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Hamiltonian Flows of Curves in $G/SO(N)$ and Vector Soliton Equations of mKdV and Sine-Gordon Type

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Abstract. The bi-Hamiltonian structure of the two known vector generalizations of the mKdV hierarchy of soliton equations is derived in a geometrical fashion from flows of non-stretching curves in Riemannian symmetric spaces $G/SO(N)$. These spaces are exhausted by the Lie groups $G = SO(N + 1), SU(N)$. The derivation of the bi-Hamiltonian structure uses a parallel frame and connection along the curve, tied to a zero curvature Maurer–Cartan form on G, and this yields the mKdV recursion operators in a geometric vectorial form. The kernel of these recursion operators is shown to yield two hyperbolic vector generalizations of the sine-Gordon equation. The corresponding geometric curve flows in the hierarchies are described in an explicit form, given by wave map equations and mKdV analogs of Schrödinger map equations.

Key words: bi-Hamiltonian; soliton equation; recursion operator; symmetric space; curve flow; wave map; Schrödinger map; mKdV map

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1 Introduction

There has been much recent interest in the close relation between integrable partial differential equations and the differential geometry of plane and space curves (see [10, 11, 12, 13, 20] for an overview and many results). The present paper studies flows of curves in Riemannian manifolds modeled by real symmetric spaces $G/SO(N)$ for arbitrary $N \geq 2$, where G is a compact semisimple Lie-group with an involutive automorphism that leaves fixed a Lie subgroup $SO(N) \subset G$. Such spaces [16] are exhausted by the groups $G = SO(N+1), SU(N)$ and describe curved G-invariant geometries that are a natural generalization of Euclidean spaces.

It is shown that if non-stretching curves are described using a moving parallel frame and an associated frame connection 1-form in $G/SO(N)$ then the frame structure equations for torsion and curvature encode $O(N-1)$-invariant bi-Hamiltonian operators. These operators will be demonstrated to produce a hierarchy of integrable flows of curves in which the frame components of the principal normal along the curve satisfy $O(N-1)$-invariant vector soliton equations. The hierarchies for both $SO(N+1)/SO(N)$, $SU(N)/SO(N)$ will be seen to possess a scaling symmetry and accordingly will be organized by the scaling weight of the flows. The 0 flow just consists of a convective (traveling wave) equation, while the $+1$ flow will be shown to give the two vector generalizations of the mKdV equation known from symmetry-integrability classifications of vector evolution equations in [27]. A recent classification analysis [5] found there are vector hyperbolic equations for which the respective vector mKdV equations are higher symmetries. These two vector hyperbolic equations will be shown to describe a -1 flow in the respective hierarchies for $SO(N+1)/SO(N)$ and $SU(N)/SO(N)$.

As further results, the Hamiltonian operators will yield explicit $O(N-1)$-invariant recursion operators for higher symmetries and higher conservation laws of the vector mKdV equations and the vector hyperbolic equations. The associated curve flows produced from these equations will describe geometric nonlinear PDEs, in particular given by wave maps and mKdV analogs of Schrödinger maps.

Previous fundamental work on vector generalizations of KdV and mKdV equations as well as their Hamiltonian structures and geometric origin appeared in [6, 7, 21, 22]. In addition, the bi-Hamiltonian structure of both vector mKdV equations was first written down in [30] from a more algebraic point of view, in a multi-component (non-invariant) notation. Special cases of two component KdV–mKdV integrable systems related to vector mKdV equations have been discussed recently in [15, 28, 25].

2 Curve flows, parallel frames, and Riemannian symmetric spaces

Let $\gamma(t,x)$ be a flow of a non-stretching curve in some n-dimensional Riemannian manifold (M,g). Write $Y=\gamma_t$ for the evolution vector of the curve and write $X=\gamma_x$ for the tangent vector along the curve normalized by $g(X,X)=1$, which is the condition that γ is non-stretching, so thus x represents arclength. In the tangent space $T_\gamma M$ of the two-dimensional surface swept out by $\gamma(t,x)$ we introduce orthonormal frame vectors e_a and connection 1-forms $\omega^{ab}=\omega^{[ab]}$ related through the Riemannian covariant derivative operator ${}^g\nabla$ in the standard way [17]:

$$
{}^g\nabla_x e_a = (X\rfloor\omega_a{}^b)e_b, \qquad {}^g\nabla_t e_a = (Y\rfloor\omega_a{}^b)e_b.
$$

(Throughout, $a,b=1,\ldots,n$ denote frame indices which get raised and lowered by the Euclidean metric $\delta_{ab}=\mathrm{diag}(+1,\ldots,+1)$). Now choose the frame along the curve to be parallel [9], so it is adapted to γ via

$$
e_a := X \quad (a=1), \qquad (e_a)_\perp \quad (a=2,\ldots,n)
$$

where $g(X,(e_a)_\perp)=0$, such that the covariant derivative of each of the $n-1$ normal vectors $(e_a)_\perp$ in the frame is tangent to γ,

$$
{}^g\nabla_x (e_a)_\perp = -v_a X \tag{1}
$$

holding for some functions v_a, while the covariant derivative of the tangent vector X in the frame is normal to γ,

$$
{}^g\nabla_x X = v^a (e_a)_\perp. \tag{2}
$$

Equivalently, along γ the connection 1-forms of the parallel frame are given by the skew matrix $\omega_x{}^{ab} := X\rfloor\omega^{ab} = 2e_x{}^{[a}v^{b]}$ where $e_x{}^a = g(X,e^a)$ is the row matrix of the frame in the tangent direction. In matrix notation we have

$$
e_x{}^a = (1,\vec{0}), \qquad \omega_{x\,a}{}^b = \begin{pmatrix} 0 & v^b \\ -v_a & \mathbf{0} \end{pmatrix}, \tag{3}
$$

with $\vec{0}$, $\mathbf{0}$ respectively denoting the $1\times(n-1)$ zero row-matrix and $(n-1)\times(n-1)$ zero skew-matrix. (Hereafter, upper/lower frame indices will represent row/column matrices.) This matrix description (3) of a parallel frame has a purely algebraic characterization: $e_x{}^a$ is a fixed unit vector in $\mathbb{R}^n$ preserved by a $SO(n-1)$ rotation subgroup of the local frame structure group $SO(n)$, while $\omega_{x\,a}{}^b$ belongs to the orthogonal complement of the corresponding rotation subalgebra $\mathfrak{so}(n-1)$ in the Lie algebra $\mathfrak{so}(n)$ of $SO(n)$.

The curve flow has associated to it the pullback of the Cartan structure equations [17] expressing that the covariant derivatives ${}^g\nabla_x := X \rfloor {}^g\nabla$ along the curve and ${}^g\nabla_t := Y \rfloor {}^g\nabla$ along the flow have vanishing torsion

$$ {}^g\nabla_x \gamma_t - {}^g\nabla_t \gamma_x = [X, Y] = 0 \tag{4} $$

and carry curvature determined from the metric g,

$$ [{}^g\nabla_x, {}^g\nabla_t] = R(X, Y) \tag{5} $$

given by the Riemann tensor $R(X, Y)$ which is a linear map on T_xM depending bilinearly on X, Y. In frame components the torsion and curvature equations look like [17]

$$ 0 = D_x e_t{}^a - D_t e_x{}^a + e_t{}^b \omega_{x b}{}^a - e_x{}^b \omega_{t b}{}^a, \tag{6} $$

$$ R_a{}^b(X, Y) = D_t \omega_{x a}{}^b - D_x \omega_{t a}{}^b + \omega_{t a}{}^c \omega_{x c}{}^b - \omega_{x a}{}^c \omega_{t c}{}^b. \tag{7} $$

Here $e_t{}^a := g(Y, e^a)$ and $\omega_{t a}{}^b := Y \rfloor \omega_a{}^b = g(e^b, {}^g\nabla_t e_a)$ are respectively the frame row-matrix and connection skew-matrix in the flow direction, and $R_a{}^b(X, Y) := g(e^b, [{}^g\nabla_x, {}^g\nabla_t] e_a)$ is the curvature matrix.

As outlined in [2, 21], these frame equations (6) and (7) directly encode a bi-Hamiltonian structure based on geometrical variables when the geometry of M is characterized by having its frame curvature matrix $R_a{}^b(e_c, e_d)$ be constant on M. In this situation the Hamiltonian variable is given by the principal normal $v := {}^g\nabla_x X = v^a (e_a)_\perp$ in the tangent direction of γ, while the principal normal in the flow direction $\varpi := {}^g\nabla_t X = \varpi^a (e_a)_\perp$ represents a Hamiltonian covector field, and the normal part of the flow vector $h_\perp := Y_\perp = h^a (e_a)_\perp$ represents a Hamiltonian vector field[1]. In a parallel frame these variables v^a, ϖ^a, h^a are encoded respectively in the top row of the connection matrices $\omega_{x a}{}^b$, $\omega_{t a}{}^b$, and in the row matrix $(e_t{}^a)_\perp = e_t{}^a - h_\parallel e_x{}^a$ where $h_\parallel := g(Y, X)$ is the tangential part of the flow vector.

