On Regularized Solution for BBGKY Hierarchy of One-Dimensional Infinite System

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Abstract. We construct a regularized cumulant (semi-invariant) representation of a solution of the initial value problem for the BBGKY hierarchy for a one-dimensional infinite system of hard spheres interacting via a short-range potential. An existence theorem is proved for the initial data from the space of sequences of bounded functions.

Key words: BBGKY hierarchy; cumulant; regularized solution

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1 Introduction

Last ten years new functional-analytical methods of investigation of the initial value problem to the BBGKY hierarchy for infinite systems have been developed in the books [1, 2, 3, 4]. As is well known when constructing a solution of the BBGKY hierarchy of such systems, considerable analytical difficulties arise [5, 6, 7, 8]. Until recently, existence theorems have been proved for one-dimensional systems in the case of a certain class of short-range interaction potentials and three-dimensional case only for hard spheres systems [1].

While constructing the solution of the initial value problem for the BBGKY hierarchy of the classical systems of particles with the initial data from the space of sequences of bounded functions, one is faced with certain difficulties related to divergence of integrals with respect to configuration variables in each term of an expansion of the solution [5] (see also [1, 2]). The same problem arises also in the case of the cumulant representation of the solution stated in [9].

In this paper, we propose a regularization method for the solution of the BBGKY hierarchy in the cumulant representation. Due to this method, the structure of the solution expansions guarantees the mutual compensation of the divergent integrals in every term of the series. We establish convergence conditions for the series of the solution and prove an existence theorem of a local in time weak solution of the BBGKY hierarchy for the initial data from the space of sequences of functions which are bounded with respect to the configuration variables and exponentially decreasing with respect to the momentum variables.

2 Initial value problem for BBGKY hierarchy

Let us consider a one-dimensional system of identical particles (intervals with length σ and unit mass $m = 1$) interacting as hard spheres via a short range pair potential Φ. Every particle i is characterized by phase coordinates $(q_i, p_i) \equiv x_i \in \mathbb{R}\times\mathbb{R}$, $i \geq 1$. For the configurations $q_i \in \mathbb{R}^1$ of such a system (q_i is the position of the center of the ith particle), the following inequalities must be satisfied: $|q_i - q_j| \geq \sigma$, $i \neq j \geq 1$. The set $W_n \equiv \{\{q_1, \dots, q_n\} \mid \exists (i,j),\, i \neq j \in \{1, \dots, n\} : |q_i - q_j| < \sigma\}$ defines the set of forbidden configurations in the phase space of a system of n

particles. The phase trajectories of such hard sphere system are determined almost everywhere in the phase space $\{x_1,\ldots,x_n\}\in\mathbb{R}^n\times(\mathbb{R}^n\setminus W_n)$, namely, outside a certain set $\mathcal{M}_n^0$ of the Lebesgue measure zero [1]. The initial data $\{x_1,\ldots,x_n\}\in\mathbb{R}^n\times(\mathbb{R}^n\setminus W_n)$ belong to the set $\mathcal{M}_n^0$ if (a) there is more than one pair collision at the same moment of time $t\in(-\infty,+\infty)$ or (b) infinitely many collisions occur within a finite time interval.

We assume that the interaction between the hard spheres is given by a potential Φ with a finite range R such that the following conditions are satisfied:

$$\begin{aligned}
&(a)\quad \Phi\in C^2([\sigma,R]),\quad 0<\sigma<R<\infty,\\
&(b)\quad \Phi(|q|)=\begin{cases}+\infty, & |q|\in[0,\sigma),\\ 0, & |q|\in(R,\infty),\end{cases}\\
&(c)\quad \Phi'(\sigma+0)=0.
\end{aligned}\tag{1}$$

We note that conditions (1) imply the estimate

$$\Big|\sum_{i<j=1}^{n}\Phi(q_i-q_j)\Big|\leq bn,\qquad b\equiv\sup_{q\in[\sigma,R]}\big|\Phi(q)\big|\big(\big[\tfrac{R}{\sigma}\big]\big),\tag{2}$$

where $\big[\frac{R}{\sigma}\big]$ is the integer part of the number $\frac{R}{\sigma}$.

