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Finding Liouvillian First Integrals of Rational ODEs of Any Order in Finite Terms

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Abstract. It is known, due to Mordukhai-Boltovski, Ritt, Prelle, Singer, Christopher and others, that if a given rational ODE has a Liouvillian first integral then the corresponding integrating factor of the ODE must be of a very special form of a product of powers and exponents of irreducible polynomials. These results lead to a partial algorithm for finding Liouvillian first integrals. However, there are two main complications on the way to obtaining polynomials in the integrating factor form. First of all, one has to find an upper bound for the degrees of the polynomials in the product above, an unsolved problem, and then the set of coefficients for each of the polynomials by the computationally-intensive method of undetermined parameters. As a result, this approach was implemented in CAS only for first and relatively simple second order ODEs. We propose an algebraic method for finding polynomials of the integrating factors for rational ODEs of any order, based on examination of the resultants of the polynomials in the numerator and the denominator of the right-hand side of such equation. If both the numerator and the denominator of the right-hand side of such ODE are not constants, the method can determine in finite terms an explicit expression of an integrating factor if the ODE permits integrating factors of the above mentioned form and then the Liouvillian first integral. The tests of this procedure based on the proposed method, implemented in Maple in the case of rational integrating factors, confirm the consistence and efficiency of the method.

Key words: differential equations; exact solution; first integral; integrating factor

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1 The Prelle–Singer method

μ is an integrating factor for an nth order ODE in solved form

$$\frac{d^n y_0}{dx^n} - f(x, y_0, y_1, \dots, y_{n-1}) = 0, \tag{1}$$

where $y_0 = y(x)$, $y_j = \frac{d^j y(x)}{dx^j}$, by standard definition [1], if $\mu(y_n - f)$ is a total derivative of some function $\zeta(x, y_0, y_1, \dots, y_{n-1})$, that is,

$$\mu\left(\frac{d^n y_0}{dx^n} - f\right) = \frac{\partial \zeta}{\partial x} + \sum_{j=0}^{n-1} y_{j+1}\frac{\partial \zeta}{\partial y_j}. \tag{2}$$

Let

$$\mu = \frac{\partial \zeta}{\partial y_{n-1}}. \tag{3}$$

Then (2) becomes

$$D(\zeta) + f\frac{\partial \zeta}{\partial y_{n-1}} = 0, \tag{4}$$

where

$$D = \frac{\partial}{\partial x} + \sum_{j=0}^{n-2} y_{j+1}\frac{\partial}{\partial y_j}.$$

We conclude that the integrating factor of ODE (1) is $\mu = \frac{\partial \zeta}{\partial y_{n-1}}$, where the function ζ is *a first integral*, a solution of the linear first-order PDE (4).

If we eliminate the function ζ from the system (3), (4), we obtain a PDE system for the integrating factor. *One* of the equations of the PDE system obtained by double differentiation of (4) with respect to y_{n-1} is as follows ($n \geq 2$):

$$D\left(\frac{\partial \mu}{\partial y_{n-1}}\right) + \frac{\partial^2 (f\mu)}{\partial y_{n-1}^2} + 2\frac{\partial \mu}{\partial y_{n-2}} = 0. \tag{5}$$

Assuming that an integrating factor is known, it is well-known that we are able to obtain corresponding first integral by quadratures. If we know n independent first integrals for ODE (1), we can then find its general solution, at least in implicit form. As it is seen from (3), the algebraic structure of the integrating factor is simpler than the structure of the first integral, so in some cases finding the integrating factors is an easier problem.

The most beautiful way of finding integrating factors is the Darboux method and its refinements. If f in (1) is a rational function with respect to all variables $x, y_0, y_1, \ldots, y_{n-1}$, we will call such an ODE a rational one. In 1878, Darboux [2] proved that if a rational ODE has at least $m(m+1)/2$, where $m = \max(\text{order}(\text{numerator}(f), \text{order}(\text{denominator}(f))$ invariant algebraic curves P_i, now known as Darboux polynomials, then it has a first integral or an integrating factor of the form $\prod_i P_i^{a_i}$ for suitable complex constants a_i. So, in principle, the problem of finding an integrating factor is reduced here to the problem of finding Darboux polynomials.