A wide class of Riemannian manifolds (M, g) in which the frame curvature matrix $R_a{}^b(e_c, e_d)$ is constant on M consists of the symmetric spaces $M = G/H$ for compact semisimple Lie groups $G \supset H$ (such that H is invariant under an involutive automorphism of G). In such spaces the Riemannian curvature tensor and the metric tensor are covariantly constant and G-invariant [17], which implies constancy of the curvature matrix $R_a{}^b(e_c, e_d)$. The metric tensor g on M is given by the Cartan–Killing inner product $\langle \cdot, \cdot \rangle$ on $T_xG \simeq \mathfrak{g}$ restricted to the Lie algebra quotient space $\mathfrak{p} = \mathfrak{g}/\mathfrak{h}$ with $T_xH \simeq \mathfrak{h}$, where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ decomposes such that $[\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$ (corresponding to the eigenspaces of the adjoint action of the involutive automorphism of G that leaves H invariant). A complete classification of symmetric spaces is given in [16]; their geometric properties are summarized in [17]. In these spaces H acts as a gauge group so consequently the bi-Hamiltonian structure encoded in the frame equations will be invariant under the subgroup of H that leaves X fixed[2].

Thus in order to obtain $O(N-1)$-invariant bi-Hamiltonian operators, as sought here, we need the group $O(N-1)$ to be the isotropy subgroup in H leaving X fixed. Hence we restrict attention to the symmetric spaces $M = G/SO(N)$ with $H = SO(N) \supset O(N-1)$. ¿From the classification in [16] all examples of these spaces are exhausted by $G = SO(N+1), SU(N)$. The example $M = SO(N+1)/SO(N) \simeq S^N$ is isometric to the N-sphere, which has constant curvature. In this symmetric space, the encoding of bi-Hamiltonian operators in terms of geometric variables has been worked out in [2] using the just intrinsic Riemannian geometry of the N-sphere, following closely the ideas in [20, 21]. An extrinsic approach based on Klein geometry [26, 5] will be used here, as it applicable to both symmetric spaces $SO(N+1)/SO(N)$ and $SU(N)/SO(N)$.

[1] See [14, 18] and the appendix of [2] for a summary of Hamiltonian theory relevant to PDE systems.

[2] More details will be given elsewhere [3].

In a Klein geometry the left-invariant $\mathfrak{g}$-valued Maurer–Cartan form on the Lie group G is identified with a zero-curvature connection 1-form ω_G called the Cartan connection [26]. Thus

$$0 = d\omega_G + \frac{1}{2}[\omega_G, \omega_G],$$

where d is the total exterior derivative on the group manifold G. Through the Lie algebra decomposition $\mathfrak{g} = \mathfrak{so}(N) \oplus \mathfrak{p}$ with $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{so}(N)$ and $[\mathfrak{so}(N), \mathfrak{p}] \subset \mathfrak{p}$, the Cartan connection determines a Riemannian structure on the quotient space $M = G/SO(N)$ where G is regarded [26] as a principal $SO(N)$ bundle over M. Fix any local section of this bundle and pull-back ω_G to give a $\mathfrak{g}$-valued 1-form ${}^{\mathfrak{g}}\omega$ at x in M. The effect of changing the local section is to induce a $SO(N)$ gauge transformation on ${}^{\mathfrak{g}}\omega$. Let σ denote an involutive automorphism of $\mathfrak{g}$ such that $\mathfrak{so}(N)$ is the eigenspace $\sigma = +1$, $\mathfrak{p}$ is the eigenspace $\sigma = -1$. We consider the corresponding decomposition of ${}^{\mathfrak{g}}\omega$: it can be shown that [26] the symmetric part

$$\omega := \frac{1}{2}({}^{\mathfrak{g}}\omega + \sigma({}^{\mathfrak{g}}\omega)) \tag{8}$$

defines a $\mathfrak{so}(N)$-valued connection 1-form for the group action of $SO(N)$ on the tangent space $T_xM \simeq \mathfrak{p}$, while the antisymmetric part

$$e := \frac{1}{2}({}^{\mathfrak{g}}\omega - \sigma({}^{\mathfrak{g}}\omega)) \tag{9}$$

defines a $\mathfrak{p}$-valued coframe for the Cartan–Killing inner product $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$ on $T_xG \simeq \mathfrak{g}$ restricted to $T_xM \simeq \mathfrak{p}$. This inner product $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$ provides a Riemannian metric

$$g = \langle e \otimes e\rangle_{\mathfrak{p}}$$

on $M = G/SO(N)$, such that the squared norm of any vector $X \in T_xM$ is $|X|^2_g = g(X, X) = \langle X\lrcorner e, X\lrcorner e\rangle_{\mathfrak{p}}$.

Moreover there is a G-invariant covariant derivative ∇ associated to this structure whose restriction to the tangent space $T_\gamma M$ for any curve flow $\gamma(t, x)$ in $M = G/SO(N)$ is defined via

$$\nabla_x e = [e, \gamma_x\lrcorner\omega] \qquad \text{and} \qquad \nabla_t e = [e, \gamma_t\lrcorner\omega]. \tag{10}$$

These derivatives ∇_x, ∇_t obey the Cartan structure equations (4) and (5), namely they have zero torsion

$$0 = (\nabla_x\gamma_t - \nabla_t\gamma_x)\lrcorner e = D_x e_t - D_t e_x + [\omega_x, e_t] - [\omega_t, e_x] \tag{11}$$

and carry G-invariant curvature

$$R(\gamma_x, \gamma_t)e = [\nabla_x, \nabla_t]e = D_x\omega_t - D_t\omega_x + [\omega_x, \omega_t] = -[e_x, e_t], \tag{12}$$

where

$$e_x := \gamma_x\lrcorner e, \qquad e_t := \gamma_t\lrcorner e, \qquad \omega_x := \gamma_x\lrcorner\omega, \qquad \omega_t := \gamma_t\lrcorner\omega.$$

The G-invariant covariant derivative and curvature on $T_\gamma M$ are thus seen to coincide with the Riemannian ones determined from the metric g. More generally, in this manner [26] the relations (8) and (9) canonically solder a Klein geometry onto a Riemannian symmetric-space geometry.

Geometrically, e_x and ω_x represent the tangential part of the coframe and the connection 1-form along γ. For a non-stretching curve γ, where x is the arclength, note e_x has unit norm in the inner product, $\langle e_x, e_x\rangle_{\mathfrak{p}} = 1$. This implies $\mathfrak{p}$ has a decomposition into tangential and normal subspaces $\mathfrak{p}_\parallel$ and $\mathfrak{p}_\perp$ with respect to e_x such that $\langle e_x, \mathfrak{p}_\perp\rangle_{\mathfrak{p}} = 0$, with $\mathfrak{p} = \mathfrak{p}_\perp \oplus \mathfrak{p}_\parallel$ and $\mathfrak{p}_\parallel \simeq \mathbb{R}$.

Remark 1. A main insight now, generalizing the results in [21, 2], is that the Cartan structure equations on the surface swept out by the curve flow $\gamma(t,x)$ in $M = G/SO(N)$ will geometrically encode $O(N-1)$-invariant bi-Hamiltonian operators if the gauge (rotation) freedom of the group action $SO(N)$ on e and ω is used to fix them to be a parallel coframe and its associated connection 1-form related by the Riemannian covariant derivative. The groups $G = SO(N+1)$ and $G = SU(N)$ will produce a different encoding except when $N = 2$, since in that case $T_xM \simeq \mathfrak{so}(3)/\mathfrak{so}(2) \simeq \mathfrak{su}(2)/\mathfrak{so}(2)$ is the same tangent space for $M = SO(3)/SO(2)$ and $M = SU(2)/SO(2)$ due to the Lie-algebra isomorphism $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$. This will be seen to account for the existence of the two different vector generalizations of the scalar mKdV hierarchy.

The algebraic characterization of a parallel frame for curves in Riemannian geometry extends naturally to the setting of Klein geometry, via the property that e_x is preserved by a $SO(N-1)$ rotation subgroup of the local frame structure group $SO(N)$ acting on $\mathfrak{p} \subset \mathfrak{g}$, while ω_x belongs to the orthogonal complement of the $SO(N-1)$ rotation Lie subalgebra $\mathfrak{so}(N-1)$ contained in the Lie algebra $\mathfrak{so}(N)$ of $SO(N)$. Their geometrical meaning, however, generalizes the Riemannian properties (1) and (2), as follows. Let e_a be a frame whose dual coframe is identified with the $\mathfrak{p}$-valued coframe e in a fixed orthonormal basis for $\mathfrak{p} \subset \mathfrak{g}$. Decompose the coframe into parallel/perpendicular parts with respect to e_x in an algebraic sense as defined by the kernel/cokernel of Lie algebra multiplication $[e_x, \cdot\,]_{\mathfrak{g}} = \mathrm{ad}(e_x)$. Thus we have $e = (e_C, e_{C^\perp})$ where the $\mathfrak{p}$-valued covectors e_C, $e_{C^\perp}$ satisfy $[e_x, e_C]_{\mathfrak{g}} = 0$, and $e_{C^\perp}$ is orthogonal to e_C, so $[e_x, e_{C^\perp}]_{\mathfrak{g}} \neq 0$. Note there is a corresponding algebraic decomposition of the tangent space $T_xM \simeq \mathfrak{p} = \mathfrak{g}/\mathfrak{so}(N)$ given by $\mathfrak{p} = \mathfrak{p}_C \oplus \mathfrak{p}_{C^\perp}$, with $\mathfrak{p}_\parallel \subseteq \mathfrak{p}_C$ and $\mathfrak{p}_{C^\perp} \subseteq \mathfrak{p}_\perp$, where $[\mathfrak{p}_\parallel, \mathfrak{p}_C] = 0$ and $\langle \mathfrak{p}_{C^\perp}, \mathfrak{p}_C \rangle_{\mathfrak{p}} = 0$, so $[\mathfrak{p}_\parallel, \mathfrak{p}_{C^\perp}] \neq 0$ (namely, $\mathfrak{p}_C$ is the centralizer of e_x in $\mathfrak{p} \subset \mathfrak{g}$). This decomposition is preserved by $\mathrm{ad}(\omega_x)$ which acts as an infinitesimal rotation, $\mathrm{ad}(\omega_x)\mathfrak{p}_C \subseteq \mathfrak{p}_{C^\perp}$, $\mathrm{ad}(\omega_x)\mathfrak{p}_{C^\perp} \subseteq \mathfrak{p}_C$. Hence, from equation (10), the derivative ∇_x of a covector perpendicular (respectively parallel) to e_x lies parallel (respectively perpendicular) to e_x, namely $\nabla_x e_C$ belongs to $\mathfrak{p}_{C^\perp}$, $\nabla_x e_{C^\perp}$ belongs to $\mathfrak{p}_C$. In the Riemannian setting, these properties correspond to ${}^g\nabla_x (e^a)_C = v^a{}_b (e^b)_{C^\perp}$, ${}^g\nabla_x (e^a)_{C^\perp} = -v_b{}^a (e^b)_C$ for some functions $v^{ab} = -v^{ba}$. Such a frame will be called $SO(N)$*-parallel* and defines a strict generalization of a Riemannian parallel frame whenever $\mathfrak{p}_C$ is larger than $\mathfrak{p}_\parallel$.