The evolution of states of the system under consideration is described by the initial value problem for the BBGKY hierarchy [1, 2]

$$\begin{aligned}
\frac{\partial}{\partial t}F_s(t,x_1,\ldots,x_s)&=\{H_s,F_s(t,x_1,\ldots,x_s)\}\\
&+\int dx_{s+1}\Big\{\sum_{i=1}^{s}\Phi(q_i-q_{s+1}),F_{s+1}(t,x_1,\ldots,x_{s+1})\Big\}\\
&+\sum_{i=1}^{s}\int_0^\infty dP\,P\cdot\Big(F_{s+1}(t,x_1,\ldots,x_s,q_i+\sigma,p_i-P)\\
&-F_{s+1}(t,x_1,\ldots,q_i,p_i-P,\ldots,x_s,q_i-\sigma,p_i)\\
&+F_{s+1}(t,x_1,\ldots,x_s,q_i-\sigma,p_i+P)\\
&-F_{s+1}(t,x_1,\ldots,q_i,p_i+P,\ldots,x_s,q_i+\sigma,p_i)\Big),
\end{aligned}\tag{3}$$

$$F_s(t,x_1,\ldots,x_s)\big|_{t=0}=F_s(0,x_1,\ldots,x_s),\qquad s\geq 1,\tag{4}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket, H_s is the Hamiltonian of the s particle system, and $F(0)=\big(1,F_1(0,x_1),\ldots,F_s(0,x_1,\ldots,x_s),\ldots\big)$ is a sequence of initial s-particle distribution functions $F_s(0,x_1,\ldots,x_s)$ defined on the phase space $\mathbb{R}^s\times(\mathbb{R}^s\setminus W_s)$.

Consider the initial value problem for the BBGKY hierarchy (3), (4) with the initial data $F(0)$ from the space $L^\infty_{\xi,\beta}$ of sequences $f=\big(1,f_1(x_1),\ldots,f_n(x_1,\ldots,x_n),\ldots\big)$ of bounded functions $f_n(x_1,\ldots,x_n)$, $f_0\equiv 1$, $n\geq 0$, that are defined on the phase space $\mathbb{R}^n\times(\mathbb{R}^n\setminus W_n)$, are invariant under permutations of the arguments x_i, $i=1,\ldots,n$, and are equal to zero on the set W_n. The norm in this space is defined by the formula

$$\|f\|=\sup_{n\geq 0}\xi^{-n}\sup_{x_1,\ldots,x_n}|f_n(x_1,\ldots,x_n)|\,\exp\Big\{\beta\sum_{i=1}^{n}\frac{p_i^2}{2}\Big\},\tag{5}$$

where ξ, β are positive integers. Note that the sequences of n particle equilibrium distribution functions of infinite systems belong to the space $L^{\infty}_{\xi,\beta}$ [1, 10].

Let $Y \equiv \{x_1,\dots,x_s\}$, $X \equiv \{Y, x_{s+1},\dots,x_{s+n}\}$, namely, $X \setminus Y = \{x_{s+1},\dots,x_{s+n}\}$, and let the symbol $|X| = |Y| + |X \setminus Y| = s+n$ denote the number of the elements of the set X. By the symbol X_Y we denote the set X with the subset Y treated as a single element similar to $x_{s+1},\dots,x_{s+n}$. Let L^1_0 be the subspace of finite sequences of continuously differentiable functions with compact supports of space L^1 of sequences of integrable functions. For $F(0)$ from L^1_0, and hence for all $F(0) \in L^1_0 \cap L^{\infty}_{\xi,\beta}$ it was proved in [8, 9] that the solution $F(t) = \big(1, F_1(t,x_1),\dots,F_s(t,x_1,\dots,x_s),\dots\big)$ of the initial value problem (3), (4) is determined by the series expansion

