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q-Deformed Bi-Local Fields II

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Abstract. We study a way of q-deformation of the bi-local system, the two particle system bounded by a relativistic harmonic oscillator type of potential, from both points of view of mass spectra and the behavior of scattering amplitudes. In our formulation, the deformation is done so that P^2, the square of center of mass momentum, enters into the deformation parameters of relative coordinates. As a result, the wave equation of the bi-local system becomes nonlinear with respect to P^2; then, the propagator of the bi-local system suffers significant change so as to get a convergent self energy to the second order. The study is also made on the covariant q-deformation in four dimensional spacetime.

Key words: q-deformation; bi-local system; harmonic oscillator; nonlinear wave equation

2000 Mathematics Subject Classification: 32G07; 81R50; 81R60

1 Introduction

The aim of studying non-local field theories proposed by Yukawa [1] was originally twofold: firstly, to derive characteristic properties of elementary particles such as the mass spectrum of hadrons, from their extended structure; secondly, to deal with the divergence difficulty, which is inherent in local field theories with local interactions. The bi-local field theory [2] was the first attempt by Yukawa following this line of thought. For such fields, the first aim has been addressed in many works in the context of effective relativistic two-particle systems of quark and anti-quark bound systems. In particular, two-particle systems bounded by a relativistic harmonic oscillator potential were useful in deriving the linear mass-square spectrum associated with the Regge behavior in their scattering amplitude [3, 4]. Contrastingly, there has been little success in the pursuit of the second aim, mainly because the bi-local fields are reduced to a superposition of an infinite number of local fields with different masses, although some people have claimed that the second order self-energy becomes convergent associated with the direction of the center-of-mass momentum. In addition, the problems of the unitarity of the scattering matrix and causality are also serious for those fields, because a bi-local system in general allows time-like relative motion. Usually, such a degree of freedom is frozen by an additional subsidiary condition [5]. However, this is not always successful for interacting cases. This situation may be different from that of string models that are characterized by the Virasoro condition associated with the parameterization invariance in such an extended model; nevertheless the study of bi-local field theories does not come to end, since a small change in models, sometimes, will cause a significant change in their physical properties.

Given the situation described above, the purpose of this paper is to study the q-deformation [6, 7, 8, 9] of a bi-local system characterized by a relativistic harmonic oscillator potential, because the q-deformation is well-defined for harmonic oscillator systems. In a previous paper [10], we studied a q-deformed 5-dimensional spacetime such that the extra dimension generates a harmonic oscillator type of potential for particles embedded in that spacetime. Then, the propagator of the particles in that spacetime acquires a significant convergent property by requiring that the 4-dimensional spacetime variables and the extra dimensional spacetime variable are mixed by the deformation. It then happens that the 4-dimensional spacetime variables do not commute with the fifth space variables. We can expect that the same situation arises in the bi-local system, if we carry out the deformation of the relative variables in this system along the line of the q-deformed 5-dimensional spacetime. This extension of the bi-local systems was done in [11]; and, we could show that the bi-local system was free from the divergence in one-loop level. In this stage, however, we could not find the way of q-deformation being consistent with the Lorentz covariance. In this paper, thus, we intend adding a new sight for the covariant deformation, in addition to the review of the q-deformed bi-local fields.

In the next section, we formulate the bi-local system. In Section 3 we construct the q-deformed bi-local system with the q-deformed relative coordinates. In this case, we define the deformation so as to get a Lorentz invariant resultant wave equation. In Section 4, the interaction of the bi-local field is discussed in the context of calculating Feynman diagrams. Some scattering amplitudes between the bi-local system and external scalar fields are also studied by considering their Regge behavior. A second order self-energy diagram is also calculated to study the convergence of the model. Section 5 is devoted to summary and discussion. In Appendix A, several ways of q-deformation in N-dimensional oscillator variables are discussed to find a possible way to the covariant deformation. In Appendix B, we add a new attempt to construct a phase space action of the q-deformed bi-local system. Appendix C is devoted to giving of the outline for calculating of a one-loop diagram corresponding to the self-energy of the q-deformed bi-local field.

