On the Essential Spectrum of Many-Particle Pseudorelativistic Hamiltonians with Permutational Symmetry Account

Grigorii ZHISLIN

Radiophysical Research Institute, 25/14 Bol'shaya Pechorskaya Str., Nizhny Novgorod, 603950 Russia
E-mail: *greg@nirfi.sci-nnov.ru*

Received October 27, 2005, in final form February 07, 2006; Published online February 20, 2006
Original article is available at http://www.emis.de/journals/SIGMA/2006/Paper024/

Abstract. In this paper we formulate our results on the essential spectrum of many-particle pseudorelativistic Hamiltonians without magnetic and external potential fields in the spaces of functions, having arbitrary type α of the permutational symmetry. We discover location of the essential spectrum for all α and for some cases we establish new properties of the lower bound of this spectrum, which are useful for study of the discrete spectrum.

Key words: pseudorelativistic Hamiltonian; many-particle system; permutational symmetry; essential spectrum

2000 Mathematics Subject Classification: 35P20; 35Q75; 46N50; 47N50; 70H05; 81Q10

In this paper we formulate our results on the essential spectrum of many-particle pseudorelativistic Hamiltonians without magnetic and external potential fields in spaces of functions, having arbitrary type α of the permutational symmetry. We discover the location of the essential spectrum for all α (Theorem 1) and for some cases we establish new properties of the lower bound of this spectrum, which are useful for study of the discrete spectrum (Lemma 1).

Before this work similar results on the essential spectrum were obtained in [1, 2], but in [2] not arbitrary α were considered, and the construction of the operator of the relative motion was not invariant with respect to the permutations of identical particles in contrast to our approach (in this respect connection of our results with [2] is the same, as connection [5] with [7]); in [1] more extensive class of pseudorelativistic Hamiltonians was studied as compared to [2] and to this paper, but in [1] the permutational symmetry was not considered. Moreover, our Lemma 1 is new.

1. Let $Z_1 = \{0, 1, \ldots, n\}$ be the quantum system of $(n+1)$ particles, m_i, $r_i = (x_i, y_i, z_i)$ and p_i be the mass, the radius-vector and the momentum of i-th particle. Pseudorelativistic (PR) energy operator of Z_1 can be written in the form

$$\mathcal{H}' = K'(r) + V(r),$$

where $r = (r_0, r_1, \ldots, r_n)$,

$$K'(r) = \sum_{j=0}^{n} \sqrt{-\Delta_j + m_j^2}^{\,1}, \qquad V(r) = V_0(r) = \frac{1}{2} \sum_{i,j=0,\, i\neq j}^{n} V_{ij}(|r_{ij}|),$$

$\Delta_j = \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial z_j^2}$, $V_{ij}(|r_{ij}|) = V_{ji}(|r_{ji}|)$ be the real potential of the interaction i-th and j-th particles, $r_{ij} = r_i - r_j$, $V_{ij}(|r_1|) \in \mathcal{L}_{2,\mathrm{loc}}(\mathbb{R}^3)$, $V_{ij}(|r_1|) \to 0$ at $|r_1| \to \infty$, and $V_{ij}(|r_{ij}|)$ are such

[1] We have chosen the unit system so the Plank constant and the light velocity are equal to 1.

that for some $\varepsilon_0 > 0$ operator $\mathcal{H}'$ is semibounded from below for $V(r) = (1+\varepsilon_0)V_0(r)$. If the system Z_1 is a molecule, the last condition means that we may consider only the molecules consisting of atoms of such elements whose number in Mendeleev periodic table is smaller than 85 [2, 3].

The operator $\mathcal{H}'$ is not local: in the coordinate space operators $\sqrt{-\Delta_j + m_j^2}$ are integral operators, in the momentum representation multiplicators $V_{ij}(|r_{ij}|)$ turn into integral operators. But in the momentum space the operators $\sqrt{-\Delta_j + m_j^2}$ are multiplication operators. Actually, let $p_j = (p_{j1}, p_{j2}, p_{j3})$, $p = (p_0, \dots, p_n)$, $\varphi(r) \in \mathcal{L}_2(\mathbb{R}^{3n+3})$, and $\overline{\varphi}(p)$ be Fourier-transform of $\varphi(r)$:

$$\overline{\varphi}(p) = \frac{1}{(\sqrt{2\pi}\,)^{3n+3}} \int_{\mathbb{R}^{3n+3}} \varphi(r)\, e^{i(p,r)}\, dr,$$

then

$$\sqrt{-\Delta_j + m_j^2}\,\varphi(r) = \sqrt{p_j^2 + m_j^2}\,\overline{\varphi}(p).$$

