(z,t)\exp\eta(t)u_{xxz} + g(z,t) = 0. \end{aligned} \tag{9}$$

In this paper, $(\cdot)'$ denotes the ordinary derivative with respect to the independent variable. Apparently, equations (8) and (9) are (more general) higher-dimensional integrable versions of the $(1+1)$ dimensional KdV equations with variable coefficients appeared in [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 38]. It is easy to check by suitable choice of coefficient functions after the dimensional reduction $\partial_z = \partial_x$.

3 A general KdV equation in $(2+1)$ dimensions

We consider a special but interesting case of equation (8) in this section.

Setting the following condition of variable coefficients:

$$\alpha(z,t) - \beta(t) = \frac{3}{4}\frac{G'(t)}{G(t)}, \quad c(z,t) = 0, \quad d(t) = 1, \quad f(t) = \frac{1}{4}, \quad g(z,t) = 0,$$

equation (8) becomes the following $(2+1)$ dimensional equation [56]:

$$u_t + uu_z + \frac{1}{2}u_x\partial_x^{-1}u_z + \frac{1}{4}u_{xxz} - \frac{G'(t)}{G(t)}u - \frac{x}{2}\frac{G'(t)}{G(t)}u_x = 0. \tag{10}$$

Here $G(t)$ is an arbitrary function of the temporal variable t. We would like to call equation (10) a general Calogero–Bogoyavlenskii–Schiff (gCBS) equation. Because equation (10) is a higher-dimensional integrable version of the general KdV (gKdV) equation [28, 38][1]:

$$u_t + \frac{3}{2}uu_x + \frac{1}{4}u_{xxx} - \frac{G'(t)}{G(t)}u - \frac{x}{2}\frac{G'(t)}{G(t)}u_x = 0. \tag{11}$$

And using the *Lax-pair Generating Technique* [48, 57, 58], we obtain a pair of linear operators associated with equation (10) given by

$$L = \frac{1}{G(t)}\big(\partial_x^2 + u\big) - \lambda \equiv \frac{1}{G(t)}L_{\mathrm{GKdV}} - \lambda, \tag{12}$$

$$T = \partial_z L_{\mathrm{GKdV}} + \frac{1}{2}\left(\partial_x^{-1}u_z - x\frac{G'(t)}{G(t)}\right)\partial_x - \frac{1}{4}\left(u_z - \frac{G'(t)}{G(t)}\right) + \partial_t. \tag{13}$$

The compatibility condition[2] of the operators L and T[3] is defined as

$$[L,T] \equiv LT - TL = 0, \tag{14}$$

which gives corresponding integrable equations. Notice here that $\lambda = \lambda(z,t)$ is the spectral parameter and satisfies the non-isospectral condition [48, 56, 59, 60]:

$$\lambda_t = \lambda\lambda_z.$$

[1]Equation (10) is reduced to equation (11) by the dimensional reduction $\partial_z = \partial_x$.
[2]This is often called the Lax equation.
[3]This pair, namely, is the Lax pair for equation (10).

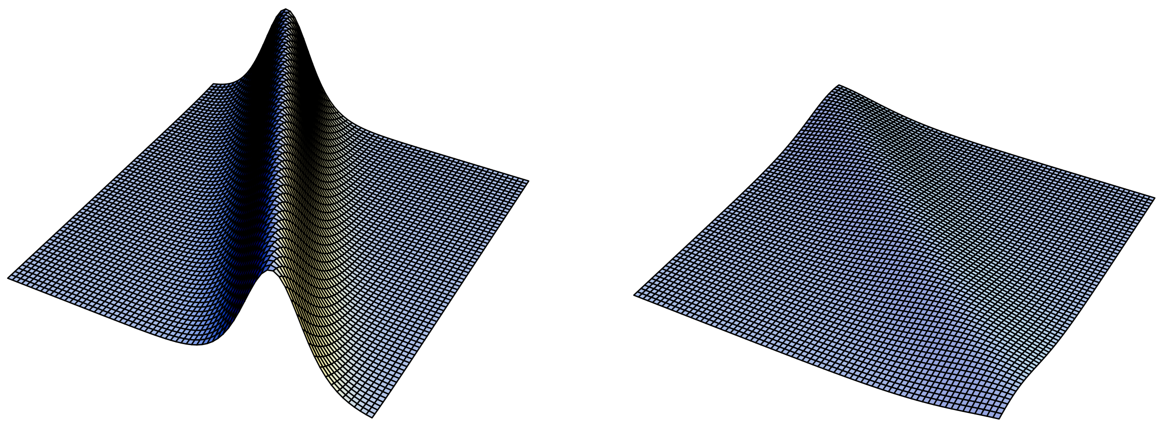

Figure 1. One line-soliton ($t = 1 \to t = 5$).