2 Bi-local field theories

The bi-local theory obtains a potential approach to the bound state of relativistic two particle system. Then the classical action of an equal mass two-particle system that leads to the standard bi-local field equations is given by [2]

$$S = \int d\tau \frac{1}{2} \sum_{i=1}^{2} \left\{ \frac{\dot{x}^{(i)2}}{e_i} + e_i(m^2 + V(\bar{x})) \right\},$$

where $\bar{x} = x^{(1)} - x^{(2)}$ is the relative coordinators of the system. Here e_i's are einbein that guarantee the invariance of S under the reparametrization of τ. By taking the variation of S with respect to e_i, we obtain the equal-mass constraints

$$\frac{\delta S}{\delta e_i} = -\left\{ p^{(i)2} - (m^2 + V(\bar{x})) \right\} = 0, \qquad i = 1, 2,$$

where $p^{(i)}_\mu = \frac{\delta S}{\delta x^{(i)\mu}}$ are the momenta conjugate to $x^{(i)\mu}$. These constraints can be rewritten as

$$\frac{1}{2} P^2 + 2(\bar{p}^2 - m^2) - 2V(\bar{x}) = 0, \tag{1}$$

$$P \cdot p = 0, \tag{2}$$

where $P = p^{(1)} + p^{(2)}$ and $\bar{p} = \frac{1}{2}(p^{(1)} - p^{(2)})$ are the total momentum and the relative momentum of the bi-local system, respectively. The variables X_μ satisfying the canonical commutation relations $[P_\mu, X_\nu] = ig_{\mu\nu}$, $[\bar{p}_\mu, X_\nu] = 0$ are determined uniquely to be $X_\mu = \frac{1}{2}\left(x_\mu^{(1)} + x_\mu^{(2)}\right)$, which are nothing but the center of mass coordinates of the equal-mass bi-local system[1].

Now, one can verify that these constraints are compatible for the covariant harmonic oscillator potential $V(\bar{x}) = -k^2\bar{x}^2$ in the following sense: In q-number theory, we can introduce the oscillator variables defined by

$$\bar{x} = \sqrt{\frac{1}{2k}}\,(a^\dagger + a), \qquad \bar{p} = i\sqrt{\frac{k}{2}}\,(a^\dagger - a).$$

Then reading equation (2) in the sense of expectation $\langle\phi|P\cdot p|\phi\rangle = 0$ as in the Gupta–Bleuler formalism in QED, equations (1) and (2) can be understood respectively as the master wave equation of the bi-local system and the physical state condition; that is, we can put

$$\left(\alpha' P^2 + \frac{1}{2}\{a_\mu^\dagger, a^\mu\} - \omega\right)|\phi\rangle = 0, \tag{3}$$

$$P^\mu a_\mu|\phi\rangle = 0, \tag{4}$$

where $\alpha' = \frac{1}{8\kappa}$ and $\omega = \frac{m^2}{2k}$. In this stage, the compatibility between equations (3) and (4) becomes clear. After the second quantization of physical states $\{|\phi\rangle\}$, those become the q-number fields, so called 'the bi-local field'.

We also note that subsidiary condition (2) defines the bi-local field associated with the indefinite metric formalism of the Lorentz group. If we read equation (2) as $P_\mu a^{\mu\dagger}|\phi\rangle = 0$ instead of (4), then we have a definite metric formalism of bi-local field theories. The former has a similarity to the string models and the latter shows an interesting form factor as meson-like bound states [5].