Let

$$T_j'(p_j) = \sqrt{p_j^2 + m_j^2}, \qquad T'(p) = \sum_{j=0}^{n} T_j'(p_j).$$

Now we can rewrite operators $\mathcal{H}'$ using mixed form writing:

$$\mathcal{H}' = T'(p) + V(r),$$

where operators $T'(p)$ and $V(r)$ act in the momentum and in the coordinate spaces respectively.

2. The operator $\mathcal{H}'$ corresponds to the energy of the whole system motion. But for applications it is interesting to know the spectrum of the operator corresponding to the *relative* motion energy. To get such operator for nonrelativistic (NR) case one separate the center-of-mass motion, but for pseudorelativistic (PR) case it is impossible. To construct the operator of the relative motion from PR operator $\mathcal{H}'$, we reduce the operator $\mathcal{H}'$ to any fixed eigenspace of operator of the total momentum [2]. Let $\xi_0 = (\xi_{01}, \xi_{02}, \xi_{03})$ be the center-of-mass radius-vector:

$$\xi_0 = \sum_{j=0}^{n} m_j r_j / M, \qquad M = \sum_{j=0}^{n} m_j,$$

$q_j = r_j - \xi_0$ be the relative coordinates of j-th particle, $j = 0, 1, \dots, n$, $q = (q_0, \dots, q_n)$. We take q, ξ_0 as the new coordinates of the particles from Z_1. Let us note that vectors $q_0, \dots, q_n$ are dependent: they belong to the space

$$R_0 = \left\{ q' \mid q' = (q_0', \dots, q_n'),\ \ \sum_{j=0}^{n} m_j q_j' = \theta = (0,0,0) \right\}$$

of relative motion. On the other hand, if $q' = (q_0', \dots, q_n') \in R_0$ and ξ_0' is an arbitrary fixed vector from $\mathbb{R}^3$, we may consider q_j' and ξ_0' as the relative coordinates of the point $r_j' = q_j' + \xi_0'$, $j = 0, 1, \dots, n$ and the center-of-mass position of Z_1 respectively. It is easy to see that Fourier-conjugate coordinates to q_j are the same p_j as for r_j, and Fourier-conjugate coordinate for ξ_0 is $\mathcal{P}_0 = (\mathcal{P}_{01}, \mathcal{P}_{02}, \mathcal{P}_{03}) = \sum\limits_{j=0}^{n} p_j$.

Let us consider the operators

$$L_{0s} = \frac{1}{i}\,\frac{d}{d\xi_{0s}}, \qquad s = 1, 2, 3.$$

In the momentum space these operators are multiplication operators

$$\overline{L}_{0s} = \mathcal{P}_{0s}.$$

It follows from above that the operators $L_{0s}\{\overline{L}_{0s}\}$ commute with $\mathcal{H}'$. So any eigenspaces of the operators L_{0s} are invariant for $\mathcal{H}'$. Let $-Q_{0s}$ be a real eigenvalue of the operator L_{0s}, W_{0s} be corresponding eigenspace and

$$W_0 = W_{01} \cap W_{02} \cap W_{03}.$$

The space W_0 is invariant for $\mathcal{H}'$. Evidently

$$W_0 = \left\{ (2\pi)^{-3/2}\, e^{-i(Q_0,\xi_0)}\, \varphi(q) \right\}^2, \qquad \overline{W}_0 = \left\{ \overline{\varphi}(p) \prod_{s=1}^{3} \delta(\mathcal{P}_{0s} - Q_{0s}) \right\},$$

where $Q_0 = (Q_{01}, Q_{02}, Q_{03})$, $\varphi(q)$ is an arbitrary function, $\varphi(q) \in \mathcal{L}_2(R_0)$, and $\overline{W}_0$ is Fourier-image of W_0.