3 Bi-Hamiltonian operators and vector soliton equations for $SO(N+1)/SO(N) \simeq S^N$

Recall $\mathfrak{so}(k)$ is a real vector space isomorphic to the Lie algebra of $k \times k$ skew-symmetric matrices. So the tangent space $T_xM = \mathfrak{so}(N+1)/\mathfrak{so}(N)$ of the Riemannian manifold $M = SO(N+1)/SO(N)$ is isomorphic to $\mathfrak{p} \simeq \mathbb{R}^N$, as described by the following canonical decomposition

$$\begin{pmatrix} 0 & p \\ -p^T & \mathbf{0} \end{pmatrix} \in \mathfrak{p} \subset \mathfrak{so}(N+1) = \mathfrak{g}, \qquad \mathbf{0} \in \mathfrak{so}(N) = \mathfrak{h}, \qquad p \in \mathbb{R}^N$$

parameterized by the N component vector p. The Cartan–Killing inner product on $\mathfrak{g}$ is given by the trace of the product of an $\mathfrak{so}(N+1)$ matrix and a transpose $\mathfrak{so}(N+1)$ matrix, multiplied by a normalization factor $\frac{1}{2}$. The norm-squared on the quotient space $\mathfrak{p}$ thereby reduces to the ordinary dot product of vectors p:

$$\left\langle \begin{pmatrix} 0 & p \\ -p^T & \mathbf{0} \end{pmatrix}, \begin{pmatrix} 0 & p \\ -p^T & \mathbf{0} \end{pmatrix} \right\rangle = \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} 0 & p \\ -p^T & \mathbf{0} \end{pmatrix}^T \begin{pmatrix} 0 & p \\ -p^T & \mathbf{0} \end{pmatrix} \right) = p \cdot p.$$

Note the Cartan–Killing inner product provides a canonical identification between $\mathfrak{p} \simeq \mathbb{R}^N$ and its dual $\mathfrak{p}^* \simeq \mathbb{R}^N$.

Let $\gamma(t,x)$ be a flow of a non-stretching curve in $M = SO(N{+}1)/SO(N) \simeq S^N$. We introduce a $SO(N)$-parallel coframe $e \in T^*_\gamma M \otimes \mathfrak{p}$ and its associated connection 1-form $\omega \in T^*_\gamma M \otimes \mathfrak{so}(N)$ along γ by putting[3]

$$e_x = \gamma_x \rfloor e = \begin{pmatrix} 0 & (1,\vec{0}) \\ -(1,\vec{0})^T & \mathbf{0} \end{pmatrix} \in \mathfrak{p}, \qquad (1,\vec{0}) \in \mathbb{R}^N, \qquad \vec{0} \in \mathbb{R}^{N-1} \tag{13}$$

and

$$\omega_x = \gamma_x \rfloor \omega = \begin{pmatrix} 0 & (0,\vec{0}) \\ (0,\vec{0})^T & \boldsymbol{\omega_x} \end{pmatrix} \in \mathfrak{so}(N+1),$$

where

$$\boldsymbol{\omega_x} = \begin{pmatrix} 0 & \vec{v} \\ -\vec{v}^T & \mathbf{0} \end{pmatrix} \in \mathfrak{so}(N), \qquad \vec{v} \in \mathbb{R}^{N-1}, \qquad \mathbf{0} \in \mathfrak{so}(N-1).$$

Note the form of e_x indicates the coframe e is adapted to γ, with $(1,\vec{0})$ representing a choice of a constant unit-norm vector in $\mathfrak{p} \simeq \mathbb{R}^N$, so $\langle e_x, e_x \rangle_{\mathfrak{p}} = (1,\vec{0}) \cdot (1,\vec{0}) = 1$. All such choices are equivalent through the $SO(N)$ rotation gauge freedom on the coframe and connection 1-form. Consequently, there is a decomposition of $SO(N+1)/SO(N)$ matrices

$$\begin{pmatrix} 0 & p \\ -p^T & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 & (p_\parallel,\vec{0}) \\ -(p_\parallel,\vec{0})^T & \mathbf{0} \end{pmatrix} + \begin{pmatrix} 0 & (0,\vec{p}_\perp) \\ -(0,\vec{p}_\perp)^T & \mathbf{0} \end{pmatrix} \in \mathfrak{p}$$

into tangential and normal parts relative to e_x via a corresponding decomposition of vectors given by

$$p = (p_\parallel, \vec{p}_\perp) \in \mathbb{R}^N$$

relative to $(1,\vec{0})$. Thus $p_\parallel$ is identified with $\mathfrak{p}_\parallel = \mathfrak{p}_C$, and $\vec{p}_\perp$ with $\mathfrak{p}_\perp = \mathfrak{p}_{C^\perp}$.

In the flow direction we put

$$e_t = \gamma_t \rfloor e = \begin{pmatrix} 0 & (h_\parallel,\vec{h}_\perp) \\ -(h_\parallel,\vec{h}_\perp)^T & \mathbf{0} \end{pmatrix} \in \mathfrak{p}, \qquad (h_\parallel,\vec{h}_\perp) \in \mathbb{R}^N, \qquad \vec{h}_\perp \in \mathbb{R}^{N-1}$$

and

$$\omega_t = \gamma_t \rfloor \omega = \begin{pmatrix} 0 & (0,\vec{0}) \\ (0,\vec{0})^T & \boldsymbol{\omega_t} \end{pmatrix} \in \mathfrak{so}(N+1), \tag{14}$$

where

$$\boldsymbol{\omega_t} = \begin{pmatrix} 0 & \vec{\varpi} \\ -\vec{\varpi}^T & \boldsymbol{\Theta} \end{pmatrix} \in \mathfrak{so}(N), \qquad \vec{\varpi} \in \mathbb{R}^{N-1}, \qquad \boldsymbol{\Theta} \in \mathfrak{so}(N-1). \tag{15}$$

The components $h_\parallel$, $\vec{h}_\perp$ correspond to decomposing $e_t = g(\gamma_t,\gamma_x)e_x + (\gamma_t)_\perp \rfloor e_\perp$ into tangential and normal parts relative to e_x. We now have

$$[e_x, e_t] = -\begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{h}_\perp \end{pmatrix} \in \mathfrak{so}(N+1), \qquad \boldsymbol{h}_\perp = \begin{pmatrix} 0 & \vec{h}_\perp \\ -\vec{h}_\perp{}^T & \mathbf{0} \end{pmatrix} \in \mathfrak{so}(N),$$

[3] Note ω is related to e by the Riemannian covariant derivative (10) on the surface swept out by the curve flow $\gamma(t,x)$.

$$[\omega_t, e_t] = -\begin{pmatrix} 0 & (0,\vec{\varpi}) \\ -(0,\vec{\varpi})^T & 0 \end{pmatrix} \in \mathfrak{p}_\perp,$$
$$[\omega_x, e_t] = -\begin{pmatrix} 0 & (-\vec{v}\cdot\vec{h}_\perp, h_\parallel\vec{v}) \\ -(-\vec{v}\cdot\vec{h}_\perp, h_\parallel\vec{v})^T & \mathbf{0} \end{pmatrix} \in \mathfrak{p}.$$

Hence the curvature equation (12) reduces to

$$D_t\vec{v} - D_x\vec{\varpi} - \vec{v}\lrcorner\mathbf{\Theta} = -\vec{h}_\perp, \tag{16}$$
$$-D_x\mathbf{\Theta} + \vec{v}\otimes\vec{\varpi} - \vec{\varpi}\otimes\vec{v} = 0, \tag{17}$$

while the torsion equation (11) yields

$$0 = D_x h_\parallel + \vec{v}\cdot\vec{h}_\perp, \tag{18}$$
$$0 = \vec{\varpi} - h_\parallel\vec{v} + D_x\vec{h}_\perp. \tag{19}$$