3 q-deformed bi-local system

The q-deformed one-dimensional harmonic oscillator is defined by the oscillators and the number operator satisfying

$$a_q a_q^\dagger - q a_q^\dagger a_q = q^{-N}, \qquad [N, a_q^\dagger] = a_q^\dagger,$$

The oscillator $(a_q, a_q^\dagger)$ can be realized from the standard harmonic oscillator $(a, a^\dagger)$, the $(a_q, a_q^\dagger)$ with $q = 1$, through the mapping

$$a_q = a\sqrt{\frac{[N]_q}{N}}, \qquad N = a^\dagger a, \qquad [N]_q = \frac{\sinh(N\log q)}{\sinh(\log q)}, \tag{5}$$

There are several ways of q-deformation in four-dimensional harmonic oscillators; and, the following is a similar way to (5) in constructing q-deformed oscillators:

$$a_{q\mu} = a_\mu\sqrt{\frac{[N]_q}{N_\mu}}, \qquad [N]_q = \frac{\sinh\{(N+\beta+\frac{1}{2})\log q\}}{\sinh(\frac{1}{2}\log q)},$$
$$N = -a_\mu^\dagger a^\mu, \qquad N_0 = -a_0^\dagger a^0, \qquad N_i = a_i^\dagger a^i.$$

Here, the β may be an operator commuting with $(a_\mu, a_\mu^\dagger)$; and so, it may contain the center of mass momentum P_μ. Interesting result is obtained by putting $\beta = \beta_0 - \frac{3}{4}\alpha' P^2$ by considering the covariance of a resultant wave equation. Then, the master wave equation becomes

$$\left(\alpha' P^2 + \frac{1}{2}\{a_{q\mu}, a_q^{\mu\dagger}\}(P^2, N) - \omega\right)|\Phi\rangle. = 0, \tag{6}$$

[1] $\mathrm{diag}\,(g_{\mu\nu}) = (+\,-\,-\,-)$.

from which we can obtain the mass-square like operator

$$m^2 = -\frac{1}{2\alpha'}\left\{a_{q\mu}, a_q^{\mu\dagger}\right\}(P^2) + \frac{\omega}{\alpha'} = -\frac{2}{\alpha'}\frac{\sinh[(N+\beta+\frac{1}{2})\log q]}{\sinh[\frac{1}{2}\log q]} + \frac{\omega}{\alpha'}.$$

It should be noticed that the m^2 operator is a Lorentz scalar in spite of that the mapping breaks the Lorentz covariance. However, if we do not worry about the complexity of $[N]$, we can adopt a covariant mapping, which yields the same master wave equation as (6). We will discuss separately this problem in Appendix A, since such a mapping is important from the viewpoint of the Lorentz covariance.

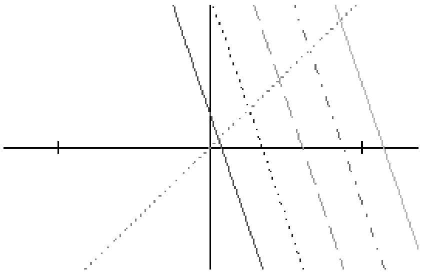

Figure 1. The line $y = x$ vs $y = -\frac{1}{2\alpha'}\{a_q, a_q^\dagger\} + \frac{\omega}{\alpha'}$ with $n = 0, 1, 2, \ldots$. The mass-square eigenvalues are represented by the x coordinates of the intersection.

Now, the mass square eigenvalues $\{m_n^2\}$ are obtained by solving the non-linear equation (6) with respect to P^2 for each eigenvalue n $(= 0, 1, 2, \ldots)$ of N; that is, those are solutions of $m_n^2 = -\frac{1}{\alpha'}\{a_q, a_q^\dagger\}(m_n^2) + \frac{\omega}{\alpha'}$. It should be noted that the q-deformed bi-local field is free from space-like solutions for $\alpha > 0$ as can be seen from Fig. 1.

The above discussion implies that the free bi-local field is similar to local free fields in the sense that its spacetime development can be determined by the Cauchy data without contradicting causality. Also, the Feynman propagator

$$G(P^2, N) = \left(P^2 + \frac{1}{2\alpha'}\{a_{q\mu}, a_q^{\mu\dagger}\} - \frac{\omega}{\alpha'} + i\epsilon\right)^{-1}$$

decreases exponentially as N, $|P^2| \to \infty$. As shown in the next section, this enables us to obtain a finite-vacuum-loop amplitude, in contrast to the usual local field theories. This property was observed firstly in [10].