Let us rewrite operator $\mathcal{H}'$ using the coordinates q, ξ_0 $\{p, \mathcal{P}\}$ and reduce it to the subspace $W_0\{\overline{W}_0\}$. Then we obtain the operator $\mathcal{H}'$ in the form

$$H_0' = T'(p, Q_0) + V(q),$$

where

$$T'(p, Q_0) = T'(p),$$

but with the condition

$$\sum_{j=0}^{n} p_j = Q_0; \tag{1}$$

$$V(q) = \frac{1}{2} \sum_{i,j=0,\, i\neq j}^{n} V_{ij}(|q_i - q_j|), \qquad q_i - q_j = r_i - r_j.$$

We see that H_0' depends on the relative coordinates q, their momenta p and the total momentum value Q_0. So if we fix Q_0 we obtain the operator, which can be considered as the operator of the relative motion. We shall study this operator in the space $\mathcal{L}_2(R_0)$ with condition (1) for momenta.

For technical reasons it is convenient to take

$$T_j(p_j) = T_j'(p_j) - m_j$$

instead of $T_j'(p_j)$ and

$$T(p, Q_0) = \sum_{j=0}^{n} T_j(p_j)$$

instead of $T'(p;\, Q_0)$. So the subject of our study is operator

$$H_0 = T(p;\, Q_0) + V(q) \tag{2}$$

[2] The coefficient $(2\pi)^{-3/2}$ in front of the $e^{-i(Q_0,\xi_0)}$ plays the role of "normalizing factor": Fourier-image of $(2\pi)^{-3/2}\, e^{-i(Q_0,\xi_0)}$ is $\prod_{s=1}^{s} \delta(\mathcal{P}_{0s} - Q_{0s})$ without any factor.

(with condition (1)). The operator H_0 is bounded from below on $C_0^\infty(R_0)$. We extend it to a self-adjoint one using Friedrichs extension, and save the notation H_0 for the obtained operator.

Let us note that instead of the dependent coordinates $q_0, \ldots, q_n$ we could introduce independent relative coordinates (and their momenta) similar to [2], but such approach generates difficulties, when one takes into account the permutational symmetry (see § **5**), and we do not use this approach.

3. We shall study spectrum of the operator H_0 not in the whole space $\mathcal{L}_2(R_0)$, but in the subspaces of functions from $\mathcal{L}_2(R_0)$, having the fixed types of permutational symmetry. We do this

i) to satisfy the Pauli exclusion principle,

ii) to obtain additional information about the structure of the spectrum H_0.

We denote by S and α the group of the permutations of all identical particles of Z_1 and an arbitrary type of irreducible representation of S respectively. Let us determine the operators T_g, $g \in S$ by relations

$$T_g\,\varphi(q) = \varphi(g^{-1}q), \qquad g \in S$$

and put

$$P^{(\alpha)} = \frac{l_\alpha}{|S|}\sum_{g\in S}\overline{\chi}_g^{(\alpha)}\,T_g, \qquad B^{(\alpha)} = P^{(\alpha)}\,\mathcal{L}_2(R_0),$$

where $\chi_g^{(\alpha)}$ is the character of the element $g \in S$ in the irreducible representation of the type α, l_α is the dimension of this representation, $|S|$ is the number of elements of S. The operator $P^{(\alpha)}$ is the projector in $\mathcal{L}_2(R_0)$ on the subspace $B^{(\alpha)} = B^{(\alpha)}(R_0)$ of functions, which are transformed by the operators T_g, $g \in S$, according to the representation of the type α [6]. Evidently $P^{(\alpha)}H_0 = H_0P^{(\alpha)}$. Let $H_0^{(\alpha)} = H_0P^{(\alpha)}$. $H_0^{(\alpha)}$ be the restriction of the operator H_0 to the subspace $B^{(\alpha)}$ of functions, having the permutational symmetry of the type α.

In this paper we discover location of the essential spectrum $s_{\mathrm{ess}}\big(H_0^{(\alpha)}\big)$ of the operator $H_0^{(\alpha)}$.