In 1983, Prelle and Singer [3] proved the essential theorem that *all first order* ODEs, which possess *elementary* first integrals, have an integrating factor of the above mentioned form $\prod_i P_i^{a_i}$. Their result unifies and generalizes a number of results originally due to Mordukhai-Boltovski [4], Ritt [5] and others. Singer in [6] refines this result for ODEs, which possess *Liouvillian* first integral.

The following theorem is a corollary of the Proposition 2.2 [6] and of the definition of the integrating factor (3) (see also [7]):

Theorem 1. *If the n-th order ODE*

$$\frac{d^n y_0}{dx^n} = \frac{A(x, y_0, y_1, \ldots, y_{n-1})}{B(x, y_0, y_1, \ldots, y_{n-1})}, \tag{6}$$

where $A, B \in K[x, y_0, y_1, \ldots, y_{n-1}]$ are polynomials, has a local Liouvillian first integral, then there exists an integrating factor of ODE (6) *of the form*

$$\mu = \prod_i P_i^{a_i} \exp\left(b_0 \prod_j Q_j^{b_j}\right), \tag{7}$$

where P_i's, Q_j's $\in K[x, y_0, y_1, \ldots, y_{n-1}]$ are irreducible polynomials, a_i's, b_0 are constants and b_j's are integers.

So, to find an integrating factor to ODE (6) in the conditions of Theorem 1, we first should obtain the sets of polynomials P_i and Q_j, and if we succeed in this task then we can obtain the constants a_i's and b_j's from the full PDE system for the integrating factor.

To consider the ways to find the sets of polynomials P_i and Q_j, let us substitute the integrating factor in the form (7) into (5). After some rearrangement, we arrive at

$$L_1 B = 2A\left(\frac{\partial B}{\partial y_{n-1}}\right)^2 \prod_i P_i^2 \prod_j Q_j^2,$$

where L_1 is a polynomial. As A and B are relatively prime polynomials, we conclude that B divides one of the polynomials P_i or Q_j. Suppose that

$$\begin{aligned} A &= A_1(y_{n-1})A_2(x, y_0, y_1, \ldots, y_{n-1}),\\ B &= B_1(y_{n-1})B_2(x, y_0, y_1, \ldots, y_{n-2})B_3(x, y_0, y_1, \ldots, y_{n-1}) \end{aligned}$$

and

$$\mu = A_1^{\varepsilon_0} B_1^{\varepsilon_1} B_2^{\varepsilon_2} B_3^{\varepsilon_3} \prod_i P_i^{a_i} \exp\left(b_0 A_1^{\eta_0} B_1^{\eta_1} B_2^{\eta_2} B_3^{\eta_3} \prod_j Q_j^{b_j}\right).$$

Substituting these expressions for A, B and μ into (5), we can conclude, by considering divisibility of polynomials, that it is necessary that $\varepsilon_3 = 1$ and $\eta_3 = 0$. That is, if $\frac{\partial P_i}{\partial y_{n-1}} \equiv 0$ or $D(P_i) \equiv 0$ ($\frac{\partial Q_j}{\partial y_{n-1}} \equiv 0$ or $D(Q_j) \equiv 0$) then such polynomials are factors of A or B, and they can be easily *selected* and *included* into the set of candidates for the integrating factor structure.

Without loss of generality, we will consider an integrating factor of the form

$$\mu = B \prod_i P_i^{a_i} \exp\left(b_0 \prod_j Q_j^{b_j}\right). \tag{8}$$

and will focus our attention on the cases when $\frac{\partial P_i}{\partial y_{n-1}} \neq 0$ and $D(P_i) \neq 0$ ($\frac{\partial Q_j}{\partial y_{n-1}} \neq 0$ and $D(Q_j) \neq 0$).

