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Geodesic Flow and Two (Super) Component Analog of the Camassa–Holm Equation

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Abstract. We derive the 2-component Camassa–Holm equation and corresponding $N = 1$ super generalization as geodesic flows with respect to the H^1 metric on the extended Bott–Virasoro and superconformal groups, respectively.

Key words: geodesic flow; diffeomorphism; Virasoro orbit; Sobolev norm

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1 Introduction

About ten years ago, Rosenau, [26], introduced a class of solitary waves with compact support as solutions of certain wave equations with nonlinear dispersion. It was found that the solutions of such systems unchanged from collision and were thus called *compactons*. The discovery that solitons may compactify under nonlinear dispersion inspired further investigation of the role of nonlinear dispersion. It has been known for some time that nonlinear dispersion causes wave breaking or lead to the formation of corners or cusps. Beyond compactons, a wide variety of other exotic non-analytic solutions, including peakons, cuspon, mesaons, etc., have been found in to exist in a variety of models that incorporate nonlinear dispersion [15].

We will study integrable evolution equations appearing in bi-Hamiltonian form

$$u_t = J_1 \frac{\delta H_1}{\delta u} = J_2 \frac{\delta H_0}{\delta u}, \qquad n = 0, 1, 2, \dots, \tag{1}$$

where J_1 and J_2 are compatible Hamiltonian operators. The initial Hamiltonians H_0, H_1 are the first two in a hierarchy of conservation laws whose corresponding bi-Hamiltonian flows are successively generated by the recursion operator

$$\mathcal{R} = J_2 J_1^{-1}.$$

We refer the reader to [21] for the basic facts about bi-Hamiltonian systems.

In an earlier work, the second author showed with Rosenau [22] that a simple scaling argument shows that most integrable bi-Hamiltonian systems are governed by tri-Hamiltonian structures. They formulated a method of "tri-Hamiltonian duality", in which a recombination of the Hamiltonian operators leads to integrable hierarchies endowed with nonlinear dispersion that supports compactons or peakons. A related construction can be found in the contemporaneous paper of Fuchssteiner [8].

The tri-Hamiltonian formalism can be best described through examples. The Korteweg–de Vries equation

$$u_t = u_{xxx} + 3uu_x,$$

can be written in bi-Hamiltonian form (1) using the two compatible Hamiltonian operators

$$J_1 = D, \qquad J_2 = D^3 + uD + Du, \qquad \text{where} \quad D \equiv \frac{d}{dx}$$

and

$$H_1 = \frac{1}{2}\int u^2\, dx, \qquad H_2 = \frac{1}{2}\int \left(-u_x^2 + u^3\right)\, dx.$$

The tri-Hamiltonian duality construction is implemented as follows:

- A simple scaling argument shows that J_2 is in fact the sum of two compatible Hamiltonian operators, namely $K_2 = D^3$ and $K_3 = uD + Du$, so that $K_1 = J_1, K_2, K_3$ form a triple of mutually compatible Hamiltonian operators.
- Thus, when we can recombine the Hamiltonian triple by transferring the leading term D^3 from J_2 to J_1, thereby constructing the Hamiltonian pairs $\widehat{J}_1 = K_2 \pm K_1 = D^3 \pm D$. The resulting self-adjoint operator $S = 1 \pm D^2$ is used to define the new field variable $\rho = Su = u \pm u_{xx}$.
- Finally, the second Hamiltonian structure is constructed by replacing u by ρ in the remaining part of the original Hamiltonian operator K_3, so that $\widehat{J}_2 = \rho D + D\rho$. Note that this change of variables does not affect $\widehat{J}_1$.

As a result of this procedure, we recover the tri-Hamiltonian dual of the KdV equation

$$\rho_t = \widehat{J}_1 \frac{\delta \widehat{H}_2}{\delta \rho} = \widehat{J}_2 \frac{\delta \widehat{H}_1}{\delta \rho}, \tag{2}$$

where

$$\widehat{H}_1 = \frac{1}{2}\int u\rho\, dx = \frac{1}{2}\int \left(u^2 \mp u_x^2\right)\, dx, \qquad \widehat{H}_2 = \frac{1}{2}\int \left(u^3 \mp uu_x^2\right)\, dx.$$

In this case, (2) reduces to the celebrated Camassa–Holm equation [2, 3]:

$$u_t \pm u_{xxt} = 3uu_x \pm \left(uu_{xx} + \tfrac{1}{2}u_x^2\right)_x.$$

Remark 1. The choice of plus sign leads to an integrable equation which supports compactons, whereas the minus sign is the water wave model derived by Camassa–Holm, whose solitary wave solutions have a sharp corner at the crest.