4. Let $Z_2 = (D_1, D_2)$ be an arbitrary decomposition of the initial system Z_1 into 2 non-empty clusters D_1 and D_2 without common elements:

$$D_1 \cup D_2 = Z_1, \qquad D_1 \cap D_2 = \varnothing$$

and

$$H(Z_2) = T(p, Q_0) + V(q; Z_2), \tag{3}$$

where

$$V(q; Z_2) = \frac{1}{2}\sum_{s=1}^{2}\;\sum_{i,j\in D_s,\, i\neq j} V_{ij}(|q_j - q_i|).$$

$H(Z_2)$ is the PR energy operator of compound system Z_2, consisting of non interacting (one with other) clusters D_1, D_2 with the same condition (1) for the total momentum as for Z_1:

$$\sum_{i=0}^{n} p_i = Q_0.$$

Let $S[D_s]$ be the group of the permutations of all identical particles from D_s, $s = 1, 2$, $\hat{g}$ be the permutation $D_1 \leftrightarrow D_2$ if these clusters are identical ($D_1 \sim D_2$). We put

$$S_0(Z_2) = S[D_1] \times S[D_2],$$
$$S(Z_2) = S_0(Z_2) \quad \text{if} \quad D_1 \nsim D_2,$$
$$S(Z_2) = \hat{S}(Z_2) = S_0(Z_2) \cup S_0(Z_2)\hat{g} \quad \text{if} \quad D_1 \sim D_2.$$

$S(Z_2)$ is the group of the permutational symmetry of the compound system Z_2. It is clear that $S_0(Z_2) \subseteq S(Z_2) \subseteq S$.

Let $F(\alpha; Z_2) = \{\alpha'\}\{F_0(\alpha; Z_2) = \{\check{\alpha}\}\}$ be the set of all types $\alpha'\{\check{\alpha}\}$ of the group $S(Z_2)\{S_0(Z_2)\}$ irreducible representations, which are contained in the group S irreducible representation $D_g^{(\alpha)}$ of the type α after reducing $D_g^{(\alpha)}$ from S to $S(Z_2)\{S_0(Z_2)\}$. For $\forall\ \alpha'\{\check{\alpha}\}$ we determine the projector $P^{(\alpha')}(Z_2)\{P^{(\check{\alpha})}(Z_2)\}$ on the subspace of functions $\varphi(q)$, which are transformed by operators T_g

$$T_g\, \varphi(q) = \varphi(g^{-1}q), \qquad g \in S(Z_2), \qquad \{g \in S_0(Z_2)\}$$

according to the group $S(Z_2)\{S_0(Z_2)\}$ irreducible representation of the type $\alpha'\{\check{\alpha}\}$.

Let $\gamma = \alpha'$ or $\gamma = \check{\alpha}$; obviously if $P^{(\gamma)}(Z_2)\,\varphi(q) = \varphi(q)$, then $P^{(\gamma)}(Z_2)\,\overline{\varphi}(p) = \overline{\varphi}(p)$. We set

$$P(\alpha; Z_2) = \sum_{\alpha' \in F(\alpha; Z_2)} P^{(\alpha')}(Z_2), \qquad \check{P}(\alpha; Z_2) = \sum_{\check{\alpha} \in F_0(\alpha; Z_2)} P^{(\check{\alpha})}(Z_2),$$
$$H(\alpha;\, Z_2) = H(Z_2) P(\alpha; Z_2), \qquad \check{H}(\alpha; Z_2) = H(Z_2)\, \check{P}(\alpha; Z_2).$$

The operator $H(\alpha; Z_2)\{\check{H}(\alpha; Z_2)\}$ is the restriction of the operator $H(Z_2)$ (see (3)) to the subspace $B(\alpha; Z_2) = P(\alpha; Z_2)\, \mathcal{L}_2(R_0)$ $\{\check{B}(\alpha; Z_2) = \check{P}(\alpha; Z_2)\, \mathcal{L}_2(R_0)\}$. Let

$$\mu^{(\alpha)} = \min_{Z_2} \inf H(\alpha; Z_2).$$

It is possible to prove that

$$\mu^{(\alpha)} = \min_{Z_2} \inf \check{H}(\alpha; Z_2). \tag{4}$$

We denote by $A(\alpha)$ the set of all Z_2, for which

$$\inf \check{H}(\alpha; Z_2) = \min_{Z_2'} \inf \check{H}(\alpha; Z_2');$$

then

$$\mu^{(\alpha)} = \inf \check{H}(\alpha; Z_2), \qquad Z_2 \in A(\alpha). \tag{5}$$

5. Our main result is the following theorem

Theorem 1. *Essential spectrum $s_{\mathrm{ess}}\big(H_0^{(\alpha)}\big)$ of the operator $H_0^{(\alpha)}$ consists of all points half-line $[\mu^{(\alpha)}, +\infty)$.*

Let us compare Theorem 1 with the corresponding results in [2].