For the case of $a_k \neq 1$, substitution of (8) into (5) leads to

$$L_2 P_k = a_k(a_k - 1)\frac{\partial P_k}{\partial y_{n-1}} \prod_{\substack{i\\ i\neq k}} P_i^2 \prod_j Q_j^2 \left[B D(P_k) + A\frac{\partial P_k}{\partial y_{n-1}}\right] \tag{9}$$

and similarly for the case of $b_k < 0$

$$L_3 Q_k = b_k^2 \frac{\partial Q_k}{\partial y_{n-1}} \prod_i P_i^2 \prod_{\substack{j\\ j\neq k}} Q_j^{2(b_j+1)} \left[B D(Q_k) + A\frac{\partial Q_k}{\partial y_{n-1}}\right], \tag{10}$$

where L_2 and L_3 are polynomials.

All components involved in (9) and (10) are polynomials. We assume that P_i and Q_j are irreducible, relatively prime polynomials, so $\gcd\big(P_i, \frac{\partial P_i}{\partial y_{n-1}}\big)$ and $\gcd\big(Q_i, \frac{\partial Q_i}{\partial y_{n-1}}\big)$ are constants, where greatest common divisor is denoted by gcd. We can conclude that ($f \mid g$ means that f divides g)

$$P_i \mid B D(P_i) + A\frac{\partial P_i}{\partial y_{n-1}} \tag{11}$$

and

$$Q_j \mid BD(Q_j) + A\frac{\partial Q_j}{\partial y_{n-1}}. \tag{12}$$

The basic significance of relations (11) and (12) is that we can investigate each of the polynomials P_i or Q_j *independently.* Properties (11) and (12) constitute the basis of the so-called Prelle–Singer method, which originates from Darboux [2], to obtain the integrating factors and first integrals of ODEs of type (6). Supposing that

$$P_k = \sum_{i=0}^{N}\sum_{j_0=0}^{N}\cdots\sum_{j_{n-1}=0}^{N} c_{kij_0\cdots j_{n-1}} x^i y_0^{j_0}\cdots y_{n-1}^{j_{n-1}} \tag{13}$$

and substituting P_k into (11), we can in principle obtain $c_{kij_0\cdots j_{n-1}}$'s and as a result P_k (or similarly Q_k) in explicit form.

To do so, we first have to establish an upper bound N for the degrees of the polynomial P_i (or Q_k). It is known that N exists, but there is no constructive method to evaluate it so far. It is the principal weakness of the Prelle–Singer method. In the existing implementations of the Prelle–Singer method, an upper bound N is predefined by a user of the method.

With an increase of equation order the number of undetermined variables $c_{kij_0\cdots j_{n-1}}$ and, correspondingly, the dimension of the system of algebraic equations for these constants increases. As a result, solving of this system becomes problematical. This approach was implemented in CAS only for first and relatively *simple second* order ODEs [8, 9, 10, 11, 12, 13, 14, 15], and as a rule the method can not be used for $N > 4$. In contrast, the modified method proposed in the next section does not suffer from these restrictions.

We must also mention that a rational ODE does not always have a local Liouvillian first integral. While generalizations of the Darboux integrability theory allow in some ("integrable") cases to find non-Liouvillian first integrals (see, e.g. [16, 17, 18, 19, 20]), we consider only Liouvillian first integral theory.

2 The method of resultants

Let us investigate closely the relation (11) or equivalently (12). Let $R_z(f, g)$ be the resultant of polynomials f and g with respect to the indeterminate z. In the sequel, we will consider only the case when both A and B are not constants and there are such $z \in (x, y_0, y_1, \ldots, y_{n-1})$ that $R_z(A, B)$ is not a constant.