Here $\otimes$ denotes the outer product of pairs of vectors ($1\times N$ row matrices), producing $N\times N$ matrices $\vec{A}\otimes\vec{B} = \vec{A}^T\vec{B}$, and $\lrcorner$ denotes multiplication of $N\times N$ matrices on vectors ($1\times N$ row matrices), $\vec{A}\lrcorner(\vec{B}\otimes\vec{C}) = (\vec{A}\cdot\vec{B})\vec{C}$ which is the transpose of the standard matrix product on column vectors, $(\vec{B}\otimes\vec{C})\vec{A} = (\vec{C}\cdot\vec{A})\vec{B}$. To proceed we use equations (17) and (18) to eliminate

$$\mathbf{\Theta} = -D_x^{-1}(\vec{\varpi}\otimes\vec{v} - \vec{v}\otimes\vec{\varpi}), \qquad h_\parallel = -D_x^{-1}(\vec{v}\cdot\vec{h}_\perp) \tag{20}$$

in terms of the variables $\vec{v}$, $\vec{h}_\perp$, $\vec{\varpi}$. Then equation (16) gives a flow on $\vec{v}$,

$$\vec{v}_t = D_x\vec{\varpi} - \vec{v}\lrcorner D_x^{-1}(\vec{\varpi}\otimes\vec{v} - \vec{v}\otimes\vec{\varpi}) - \chi\vec{h}_\perp$$

with

$$\vec{\varpi} = -D_x^{-1}(\vec{v}\cdot\vec{h}_\perp)\vec{v} - D_x\vec{h}_\perp$$

obtained from equation (19). Here $\chi = 1$ represents the Riemannian scalar curvature of $SO(N+1)/SO(N) \simeq S^N$ (see [2]). In these equations we read off the operators

$$\mathcal{H} = D_x + \vec{v}\lrcorner D_x^{-1}(\vec{v}\wedge\ ), \qquad \mathcal{J} = D_x + D_x^{-1}(\vec{v}\cdot\ )\vec{v},$$

where $\vec{A}\wedge\vec{B} = \vec{A}\otimes\vec{B} - \vec{B}\otimes\vec{A}$. The results in [21] prove the following properties of $\mathcal{H}$, $\mathcal{J}$.

Theorem 1. *$\mathcal{H}$, $\mathcal{J}$ are compatible $O(N-1)$-invariant Hamiltonian cosymplectic and symplectic operators with respect to the Hamiltonian variable $\vec{v}$. Consequently, the flow equation takes the Hamiltonian form*

$$\vec{v}_t = \mathcal{H}(\vec{\varpi}) - \chi\vec{h}_\perp = \mathcal{R}(\vec{h}_\perp) - \chi\vec{h}_\perp, \qquad \vec{\varpi} = \mathcal{J}(\vec{h}_\perp)$$

where $\mathcal{R} = \mathcal{H}\circ\mathcal{J}$ is a hereditary recursion operator.

On the x-jet space of $\vec{v}(t,x)$, the variables $\vec{h}_\perp$ and $\vec{\varpi}$ have the respective meaning of a Hamiltonian vector field $\vec{h}_\perp\lrcorner\partial/\partial\vec{v}$ and covector field $\vec{\varpi}\lrcorner d\vec{v}$ (see the Appendix of [2]). Thus the recursion operator[4]

$$\begin{aligned}\mathcal{R} &= D_x(D_x + D_x^{-1}(\vec{v}\cdot\ )\vec{v}) + \vec{v}\lrcorner D_x^{-1}(\vec{v}\wedge D_x\ ) \\ &= D_x^2 + |\vec{v}|^2 + D_x^{-1}(\vec{v}\cdot\ )\vec{v}_x - \vec{v}\lrcorner D_x^{-1}(\vec{v}_x\wedge\ )\end{aligned} \tag{21}$$

[4]This $O(N-1)$-invariant form of the recursion operator appeared already in [1].

generates a hierarchy of commuting Hamiltonian vector fields $\vec{h}_\perp^{(k)}$ starting from $\vec{h}_\perp^{(0)} = \vec{v}_x$ given by the infinitesimal generator of x-translations in terms of arclength x along the curve.

The adjoint operator $\mathcal{R}^*$ generates a related hierarchy of involutive Hamiltonian covector fields $\vec{\varpi}^{(k)} = \delta H^{(k)}/\delta\vec{v}$ in terms of Hamiltonians $H = H^{(k)}(\vec{v}, \vec{v}_x, \vec{v}_{2x}, \dots)$ starting from $\vec{\varpi}^{(0)} = \vec{v}$, $H^{(0)} = \frac{1}{2}|\vec{v}|^2$. These hierarchies are related by $\vec{h}_\perp^{(k)} = \mathcal{H}(\vec{\varpi}^{(k)})$, $\vec{\varpi}^{(k+1)} = \mathcal{J}(\vec{h}_\perp^{(k)})$, $k = 0, 1, 2, \dots$. Both hierarchies share the mKdV scaling symmetry $x \to \lambda x$, $\vec{v} \to \lambda^{-1}\vec{v}$, under which we see $\vec{h}_\perp^{(k)}$ and $H^{(k)}$ have scaling weight $2 + 2k$, while $\vec{\varpi}^{(k)}$ has scaling weight $1 + 2k$.

Corollary 1. *Associated to the recursion operator $\mathcal{R}$ there is a corresponding hierarchy of commuting bi-Hamiltonian flows on $\vec{v}$ given by $O(N-1)$-invariant vector evolution equations*

$$\vec{v}_t = \vec{h}_\perp^{(k+1)} - \chi\vec{h}_\perp^{(k)} = \mathcal{H}(\delta H^{(k,\chi)}/\delta\vec{v}) = \mathcal{J}^{-1}(\delta H^{(k+1,\chi)}/\delta\vec{v}), \qquad k = 0, 1, 2, \dots \tag{22}$$

with Hamiltonians $H^{(k+1,\chi)} = H^{(k+1)} - \chi H^{(k)}$, where $\mathcal{H}, \mathcal{J}^{-1}$ are compatible Hamiltonian operators. An alternative (explicit) Hamiltonian operator for these flows is $\mathcal{E} := \mathcal{H} \circ \mathcal{J} \circ \mathcal{H} = \mathcal{R} \circ \mathcal{H}$.

Remark 2. Using the terminology of [5], $\vec{h}_\perp^{(k)}$ will be said to produce a $+(k+1)$ flow on $\vec{v}$. This differs from the terminology in [2] which refers to equation (22) as the $+k$ flow.

The $+1$ flow given by $\vec{h}_\perp = \vec{v}_x$ yields

$$\vec{v}_t = \vec{v}_{3x} + \frac{3}{2}|\vec{v}|^2\vec{v}_x - \chi\vec{v}_x \tag{23}$$

which is a vector mKdV equation up to a convective term that can be absorbed by a Galilean transformation $x \to x - \chi t$, $t \to t$. The $+(k+1)$ flow gives a vector mKdV equation of higher order $3 + 2k$ on $\vec{v}$.

There is also a 0 flow $\vec{v}_t = \vec{v}_x$ arising from $\vec{h}_\perp = 0$, $h_\parallel = 1$, which falls outside the hierarchy generated by $\mathcal{R}$.

All these flows correspond to geometrical motions of the curve $\gamma(t, x)$, given by

$$\gamma_t = f(\gamma_x, \nabla_x\gamma_x, \nabla_x^2\gamma_x, \dots)$$

subject to the non-stretching condition

$$|\gamma_x|_g = 1.$$

The equation of motion is obtained from the identifications $\gamma_t \leftrightarrow e_t$, $\nabla_x\gamma_x \leftrightarrow \mathcal{D}_x e_x = [\omega_x, e_x]$, and so on, using $\nabla_x \leftrightarrow D_x + [\omega_x, \cdot] = \mathcal{D}_x$. These identifications correspond to $T_xM \leftrightarrow \mathfrak{p}$ as defined via the parallel coframe. Note we have

$$[\omega_x, e_x] = -\begin{pmatrix} 0 & (0, \vec{v}) \\ -(0, \vec{v})^T & \mathbf{0} \end{pmatrix},$$

$$[\omega_x, [\omega_x, e_x]] = -\begin{pmatrix} 0 & (|\vec{v}|^2, \vec{0}) \\ -(|\vec{v}|^2, \vec{0})^T & \mathbf{0} \end{pmatrix} = -|\vec{v}|^2 e_x,$$

and so on. In particular, for the $+1$ flow,

$$\vec{h}_\perp = \vec{v}_x, \qquad h_\parallel = -D_x^{-1}(\vec{v} \cdot \vec{v}_x) = -\frac{1}{2}|\vec{v}|^2,$$

thus

$$e_t = \begin{pmatrix} 0 & (h_\parallel, \vec{h}_\perp) \\ -(h_\parallel, \vec{h}_\perp)^T & \mathbf{0} \end{pmatrix} = -\frac{1}{2}|\vec{v}|^2 \begin{pmatrix} 0 & (1, \vec{0}) \\ -(1, \vec{0})^T & \mathbf{0} \end{pmatrix} + \begin{pmatrix} 0 & (0, \vec{v}_x) \\ -(0, \vec{v}_x)^T & \mathbf{0} \end{pmatrix}$$

$$= -D_x[\omega_x, e_x] + \frac{1}{2}[\omega_x, [\omega_x, e_x]] = -\mathcal{D}_x[\omega_x, e_x] - \frac{3}{2}|\vec{v}|^2 e_x.$$

We identify the first term as $-\nabla_x(\nabla_x\gamma_x) = -\nabla_x^2\gamma_x$. For the second term we observe $|\vec{v}|^2 = \langle[\omega_x, e_x], \omega_x, e_x]\rangle_{\mathfrak{p}} \leftrightarrow g(\nabla_x\gamma_x, \nabla_x\gamma_x) = |\nabla_x\gamma_x|_g^2$ since the Cartan–Killing inner product corresponds to the Riemannian metric g. Hence we have $e_t \leftrightarrow -(\nabla_x^2\gamma_x + \frac{3}{2}|\nabla_x\gamma_x|_g^2\gamma_x)$. This describes a geometric nonlinear PDE for $\gamma(t,x)$,

$$-\gamma_t = \nabla_x^2\gamma_x + \frac{3}{2}|\nabla_x\gamma_x|_g^2\gamma_x \tag{24}$$

which is referred to as the *non-stretching mKdV map equation* on the symmetric space $M = SO(N+1)/SO(N) \simeq S^N$. A different derivation using just the Riemannian geometry of S^N was given in [2]. Since in the tangent space $T_x S^N \simeq \mathfrak{so}(N+1)/\mathfrak{so}(N)$ the kernel of $[e_x, \cdot\,]$ is spanned by e_x, a parallel frame in the setting of the Klein geometry of $SO(N+1)/SO(N)$ is precisely the same as in the Riemannian geometry of S^N.