$$F_{|Y|}(t,Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{\mathbb{R}^n \times (\mathbb{R}^n \setminus W_n)} d(X \setminus Y) \sum_{\mathtt{P}:\, X_Y = \bigcup_l X_l} (-1)^{|\mathtt{P}|-1} (|\mathtt{P}|-1)! \times \prod_{X_l \subset \mathtt{P}} S_{|X_l|}(-t, X_l)\, F_{|X|}(0, X), \qquad |X \setminus Y| \geq 0, \tag{6}$$

where $\sum\limits_{\mathtt{P}}$ is the sum over all possible partitions $\mathtt{P}$ of the set X_Y into $|\mathtt{P}|$ mutually disjoint nonempty subsets $X_l \subset X_Y$, $X_k \cap X_l = \varnothing$, $k \neq l$, such that the entire set Y is contained in one of the subsets X_l.

On the set of sequences $f \in L^1_0 \cap L^{\infty}_{\xi,\beta}$, the evolution operator $S_{|X_l|}(-t, X_l)$ from expansion (6) is given by the formula

$$\begin{aligned} \big(S(-t)f\big)_{|X_l|}(X_l) &= \big(S_{|X_l|}(-t) f_{|X_l|}\big)(X_l) \equiv S_{|X_l|}(-t, X_l)\, f_{|X_l|}(X_l) \\ &= \begin{cases} f_{|X_l|}\big(\mathtt{X}_1(-t, X_l),\dots,\mathtt{X}_{|X_l|}(-t, X_l)\big), & \text{if } x \in \big(\mathbb{R}^{|X_l|} \times (\mathbb{R}^{|X_l|} \setminus W_{|X_l|})\big) \setminus \mathcal{M}^0_{|X_l|}, \\ 0, & \text{if } x \in \mathbb{R}^{|X_l|} \times W_{|X_l|}, \end{cases} \end{aligned} \tag{7}$$

where $\mathtt{X}_1(-t, X_l)$, $i = 1,\dots,|X_l|$, is the solution of the initial value problem for the Hamilton equations of the system of $|X_l|$ particles with initial data $\mathtt{X}_i(0, X_l) = x_i$ ($S_{|X_l|}(0) = I$ is the identity operator). Under conditions (1) on the potential Φ, the evolution operator (7) exists for $t \in (-\infty, +\infty)$; its properties are described in [1].

In the nth term of expansion (6), the form of the integrands is constructed by using the cumulant of order $1+n$ for the evolution operators (7):

$$\sum_{\mathtt{P}:\, X_Y = \bigcup_l X_l} (-1)^{|\mathtt{P}|-1} (|\mathtt{P}|-1)! \prod_{X_l \subset \mathtt{P}} S_{|X_l|}(-t, X_l) \equiv \mathfrak{A}_{1+|X \setminus Y|}(t, X_Y), \qquad |X \setminus Y| \geq 0. \tag{8}$$

Here, the notation from formula (6) is used. Note that the order of the cumulant $\mathfrak{A}_{1+|X \setminus Y|}(t)$ is determined by the number of elements of the set X_Y (in this case, by $1 + |X \setminus Y|$ elements).

The simplest examples ($1 + |X \setminus Y| = 1, 2, 3$) of the evolution operator $\mathfrak{A}_{1+n}(t)$ (8) have the form [9, 11, 12]

$$\mathfrak{A}_1(t, Y) = S_s(-t, Y), \tag{8a}$$

$$\mathfrak{A}_2(t, Y, x_{s+1}) = S_{s+1}(-t, Y, x_{s+1}) - S_s(-t, Y)\, S_1(-t, x_{s+1}), \tag{8b}$$

$$\begin{aligned} \mathfrak{A}_3(t, Y, x_{s+1}, x_{s+2}) &= S_{s+2}(-t, Y, x_{s+1}, x_{s+2}) - S_{s+1}(-t, Y, x_{s+1})\, S_1(-t, x_{s+2}) \\ &\quad - S_{s+1}(-t, Y, x_{s+2})\, S_1(-t, x_{s+1}) - S_s(-t, Y)\, S_2(-t, x_{s+1}, x_{s+2}) + \\ &\quad + 2! S_s(-t, Y)\, S_1(-t, x_{s+1})\, S_1(-t, x_{s+2}). \end{aligned} \tag{8c}$$

Thus, the cumulant representation of the solution (6) of the initial value problem for the BBGKY hierarchy (3), (4) is determined by the cumulants $\mathfrak{A}_{1+|X \setminus Y|}(t)$ (8) for the evolution operators $S_{|X_l|}(-t, X_l)$ (7).