The following Theorem 2 and Corollary are our main results:

Theorem 2. *If a polynomial P_i satisfies the condition* (11) *then for any indeterminate $z \in (x, y_0, y_1, \ldots, y_{n-1})$ the resultant $R_z(P_i, B)$ must divide the resultant $R_z(A, B)$.*

Proof. We may write $P_i L = BD(P_i) + A\frac{\partial P_i}{\partial y_{n-1}}$ for some polynomial L. Taking resultants of both sides of this equation with B and with respect to an indeterminate $z \in (x, y_0, y_1, \ldots, y_{n-1})$, we get the following polynomial equation

$$R_z(P_i, B)R_z(L, B) = R_z(A, B)R_z\left(\frac{\partial P_i}{\partial y_{n-1}}, B\right).$$

As P_i and $\frac{\partial P_i}{\partial y_{n-1}}$ have no common roots, $R_z(P_i, B)$ does not divide $R_z\big(\frac{\partial P_i}{\partial y_{n-1}}, B\big)$. Therefore,

$$R_z(P_i, B) \mid R_z(A, B) \tag{14}$$

and it is obvious that we can choose $R_z(P_i, B)$ as a divisor of $R_z(A, B)$ up to a constant factor. ∎

How can we recover P_i if we know the resultant $R_z(P_i, B)$? First of all, we note that the case when A/B depends on only one indeterminate is trivial. As we have mentioned above, if P_i depends only on one indeterminate, then such polynomials are factors of A or B and we have already included them into the set of candidates. So, we assume that P_i depends at least on two indeterminates and there exists such z that $R_z(P_i, B)$ is a *polynomial*. We do not insist that $R_z(P_i, B)$ is irreducible.

It is known from elementary facts about resultants that there exist such polynomials $\alpha_i, \beta_i \in K[x, y_0, y_1, \dots, y_{n-1}]$ that $\beta_i\, P_i = \alpha_i\, B + R_z(P_i, B)$.

As a corollary of Theorem 2, we have the following result. Assuming that P_i's (and Q_j's) are irreducible polynomials we conclude that

Corollary. *If polynomials P_i (and respectively Q_j) satisfy* (14) *then the following holds*

$$P_{i\,\mathrm{hyp}} = \alpha_i\, B + c_i\, [\text{one of nonconstant factors of } R_z(A, B)], \tag{15}$$

where $P_{i\,\mathrm{hyp}} = \beta_i\, P_i$, β_i is some polynomial, $\alpha_i \in (-1, 0, 1)$ and c_i are constants.

Proof. If $\alpha_i \neq 0$ identically, then for any $z \in (x, y_0, y_1, \dots, y_{n-1})$ from

$$\begin{aligned} R_z(\beta_i P_i - \alpha_i B, P_i) &= \gamma R_z(\alpha_i, P_i) R_z(B, P_i) \\ &= c_i R_z([\text{one of nonconstant factors of } R_z(A, B)], B) = R_z(B, P_i), \end{aligned}$$

where $\gamma \in (-1, 1)$, we conclude that $R_z(\alpha_i, P_i) = 1/\gamma$. So as P_i depends at least on two indeterminates then α_i must be a constant and $\alpha_i \in (-1, 0, 1)$.

Now we are able to modify the Prelle–Singer method by replacing the hypothesis (13) with (15) and finding constants α_i and c_i from the following equation

$$R_z\left(BD(P_{i\,\mathrm{hyp}}) + A\frac{\partial P_{i\,\mathrm{hyp}}}{\partial y_{n-1}}, P_{i\,\mathrm{hyp}}\right) = 0.$$

Then P_i is a irreducible factor of $P_{i\,\mathrm{hyp}}$ which satisfies (11).

Thus, we can use algebraic, not differential, relations for obtaining polynomials as in the original Darboux (and Prelle–Singer) approach. First of all, we *bypass* the principal weakness of the Prelle–Singer method as we do not need to establish an upper bound N for the degrees of polynomial P_i here. Second, we have the advantage that finding only two undetermined constants α_i and c_i in the $P_{i\,\mathrm{hyp}}$ structure is much easier than finding the set of $c_{kij_0\cdots j_{n-1}}$'s. ■