First, in [2] a similar result was proved only for one of simplest types α of the permutational symmetry (for α corresponding to one-column Young scheme), while here we assume arbitrary α.

Second, we use more natural, simple and transparent approach for taking symmetry into account, compared to [2]. Actually, we apply relative coordinates q_i with respect to center-of-mass position ξ_0: $q_i = r_i - \xi_0$, $i = 0, 1, \ldots, n$ and so the transposition g_j: $r_j \leftrightarrow r_0$ of j-th and

0-th particles results in the transposition of q_j and q_0 only, but just as all other coordinates q_i, $i \neq j$, $i \neq 0$, are without any change. In [2] relative coordinates $\tilde{q}_i$ are taken with respect to the position of 0-th particle: $\tilde{q}_i = r_i - r_0$, $i = 1, 2, \ldots, n$ and this choice implies changing of all $\tilde{q}_i$ under transposition g_j. Namely, $T_{g_j}\,\psi(\tilde{q}) = \psi(g_j^{-1}\,\tilde{q}) = \psi(\hat{q})$, where $\tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_n)$, $\hat{q} = (\hat{q}_1, \ldots \hat{q}_n)$, $\hat{q}_i = \tilde{q}_i - \tilde{q}_j$, $i \neq j$, $\hat{q}_j = -\tilde{q}_j$. Such situation is not realized only if the system Z_1 contains a particle, which is not identical to any other particle from Z_1 (and if we index this particle by number 0), but there is no such exceptional particle in the most number of molecules. Completing the second remark, we can note, roughly speaking, that our approach for taking permutational symmetry into account follows [5], while authors [2] follow [7].

6. We do not write here the proof of the Theorem 1, since the significant part of this proof will be needed for the study of the discrete spectrum $s_d\big(H_0^{(\alpha)}\big)$ of the operator $H_0^{(\alpha)}$ (this study is not finished), so we shall publish the full proof of the Theorem 1 later (together with the results on the discrete spectrum). But here we shall do some preparations for our next paper. Namely, we shall obtain from (4), (5) the other formula for $\mu^{(\alpha)}$, which is more convenient for the investigation of the structure $s_d\big(H_0^{(\alpha)}\big)$. To do it first of all we transform the expression of the operator $H(Z_2)$ for fixed $Z_2 = (D_1, D_2)$. We introduce clusters D_s center-of-mass coordinates

$$\xi_s = (\xi_{s1}, \xi_{s2}, \xi_{s3}) = \sum_{j\in D_s} r_j m_j / M_s, \qquad M_s = \sum_{j\in D_s} m_j,$$

the relative coordinates $q_j(Z_2) = r_j - \xi_s$, $j \in D_s$, of the particles from D_s with respect to center-of-mass position and the vector $\eta = \xi_2 - \xi_1$. Evidently, $q_j(Z_2) = q_j + \xi_0 - \xi_s$, where $\xi_0 - \xi_1 = M_2\eta/M$, $\xi_0 - \xi_2 = -M_1\eta/M$.

The coordinates $q(Z_2) = \big(q_0(Z_2), \ldots, q_n(Z_2)\big)$ are not independent, since $\sum\limits_{j\in D_s} m_j\, q_j(Z_2) = \theta$, $s = 1, 2$. It is easy to see that Fourier-conjugate coordinates to $q_j(Z_2)$ are the same p_j that were introduced before. Let $\mathcal{P}_s = \sum\limits_{j\in D_s} p_j$. Then Fourier-conjugate coordinates to η are

$$\mathcal{P}_\eta = (\mathcal{P}_{\eta 1}, \mathcal{P}_{\eta 2}, \mathcal{P}_{\eta 3}) = (\mathcal{P}_2 M_1 - \mathcal{P}_1 M_2)/M \tag{6a}$$