4 Interaction of the bi-local field

We here discuss, shortly, on the interaction caused by the three vertex of bi-local fields. The symmetric three vertex in Fig. 2 can be naturally defined by [12, 13]

$$\left.\begin{array}{l}\left[x_\mu^{(1)}(b) - x_\mu^{(2)}(a)\right]|V\rangle = 0 \\ \left[p_\mu^{(1)}(b) + p_\mu^{(2)}(a)\right]|V\rangle = 0\end{array}\right\} (a, b, c \text{ cyclic}). \tag{7}$$

The vertex function $|V\rangle$ determined by these equations, however, does not satisfy the physical state condition $P\cdot a(k)|V\rangle = 0$, $(k = a, b, c)$; in order to get a physical vertex, thus, we have to put $|V_{\rm phys}\rangle = \Lambda_a\Lambda_b\Lambda_c|V\rangle$ with projection operators defined by $P\cdot a(k)\Lambda_k = 0$, $\Lambda_k^2 = \Lambda_k$, $(k = a, b.c)$.

Using the vertex function, we can calculate the second order scattering amplitudes in Fig. 4 $\sim$ Fig. 7. In the vertices of those amplitudes, it is sufficient to put one of bi-local fields in Fig. 2, say the particle 'b', in the ground state, which can be identified with an external scalar field as in Fig. 3. Then the vertex conditions (7) become

$$[x^{(1)}(a) - x^{(2)}(c)]|V\rangle = [x^{(2)}(a) - x^{(1)}(c)]|V\rangle = 0,$$
$$[p^{(1)}(a) + p^{(2)}(c)]|V\rangle = [p^{(2)}(a) + p^{(1)}(c)]|V\rangle = 0.$$

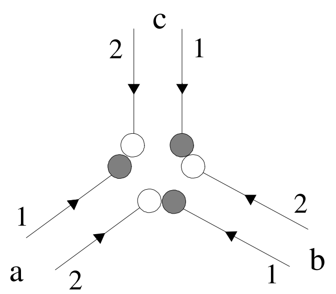

Figure 2. The a, b, c designate three bi-local systems; and 1, 2 are constituents in each bi-local system.

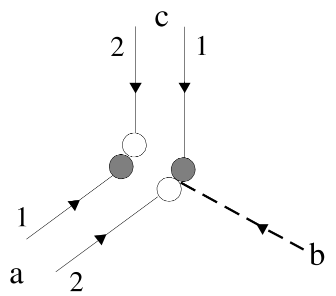

Figure 3. The dashed line denotes the local scalar field corresponding to the ground state of b.

These equations with additional physical state conditions can be solved easily to give

$$|V\rangle = g\delta^{(4)}(P(a) + P(b) + P(c))$$
$$\times \exp\left[-\frac{i}{2}\sqrt{\frac{1}{2k}}(a(a)^\dagger - a(c)^\dagger)_\perp \cdot P(b) + a(a)^\dagger_\perp \cdot a(c)^\dagger_\perp\right]|0\rangle.$$

where $a_{\perp\mu} = O_{\mu\nu}a^\nu$ with $O_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu \cdot p_\nu}{p^2}$. With aid of this vertex function, one can calculate the second order scattering amplitudes of bi-local fields. In particular, for a simpler case with ground states a, a', it is found that the s-channel scattering amplitude decreases rapidly for $s \to \infty$ with t fixed and the t-channel scattering amplitude shows the Regge behavior for $t \to \infty$ with s fixed. This means that the t-channel amplitude in Fig. 5 obtained by interchanging s and t from the second-order scattering amplitude exhibits Regge behavior for large s, with t fixed [11].

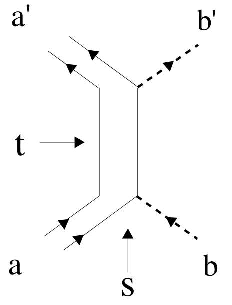

Figure 4. A second-order scattering amplitude of a bi-local system by two external fields with $s = (P(a) + P(b))^2$ and $t = (P(a) - P(a'))^2$.