Example. Let us consider a relatively simple second order ODE

$$y'' = \frac{y'^3 + y'(x-2) - y}{y'^2 + (2y'-1)(x+y) - x}.$$

For this ODE, $R_x(A, B) = -(2y_1^2 + y_1 - 2)(y_1^2 - 2y_1 - y_0)$ and we can form the following hypothesis $P_{\mathrm{hyp}} = \alpha(y_1^2 + (2y_1 - 1)(x + y_0) - x) + c(y_1^2 - 2y_1 - y_0)$. Calculation of the following resultant leads to $R_{y_1}(BD(P_{\mathrm{hyp}}) + A\frac{\partial P_{\mathrm{hyp}}}{\partial y_1}, P_{\mathrm{hyp}}) = \alpha^2(\alpha - c)L = 0$, where L is some polynomial. So, there are two cases, when $\alpha = 0$ then $P_{1\,\mathrm{hyp}} = c(y_1^2 - 2y_1 - y_0)$, and when $\alpha = c$ then $P_{2\,\mathrm{hyp}} = 2c(y_1 - 1)(y_1 + y_0 + x)$.

By substitution of the irreducible factors of $P_{1\,\mathrm{hyp}}$ and $P_{2\,\mathrm{hyp}}$ to (11), we select the following candidates for the integrating factor structure: $P_1 = (y_1^2 - 2y_1 - y_0)$ and $P_2 = (y_1 + y_0 + x)$. So the hypothesis of μ is

$$\mu_{\mathrm{hyp}} = (y_1^2 + (2y_1 - 1)(x + y_0) - x)(y_1 + y_0 + x)^{X1}(y_1^2 - 2y_1 - y_0)^{X2}.$$

After substitution of this $\mu_{\rm hyp}$ into the PDE system for the integrating factor, we obtain that $X2 = -(X1+2)$ so an integrating factor of given ODE is

$$\mu = (y'^2 + (2y'-1)(x+y) - x)(y'+y+x)^{X1}(y'^2 - 2y' - y)^{-(X1+2)},$$

where $X1$ is an *arbitrary* constant.

Examinations of the cases when $a_k = 1$ and $b_k > 0$, are more difficult. We are not able to give here the full analysis of the problem which necessitates consideration of many subcases. *For brevity*, we will only demonstrate that the method of resultants in principle enables us to obtain all the needed polynomials in a finite number of steps. The main tool here is usage of *repeated resultants*.

For compactness, we will use the following notation for the repeated resultants

$$R_{z_1,z_2}(f,(g,h)) = R_{z_2}(R_{z_1}(f,h), R_{z_1}(g,h)).$$

For the case $a_k = 1$, we obtain the following equation similar to (9):

$$\begin{aligned} L_4 P_k = &\prod_{\substack{i\\ i\neq k}} \tilde{P}_i \prod_j \tilde{Q}_j (M_1 A + M_2 B) \\ &+ \prod_{\substack{i\\ i\neq k}} \tilde{P}_i^2 \prod_j \tilde{Q}_j^2 \left[\left[2\frac{\partial A}{\partial y_{n-1}} + D(B)\right]\frac{\partial P_k}{\partial y_{n-1}} + D(P_k)\frac{\partial B}{\partial y_{n-1}}\right], \end{aligned}$$

where L_4, M_1, M_2 are some polynomials, and $\tilde{P}_i$ $(i \neq k)$, $\tilde{Q}_j$ are only such polynomials from the integrating factor structure which *do not satisfy* conditions (11) and (12). So for this case we can obtain the following properties ($\frac{\partial P_k}{\partial y_{n-1}} \neq 0$):

$$\begin{aligned} R_{z_1,z_2}(P_k,(A,B)) \mid R_{z_1,z_2}\Bigg(&\prod_{\substack{i\\ i\neq k}} \tilde{P}_i^2 \prod_j \tilde{Q}_j^2 \left[\left[2\frac{\partial A}{\partial y_{n-1}} + D(B)\right]\frac{\partial P_k}{\partial y_{n-1}} \right.\\ &\left. + D(P_k)\frac{\partial B}{\partial y_{n-1}}\right], (A,B)\Bigg) \end{aligned} \tag{16}$$

and further

$$\begin{aligned} R_{z_1,z_2,z_3}\left(P_k, \left(A, B, \frac{\partial B}{\partial y_{n-1}}\right)\right) \mid R_{z_1,z_2,z_3}\Bigg(&\prod_{\substack{i\\ i\neq k}} \tilde{P}_i^2 \prod_j \tilde{Q}_j^2 \frac{\partial P_k}{\partial y_{n-1}}\left[2\frac{\partial A}{\partial y_{n-1}}\right. \\ &\left. + D(B)\right], \left(A, B, \frac{\partial B}{\partial y_{n-1}}\right)\Bigg). \end{aligned} \tag{17}$$

From (16) we can obtain the hypothesis for P_k and then with the help of (17) all its undetermined constants.