The Ito equation. Let us next study the Ito equation [14],

$$u_t = u_{xxx} + 3uu_x + vv_x,$$
$$v_t = (uv)_x,$$

which is a prototypical example of a two-component KdV equation. This can also be expressed in bi-Hamiltonian form using the following two Hamiltonian operators

$$J_1 = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}, \qquad J_2 = \begin{pmatrix} D^3 + uD + Du & vD \\ Dv & 0 \end{pmatrix},$$

with Hamiltonians

$$H_1 = \frac{1}{2}\int \left(u^2 + v^2\right)\, dx, \qquad H_2 = \frac{1}{2}\int \left(u^3 + uv^2 - u_x^2\right)\, dx.$$

Again, a simple scaling argument is used to split

$$J_2 = \begin{pmatrix} D^3 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} uD + Du & vD \\ Dv & 0 \end{pmatrix}$$

as a sum of two compatible Hamiltonian operators. To construct the dual, we transfer the leading term D^3 from the first Hamiltonian operator to the second. We obtain the first Hamiltonian operator for the new equation

$$\widehat{J}_1 = \begin{pmatrix} D \pm D^3 & 0 \\ 0 & D \end{pmatrix} \equiv D \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the new variables are defined as

$$\begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The second Hamiltonian structure follows from the truncated part of the original Hamiltonian operator J_2, so that

$$\widehat{J}_2 = \begin{pmatrix} \rho D + D\rho & vD \\ Dv & 0 \end{pmatrix}$$

with

$$\widehat{H}_1 = \frac{1}{2}\int \left(u\rho + v^2\right) dx, \qquad \widehat{H}_2 = \frac{1}{2}\int \left(u^3 + uv^2 \mp uu_x^2\right) dx.$$

The dual system (2) takes the explicit form

$$\begin{aligned} &u_t \pm u_{xxt} = 3uu_x + vv_x + \left(uu_{xx} + \tfrac{1}{2}u_x^2\right)_x, \\ &v_t = (uv)_x. \end{aligned}$$

Motivation. The Camassa–Holm equation was derived physically as a shallow water wave equation by Camassa and Holm [2, 3, 13], and identified as the geodesic flow on the group of one-dimensional volume-preserving diffeomorphisms under the H^1 metric. The multi-dimensional analogs lead to important alternative models to the classical Euler equations of fluid mechanics. Later, Misiolek [20] showed that, like the KdV equation [23], it can also be characterized as a geodesic flow on the Bott–Virasoro group.

Recently, a 2-component generalization of the Camassa–Holm equation has drawn a lot of interest among researchers. Chen, Dubrovin, Falqui, Grava, Liu and Zhang (the group at SISSA) have been working on multi-component analogues, using reciprocal transformations and studying their effect on the Hamiltonian structures, [4, 7, 16]. They show that the 2-component system cited above admits peakons, albeit of a different shape owing to the difference in the corresponding Green's functions. Another two-component generalization also appeared recently as the bosonic sector of the extended $N = 2$ supersymmetric Camassa–Holm equation [25].

Following Ebin–Marsden [6], we enlarge Diff (S^1) to a Hilbert manifold $\mathrm{Diff}^s(S^1)$, the diffeomorphisms of the Sobolev class H^s. This is a topological space. If $s > n/2$, it makes sense to talk about an H^s map from one manifold to another. Using local charts, one can check whether the derivations of order $\leq s$ are square integrable. The Lie algebra of $\mathrm{Diff}^s(S^1)$ is denoted by $\mathrm{Vect}^s(S^1)$.

In this paper we show that a 2-component generalization of the Camassa–Holm equation and its super analog also follow from the geodesic with respect to the H^1 metric on the semidirect product space $\mathrm{Diff}^s(S^1) \ltimes C^\infty(S^1)$ and its supergroup respectively. In fact, it is known that numerous coupled KdV equations [10, 11, 12] follow from geodesic flows of the right invariant L^2 metric on the semidirect product group $\mathrm{Diff}\,\widehat{(S^1) \ltimes C^\infty}(S^1)$ [1, 18].