Let here mention exact solutions of the gCBS equation (10). As a result of using of the following transformation:

$$\bar{x} = x\sqrt{G(t)}, \quad \bar{z} = z, \quad \bar{t} = \partial_t^{-1} G(t), \quad \bar{u}(\bar{x}, \bar{z}, \bar{t}) = \frac{u(x, z, t)}{G(t)}, \tag{15}$$

equation (10) becomes the canonical form of the CBS equation (2) in terms of $\bar{u} = \bar{u}(\bar{x}, \bar{z}, \bar{t})$ [61]:

$$\bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{z}} + \frac{1}{2}\bar{u}_{\bar{x}}\partial_{\bar{x}}^{-1}\bar{u}_{\bar{z}} + \frac{1}{4}\bar{u}_{\bar{x}\bar{x}\bar{z}} = 0. \tag{16}$$

The N line-soliton solutions with $\tau_N = \tau_N(\bar{x}, \bar{z}, \bar{t})$ for equation (16) were expressed as

$$\begin{aligned}
&\bar{u} = 2(\log \tau_N)_{\bar{x}\bar{x}}, \\
&\tau_N = 1 + \sum_{n=1}^{N} \sum_{{}_N C_n} A_{i_1 \cdots i_n} \exp(\lambda_{i_1} + \cdots + \lambda_{i_n}), \\
&\lambda_j = p_j \bar{x} + q_j \bar{z} + r_j \bar{t} + s_j, \quad j = 1, 2, \ldots, N, \\
&r_j = -\frac{p_j^2 q_j}{4}, \\
&A_{i_1 \cdots i_n} \equiv A_{i_1, i_2} \cdots A_{i_1, i_n} \cdots A_{i_{n-1}, i_n}, \\
&A_{ij} = \left(\frac{p_i - p_j}{p_i + p_j}\right)^2,
\end{aligned}$$

where the summation ${}_N C_n$ indicates summation over all possible combinations of n elements taken from N, and symbols s_j always denote arbitrary constants [52]. So exact solutions of equation (10) can be presented via the transformation (15). We would like to illustrate line-soliton solutions with $G(t) = 1/t$. Figs. 1 and 2 are time evolutions of one line-soliton (with p_1 and q_1) and two line-soliton[4] (with p_1, p_2, q_1 and q_2) solutions. And a V-soliton type solution appears on Fig. 3 setting $p_1 = p_2$ in two line-soliton solutions.

A modified equation corresponding to the gCBS equation (10) will be given in next section.

4 A modified general KdV equation in $(2 + 1)$ dimensions

We present a higher-dimensional integrable versions of the modified gKdV (mgKdV) equation [29] from the Lax pair (12) and (13) as follows.

[4] $p_1 \neq p_2$.