The convective term $|\nabla_x\gamma_x|_g^2\gamma_x$ can be written in an alternative form using the Klein geometry of $SO(N+1)/SO(N) \simeq S^N$. Let $\mathrm{ad}(\cdot)$ denote the standard adjoint representation acting in the Lie algebra $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{so}(N)$. We first observe

$$\mathrm{ad}([\omega_x, e_x])e_x = \begin{pmatrix} 0 & (0,\vec{0}) \\ (0,\vec{0})^T & \boldsymbol{v} \end{pmatrix} \in \mathfrak{so}(N+1),$$

where

$$\boldsymbol{v} = -\begin{pmatrix} 0 & \vec{v} \\ -\vec{v}^T & \mathbf{0} \end{pmatrix} \in \mathfrak{so}(N).$$

Applying $\mathrm{ad}([\omega_x, e_x])$ again, we obtain

$$\mathrm{ad}([\omega_x, e_x])^2 e_x = -|\vec{v}|^2 \begin{pmatrix} 0 & (1,\vec{0}) \\ -(1,\vec{0})^T & \mathbf{0} \end{pmatrix} = -|\vec{v}|^2 e_x.$$

Hence, $|\vec{v}|^2 e_x \leftrightarrow -\chi^{-1}\mathrm{ad}(\nabla_x\gamma_x)^2\gamma_x = |\nabla_x\gamma_x|_g^2\gamma_x$, and thus the mKdV map equation is equivalent to

$$-\gamma_t = \nabla_x^2\gamma_x - \frac{3}{2}\chi^{-1}\mathrm{ad}(\nabla_x\gamma_x)^2\gamma_x. \tag{25}$$

Note here that $\mathrm{ad}(\nabla_x\gamma_x) = [\nabla_x\gamma_x, \cdot\,]$ maps $\mathfrak{p} \simeq T_x M$ into $\mathfrak{so}(N)$ and maps $\mathfrak{so}(N)$ into $\mathfrak{p} \simeq T_x M$, so $\mathrm{ad}(\nabla_x\gamma_x)^2$ is well-defined on the tangent space $T_x M \simeq \mathfrak{p}$ of $M = SO(N+1)/SO(N)$.

Higher flows on $\vec{v}$ yield higher-order geometric PDEs. The 0 flow on $\vec{v}$ directly corresponds to

$$\gamma_t = \gamma_x \tag{26}$$

which is just a convective (linear traveling wave) map equation.

There is a -1 flow contained in the hierarchy, with the property that $\vec{h}_\perp$ is annihilated by the symplectic operator $\mathcal{J}$ and hence gets mapped into $\mathcal{R}(\vec{h}_\perp) = 0$ under the recursion operator. Geometrically this flow means simply $\mathcal{J}(\vec{h}_\perp) = \vec{\varpi} = 0$ which implies $\omega_t = 0$ from equations (14), (15), (20), and hence $0 = [\omega_t, e_x] = \mathcal{D}_t e_x$ where $\mathcal{D}_t = D_t + [\omega_t, \cdot]$. The correspondence $\nabla_t \leftrightarrow \mathcal{D}_t$, $\gamma_x \leftrightarrow e_x$ immediately leads to the equation of motion

$$0 = \nabla_t\gamma_x \tag{27}$$

for the curve $\gamma(t,x)$. This nonlinear geometric PDE is precisely a wave map equation on the symmetric space $SO(N+1)/SO(N) \simeq S^N$. The resulting flow equation on $\vec{v}$ is

$$\vec{v}_t = -\chi \vec{h}_\perp, \qquad \chi = 1, \tag{28}$$

where

$$0 = \vec{\varpi} = -D_x \vec{h}_\perp + h_\parallel \vec{v}, \qquad D_x h_\parallel = \vec{h}_\perp \cdot \vec{v}.$$

Note this flow equation possesses the conservation law $0 = D_x({h_\parallel}^2 + |\vec{h}_\perp|^2)$ with

$${h_\parallel}^2 + |\vec{h}_\perp|^2 = \langle e_t, e_t \rangle_{\mathfrak{p}} = |\gamma_t|_g^2$$

corresponding to

$$0 = \nabla_x |\gamma_t|_g^2. \tag{29}$$

Thus a conformal scaling of t can be used to put $|\gamma_t|_g = 1$, and so

$$1 = {h_\parallel}^2 + |\vec{h}_\perp|^2.$$

This relation enables $h_\parallel$, $\vec{h}_\perp$ to be expressed entirely in terms of $\vec{v}$ and its derivatives through equation (28). Consequently, the wave map equation describing the -1 flow reduces to a nonlocal evolution equation

$$\vec{v}_t = -D_x^{-1}\Big(\sqrt{\chi^2 - |\vec{v}_t|^2}\vec{v}\Big), \qquad \chi = 1$$

which is equivalent to a hyperbolic vector equation

$$\vec{v}_{tx} = -\sqrt{1 - |\vec{v}_t|^2}\vec{v}. \tag{30}$$

It follows that $\vec{h}_\perp$ obeys the vector SG equation

$$\Big(\sqrt{(1 - |\vec{h}_\perp|^2)^{-1}}\vec{h}_{\perp x}\Big)_t = -\vec{h}_\perp \tag{31}$$

which has been derived previously in [8, 19, 30] from a different point of view. These equations (30) and (31) possess the mKdV scaling symmetry $x \to \lambda x$, $\vec{v} \to \lambda^{-1}\vec{v}$, where $\vec{h}_\perp$ has scaling weight 0.

The hyperbolic vector equation (30) admits the vector mKdV equation (23) as a higher symmetry, which is shown by the symmetry-integrability classification results in [5]. As a consequence of Corollary 1, we see that the recursion operator $\mathcal{R} = \mathcal{H} \circ \mathcal{J}$ generates a hierarchy of vector mKdV higher symmetries

$$\vec{v}_t^{(0)} = \vec{v}_x, \tag{32}$$

$$\vec{v}_t^{(1)} = \mathcal{R}(\vec{v}_x) = \vec{v}_{3x} + \frac{3}{2}|\vec{v}|^2\vec{v}_x, \tag{33}$$

$$\vec{v}_t^{(2)} = \mathcal{R}^2(\vec{v}_x) = \vec{v}_{5x} + \frac{5}{2}(|\vec{v}|^2\vec{v}_{2x})_x + \frac{5}{2}\left((|\vec{v}|^2)_{xx} + |\vec{v}_x|^2 + \frac{3}{4}|\vec{v}|^4\right)\vec{v}_x - \frac{1}{2}|\vec{v}_x|^2\vec{v}, \tag{34}$$

and so on, all of which commute with the -1 flow

$$\vec{v}_t^{(-1)} = \vec{h}_\perp \tag{35}$$

associated to the vector SG equation (31). Moreover the adjoint operator $\mathcal{R}^* = \mathcal{J} \circ \mathcal{H}$ generates a hierarchy of Hamiltonians

$$\begin{aligned} H^{(0)} &= \frac{1}{2}|\vec{v}|^2, \\ H^{(1)} &= -\frac{1}{2}|\vec{v}_x|^2 + \frac{1}{8}|\vec{v}|^4, \\ H^{(2)} &= \frac{1}{2}|\vec{v}_{2x}|^2 - \frac{3}{4}|\vec{v}|^2|\vec{v}_x|^2 - \frac{1}{2}(\vec{v}\cdot\vec{v}_x)^2 + \frac{1}{16}|\vec{v}|^6, \end{aligned}$$

and so on, all of which are conserved densities for the -1 flow and thereby determine higher conservation laws for the hyperbolic vector equations (30) and (31). Viewed as flows, the entire hierarchy of vector PDEs (35), (32) to (34), etc. possesses the mKdV scaling symmetry $x \to \lambda x$, $\vec{v} \to \lambda^{-1}\vec{v}$, with $t \to \lambda^{1+2k}t$ for $k = -1, 0, 1, 2$, etc. Moreover for $k \geq 0$, all these expressions will be local polynomials in the variables $\vec{v}, \vec{v}_x, \vec{v}_{xx}, \ldots$ as established by general results in [29, 24] concerning nonlocal recursion operators.