2 Preliminaries

The Lie algebra of $\mathrm{Diff}^s(S^1) \ltimes C^\infty(S^1)$ is the semidirect product Lie algebra

$$\mathfrak{g} = \mathrm{Vect}^s(S^1) \ltimes C^\infty(S^1).$$

An element of $\mathfrak{g}$ is a pair

$$\left(f(x)\frac{d}{dx}, a(x)\right), \qquad \text{where} \qquad f(x)\frac{d}{dx} \in \mathrm{Vect}^s(S^1), \qquad \text{and} \qquad a(x) \in C^\infty(S^1).$$

It is known that this algebra has a three dimensional central extension given by the non-trivial cocycles

$$\begin{aligned} \omega_1\left(\left(f(x)\frac{d}{dx}, a(x)\right), \left(g\frac{d}{dx}, b\right)\right) &= \int_{S^1} f'(x)g''(x)\,dx, \\ \omega_2\left(\left(f(x)\frac{d}{dx}, a(x)\right), \left(g\frac{d}{dx}, b\right)\right) &= \int_{S^1} \left[\, f''(x)b(x) - g''(x)a(x) \,\right] dx, \\ \omega_3\left(\left(f(x)\frac{d}{dx}, a(x)\right), \left(g\frac{d}{dx}, b\right)\right) &= 2\int_{S^1} a(x)b'(x)\,dx. \end{aligned} \tag{3}$$

The first cocycle ω_1 is the well-known Gelfand–Fuchs cocycle. The Virasoro algebra

$$\mathrm{Vir} = \mathrm{Vect}^s(S^1) \oplus \mathbb{R}$$

is the unique non-trivial central extension of $\mathrm{Vect}^s(S^1)$ based on the Gelfand–Fuchs cocycle. The space $C^\infty(S^1) \oplus \mathbb{R}$ is identified as *regular part* of the dual space to the Virasoro algebra. The pairing between this space and the Virasoro algebra is given by:

$$\left\langle\, (u(x), a)\,, \left(f(x)\frac{d}{dx}, a(x)\right) \right\rangle = \int_{S^1} u(x)f(x)\,dx + a\alpha.$$

Similarly we consider the following extension of $\mathfrak{g}$,

$$\widehat{\mathfrak{g}} = \mathrm{Vect}^s(S^1) \ltimes C^\infty(S^1) \oplus \mathbb{R}^3.$$

The commutation relation in $\widehat{\mathfrak{g}}$ is given by

$$\left[\left(f\frac{d}{dx}, a, \alpha\right), \left(g\frac{d}{dx}, b, \beta\right)\right] := \left((fg' - f'g)\frac{d}{dx}, fb' - ga', \omega\right),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$, and where $\omega = (\omega_1, \omega_2, \omega_3)$ are the cocycles.

Let

$$\widehat{\mathfrak{g}}^*_{\mathrm{reg}} = C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbb{R}^3$$

denote the *regular part* of the dual space $\widehat{\mathfrak{g}}^*$ to the Lie algebra $\widehat{\mathfrak{g}}$, under the following pairing:

$$\langle \widehat{u}\,,\, \widehat{f}\,\rangle = \int_{S^1} \big[\, f(x)u(x) + a(x)v(x)\,\big]\, dx + \alpha\cdot\gamma,$$

where $\widehat{u} = (u(x), v, \gamma) \in \widehat{\mathfrak{g}}^*_{\rm reg}$, $\widehat{f} = \big(f\frac{d}{dx}, a, \alpha\big) \in \widehat{\mathfrak{g}}$. Of particular interest are the coadjoint orbits in $\widehat{\mathfrak{g}}^*_{\rm reg}$. In this case, Gelfand, Vershik and Graev [9], have constructed some of the corresponding representations.