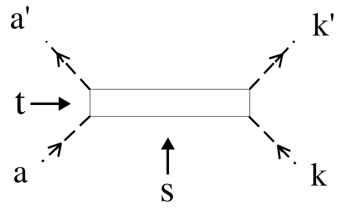

Figure 5. The t-channel amplitude obtained through interchange $s \leftrightarrow t$ from Fig. 4.

A more remarkable fact in this model is that the loop diagram in Fig. 7 corresponding to the self energy of the ground state of the bi-local system is found to be convergent [11]. Indeed, according to the magnitude of momenta in the internal line, the self energy can be written as

$$\delta m^2 \sim \delta m^2(|p|\langle|k|) + \delta m^2(|p| \gg |k|). \quad (8)$$

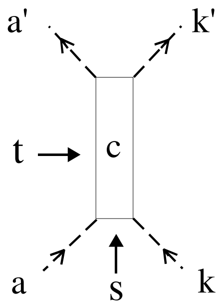

Figure 6. The s-channel scattering amplitude for four grand states.

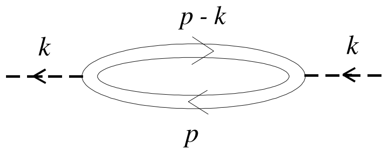

Figure 7. The self-energy for the grand state.

The first term of equations (8) is finite obviously, and the second term can be evaluated as

$$\delta m^2(|p| \gg |k|) \sim i \left(\frac{g\pi\alpha \sinh(\frac{1}{2}\log q)}{2\alpha_0' \log q} e^{(\beta_0 + \frac{1}{2})\log q} \right)^2 ,$$

Therefore, the self-energy in our model is found to be convergent by virtue of the factor $\sinh(\cdot)$ in the propagator with the deformation parameter $q \neq 1$.

5 Summary and discussion

The relativistic two-particle system, the bi-local system, bounded by a 4-dimensional harmonic oscillator potential yields a successful description of two-body meson-like states. The mass-square spectrum, then, arises from the excitation of relative variables, which are independent of the center-of-mass variables.

In this work, we have constructed a q-deformed bi-local system with harmonic oscillator type bound potential. The deformation $(a_{q\mu}(P^2), a^\dagger_{q\mu}(P^2))$ is carried out so as to include the center-of mass momentum in the deformation parameters. Then q-deformed relative coordinates are non-commutative with each other, while the center-of-mass coordinates remain as commutative variables. The formal mass square operator is Lorentz invariant, though the mapping $(a_\mu, a^\dagger_\mu) \to (a_{q\mu}, a^\dagger_{q\mu})$ spoils the covariance. However, the way of mapping giving the same mass square operator is not unique; and there is a covariant mapping in compensation of simple commutation relations of the oscillator variables.

In the q-deformed bi-local system, the mass eigenvalues are real simple zeros of master wave equation; then, the propagator becomes free from multi-pole ghosts. Further, the wave function of system acquires new properties such that the propagator of system damped rapidly as $|P^2|$ tends to infinity. We have also studied the interaction of the bi-local system with external scalar field, which are identified with the ground state of the system. Then, we could verify the following: First, the second-order t-channel scattering amplitude exhibits Regge behavior in the limit $t \to \infty$ with s fixed. Secondly, a second-order loop diagram, which corresponds to a self-energy of the system, shows convergent property providing $q \neq 1$. This is due to a characteristic property of the propagator.

To complete the q-deformed bi-local theory within the framework of field theories, it is necessary to study the higher-order diagrams in addition to the analysis on the causality. Those are interesting future problems.

A Representation of a q-oscillator

We here discuss the representation of the q-oscillator variables defined by

$$[A, A^\dagger]_q \equiv AA^\dagger - q^\alpha A^\dagger A = q^{-\alpha(N+\beta)}, \tag{9}$$

where α, β and q (≥ 1) are real parameters, and $N = a^\dagger a$ is the number operator for the ordinary oscillator variables satisfying $[a, a^\dagger] = 1$. There is a mapping between $(a, a^\dagger)$ and $(A, A^\dagger)$ such as

$$A = a\sqrt{\frac{[N]_q}{N}}, \qquad A^\dagger = \sqrt{\frac{[N]_q}{N}}a^\dagger,$$