3 Regularization of solution

For $F(0) \in L^{\infty}_{\xi,\beta}$ every term $n \equiv |X \setminus Y|$ of expansion (6) contains divergent integrals with respect to the configuration variables. Let us show that the above-stated cumulant nature of the solution expansions (6) for the initial value problem of the BBGKY hierarchy (3), (4) guarantees the compensation of the divergent integrals, i.e., the cumulants are determined terms of expansion (6) as the sum of summands with divergent integrals that compensate one another. In order to prove this fact, let us rearrange the terms of expansion (6) so that they are represented by the simplest mutually compensating groups of summands. Such procedure will be called a regularization of the solution (6). In this case, the regularization will be based on expressing cumulants of higher order in terms of the first and second order cumulants. For fixed initial data the second-order cumulants will be determined by expressions which compensate each other over a certain bounded domain.

The next lemma shows that, in the general case, the cumulant of the $(1+n)$th order $\mathfrak{A}_{1+n}(t)$, $n \geq 1$, is expressed via the first and second order cumulants.

Lemma 1. *The equality*

$$\mathfrak{A}_{1+|X\setminus Y|}(t, X_Y) = \sum_{\substack{Z\subset X\setminus Y\\ Z\neq\varnothing}} \mathfrak{A}_2(t, Y, Z) \times \sum_{\mathrm{P}: X\setminus(Y\cup Z)=\bigcup_l X_l} (-1)^{|\mathrm{P}|}(|\mathrm{P}|)! \prod_{X_l\subset \mathrm{P}} \mathfrak{A}_1(t, X_l), \qquad |X \setminus Y| \geq 1, \tag{9}$$

is true, where $\sum_{Z}$ *is the sum over all the nonempty subsets* Z *of the set* $X \setminus Y$, $Z \subset X \setminus Y$, *and the group of* $|Z|$ *particles evolves as a single element, and* $\sum_{P}$ *is the sum over all possible partitions* P *of the set* $X \setminus (Y \cup Z)$ *into* $|\mathrm{P}|$ *mutually disjoint nonempty subsets* $X_l \subset X \setminus (Y \cup Z)$, $X_k \cap X_l = \varnothing$, $k \neq l$, *such that every cluster of* $|X_l|$ *particles evolves as a single element.*

The proof of Lemma 1 is based on the verification that (9) is equal to expression (8) by taking into account the representation of a second order cumulant in terms of the first order ones.

By using Lemma 1, represent integrands of every summand from expansion (6) in terms of the first and second order cumulants. As a result, expansion (6) for the solution of the initial value problem (3), (4) takes the form

$$F_{|Y|}(t, Y) = \mathfrak{A}_1(t, Y)F_{|Y|}(0, Y) + \sum_{n=1}^{\infty} \frac{1}{n!} \int\limits_{\mathbb{R}^n\times(\mathbb{R}^n\setminus W_n)} d(X \setminus Y) \times \sum_{\substack{Z\subset X\setminus Y\\ Z\neq\varnothing}} \mathfrak{A}_2(t, Y, Z) \sum_{\mathrm{P}: X\setminus(Y\cup Z)=\bigcup_l X_l} (-1)^{|\mathrm{P}|}|\mathrm{P}|! \prod_{X_l\subset \mathrm{P}} \mathfrak{A}_1(t, X_l)\, F_{|X|}(0, X), \quad |X \setminus Y| \geq 1. \tag{10}$$