Theorem 2. *In the symmetric space $SO(N+1)/SO(N)$ there is a hierarchy of bi-Hamiltonian flows of curves $\gamma(t,x)$ described by geometric map equations. The* 0 *flow is a convective (traveling wave) map* (26)*, while the* $+1$ *flow is a non-stretching mKdV map* (24) *and the* $+2,\ldots$ *flows are higher order analogs. The kernel of the recursion operator* (21) *in the hierarchy yields the* -1 *flow which is a non-stretching wave map* (27)*.*

4 Bi-Hamiltonian operators and vector soliton equations for $SU(N)/SO(N)$

Recall $\mathfrak{su}(k)$ is a complex vector space isomorphic to the Lie algebra of $k \times k$ skew-hermitian matrices. The real and imaginary parts of these matrices respectively belong to the real vector space $\mathfrak{so}(k)$ of skew-symmetric matrices and the real vector space $\mathfrak{s}(k) \simeq \mathfrak{su}(k)/\mathfrak{so}(k)$ defined by $k \times k$ symmetric trace-free matrices. Hence $\mathfrak{g} = \mathfrak{su}(N)$ has the decomposition $\mathfrak{g} = \mathfrak{h} + \mathrm{i}\mathfrak{p}$ where $\mathfrak{h} = \mathfrak{so}(N)$ and $\mathfrak{p} = \mathfrak{s}(N)$. The Cartan–Killing inner product is given by the trace of the product of an $\mathfrak{su}(N)$ matrix and a hermitian-transpose $\mathfrak{su}(N)$ matrix, multiplied by 1/2. Note any matrix in $\mathfrak{s}(N)$ can be diagonalized under the action of the group $SO(N)$.

Let $\gamma(t,x)$ be a flow of a non-stretching curve in $M = SU(N)/SO(N)$ where we identify $T_xM \simeq \mathfrak{p}$ (dropping a factor i for simplicity[5]). We suppose there is a $SO(N)$-parallel coframe $e \in T^*_\gamma M \otimes \mathfrak{p}$ and its associated connection 1-form $\omega \in T^*_\gamma M \otimes \mathfrak{so}(N)$ along γ given by[6]

$$\begin{aligned} &e_x = \gamma_x \lrcorner e = \kappa\left(\begin{pmatrix} -1 & \vec{0} \\ \vec{0} & \mathbf{0} \end{pmatrix} + \frac{1}{N}\mathbb{I}\right) = \frac{\kappa}{N}\begin{pmatrix} 1-N & \vec{0} \\ \vec{0} & \mathbf{1} \end{pmatrix} \in \mathfrak{p}, \\ &\mathbf{0}, \mathbf{1} \in \mathfrak{u}(N-1), \qquad \vec{0} \in \mathbb{R}^{N-1} \end{aligned} \tag{36}$$

up to a normalization factor κ which we will fix shortly, and

$$\omega_x = \begin{pmatrix} 0 & \vec{v} \\ -\vec{v}^T & \mathbf{0} \end{pmatrix} \in \mathfrak{so}(N), \qquad \vec{v} \in \mathbb{R}^{N-1}. \tag{37}$$

[5]Retaining the i in this identification will change only the sign of the scalar curvature factor χ in the flow equation.

[6]As before, ω is related to e by the Riemannian covariant derivative (10) on the surface swept out by the curve flow $\gamma(t,x)$.

Since the form of e_x is a constant matrix, it indicates that the coframe is adapted to γ provided e_x has unit norm in the Cartan–Killing inner product. We have

$$\langle e_x, e_x\rangle_{\mathfrak{p}} = \frac{\kappa^2}{2N^2}\mathrm{tr}\begin{pmatrix}(1-N)^2 & 0\\ 0 & \mathbf{1}\end{pmatrix} = \kappa^2(N-1)/(2N) = 1 \tag{38}$$

after fixing $\kappa^2 = 2N(N-1)^{-1}$. As a consequence, all matrices in $\mathfrak{p} = \mathfrak{s}(N)$ will have a canonical decomposition into tangential and normal parts relative to e_x,

$$\begin{pmatrix}(N^{-1}-1)p_{\|} & \vec{p}_{\perp}\\ \vec{p}_{\perp}{}^T & \boldsymbol{p}_{\perp} - N^{-1}p_{\|}\mathbf{1}\end{pmatrix} = \frac{1}{N}\begin{pmatrix}(1-N)p_{\|} & \vec{0}\\ \vec{0}^T & p_{\|}\mathbf{1}\end{pmatrix} + \begin{pmatrix}0 & \vec{p}_{\perp}\\ \vec{p}_{\perp}{}^T & \boldsymbol{p}_{\perp}\end{pmatrix}$$

parameterized by the $(N-1)\times(N-1)$ matrix $\boldsymbol{p}_{\perp} \in \mathfrak{s}(N-1)$ and the N component vector $(p_{\|}, \vec{p}_{\perp}) \in \mathbb{R}^N$, corresponding to the decomposition $\mathfrak{s}(N) = \mathfrak{s}(N)_{\|} \oplus \mathfrak{s}(N)_{\perp}$ given by $\langle \mathfrak{s}(N)_{\perp}, e_x\rangle_{\mathfrak{p}} = 0$ and $\langle \mathfrak{s}(N)_{\|}, e_x\rangle_{\mathfrak{p}} = \kappa p_{\|}$ under the previous normalization of e_x. Here $(p_{\|}, \boldsymbol{p}_{\perp})$ is identified with $\mathfrak{p}_C \supset \mathfrak{p}_{\|}$, and $\vec{p}_{\perp}$ with $\mathfrak{p}_{C^{\perp}} \subset \mathfrak{p}_{\perp}$. Note $\boldsymbol{p}_{\perp}$ is empty only if $N = 2$, so consequently for $N > 2$ the $SO(N)$-parallel frame (36) and (37) is a strict generalization of a Riemannian parallel frame.

In the flow direction we put

$$\begin{aligned} e_t &= \gamma_t \lrcorner e = \kappa\left(h_{\|}\begin{pmatrix}N^{-1}-1 & \vec{0}\\ \vec{0} & N^{-1}\mathbf{1}\end{pmatrix} + \begin{pmatrix}0 & \vec{h}_{\perp}\\ \vec{h}_{\perp}{}^T & \boldsymbol{h}_{\perp}\end{pmatrix}\right),\\ &= \kappa\begin{pmatrix}(N^{-1}-1)h_{\|} & \vec{h}_{\perp}\\ \vec{h}_{\perp}{}^T & \boldsymbol{h}_{\perp} + N^{-1}h_{\|}\mathbf{1}\end{pmatrix} \in \mathfrak{p} = \mathfrak{s}(N),\\ &(h_{\|}, \vec{h}_{\perp}) \in \mathbb{R}^N, \qquad \boldsymbol{h}_{\perp} \in \mathfrak{s}(N-1)\end{aligned}$$

and

$$\omega_t = \gamma_t \lrcorner \omega = \begin{pmatrix}0 & \vec{\varpi}\\ -\vec{\varpi}^T & \boldsymbol{\Theta}\end{pmatrix} \in \mathfrak{so}(N), \qquad \vec{\varpi} \in \mathbb{R}^{N-1}, \qquad \boldsymbol{\Theta} \in \mathfrak{so}(N-1). \tag{39}$$

Note the components $h_{\|}$, $(\vec{h}_{\perp}, \boldsymbol{h}_{\perp})$ correspond to decomposing $e_t = g(\gamma_t, \gamma_x)e_x + (\gamma_t)_{\perp}\lrcorner e_{\perp}$ into tangential and normal parts relative to e_x. We thus have

$$\begin{aligned} [e_x, e_t] &= -\kappa^2\begin{pmatrix}0 & \vec{h}_{\perp}\\ -\vec{h}_{\perp}{}^T & \mathbf{0}\end{pmatrix} \in \mathfrak{so}(N),\\ [\omega_x, e_t] &= \kappa\begin{pmatrix}2\vec{h}_{\perp}\cdot\vec{v} & \vec{v}\lrcorner\boldsymbol{h}_{\perp} + h_{\|}\vec{v}\\ (\vec{v}\lrcorner\boldsymbol{h}_{\perp} + h_{\|}\vec{v})^T & -(\vec{v}\otimes\vec{h}_{\perp} + \vec{h}_{\perp}\otimes\vec{v})\end{pmatrix} \in \mathfrak{s}(N),\\ [\omega_t, e_x] &= \kappa\begin{pmatrix}0 & \vec{\varpi}\\ \vec{\varpi}^T & \mathbf{0}\end{pmatrix} \in \mathfrak{s}(N)_{\perp}.\end{aligned}$$

Now the curvature equation (12) yields

$$D_t\vec{v} - D_x\vec{\varpi} - \vec{v}\lrcorner\boldsymbol{\Theta} = \kappa^2\vec{h}_{\perp}, \tag{40}$$
$$-D_x\boldsymbol{\Theta} + \vec{v}\otimes\vec{\varpi} - \vec{\varpi}\otimes\vec{v} = 0, \tag{41}$$

which are unchanged from the case $G = SO(N+1)$ up to the factor in front of $\vec{h}_{\perp}$. The torsion equation (11) reduces to

$$0 = 2\kappa^{-2}D_x h_{\|} - 2\vec{v}\cdot\vec{h}_{\perp}, \tag{42}$$

$$0 = \vec{\varpi} - h_\| \vec{v} - D_x \vec{h}_\perp - \vec{v} \lrcorner \boldsymbol{h}_\perp, \tag{43}$$

which are similar to those in the case $G = SO(N+1)$, plus

$$0 = -D_x(\boldsymbol{h}_\perp + N^{-1} h_\| \mathbf{1}) + \vec{v} \otimes \vec{h}_\perp + \vec{h}_\perp \otimes \vec{v}. \tag{44}$$