where $N = a^\dagger a$. To determine $[N]_q$, let us recall $Na = a(N-1)$ and $Na^\dagger = a^\dagger(N+1)$; then, we can verify

$$AA^\dagger = [N+1]_q, \qquad A^\dagger A = [N]_q,$$

from which equation (9) can be reduced to

$$[N+1]_q - q^\alpha [N]_q = q^{-\alpha(N+\beta)}.$$

This recurrence equation can be solved easily; and we have

$$[N]_q = \frac{q^{\alpha(N+\beta)} - q^{-\alpha(N+\beta)}}{q^\alpha - q^{-\alpha}}. \tag{10}$$

with the condition $[0] = 0$.

Under this q-deformation, the Hamiltonian for ordinary oscillator $\frac{1}{2}\{a^\dagger, a\} + \beta$ will be replaced by $\frac{1}{2}\{A^\dagger, A\}$, which can be rewritten as

$$\frac{1}{2}\{A^\dagger, A\} = \frac{1}{2}\left([N]_q + [N+1]_q\right) = \frac{1}{2}\frac{\sinh\left[\alpha(N+\beta+\frac{1}{2})\log q\right]}{\sinh(\frac{1}{2}\alpha\log q)}.$$

Indeed, one can verify that $\frac{1}{2}\{A, A^\dagger\} \to \frac{1}{2}\{a^\dagger, a\} + \beta$ according as as $q \to 1$. This implies that β plays the role of an additional term in the zero-point energy; and, the α can be absorbed into the definition of q by the substitution $q^\alpha \to q$.

The q-deformation can be extended to D-dimensional oscillator variables defined by $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$ $(i, j = 1, 2, \ldots, D)$. We note that there is no mapping $A_i(a, a^\dagger)$ satisfying

$$[A_i, A_j^\dagger]_q = \delta_{ij} f(N) \qquad \text{and} \qquad [A_i, A_j]_q = [A_i^\dagger, A_j^\dagger]_q = 0,$$

where $N = \sum_i a_i^\dagger a_i$. Indeed, since $A_j[A_i, A_j^\dagger]_q = q^{2\alpha}[A_i, A_j^\dagger]_q A_j + q^\alpha [A_i, f(N)]$ for $i \neq j$, we have $[A_i, f(N)] = 0$, which holds only when A_i is a function of N, that is, a function without vector indices $\{i\}$. The mappings listed below, however, may be useful for the model building.

Case (i): $A_i = a_i\sqrt{\frac{[N_i]_q}{N_i}}$. The $\{A_i\}$ satisfy the simple algebra

$$[A_i, A_j^\dagger]_q = \delta_{ij} q^{-\alpha(N_i+\beta)}.$$

However, the mapping does not preserve $U(D)$ symmetry, even for $\sum_i \{A_i, A_i^\dagger\}$.

Case (ii): $A_i = a_i\sqrt{\frac{[N]_q}{N}}$. This mapping preserves the $U(D)$ vector property of A_i, while the algebra of $\{A_i\}$ is modified so that

$$A_i A_j^\dagger - \left(\frac{[N+1]_q}{[N]_q}\frac{N}{N+1}\right) A_j^\dagger A_i = \frac{[N+1]_q}{N+1}\delta_{ij}.$$

Further, with $[N]$ in equation (10), the Hamiltonian becomes a summed form:

$$\frac{1}{2}\sum_i \{A_i^\dagger, A_i\} = \frac{1}{2}\frac{\sinh\left[\alpha(N+\beta+\frac{1}{2})\log q\right]}{\sinh(\frac{1}{2}\alpha\log q)} + \frac{1}{2}\frac{D+1}{N+1}\frac{\sinh\left[\alpha(N+\beta+1)\log q\right]}{\sinh(\alpha\log q)}.$$

We shall bring up this case later, again, since this case is interesting in the viewpoint of the covariant formulation of a q-deformed bi-local model.