For the initial data $F(0) \in L^1 \cap L^{\infty}_{\xi,\beta}$, the equality

$$\int\limits_{\mathbb{R}^n\times(\mathbb{R}^n\setminus W_n)} d(X \setminus Y) \sum_{\substack{Z\subset X\setminus Y\\ Z\neq\varnothing}} \mathfrak{A}_2(t, Y, Z) \sum_{\mathrm{P}: X\setminus(Y\cup Z)=\bigcup_l X_l} (-1)^{|\mathrm{P}|}|\mathrm{P}|! \prod_{X_l\subset \mathrm{P}} \mathfrak{A}_1(t, X_l)\, F_{|X|}(0, X)$$
$$= \int\limits_{\mathbb{R}^n\times(\mathbb{R}^n\setminus W_n)} d(X \setminus Y) \sum_{\substack{Z\subset X\setminus Y\\ Z\neq\varnothing}} (-1)^{|X\setminus(Y\cup Z)|}\mathfrak{A}_2(t, Y, Z)\, F_{|X|}(0, X), \qquad |X \setminus Y| \geq 1, \tag{11}$$

is true. Here, we have used the Liouville theorem [1] and taken into account the relation

$$\sum_{k=1}^{m}(-1)^k k!\, s(m,k)=(-1)^m, \qquad m\geq 1, \tag{12}$$

where $s(m,k)$ is the Stirling number of the second kind defined as the number of all distinct partitions of a set containing m elements into k subsets.

Thus, by virtue of equality (11), the expansion (10) for the solution of the initial value problem for the BBGKY hierarchy (3), (4) takes the form

$$\begin{aligned} &F_{|Y|}(t,Y)=\mathfrak{A}_1(t,Y)F_{|Y|}(0,Y) \\ &+\sum_{n=1}^{\infty}\frac{1}{n!}\int\limits_{\mathbb{R}^n\times(\mathbb{R}^n\setminus W_n)} d(X\setminus Y)\sum_{\substack{Z\subset X\setminus Y\\ Z\neq\varnothing}}(-1)^{|X\setminus(Y\cup Z)|}\mathfrak{A}_2(t,Y,Z)\,F_{|X|}(0,X), \qquad |X\setminus Y|\geq 1, \end{aligned} \tag{13}$$

where the notation from (9) is used.

Representation (13) will be called a regularized cumulant representation of the solution of the initial value problem for the BBGKY hierarchy (3), (4).

4 Existence theorem

Taking into account the invariance of the Gibbs distributions (the Maxwellian distribution can be extended to become a Gibbs distribution) with respect to the action of evolution operators $S\,(-t)$ (7) and using the condition (2) and the relation (9) we estimate the integrands in expansion (13).

Lemma 2. *If $F(0)\in L^{\infty}_{\xi,\beta}$ then the inequality*

$$\begin{aligned} &\Big|\sum_{\substack{Z\subset X\setminus Y\\ Z\neq\varnothing}}(-1)^{|X\setminus(Y\cup Z)|}\mathfrak{A}_2(t,Y,Z)\,F_{|X|}(0,X)\Big| \\ &\qquad\leq 2\,\|F(0)\|\,(\xi\,\mathrm{e}^{2\beta b})^s\,(\xi\,\mathrm{e}^{2\beta b})^n\exp\Big\{-\beta\sum_{i=1}^{s+n}\frac{p_i^2}{2}\Big\} \end{aligned} \tag{14}$$

holds, where the notation from formulae (2), (5) *and* (9) *is used.*

As a consequence of Lemma 2, the following existence theorem is true.

Theorem. *If $F(0)\in L^{\infty}_{\xi,\beta}$ is a sequence of nonnegative functions, then for $\xi<\frac{\mathrm{e}^{-2\beta b-1}}{2\tilde{C}_1}\sqrt{\frac{\beta''}{2\pi}}$ and $t\in[0,t_0)$, where $t_0=\frac{1}{\tilde{C}_2}\Big(\frac{\mathrm{e}^{-2\beta b-1}}{2\xi}\sqrt{\frac{\beta''}{2\pi}}-\tilde{C}_1\Big)$, $\tilde{C}_1=\max(2R,1)$, $\tilde{C}_2=\max\big(2(4b+1),\frac{2}{\beta'}\big)$, $\beta=\beta'+\beta''$, $b\equiv\sup\limits_{q\in[\sigma,R]}\big|\Phi(q)\big|\big(\big[\frac{R}{\sigma}\big]\big)$ and $\big[\frac{R}{\sigma}\big]$ is the integer part of the number $\frac{R}{\sigma}$, there exists a unique weak solution of the initial value problem for the BBGKY hierarchy* (3), (4)*, namely, the sequence $F(t)\in L^{\infty}_{\xi,\beta}$ of nonnegative functions $F_s(t)$ determined by expansion* (13).