Proceeding as before, we use equations (41), (42), (44) to eliminate

$$\begin{aligned} \boldsymbol{\Theta} &= D_x^{-1}(\vec{v} \otimes \vec{\varpi} - \vec{\varpi} \otimes \vec{v}), \\ h_\| &= \kappa^2 D_x^{-1}(\vec{v} \cdot \vec{h}_\perp), \\ \boldsymbol{h}_\perp &= D_x^{-1}(2(1-N)^{-1} \vec{v} \cdot \vec{h}_\perp \mathbf{1} + \vec{v} \otimes \vec{h}_\perp + \vec{h}_\perp \otimes \vec{v}) \end{aligned} \tag{45}$$

in terms of the variables $\vec{v}$, $\vec{h}_\perp$, $\vec{\varpi}$. Then equation (40) gives a flow on $\vec{v}$,

$$\vec{v}_t = D_x \vec{\varpi} + \vec{v} \lrcorner D_x^{-1}(\vec{v} \otimes \vec{\varpi} - \vec{\varpi} \otimes \vec{v}) + \kappa^2 \vec{h}_\perp$$

with

$$\vec{\varpi} = D_x \vec{h}_\perp + 2D_x^{-1}(\vec{v} \cdot \vec{h}_\perp)\vec{v} + \vec{v} \lrcorner D_x^{-1}(\vec{v} \otimes \vec{h}_\perp + \vec{h}_\perp \otimes \vec{v})$$

obtained from equation (43) after we combine $h_\| \vec{v}$ terms. We thus read off the operators

$$\mathcal{H} = D_x + \vec{v} \lrcorner D_x^{-1}(\vec{v} \wedge\ ), \qquad \mathcal{J} = D_x + 2D_x^{-1}(\vec{v} \cdot\ )\vec{v} + \vec{v} \lrcorner D_x^{-1}(\vec{v} \odot\ ), \tag{46}$$

where $\vec{A} \wedge \vec{B} = \vec{A} \otimes \vec{B} - \vec{B} \otimes \vec{A}$ and $\vec{A} \odot \vec{B} = \vec{A} \otimes \vec{B} + \vec{B} \otimes \vec{A}$.

The results in Theorem 1 and Corollary 1 carry over verbatim (with the same method of proof used in [21]) for the operators $\mathcal{H}$ and $\mathcal{J}$ here, up to a change in the scalar curvature factor

$$\chi = -\kappa^2 = 2N/(1-N)$$

connected with the Riemannian geometry of $SU(N)/SO(N)$. In particular, $\mathcal{R} = \mathcal{H} \circ \mathcal{J}$ yields a hereditary recursion operator

$$\begin{aligned} \mathcal{R} &= D_x(D_x + 2D_x^{-1}(\vec{v} \cdot\ )\vec{v} + \vec{v} \lrcorner D_x^{-1}(\vec{v} \odot\ )) + \vec{v} \lrcorner D_x^{-1}(\vec{v} \wedge (D_x + \vec{v} \lrcorner D_x^{-1}(\vec{v} \odot\ ))) \\ &= D_x^2 + 2(|\vec{v}|^2 + (\vec{v} \cdot\ )\vec{v}) + 2D_x^{-1}(\vec{v} \cdot\ )\vec{v} + \vec{v}_x \lrcorner D_x^{-1}(\vec{v} \odot\ ) \\ &\quad + \vec{v} \lrcorner D_x^{-1}(\vec{v} \wedge (\vec{v} \lrcorner D_x^{-1}(\vec{v} \odot\ )) - \vec{v}_x \wedge\ ) \end{aligned} \tag{47}$$

generating a hierarchy of $O(N-1)$-invariant commuting bi-Hamiltonian flows on $\vec{v}$, corresponding to commuting Hamiltonian vector fields $\vec{h}_\perp^{(k)} \lrcorner \partial/\partial \vec{v}$ and involutive covector fields $\vec{\varpi}^{(k)} \lrcorner d\vec{v}$, $k = 0, 1, 2, \dots$ starting from $\vec{h}_\perp^{(0)} = \vec{v}_x$, $\vec{\varpi}^{(0)} = \vec{v}$. In the terminology of [5], $\vec{h}_\perp^{(k)}$ is said to produce the $+(k+1)$ flow equation (22) on $\vec{v}$ (cf. Remark 2). Note these flows admit the same mKdV scaling symmetry $x \to \lambda x$, $\vec{v} \to \lambda^{-1} \vec{v}$ as in the case $SO(N+1)/SO(N)$. They also have similar recursion relations $\vec{h}_\perp^{(k)} = \mathcal{H}(\vec{\varpi}^{(k)})$, $\vec{\varpi}^{(k+1)} = \mathcal{J}(\vec{h}_\perp^{(k)}) = \delta H^{(k+1)}/\delta \vec{v}$, $k = 0, 1, 2, \dots$, in terms of Hamiltonians $H = H^{(k)}(\vec{v}, \vec{v}_x, \vec{v}_{2x}, \dots)$.

The $+1$ flow is given by $\vec{h}_\perp = \vec{v}_x$, yielding

$$\vec{v}_t = \vec{v}_{3x} + 3|\vec{v}|^2 \vec{v}_x + 3(\vec{v} \cdot \vec{v}_x)\vec{v} - \chi \vec{v}_x. \tag{48}$$

Up to the convective term, which can be absorbed by a Galilean transformation, this is a different vector mKdV equation compared to the one arising in the case $SO(N+1)/SO(N)$ for $N > 2$. The $+(k+1)$ flow yields a higher order version of this equation (48).

The hierarchy of flows corresponds to geometrical motions of the curve $\gamma(t,x)$ obtained in a similar fashion to the ones in the case $SO(N+1)/SO(N)$ via identifying $\gamma_t \leftrightarrow e_t$, $\gamma_x \leftrightarrow e_x$, $\nabla_x\gamma_x \leftrightarrow [\omega_x, e_x] = \mathcal{D}_x e_x$, and so on as before, where $\nabla_x \leftrightarrow \mathcal{D}_x = D_x + [\omega_x, e_x]$. Note here we have

$$[\omega_x, e_x] = \kappa \begin{pmatrix} 0 & \vec{v} \\ \vec{v}^T & \mathbf{0} \end{pmatrix}, \qquad [\omega_x, [\omega_x, e_x]] = 2\kappa \begin{pmatrix} |\vec{v}|^2 & \vec{0} \\ \vec{0} & -\vec{v}\otimes\vec{v} \end{pmatrix},$$

and so on. In addition,

$$\mathrm{ad}([\omega_x, e_x])e_x = \kappa^2 \begin{pmatrix} 0 & \vec{v} \\ -\vec{v}^T & \mathbf{0} \end{pmatrix},$$
$$\mathrm{ad}([\omega_x, e_x])^2 e_x = -2\kappa^3 \begin{pmatrix} |\vec{v}|^2 & \vec{0} \\ \vec{0} & -\vec{v}\otimes\vec{v} \end{pmatrix} = \chi[\omega_x, [\omega_x, e_x]].$$

Thus, for the $+1$ flow,

$$\vec{h}_\perp = \vec{v}_x, \qquad h_\parallel = \frac{1}{2}\kappa^2|\vec{v}|^2, \qquad \boldsymbol{h}_\perp = \vec{v}\otimes\vec{v} + (1-N)^{-1}|\vec{v}|^2\mathbf{1},$$

we obtain (through equation (38))

$$e_t = \kappa \begin{pmatrix} (N^{-1}-1)h_\parallel & \vec{h}_\perp \\ \vec{h}_\perp{}^T & \boldsymbol{h}_\perp + N^{-1}h_\parallel\mathbf{1} \end{pmatrix} = \kappa \begin{pmatrix} -|\vec{v}|^2 & \vec{v}_x \\ \vec{v}_x{}^T & \vec{v}\otimes\vec{v} \end{pmatrix}$$
$$= D_x[\omega_x, e_x] - \frac{1}{2}[\omega_x, [\omega_x, e_x]].$$

Then writing these expressions in terms of $\mathcal{D}_x$ and $\mathrm{ad}([\omega_x, e_x])$, we get

$$e_t = \mathcal{D}_x[\omega_x, e_x] - \frac{3}{2}\chi^{-1}\mathrm{ad}([\omega_x, e_x])^2 e_x \leftrightarrow \nabla_x^2\gamma_x - \frac{3}{2}\chi^{-1}\mathrm{ad}(\nabla_x\gamma_x)^2\gamma_x.$$

Thus, up to a sign, $\gamma(t,x)$ satisfies a geometric nonlinear PDE given by the non-stretching mKdV map equation (25) on the symmetric space $SU(N)/SO(N)$. The higher flows on $\vec{v}$ determine higher order map equations for γ.

The 0 flow as before is $\vec{v}_t = \vec{v}_x$ arising from $\vec{h}_\perp = 0, h_\parallel = 1$, which corresponds to the convective (traveling wave) map (26).

There is also a -1 flow contained in the hierarchy, with the property that $\vec{h}_\perp$ is annihilated by the symplectic operator $\mathcal{J}$ and hence lies in the kernel $\mathcal{R}(\vec{h}_\perp) = 0$ of the recursion operator. The geometric meaning of this flow is simply $\mathcal{J}(\vec{h}_\perp) = \vec{\varpi} = 0$ implying $\omega_t = 0$ from equations (39) and (45) so $0 = [\omega_t, e_x] = \mathcal{D}_t e_x$ where $\mathcal{D}_t = D_t + [\omega_t, \cdot]$. Thus, as in the case $SO(N+1)/SO(N)$, we see from the correspondence $\nabla_t \leftrightarrow \mathcal{D}_t$, $\gamma_x \leftrightarrow e_x$ that $\gamma(t,x)$ satisfies a nonlinear geometric PDE given by the wave map equation (27) on the symmetric space $SU(N)/SO(N)$.