where by (1)

$$\mathcal{P}_1 + \mathcal{P}_2 = Q_0. \tag{6b}$$

We consider $q(Z_2)$ and η as new coordinates of particles from Z_1 and denote the operator $H(Z_2)$ in new coordinates by $H_0(Z_2)$. According to consideration above and since $q_i - q_j = q_i(Z_2) - q_j(Z_2)$, $i, j \in D_s$, $s = 1, 2$, we have

$$H(Z_2) = H_0(Z_2) = T(p, Q_0, \mathcal{P}_\eta) + V\big(q(Z_2); Z_2\big) \tag{7}$$

where the operator (7) has the same form as the operator (3), but the conditions (6) have to be satisfied.

Let us introduce spaces

$$R_0(Z_2) = \left\{ q(Z_2) \mid q(Z_2) = \big(q_0(Z_2), \ldots, q_n(Z_2)\big), \quad \sum_{j\in D_s} m_j\, q_j(Z_2) = \theta, \quad s = 1, 2 \right\},$$

$$R_\eta = \{\eta \mid \eta = (\eta_1, \eta_2, \eta_3)\}, \qquad R_{0,\eta}(Z_2) = R_0(Z_2) \oplus R_\eta,$$

$$\mathcal{L}_2\big(R_{0,\eta}(Z_2)\big) = \left\{ \varphi\big(q(Z_2), \eta\big) \;\Big|\; \int_{R_{0,\eta}} |\varphi|^2 dq(Z_2)\, d\eta < +\infty \right\}.$$

In the space $\mathcal{L}_2\big(R_{0,\eta}(Z_2)\big)$ we determine operators $P_0^{(\breve{\alpha})}(Z_2)$ similarly to operators $P^{(\breve{\alpha})}(Z_2)$, but now the operators T_g, $g \in S_0(Z_2)$, are defined on functions $\varphi\big(q(Z_2), \eta\big)$ and $\overline{\varphi}(p, \mathcal{P}_\eta)$ by relations

$$T_g\, \varphi\big(q(Z_2), \eta\big) := \varphi\big(g^{-1}\, q(Z_2), \eta\big), \qquad T_g\, \overline{\varphi}(p, \mathcal{P}_\eta) = \overline{\varphi}(g^{-1} p, \mathcal{P}_\eta).$$

Here we took into account that $g^{-1}\eta = \eta$ and $g^{-1}\mathcal{P}_\eta = \mathcal{P}_\eta$ for $\forall\, \eta, \mathcal{P}_\eta$, $\forall\, g \in S_0(Z_2)$.

Let us

$$\check{P}_0(\alpha; Z_2) = \sum_{\breve{\alpha} \in F_0(\alpha; Z_2)} P_0^{(\breve{\alpha})}(Z_2), \qquad \check{H}_0(\alpha;\, Z_2) = H_0(Z_2)\, \check{P}_0(\alpha; Z_2).$$

According to (5),

$$\mu^{(\alpha)} = \inf \check{H}_0(\alpha; Z_2), \qquad Z_2 \in A(\alpha),$$

where the operator $\check{H}_0(\alpha; Z_2)$ is considered in the space $\mathcal{L}_2(R_{0,\eta})$. Since the operator $T(p, Q_0, \mathcal{P}_\eta)$ is a multiplication operator and the potential $V\big(q(Z_2); Z_2\big)$ does not depend on η, we may consider the operator $\check{H}_0(\alpha; Z_2) \equiv \check{H}_0(\alpha; Z_2; \mathcal{P}_\eta)$ in the space $\mathcal{L}_2\big(R_0(Z_2)\big)$ at the arbitrary fixed $\mathcal{P}_\eta = Q$. Then

$$\mu^{(\alpha)} = \inf_Q \inf \check{H}_0(\alpha; Z_2; Q), \qquad Z_2 \in A(\alpha). \tag{8}$$