Let us introduce H^1 inner product on the algebra $\widehat{\mathfrak{g}}$

$$\langle \widehat{f}\,,\, \widehat{g}\,\rangle_{H^1} = \int_{S^1} \big[\, f(x)g(x) + a(x)b(x) + \partial_x f(x)\partial_x g(x)\,\big]\, dx + \alpha\cdot\beta,$$

where

$$\widehat{f} = \left(f\frac{d}{dx}, a, \alpha\right), \qquad \widehat{g} = \left(g\frac{d}{dx}, b, \beta\right).$$

Now we compute:

Lemma 1. *The coadjoint operator with respect to the H^1 inner product is given by*

$$\mathrm{ad}^*_{\hat f}\begin{pmatrix} u \\ v\end{pmatrix} = \begin{pmatrix} (1-\partial^2)^{-1}[2f'(x)(1-\partial_x^2)u(x) + f(x)(1-\partial_x^2)u'(x) + a'v(x)] \\ f'v(x) + f(x)v'(x)\end{pmatrix}.$$

Proof. Since we have identified $\mathfrak{g}$ with $\mathfrak{g}^*$, it follows from the definition that

$$\begin{aligned}\langle\, \mathrm{ad}^*_{\hat f}\,\widehat{u}\,,\, \widehat{g}\,\rangle_{H^1} &= \langle\, \widehat{u}\,,\, [\widehat{f},\widehat{g}]\,\rangle_{H^1} \\ &= -\int_{S^1}\big[\,(fg'-f'g)u - (fb'-ga')v - \partial_x(fg'-f'g)\partial_x u\,\big]dx.\end{aligned}$$

After computing all the terms by integrating by parts and using the fact that the functions $f(x)$, $g(x)$, $u(x)$ and $a(x)$, $b(x)$, $v(x)$ are periodic, the right hand side can be expressed as above.

Let us compute now the left hand side:

$$\begin{aligned}\mathrm{ad}^*_{\hat f}\begin{pmatrix} u \\ v\end{pmatrix} &= \int_{S^1}\big[\,(\mathrm{ad}^*_{\hat f}\,u)g + (\mathrm{ad}^*_{\hat f}\,u)'g' + (\mathrm{ad}^*_{\hat f}\,v)b\,\big]\, dx \\ &= \int_{S^1}\big[\,[(1-\partial^2)\mathrm{ad}^*_{\hat f}\,u]g + (\mathrm{ad}^*_{\hat f}\,v)b\,\big]\, dx = \big\langle((1-\partial^2)\mathrm{ad}^*_{\hat f}\,u, (\mathrm{ad}^*_{\hat f}\,v)), (g,b)\big\rangle.\end{aligned}$$

Thus by equating the the right and left hand sides, we obtain the desired formula. ∎

We conclude that the Hamiltonian operator arising from the induced Lie–Poisson structure is

$$\begin{pmatrix} D\rho + \rho D & vD \\ Dv & 0\end{pmatrix},$$

where $\rho = (1-\partial_x^2)u$. We conclude that

Theorem. *A curve*

$$\widehat{c}(t) = \left(u(x,t)\frac{d}{dx}, v(x,t), \gamma\right) \subset \mathfrak{g}$$

defines a geodesic in the H^1 metric if and only if

$$\begin{aligned}&u_t - u_{xxt} = 3uu_x + vv_x - \big(uu_{xx} + \tfrac12 u_x^2\big)_x, \\ &v_t = 2(uv)_x.\end{aligned}$$

3 Geodesic flow and superintegrable systems

The first and foremost characteristic property of a superalgebra is that all the additive groups of its basic and derived structures are $\mathbb{Z}_2$ graded. A *vector superspace* is a $\mathbb{Z}_2$ graded vector space $V = V_B \oplus V_F$. An element v of V_B (resp. V_F) is said to be even or bosonic (resp. odd or fermionic). The super-commutator of a pair of elements $v, w \in V$ is defined to be the element

$$[v, w] = vw - (-1)^{\bar{v}\bar{w}} wv.$$

The generalized Neveu–Schwartz superalgebra [24] is composed of two parts: the bosonic (even) and the fermionic (odd). These are given by

$$S\mathfrak{g}_B = \mathrm{Vect}^s(S^1) \oplus C^\infty(S^1) \oplus \mathbb{R}^3, \qquad S\mathfrak{g}_F = C^\infty(S^1) \oplus C^\infty(S^1).$$

There are *three* different actions:

(A) the action of the bosonic part on the bosonic part, discussed earlier.