Proof. Let particles interact via a short-range pair potential that satisfies conditions (1). We assume that, at the initial instant, the configuration coordinates q_i, $i=1,2,\ldots,s$, of the particles constituting the cluster Y take values in a compact set of those $|Y|$ intervals l_i with length $|l_Y|$

that $q_i \in l_i$. Then if during the time interval $[0,t)$ none of the particles of any cluster $Z \subset X \setminus Y$ interacts with the particles of cluster Y, then the operator equality holds

$$S_{|Y\cup Z|}(-t,Y,Z) = S_{|Y|}(-t,Y)S_{|Z|}(-t,Z),$$

and, as a result, we have

$$\mathfrak{A}_2(t,Y,Z)F_{|X|}(0,X) = 0.$$

Therefore, in this case, the integrands in the nth term of expansion (13) are equal to zero. Since $Z \subset X \setminus Y$, it follows that in the expansion (13) the domain $\mathbb{R}^n \setminus W_n$ of integration with respect to the configuration variables is determined by n bounded intervals where the particles of the cluster Y during the time interval $[0,t)$ interact with particles of the cluster Z that contains the maximal number of particles, namely, with the particles of the cluster $X \setminus Y$. Thus, the domain $\mathbb{R}^n \setminus W_n$ of integration estimates the following finite value

$$V(t) \leq \Big(C + C_0 t + (C_1 + C_2 t)n + t\sum_{i=s+1}^{s+n} p_i^2\Big)^n, \tag{15}$$

where $C \equiv |l_Y| + 2sR$, $C_0 \equiv 2s(4b+1) + \sum\limits_{i=1}^{s} p_i^2$, $C_1 \equiv 2R$, and $C_2 \equiv 2(4b+1)$.

In view of estimates (14) and (15), we obtain

$$\begin{aligned} |F_{|Y|}(t,Y)| \leq 2\,\|F(0)\|\ (\xi \mathrm{e}^{2\beta b})^s \exp\Big\{-\beta\sum_{i=1}^{s}\frac{p_i^2}{2}\Big\}\sum_{n=0}^{\infty}\frac{1}{n!}(\xi\mathrm{e}^{2\beta b})^n \\ \times \int_{\mathbb{R}^n} dp_{s+1}\cdots dp_{s+n}\exp\Big\{-\beta\sum_{i=s+1}^{s+n}\frac{p_i^2}{2}\Big\}\Big(C + C_0 t + (C_1 + C_2 t)n + t\sum_{i=s+1}^{s+n} p_i^2\Big)^n. \end{aligned} \tag{16}$$

Taking in account the relation

$$\begin{aligned} &\Big((C + C_0 t) + (C_1 + C_2 t)n + t\sum_{i=s+1}^{s+n} p_i^2\Big)^n \\ &= \sum_{k=0}^{n}\frac{n!}{k!}(C + C_0 t)^k \sum_{r=0}^{n-k}\frac{1}{r!}\big((C_1 + C_2 t)n\big)^r \frac{1}{(n-k-r)!}t^{n-k-r}\Big(\sum_{i=s+1}^{s+n} p_i^2\Big)^{n-k-r} \end{aligned} \tag{17}$$

and the inequality

$$\Big(\sum_{i=s+1}^{s+n} p_i^2\Big)^{n-k-r} \exp\Big\{-\beta'\sum_{i=s+1}^{s+n}\frac{p_i^2}{2}\Big\} \leq (n-k-r)!\Big(\frac{2}{\beta'}\Big)^{n-k-r},$$

we compute the integrals with respect to the momentum variables in the right-hand side of (16):

$$\int\limits_{\mathbb{R}^n} dp_{s+1}\cdots dp_{s+n}\exp\Big\{-\beta''\sum_{i=s+1}^{s+n}\frac{p_i^2}{2}\Big\} = \Big(\frac{2\pi}{\beta''}\Big)^{\frac{n}{2}}, \qquad \beta = \beta' + \beta''. \tag{18}$$