Operator $\check{H}(\alpha; Z_2; Q)$ depends on Q continuously and

$$\lim_{|Q| \to +\infty} \inf \check{H}_0(\alpha; Z_2; Q) = +\infty,$$

since if $|Q| \to +\infty$, then at least for one j it holds $|p_j| \to \infty$ and consequently $T(p, Q_0, Q) \to +\infty$. So there exists a compact set $\Gamma(\alpha; Z_2)$ of such vectors $Q \in \mathbb{R}^3$ that

$$\mu^{(\alpha)} = \inf \check{H}_0(\alpha; Z_2; Q), \qquad Q \in \Gamma(\alpha; Z_2), \qquad Z_2 \in A(\alpha).$$

7. Unfortunately, in the general case we know nothing about finiteness or infiniteness of the number of the set $\Gamma(\alpha; Z_2)$ elements. But we can prove the following assertion

Lemma 1. *Let for some open region $W \subset \mathbb{R}^3$, $\Gamma(\alpha; Z_2) \subset W$,*

i) *$\lambda(\alpha; Z_2; Q) := \inf \check{H}_0(\alpha; Z_2; Q)$ is the point of the discrete spectrum of the operator $\check{H}_0(\alpha; Z_2; Q)$ for $Q \in W$,*

ii) *there is such $\breve{\alpha}_0$, which does not depend on Q, that the representation $g \to T_g$, $g \in S_0(Z_2)$ in the eigenspace $U(\alpha; Z_2; Q)$ of the operator $\check{H}_0(\alpha; Z_2; Q)$, corresponding to its eigenvalue $\lambda(\alpha; Z_2; Q)$, has ONE irreducible component of the type $\breve{\alpha}_0$ for each $Q \in W$.*

Then the set $\Gamma(\alpha; Z_2)$ is finite.

Proof. Let $\check{B}_0(\alpha; Z_2) = \check{P}_0(\alpha; Z_2)\mathcal{L}_2\big(R_0(Z_2)\big)$. Since

$$\check{P}_0(\alpha; Z_2) = P_0^{(\breve{\alpha}_0)}(Z_2) + \sum_{\breve{\alpha} \in F_0(\alpha; Z_2), \breve{\alpha} \neq \breve{\alpha}_0} P_0^{(\breve{\alpha})}(Z_2),$$

then

$$B_0^{(\breve{\alpha}_0)}(Z_2) := P_0^{(\breve{\alpha}_0)}(Z_2)\, \check{B}_0(\alpha; Z_2) = P_0^{(\breve{\alpha}_0)}(Z_2)\, \mathcal{L}_2\big(R_0(Z_2)\big).$$

It follows from the conditions i), ii) that in the space

$$U^{(\check{\alpha}_0)} = U(\alpha; Z_2; Q) \cap B_0^{(\check{\alpha}_0)}(Z_2) \equiv P_0^{(\check{\alpha}_0)}\, U(\alpha; Z_2; Q)$$

the representation $g \to T_g$, $g \in S_0(Z_2)$ is irreducible and has the type $\check{\alpha}_0$.

Let $P_{01}^{(\check{\alpha}_0)}$ be the projector in $B_0^{(\check{\alpha}_0)}(Z_2)$ on the space $B_{01}^{(\check{\alpha}_0)}(Z_2)$ of functions, which belong to the first line of the group $S_0(Z_2)$ irreducible representation of the type $\check{\alpha}_0$.

Then the space $B_{01}^{(\check{\alpha}_0)}(Z_2)$ is invariant under the operator $H_0(Z_2)$ and in this space the minimal eigenvalue $\lambda(\alpha; Z_2; Q)$ of the operator $H_0(Z_2)$ is nondegenerated, since the corresponding eigenspace $P_{01}^{(\check{\alpha}_0)}\, U^{(\check{\alpha}_0)}$ is one-dimensional. In other words, the minimal eigenvalue of the operator $P_{01}^{(\check{\alpha}_0)}\, H_0(\alpha; Z_2; Q)$ is nondegenerated at $\forall\, Q \in W$. But if $\lambda(\alpha; Z_2; Q)$ is nondegenerated, then $\lambda(\alpha; Z_2; Q)$ is analytical function of Q, since the operator $H_0(Z_2)$ is analytical function on Q [4]. That is why there is only finite number of such vectors Q, for which