Case (iii): $A_i = a_i\sqrt{\frac{[N]_q}{N_i}}$. We can then verify that the $\{A_i\}$ satisfy a $U(D)$ covariant algebra such as

$$[A_i, A_j^\dagger]_q = q^{-\alpha(N+\beta)} A_i [N]_q^{-1} A_j^\dagger,$$

although the mapping spoils that symmetry; and the Hamiltonian becomes an invariant form, which is used in Section 3:

$$\frac{1}{2}\sum_i \{A_i^\dagger, A_i\} = \frac{D}{2}\frac{\sinh\left[\alpha(N+\beta+\frac{1}{2})\log q\right]}{\sinh(\frac{1}{2}\alpha\log q)}. \tag{11}$$

We now turn to the covariant mapping (ii) to note that there is a function $[N]$ related with a given Hamiltonian $H(N)$ in principle. In order to show this, we first write $\frac{1}{2}\sum_i\{A_i^\dagger, A_i\} = H(N)$ as

$$[N+1] + \frac{N+1}{N+D}[N] = \frac{2(N+1)}{N+D}H(N),$$

in which N may be read as an integer. Then using the function $\varphi(N)$ defined by

$$\varphi(N) = \prod_{k=0}^{N}\frac{k+1}{k+D} = (D-1)B(D-1, N+2),$$

with the convention $\varphi(0) = 1$, we can get the recurrence relation

$$\frac{[N+1]}{\varphi(N)} + \frac{[N]}{\varphi(N-1)} = 2\frac{H(N)}{\varphi(N-1)} \equiv g(N).$$

This equation can be solved by the standard manner; and we obtain

$$[N] = (-1)^{N-1}\varphi(N-1)\left\{[1] + \sum_{k=1}^{N-1}(-1)^k g(k)\right\}$$

The right-hand side of this equation is determined by $H(N)$ only. Therefore, if it is necessary, we can arrive at the expression (11) starting from the covariant mapping (ii), though the $[N]$ becomes complex one.

B The phase-space action for a q-deformed bi-local model

It is well known that the phase-space action for the one-dimensional harmonic oscillator has a simple form

$$L(z,\dot{z})=\frac{i}{2}z^{*}\overset{\leftrightarrow}{\partial_t}z-\langle z|H|z\rangle,\qquad H=\frac{\hbar\omega}{2}\{a^{\dagger},a\},\tag{12}$$

where $|z\rangle=e^{-|z|^2/2+za^{\dagger}|0\rangle}$ is coherent with $\langle z|z\rangle=1$. Indeed, if we substitute $z=\sqrt{\frac{m\omega}{2}}x+\frac{i}{\sqrt{2m\omega}}p$ for (12), then the L will be reduced to the standard action of the harmonic oscillator after eliminating p. In a similar sense, for the $q=1$ bi-local model, the action can be written as

$$L=-P\cdot\dot{X}+\frac{i}{2}z_{\mu}^{*}\overset{\leftrightarrow}{\partial_\tau}z^{\mu}-eH(P,z,z^{*}),$$

where

$$H(P,z,z^{*})=\alpha' P^2+\langle\bar{z}|\frac{1}{2}\{a_{\mu}^{\dagger},a^{\mu}\}|z\rangle-\omega,\tag{13}$$

and $e(\tau)$ is the einbein, which guarantees the τ reparametrization invariance of the action. The total momentum P_μ can be eliminated from L obviously by using the constraint $\frac{\partial}{\partial P^{\mu}}L=0$. The H in the q-deformed case is simply obtained by the substitution $\frac{1}{2}\{a_{\mu}^{\dagger},a^{\mu}\}\to\frac{1}{2}\{A_{\mu}^{\dagger},A^{\mu}\}$ in equation (13).