(B) the action of the bosonic part on the fermionic part, given by

$$[\ ,\ ] : S\mathfrak{g}_B \otimes S\mathfrak{g}_F \longrightarrow S\mathfrak{g}_F,$$
$$[(f(x)\frac{d}{dx}, a(x)), (\phi(x), \alpha(x))] := \begin{pmatrix} f(x)\phi' - \frac{1}{2} f'(x)\phi(x) \\ f(x)\alpha'(x) + \frac{1}{2} f'(x)\alpha(x) - \frac{1}{2} a'(x)\phi(x) \end{pmatrix}.$$

(C) the action of the fermionic part on the fermionic part, given by

$$[\ ,\ ]_+ : S\mathfrak{g}_F \otimes S\mathfrak{g}_F \longrightarrow S\mathfrak{g}_B,$$
$$[(\phi(x), \alpha(x)), (\psi(x), \beta(x))]_+ = \left(\phi\psi\frac{d}{dx}, \phi\beta + \alpha\psi, \omega_F\right),$$

where $\omega_F = (\omega_{F1}, \omega_{F2}, \omega_{F3})$ is the fermionic cocycle, with components

$$\omega_{F1}((\phi, \alpha), (\psi, \beta)) = 2\int_{S^1} \phi'(x)\psi'(x)\, dx,$$
$$\omega_{F2}((\phi, \alpha), (\psi, \beta)) = -2\int_{S^1} (\phi'(x)\beta(x) + \psi'\alpha(x))\, dx,$$
$$\omega_{F3}((\phi, \alpha), (\psi, \beta)) = 4\int_{S^1} \alpha(x)\beta(x)\, dx.$$

The supercocycle ω_S has two parts, the bosonic and the fermionic:

$$\omega_S = \omega_B \oplus \omega_F,$$

where the bosonic part ω_B is identical to $\omega = (\omega_1, \omega_2, \omega_3)$, as given by (3).

With this in hand, we establish the supersymmetric 2-component generalization of the Camassa–Holm equation.

Definition. The H^1 pairing between the regular part of the dual space $S\widehat{\mathfrak{g}}^*$ and $S\mathfrak{g}$ is given by

$$\left\langle (u(x), v(x), \psi(x), \beta)\, ,\, (f(x)\frac{d}{dx}, a(x), \phi(x), \alpha) \right\rangle_{H^1}$$
$$= \int_{S^1} f(x)u(x)\, dx + \int_{S^1} f_x u_x\, dx + \int_{S^1} a(x)v(x)\, dx$$
$$+ \int_{S^1} \phi(x)\psi(x)\, dx + \int_{S^1} \phi_x\psi_x\, dx + \int_{S^1} \alpha(x)\beta(x)\, dx.$$

Let us compute the coadjoint action with respect to the H^1 norm.

Lemma 2.

$$\mathrm{ad}^*_{\hat{f}} \begin{pmatrix} u(x) \\ v(x) \\ \psi(x) \\ \beta(x) \end{pmatrix}$$

$$= \begin{pmatrix} (1-\partial^2)^{-1}[2f'(1-\partial^2)u(x) + (1-\partial^2)u'f + a'v + \frac{1}{2}(1-\partial^2)\psi'\phi + \frac{3}{2}(1-\partial^2)\psi\phi'] \\ f'v + fv' + \frac{1}{2}(\beta'\phi + \beta\phi') \\ (1-\partial^2)^{-1}[f(1-\partial^2)\psi' + \frac{3}{2}f'(1-\partial^2)\psi + \frac{1}{2}a'\beta + (1-\partial^2)u\phi + v\alpha] \\ f\beta' + \frac{1}{2}f'\beta + v\phi \end{pmatrix}.$$

Sketch of the proof. Using the definition of the coadjoint action

$$\langle\, \mathrm{ad}^*_{\hat{f}}\, \widehat{u}\,,\, \widehat{g}\,\rangle_{H^1} = \langle\, \widehat{f}\,,\, [\widehat{u}, \widehat{g}]\,\rangle_{H^1}$$

with

$$\widehat{f} = \begin{pmatrix} f(x) \\ a(x) \\ \phi(x) \\ \alpha(x) \end{pmatrix}, \qquad \widehat{u} = \begin{pmatrix} u(x) \\ v(x) \\ \psi(x) \\ \beta(x) \end{pmatrix}, \qquad \widehat{g} = \begin{pmatrix} g(x) \\ b(x) \\ \chi(x) \\ \gamma(x) \end{pmatrix},$$

we obtain

$$\left\langle (u, v, \psi, \beta)]\,, \begin{pmatrix} (fg' - f'g)\frac{d}{dx} + \phi\chi\frac{d}{dx} \\ fb' - ga' + \phi\gamma + \alpha\chi \\ f\chi' - \frac{1}{2}f'\chi + g\phi' - \frac{1}{2}g'\phi \\ f\gamma' + \frac{1}{2}f'\gamma - \frac{1}{2}a'\gamma + g\alpha' + \frac{1}{2}g'\alpha - \frac{1}{2}b'\phi \end{pmatrix} \right\rangle_{H^1}.$$