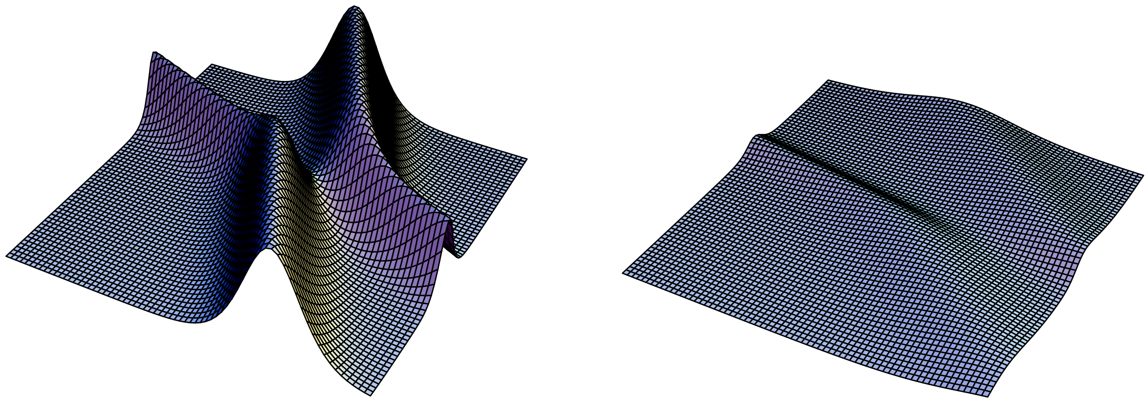

Figure 2. Two line-solitons ($t = 1 \to t = 5$).

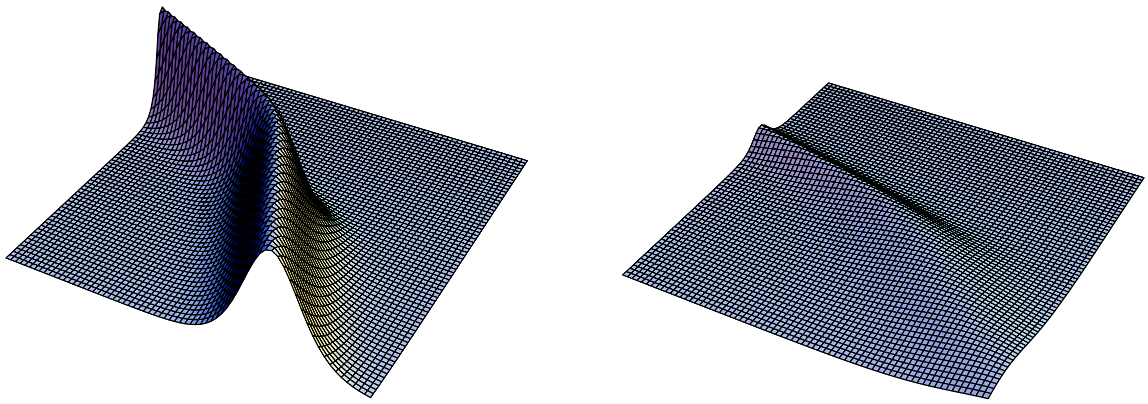

Figure 3. V-soliton ($t = 1 \to t = 5$).

Using the *Lax-pair Generating Technique*, we obtain a following Lax pair:

$$L = \frac{1}{G(t)}\left(\partial_x^2 + v\partial_x\right) - \lambda \equiv \frac{1}{G(t)} L_{\mathrm{GmKdV}} - \lambda,$$

$$T = \partial_z L_{\mathrm{GmKdV}} + \frac{1}{2}\left(\partial_x^{-1} v_z\right)\partial_x^2 + \frac{1}{2}\left(v\partial_x^{-1} v_z - \frac{1}{4}\partial_x^{-1}\left(v^2\right)_z - \frac{1}{2}v_z - \frac{G'(t)}{G(t)}\right)\partial_x + \partial_t.$$

Note here that λ satisfies a non-isospectral condition:

$$\lambda_t = \lambda^2 \lambda_z.$$

Then the Lax equation (14) gives a higher-dimensional mgKdV equation, or a modified gCBS equation for $v = v(x, z, t)$:

$$v_t - \frac{1}{4}v^2 v_z - \frac{1}{8}v_x \partial_x^{-1}\left(v^2\right)_z + \frac{1}{4}v_{xxz} - \frac{1}{2}\frac{G'(t)}{G(t)}v - \frac{x}{2}\frac{G'(t)}{G(t)}v_x = 0. \tag{17}$$