The -1 flow equation produced on $\vec{v}$ is again a nonlocal evolution equation

$$\vec{v}_t = -\chi\vec{h}_\perp, \qquad \chi = -\kappa^2 \tag{49}$$

with $\vec{h}_\perp$ satisfying

$$0 = \vec{\varpi} = D_x\vec{h}_\perp + h\vec{v} + \vec{v}\lrcorner\boldsymbol{h} \tag{50}$$

where it is convenient to introduce the variables $\boldsymbol{h} = \boldsymbol{h}_\perp + N^{-1}h_\parallel\mathbf{1}$, $h = 2\kappa^{-2}h_\parallel = \mathrm{tr}\boldsymbol{h}$ which satisfy

$$D_x h = 2\vec{v}\cdot\vec{h}_\perp, \tag{51}$$

$$D_x \boldsymbol{h} = \vec{v} \otimes \vec{h}_\perp + \vec{h}_\perp \otimes \vec{v}. \tag{52}$$

Note equations (50) to (52) combined with equation (49) together constitute a nonhomogeneous linear system on $\vec{h}_\perp$, h, $\boldsymbol{h}$, from which these variables will be uniquely determined in terms of $\vec{v}$ (and its derivatives). Compared to the case $SO(N+1)/SO(N)$, however, the presence of the additional variable $\boldsymbol{h}$ produces a quite different form here for the flow on $\vec{v}$. To proceed, the form of equations (50) to (52) suggests

$$\boldsymbol{h} = \alpha \vec{h}_\perp \otimes \vec{h}_\perp + \beta \mathbf{1} \tag{53}$$

for some expressions $\alpha(h)$, $\beta(h)$. Substitution of $\boldsymbol{h}$ into equation (52) followed by the use of equations (50) and (51) leads to $D_x \beta = 0 = \beta + \alpha^{-1} + h$ and hence we get

$$\alpha = -(h+\beta)^{-1}, \qquad \beta = \text{const.} \tag{54}$$

Next, by taking the trace of $\boldsymbol{h}$ and using equation (54), we obtain

$$|\vec{h}_\perp|^2 = N\beta(h+\beta) - (h+\beta)^2 \tag{55}$$

which enables h to be expressed in terms of $\vec{h}_\perp$ and β. To determine β we use the wave map conservation law (29) where, now,

$$|\gamma_t|_g^2 = \langle e_t, e_t \rangle_{\mathfrak{p}} = \kappa^2 (|\vec{h}_\perp|^2 + \frac{1}{2}(h^2 + |\boldsymbol{h}|^2)).$$

This corresponds to a conservation law admitted by equations (50) to (52),

$$0 = D_x \left(|\vec{h}_\perp|^2 + \frac{1}{2}(h^2 + |\boldsymbol{h}|^2) \right),$$

and as before, a conformal scaling of t can now be used to put $|\gamma_t|_g$ equal to a constant. A convenient value which simplifies subsequent expressions is $|\gamma_t|_g = 2$, so then

$$(2/\kappa)^2 = |\vec{h}_\perp|^2 + \frac{1}{2}\big(|\boldsymbol{h}|^2 + h^2\big).$$

Substitution of equations (53) to (55) into this expression yields

$$\beta^2 = (2/N)^2$$

from which we obtain via equation (55)

$$h = 2N^{-1} - 1 \pm \sqrt{1 - |\vec{h}_\perp|^2}, \qquad \alpha = |\vec{h}_\perp|^{-2}\Big(1 \mp \sqrt{1 - |\vec{h}_\perp|^2}\Big).$$

These variables then can be expressed in terms of $\vec{v}$ through the flow equation (49). Finally, we note equations (50), (53), (54) yield the additional relation

$$D_x(\sqrt{\alpha}\vec{h}_\perp) = \sqrt{1/\alpha}\vec{v}. \tag{56}$$

Hence the flow equation on $\vec{v}$ becomes

$$\vec{v}_t = \sqrt{A_\pm} D_x^{-1}\big(\sqrt{A_\pm}\vec{v}\big) \tag{57}$$

where

$$A_\pm = 1 \pm \sqrt{1 - |\vec{v}_t|^2} = |\vec{v}_t|^2 A_\mp,$$

with a factor $2\kappa^{-2}$ having been absorbed into a scaling of t.

This nonlocal evolution equation (57) for the -1 flow is equivalent to the vector SG equation

$$\big(\sqrt{A_{\mp}\vec{v}_t}\big)_x = \sqrt{A_{\pm}\vec{v}}$$

or in hyperbolic form

$$\vec{v}_{tx} = A_{\pm}\vec{v} - A_{\mp}|\vec{v}_t|^{-2}(\vec{v}\cdot\vec{v}_t)\vec{v}_t. \tag{58}$$

Alternatively, through relation (56), $\vec{h}_\perp$ obeys a vector SG equation

$$(\sqrt{\alpha}(\sqrt{\alpha}\vec{h}_\perp)_x)_t = \vec{h}_\perp. \tag{59}$$

These equations (58) and (59) possess the mKdV scaling symmetry $x \to \lambda x$, $\vec{v} \to \lambda^{-1}\vec{v}$, where $\vec{h}_\perp$ in equation (59) has scaling weight 0.

In [5] the symmetry-integrability classification results show that the hyperbolic vector equation (58) admits the vector mKdV equation (48) as a higher symmetry. ¿From the properties of the mKdV Hamiltonian operators (46) stated in Corollary 1, we see that the recursion operator (47) generates a hierarchy of vector mKdV higher symmetries

$$\vec{v}_t^{(0)} = \vec{v}_x, \tag{60}$$

$$\vec{v}_t^{(1)} = \mathcal{R}(\vec{v}_x) = \vec{v}_{3x} + 3(|\vec{v}|^2\vec{v}_x + (\vec{v}\cdot\vec{v}_x)\vec{v}), \tag{61}$$

$$\begin{aligned}\vec{v}_t^{(2)} = \mathcal{R}^2(\vec{v}_x) = \vec{v}_{5x} + 5(&|\vec{v}|^2\vec{v}_{3x} + 3(\vec{v}\cdot\vec{v}_x)\vec{v}_{2x} + (2|\vec{v}_x|^2 + 3\vec{v}\cdot\vec{v}_{2x} + 2|\vec{v}|^4)\vec{v}_x \\ &+ (3\vec{v}\cdot\vec{v}_{3x} + 2\vec{v}_x\cdot\vec{v}_{2x} + 4|\vec{v}|^2\vec{v}\cdot\vec{v}_x)\vec{v}),\end{aligned} \tag{62}$$

and so on, while the adjoint of this operator (47) generates a hierarchy of Hamiltonians

$$\begin{aligned}H^{(0)} &= \frac{1}{2}|\vec{v}|^2,\\ H^{(1)} &= -\frac{1}{2}|\vec{v}_x|^2 + \frac{1}{2}|\vec{v}|^4,\\ H^{(2)} &= -\frac{1}{2}|\vec{v}_{2x}|^2 - 2|\vec{v}|^2|\vec{v}_x|^2 - 3(\vec{v}\cdot\vec{v}_x)^2 + |\vec{v}|^6,\end{aligned}$$

and so on, all of which are conserved densities for the -1 flow and thereby determine higher conservation laws for the hyperbolic vector equation (58). Viewed as flows, the entire hierarchy of vector PDEs (60) to (62), etc., including the -1 flow

$$\vec{v}_t^{(-1)} = \vec{h}_\perp$$

associated to the vector SG equation (59), is seen to possess the mKdV scaling symmetry $x \to \lambda x$, $\vec{v} \to \lambda^{-1}\vec{v}$, with $t \to \lambda^{1+2k}t$ for $k = -1, 0, 1, 2$, etc.. Moreover for $k \geq 0$, all these expressions will be local polynomials in the variables $\vec{v}, \vec{v}_x, \vec{v}_{xx}, \ldots$ as established by results in [23] applied to the separate Hamiltonian (cosymplectic and symplectic) operators (46)[7].

Theorem 3. *In the symmetric space $SU(N)/SO(N)$ there is a hierarchy of bi-Hamiltonian flows of curves $\gamma(t,x)$ described by geometric map equations. The* 0 *flow is a convective (traveling wave) map* (26)*, while the* $+1$ *flow is a non-stretching mKdV map* (25) *and the* $+2,\ldots$ *flows are higher order analogs. The kernel of the recursion operator* (47) *in the hierarchy yields the* -1 *flow which is a non-stretching wave map* (27)*.*

[7]Due the doubly nonlocal form of the last term in the recursion operator (47), the general results in [29, 24] are not directly applicable.

5 Concluding remarks

In the Riemannian symmetric spaces $G/SO(N)$, as exhausted by the Lie groups $G = SO(N+1)$ and $G = SU(N)$, there is a hierarchy of integrable bi-Hamiltonian flows of non-stretching curves $\gamma(t,x)$, where the $+1$ flow is described by the mKdV map equation (25) and the $+2,\ldots$ flows are higher-order analogs, while the wave map equation (27) describes a -1 flow that is annihilated by the recursion operator of the hierarchy. In a parallel frame the principal normal components along γ for these flows respectively satisfy a vector mKdV equation and a vector hyperbolic equation, which are $O(N-1)$-invariant. The hierarchies for $SO(N+1)/SO(N), SU(N)/SO(N)$ coincide in the scalar case $N = 2$. Moreover the scalar hyperbolic equation in this case is equivalent to the SG equation. These results account for the existence of the two known versions of vector generalizations of the mKdV and SG equations [5].

Similar results hold for hermitian symmetric spaces $G/U(N)$. In particular, there is a hierarchy of flows of curves in such spaces yielding scalar-vector generalizations of the mKdV equation and the SG equation. A further generalization of such results for all symmetric spaces G/H will be given elsewhere [3].

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