$$\mu^{(\alpha)} = \lambda(\alpha; Z_2; Q). \qquad \blacksquare$$

8. Discussion. Theorem 1 and Lemma 1 describe the location of essential spectrum $s_{\mathrm{ess}}\big(H_0^{(\alpha)}\big)$ of the operator $H_0^{(\alpha)}$ and some properties of its lower bound respectively. Now let us consider a role of these results for the discrete spectrum study. It follows from Theorem 1 that to prove the existence of nonempty discrete spectrum $s_d\big(H_0^{(\alpha)}\big)$ of the operator $H_0^{(\alpha)}$ it is sufficient to construct such trial function ψ, $P^{(\alpha)}\psi = \psi$ that

$$\Big(H_0^{(\alpha)}\psi, \psi\Big) < \mu^{(\alpha)}(\psi, \psi), \tag{9}$$

where the number $\mu^{(\alpha)}$ is determined by the relations (5) and (8). Construction of a function ψ for (9) is important component of geometrical methods application in the study of the spectrum $s_d\big(H_0^{(\alpha)}\big)$ of operator $H_0^{(\alpha)}$.

But Theorem 1 is not a sufficient base to study the spectral asymptotics of the discrete spectrum $s_d\big(H_0^{(\alpha)}\big)$ near $\mu^{(\alpha)}$, when this spectrum is infinite. To understand the reason for that, let us consider the case when $\mu^{(\alpha)}$ is the point of the spectrum $s_d\big(H(\alpha; Z_2; Q)\big)$ for $Z_2 \in A(\alpha)$, $Q \in \Gamma(\alpha; Z_2)$ (such situation is expected for PR atoms). Then the infinite series of the eigenvalues $\lambda_k(Q)$, $k = 1, 2, \ldots$, from $s_d\big(H_0^{(\alpha)}\big)$ may exist for $\forall\ Q \in \Gamma(\alpha; Z_2)$. In this case it is possible to show that corresponding eigenfunctions ψ_k describe (when $k \to \infty$) such decomposition $Z_2 = \{C_1, C_2\}$ of the initial system Z_1, for which

$$\mathcal{P}_1 + \mathcal{P}_2 = Q_0, \qquad M_1\mathcal{P}_2 - M_2\mathcal{P}_1 = MQ$$

(see (6)). Consequently, if the set $\Gamma(\alpha; Z_2)$ is infinite, then the spectrum $s_d\big(H_0^{(\alpha)}\big)$ may consist of infinite number of the infinite series eigenvalues $\lambda_k(Q)$, $k = 1, 2, \ldots$, where all series are determined by the values Q from $\Gamma(\alpha; Z_2)$. For such situation there are no approaches to get the spectral asymptotics of $s_d\big(H_0^{(\alpha)}\big)$. Thus, it was very desirable to establish the conditions of impossibility of this situation that is the conditions of finiteness of the set $\Gamma(\alpha; Z_2)$. Namely, such conditions are given in Lemma 1 of the paper.

Acknowledgements

This investigation is supported by RFBR grant 05-01-00299.

[1] Damak M., On the spectral theory of dispersive N-body Hamiltonians, *J. Math. Phys.*, 1999, V.40, 35–48.

[2] Lewis R.T., Siedentop H., Vugalter S., The essential spectrum of relativistic multi-particle operators, *Ann. Inst. H. Poincaré Phys. Théor.*, 1997, V.67, 1–28.

[3] Lieb E., Yau H.-T., The stability and instability of relativistic matter, *Comm. Math. Phys.*, 1988, V.118, 177–213.

[4] Reed M., Simon B., Methods of modern mathematical physics. IV Analysis of operators, New York – San Francisco – London, Academic Press, 1978.

[5] Sigalov A.G., Sigal I.M., Invariant description, with respect to transpositions of identical particles, of the energy operator spectrum of quantum-mechanical systems, *Teoret. Mat. Fiz.*, 1970, V.5, 73–93 (in Russian).

[6] Wigner E.P., Group theory and its application to quantum mechanics, New York, 1959.

[7] Zhislin G., Spectrum of differential operators of quantum mechanical many-particle system in the spaces of functions of the given symmetry, *Izvest. Akad. Nauk SSSR, Ser. Mat.*, 1969, V.33, 590–649 (English transl.: *Math. USSR-Izvestia*, 1969, V.3, 559–616).