This would give us the right hand side without the $(1-\partial^2)^{-1}$ term, which appears on the left hand side:

$$\begin{aligned} \text{L.H.S.} &= \int_{S^1} (\mathrm{ad}^*_{\hat{f}}\, u) g\, dx + \int_{S^1} (\mathrm{ad}^*_{\hat{f}}\, u)' g'\, dx \int_{S^1} (\mathrm{ad}^*_{\hat{f}}\, v) b\, dx \\ &\quad + \int_{S^1} (\mathrm{ad}^*_{\hat{f}}\, \psi)\phi\, dx + \int_{S^1} (\mathrm{ad}^*_{\hat{f}}\, \psi')\phi'\, dx + \int_{S^1} (\mathrm{ad}^*_{\hat{f}}\, \beta)\alpha\, dx \\ &= \int_{S^1} [(1-\partial^2)\mathrm{ad}^*_{\hat{f}}\, u] g\, dx + \int_{S^1} (\mathrm{ad}^*_{\hat{f}}\, v) b\, dx \\ &\quad + \int_{S^1} [(1-\partial^2)\mathrm{ad}^*_{\hat{f}}\, \psi]\phi\, dx + \int_{S^1} (\mathrm{ad}^*_{\hat{f}}\, \beta)\alpha\, dx. \end{aligned}$$

Equating the right and left hand sides, we obtain the desired formula. ■

Therefore, if we use the Euler–Poincaré equation and the computational trick used in [5], we obtain the supersymmetric version of the two component Camassa–Holm equation:

$$\begin{aligned} &m_t = 2mu_x + m_x u + (vv_x) + 3\xi\xi'', \\ &v_t = 2(uv)_x + \beta'\xi' + \beta\xi'', \\ &(1-\partial^2)\xi_t = 4m\xi' + 3m'\xi + 2\xi''', \\ &\beta_t = 2u\beta' + u'\beta + 2v\xi', \end{aligned}$$

where $m = u - u_{xx}$.

Corollary 1.

$$\mathrm{ad}^*_{\hat f}\,\widehat{u} = \begin{pmatrix} 2uf'(x) + u'f + a'v + f''' + \frac{1}{2}\psi'\phi + \frac{3}{2}\psi\phi' \\ f'v + fv' + \frac{1}{2}(\beta'\phi + \beta\phi') \\ f\psi' + \frac{3}{2}f'\psi + \frac{1}{2}a'\beta + u\phi + v\alpha + 2\phi'' \\ f\beta' + \frac{1}{2}f'\beta + v\phi \end{pmatrix}.$$

In this way, we recover a supersymmetric version of Ito equation [14] given by

$$\begin{aligned} &u_t = 6uu_x + 2(vv_x) + u_{xxx} + 3\xi\xi'', \\ &v_t = 2(uv)_x + \beta'\xi' + \beta\xi'', \\ &\xi_t = 4u\xi' + 3u'\xi + 2\xi''', \\ &\beta_t = 2u\beta' + u'\beta + 2v\xi'. \end{aligned} \tag{4}$$

Corollary 2. *In the supersymmetric Ito equation* (4)*:*

(A) *if we set the super variables $\xi = \beta = 0$, we recover the Ito equation.*

(B) *If we set $v = \beta = 0$, we obtain*

$$\begin{aligned} &u_t = 6uu_x + u_{xxx} + 3\xi\xi'', \\ &\xi_t = 4u\xi' + 3u'\xi + 2\xi''', \end{aligned}$$

which is a fermionic extension of KdV equation and, modulo rescalings, is the super KdV equation of Mathieu and Manin–Radul [17, 19].

Remark 2. The physicists usually distinguish the fermionic and supersymmetric extension among each other. From the physical point of view the supersymmetry requires the invariance under the supersymmetric transformations while for the fermionic extension we have no such restriction. All extended equations considered in this paper are fermionic.

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