Then estimate (16) takes the form

$$|F_{|Y|}(t,Y)| \leq 2\,\|F(0)\|\ (\xi\mathrm{e}^{2\beta b})^s \exp\Big\{-\beta\sum_{i=1}^{s}\frac{p_i^2}{2}\Big\}$$

$$\times \sum_{n=0}^{\infty}(2\xi e^{2\beta b})^n \left(\frac{2\pi}{\beta''}\right)^{\frac{n}{2}} \sum_{k=0}^{n} \frac{(C+C_0t)^k}{k!} \sum_{r=0}^{n-k} \frac{n^r}{r!}(C_1+C_2t)^r \frac{t}{(n-k-r)!}. \qquad (16')$$

Let us put $\tilde{C_1} = \max(C_1, 1)$ and $\tilde{C_2} = \max\left(C_2, \frac{2}{\beta'}\right)$ and continue estimate (16′). For arbitrary $t \geq 0$, the inequalities $\tilde{C_1} + \tilde{C_2}t \geq 1$ and $(\tilde{C_1} + \tilde{C_2}t)\frac{2t}{\beta'} \geq 1$ are true, and, therefore,

$$(C_1+C_2t)^r \left(\frac{2t}{\beta'}\right)^{n-k-r} \leq (\tilde{C_1} + \tilde{C_2}t)^n.$$

By using the inequalities $\sum\limits_{r=0}^{n-k} \frac{n^r}{r!} \leq e^n$ and $\sum\limits_{k=0}^{n} \frac{(C+C_0t)^k}{k!} \leq e^{(C+C_0t)}$ and taking estimate (16′) into account, we obtain

$$\begin{aligned} \left|F_{|Y|}(t,Y)\right| \leq 2\left\|F(0)\right\| (\xi e^{2\beta b})^s \exp\left\{-\beta\sum_{i=1}^{s}\frac{p_i^2}{2}\right\} \\ \times e^{(C+C_0t)} \sum_{n=0}^{\infty}\left(2\xi e^{2\beta b+1}\sqrt{\frac{\beta''}{2\pi}}\right)^n (\tilde{C_1}+\tilde{C_2}t)^n. \end{aligned} \qquad (19)$$

Thus, if $\xi < \frac{e^{-2\beta b-1}}{2\tilde{C_1}}\sqrt{\frac{\beta''}{2\pi}}$ then series (19) converges for $0 \leq t < t_0 \equiv \frac{1}{\tilde{C_2}}\left(\frac{e^{-2\beta b-1}}{2\xi}\sqrt{\frac{\beta''}{2\pi}} - \tilde{C_1}\right)$. We have thus shown that, under the conditions of the theorem, series (16) converges.

Finally, using the well-known theorems of functional analysis [13] and arguing similarly to [1, 2], we show that the sequence $F(t)$ is the unique weak solution of the initial value problem for the BBGKY hierarchy (3), (4). ■

5 Conclusion

The theory of the BBGKY hierarchy is developing now since the area of its application grows [14, 15]. At present while solving the initial value problem of the BBGKY hierarchy, many mathematical problems arise [3, 5, 16, 17]. In this paper one of such problems is considered for infinite particle system employing the method of the interaction region [5]. Taking into account the cumulant representation [8, 9] we construct a new regularized representation of the solution of the BBGKY hierarchy for a one-dimensional infinite system of hard spheres interacting via a short-range potential. For the initial data from the space of sequences of functions which are bounded in configuration variables and exponentially decreasing in momentum variables, existence of the cumulant representation allows to properly regularize such expansion for the solution, determining by second order cumulant as the sum of summands with divergent integrals that compensate one another. A multidimensional case and a case of a more general interaction potential will be investigated in further contribution.

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