If there is a polynomial P_s in the integrating factor structure which satisfy (11), then finding P_k can be simplified. Since

$$R_z(P_k,P_s) \mid R_z\left(A\prod_{\substack{i\\ i\neq k}} \tilde{P}_i^2 \prod_j \tilde{Q}_j^2 \frac{\partial P_s}{\partial y_{n-1}}\left[A\frac{\partial P_k}{\partial y_{n-1}} + BD(P_k)\right], P_s\right), \tag{18}$$

then

$$R_{z_1,z_2}(P_k,(P_s,B)) \mid R_{z_1,z_2}\left(A^2\prod_{\substack{i\\ i\neq k}}\tilde{P}_i^2\prod_j\tilde{Q}_j^2\frac{\partial P_s}{\partial y_{n-1}}\frac{\partial P_k}{\partial y_{n-1}},(P_s,B)\right). \tag{19}$$

Among other things we can observe from (16) and (18) is that it is very likely that the P_k's satisfy the relation (11) even if $a_k = 1$. Also, there are relations for Q_k with $b_k > 0$, similar to (16)–(19).

The main elements of the method described above were implemented in demonstration procedures in *Maple* [21, 22] for the case of rational integrating factors.

We have tested routines in the following area: for a given ODE order we assigned the first integral ζ of the following type

$$\zeta = \frac{\prod_i N_i^{\alpha_i}}{\prod_i M_i^{\beta_i}} + \sum_i(\eta_i\ln\Phi_i + \theta_i\arctan\Psi_i),$$

where N_i, M_i, Φ_i, Ψ_i are some polynomials, $\alpha_i > 0$, $\beta_i > 0$, η_i, θ_i are some constants. Such algebraic first integral leads to an ODE of type (1) with $f = -D(\zeta)/\frac{\partial\zeta_k}{\partial y_{n-1}}$, which is potentially tractable by the proposed procedure as this ODE has a rational integrating factor $\mu = \frac{\partial\zeta_k}{\partial y_{n-1}} = \prod_i P_i^{a_i}$.

The first stage – finding candidates and hypothesis of the structure of an integrating factor – is very fast. The second stage is relatively resource consuming – finding the unknown parameters a_i of the structure of an integrating factor.

We can conclude that our prototypes are workable for rational ODEs even with symbolic constant parameters of orders from $n = 1$ to $n = 4 - 5$. Although in principle the method provides ability to find an integrating factor for almost any of the above mentioned type of ODEs, in practice the procedure heavily relies on some basic CAS functions such as *factor, simplify* and so on, and in some cases fails. In addition, different outputs are observed in different *Maple* versions for the same ODE.

As a rule, *Maple* cannot find any integrating factor for ODEs from the considered test area while proposed procedure is usually successful. Often our procedure produces more than one integrating factor. For example, the output may contain a set of μ's or μ with arbitrary parameters, and sometimes they lead to independent first integrals.

3 Conclusion

We have proposed an algebraic method for finding polynomials P_i and Q_j of the integrating factor form (7) of rational ODE (6) for any order using resultants of the numerator and denominator of the ODE's right-hand-side.

The main advantage of the method of resultants is ability to obtain the polynomials P_i and Q_j without an a priori assumption of the upper bound for their respective degrees. Our method bypasses the principal weakness of the Prelle–Singer (Darboux) method.

Moreover, the number of undetermined variables involved in calculations by our method is radically smaller than in the Prelle–Singer (Darboux) method. This property allows us to obtain polynomials P_i and Q_j of arbitrary degrees without any complications whereas the Prelle–Singer (Darboux) method can not be useful as a rule for $N > 4$.

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