Let us show that even in this q-deformed case, we can eliminate P_μ from L by using the constraint

$$\frac{\partial L}{\partial P^{\mu}}=-\dot{X}_{\mu}+e\partial_{P^{\mu}}H=0.$$

If we notice here that $(\partial_P H)^2\equiv f(P^2)$ is a function of P^2 only, the c onstraint allow us to solve P^2 in terms of $(e^{-1}\dot{X})^2$; then, we have

$$P^2=f^{-1}\left[\left(e^{-1}\dot{X}\right)^2\right]\qquad\text{and}\qquad P\cdot\dot{X}=\sqrt{f^{-1}\left[\left(e^{-1}\dot{X}\right)^2\right]\dot{X}^2}.$$

Substituting this expression for L, we finally obtain

$$L=-\sqrt{f^{-1}\left[\left(e^{-1}\dot{X}\right)^2\right]\dot{X}^2}++\frac{i}{2}z_{\mu}^{*}\overset{\leftrightarrow}{\partial_\tau}z^{\mu}+eH\left(f^{-1}\left[\left(e^{-1}\dot{X}\right)^2\right],z,\dot{z}\right).$$

One can verify that the constraint corresponding to the master wave equation is obtained by taking the derivative of L with respect to e.

C Loop amplitude

We here show the outline calculation of the loop diagram corresponding to the self-energy δm^2 for the ground state, which can be written as

$$\begin{aligned}\delta m^2&\sim g^2\int d^4p\,\mathrm{Tr}\left[G((p-k)^2,N_\perp(p-k)):e^{-\frac{i}{2}\bar{x}\cdot O(p-k)\cdot k}:G(p^2,N_\perp(p)):e^{\frac{i}{2}\bar{x}\cdot O(p)\cdot k}:\right]\\&=\delta m^2(|p|\lesssim|k|)+\delta m^2(|p|\gg|k|).\end{aligned}\tag{14}$$

The first term of the r.h.s. in equation (14) will be finite, and second term can be roughly evaluated as

$$\delta m^2(|p| \gg |k|) \sim g^2 \int d^4 p \int_C \frac{d\zeta}{2\pi i\zeta} \int_C \frac{d\zeta'}{2\pi i\zeta'} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \\ \times \mathrm{Tr}_{\mathrm{phys}} \left[\zeta^{N_\perp(p)} : e^{-\frac{i}{2}\bar{x}\cdot O(p)\cdot k} : \zeta'^{N_\perp(p)} : e^{\frac{i}{2}\bar{x}\cdot O(p)\cdot k} : \right] \zeta^{-n} G(p^2, n) \zeta'^{-n'} G(p^2, n'), \tag{15}$$

where $\mathrm{Tr}_{\mathrm{phys}}$ means the 'trace' in the physical subspace. The trace can be calculated by using the coherent state $|z\rangle = e^{-z\cdot a^\dagger}|0\rangle$ as follows

$$\mathrm{Tr}_{\mathrm{phys}}[\cdot] = \int \left(\prod_{\mu=0}^{3} \frac{d^2 z^\mu}{\pi} \right) e^{\bar{z}^*\cdot z} \langle \bar{z}| \cdots |z\rangle|_{z_\parallel = 0} \\ = \frac{1}{(1-\zeta\zeta')^4} \exp\left[-\frac{m_0^2}{2p^2} \frac{2\zeta\zeta' - (\zeta+\zeta')}{1-\zeta\zeta'} \right] \simeq \frac{1}{(1-\zeta\zeta')^4}, \tag{16}$$

where $\bar{z} = (-z^0, z^1, z^2, z^3)$ and $z_\parallel = \frac{P(P\cdot z)}{P^2}$. The last form of equation (16) seems to be singular at $\zeta = \zeta'^{-1}$ ($\zeta' = \zeta^{-1}$), which is, however, located in outside of the counter C. Further, since $G(p^2, n)$ has no poles of imaginary p^0 [11], we can evaluate the integral with respect to p in equation (15) by means of analytic continuation $p = (i\bar{p}^0, \bar{p}^i)$.

Then, approximating the lower bound of the integration with respect to $\bar{p}$ to 0, we can carried out the $\bar{p}$ integration in equation (15) explicitly; and, we obtain

$$\delta m^2(|p| \gg |k|) \sim i \left(\frac{g\pi\alpha' \sinh(\frac{1}{2}\log q)}{2\alpha_0' \log q} e^{-(\beta_0 + \frac{1}{2})\log q} \right)^2 .$$

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