One can also easily check that equation (17) can be reduced to the canonical form of the mgKdV equation for $v = v(x, t)$ [29]:

$$v_t - \frac{3}{8}v^2 v_x + \frac{1}{4}v_{xxx} - \frac{1}{2}\frac{G'(t)}{G(t)}v - \frac{x}{2}\frac{G'(t)}{G(t)}v_x = 0,$$

by the dimensional reduction $\partial_z = \partial_x$. Via the transformation as follows:

$$\bar{x} = x\sqrt{G(t)}, \quad \bar{z} = z, \quad \bar{t} = \partial_t^{-1} G(t), \quad \bar{v}(\bar{x}, \bar{z}, \bar{t}) = \frac{v(x, z, t)}{\sqrt{G(t)}}, \tag{18}$$

equation (17) becomes the modified CBS one for $\bar{v} = \bar{v}(\bar{x}, \bar{z}, \bar{t})$ [61]:

$$\bar{v}_{\bar{t}} - \frac{1}{4}\bar{v}^2\bar{v}_{\bar{z}} - \frac{1}{8}\bar{v}_{\bar{x}}\partial_{\bar{x}}^{-1}\big(\bar{v}^2\big)_{\bar{z}} + \frac{1}{4}\bar{v}_{\bar{x}\bar{x}\bar{z}} = 0, \tag{19}$$

whose N line-soliton solutions were given similarly in [52].

5 Conclusions

In mechanical, physical, mathematical and engineering sciences, nonlinear systems play an important role, especially those with variable coefficients for certain realistic situations. In this paper we have presented higher-dimensional gKdV and modified gKdV equations (10) and (17), which are integrable in the sense of the Painlevé test. Then we have found the transformations (15) and (18), which link equations (10) and (17) to the canonical forms (16) and (19), respectively. And also N line-soliton solutions to equations (10) and (17) have been given via the transformations (15) and (18), respectively.

By applying the (weak) Painlevé test[5], we are searching variable-coefficient forms of the nonlinear Schrödinger, sine-Gordon, Camassa–Holm and Degasperis–Procesi equations in $(2+1)$ dimensions and so on. Here we would like to give only the form of the $(2+1)$ dimensional nonlinear Schrödinger equation for $\phi = \phi(x, z, t)$ [66, 67, 68, 69]:

$$i\phi_t + a(x)b(t)\phi_{xz} + b(t)\phi\partial_x^{-1}(|\phi|^2)_z + \left\{c(z,t) + \partial_x^{-1}\left(\frac{1}{a(x)}\right)\right\}\phi + \frac{1}{2}b(t)a'(x)\phi_z = 0,$$

where $i^2 = -1$. The detail will be reported in [70]. We will study relations of nonlinear integrable equations with variable coefficients given in this paper to realistic situations in mechanical, physical, mathematical and engineering sciences. Though our program is going well now, there are still many things worth studying to be seen.

Finally let us mention two forms with variable coefficients of the AKNS equation for $u = u(x, t)$:

$$\begin{aligned} &u_t + \left\{\frac{d'(t)}{d(t)} - \frac{f'(t)}{f(t)} + \frac{4}{3}\big(\alpha(t) - \beta(t) + c'(t)\big)\right\}u + \frac{2}{3}x\big(\alpha(t) - \beta(t) + c'(t)\big)u_x \\ &\qquad + c(t)u_t + d(t)uu_t + \frac{d(t)}{2}u_x\partial_x^{-1}u_t + f(t)u_{xxt} + g(t) = 0, \end{aligned} \tag{20}$$

and

$$\begin{aligned} &u_t + \big(2A(t) - \eta'(t)\big)u + c(t)u_t + \big(A(t)x + B(t)\big)u_x + d(t)uu_t + \frac{d(t)}{2}u_x\partial_x^{-1}u_t \\ &\qquad + \frac{3}{2}d(t)\exp\eta(t)u_{xxt} + g(t) = 0, \end{aligned} \tag{21}$$

which are derived from equations (8) and (9) by the dimensional reduction $\partial_z = \partial_t$. Equations (20) and (21) seem to be new. However it is not certain that equations (20) and (21) are integrable[6]. We are now looking for corresponding transformations into the canonical form of the conventional $(1+1)$ dimensional AKNS equation.

[5]One can find the weak Painlevé test for the Camassa–Holm equation in [62, 63, 64, 65].

[6]One can check easily that equations (20) and (21) are integrable in the sense of the Painlevé test if $f(t) = d(t)$ for equation (20) and η being a constant for equation (21).

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