Symmetry, Integrability and Geometry: Methods and Applications Vol. 2 (2006), Paper 021, 10 pages

On the Degenerate Multiplicity of the sl_2 Loop Algebra for the 6V Transfer Matrix at Roots of Unity

Tetsuo DEGUCHI

Department of Physics, Faculty of Science, Ochanomizu University,
2-1-1 Ohtsuka, Bunkyo-Ku, Tokyo 112-8610, Japan
E-mail: *deguchi@phys.ocha.ac.jp*

Received October 31, 2005, in final form February 06, 2006; Published online February 17, 2006
Original article is available at http://www.emis.de/journals/SIGMA/2006/Paper021/

Abstract. We review the main result of cond-mat/0503564. The Hamiltonian of the XXZ spin chain and the transfer matrix of the six-vertex model has the sl_2 loop algebra symmetry if the q parameter is given by a root of unity, $q_0^{2N} = 1$, for an integer N. We discuss the dimensions of the degenerate eigenspace generated by a regular Bethe state in some sectors, rigorously as follows: We show that every regular Bethe ansatz eigenvector in the sectors is a highest weight vector and derive the highest weight $\bar{d}_k^{\pm}$, which leads to evaluation parameters a_j. If the evaluation parameters are distinct, we obtain the dimensions of the highest weight representation generated by the regular Bethe state.

Key words: loop algebra; the six-vertex model; roots of unity representations of quantum groups; Drinfeld polynomial

2000 Mathematics Subject Classification: 81R10; 81R12; 81R50; 81V70

1 Introduction

The XXZ spin chain is one of the most important exactly solvable quantum systems. The Hamiltonian under the periodic boundary conditions is given by

$$H_{XXZ} = \frac{1}{2}\sum_{j=1}^{L}\left(\sigma_j^X\sigma_{j+1}^X + \sigma_j^Y\sigma_{j+1}^Y + \Delta\sigma_j^Z\sigma_{j+1}^Z\right). \tag{1}$$

Here the XXZ anisotropy Δ is related to the q parameter by $\Delta = (q+q^{-1})/2$. The XXZ Hamiltonian (1) is derived from the logarithmic derivative of the transfer matrix of the six-vertex model, and hence they have the same set of eigenvectors.

Recently, it was explicitly shown that when q is a root of unity the XXZ Hamiltonian commutes with the generators of the sl_2 loop algebra [13]. Let q_0 be a primitive root of unity satisfying $q_0^{2N} = 1$ for an integer N. We introduce operators $S^{\pm(N)}$ as follows

$$\begin{aligned} S^{\pm(N)} = \sum_{1\le j_1<\cdots<j_N\le L} & q_0^{\frac{N}{2}\sigma^Z}\otimes\cdots\otimes q_0^{\frac{N}{2}\sigma^Z}\otimes\sigma_{j_1}^{\pm}\otimes q_0^{\frac{(N-2)}{2}\sigma^Z}\otimes\cdots\otimes q_0^{\frac{(N-2)}{2}\sigma^Z} \\ & \otimes\sigma_{j_2}^{\pm}\otimes q_0^{\frac{(N-4)}{2}\sigma^Z}\otimes\cdots\otimes\sigma_{j_N}^{\pm}\otimes q_0^{-\frac{N}{2}\sigma^Z}\otimes\cdots\otimes q_0^{-\frac{N}{2}\sigma^Z}. \end{aligned}$$

They are derived from the Nth power of the generators $S^{\pm}$ of the quantum group $U_q(sl_2)$ or $U_q(\hat{sl}_2)$. We also define $T^{(\pm)}$ by the complex conjugates of $S^{\pm(N)}$, i.e. $T^{\pm(N)} = \left(S^{\pm(N)}\right)^*$. The operators, $S^{\pm(N)}$ and $T^{\pm(N)}$, generate the sl_2 loop algebra, $U(L(sl_2))$, in the sector

$$S^Z \equiv 0 \pmod N. \tag{2}$$

Here the value of the total spin S^Z is given by an integral multiple of N. In the sector (2), the operators $S^{\pm(N)}$ and $T^{\pm(N)}$ (anti-)commute with the transfer matrix of the six-vertex model $\tau_{6V}(v)$, and they commute with the Hamiltonian of the XXZ spin chain [13]:

$$[S^{\pm(N)}, H_{XXZ}] = [T^{\pm(N)}, H_{XXZ}] = 0.$$

One of the most important physical questions is to obtain the degenerate multiplicity of the sl_2 loop algebra. Let us denote by $|R\rangle$ a regular Bethe state with R down spins. Here we define regular Bethe states by such Bethe ansatz eigenvectors that are constructed from finite and distinct solutions of the Bethe ansatz equations, whose precise definition will be given in Section 2. For any given regular Bethe state in the sector (2), we may have the following degenerate eigenvectors of the XXZ Hamiltonian

$$S^{-(N)}|R\rangle, \quad T^{-(N)}|R\rangle, \quad \left(S^{-(N)}\right)^2|R\rangle, \quad T^{-(N)}S^{+(N)}T^{-(N)}|R\rangle, \quad \ldots.$$

However, it is nontrivial how many of them are linearly independent. The number should explain the degree of the spectral degeneracy. We thus want to know the dimensions of the degenerate eigenspace generated by the Bethe state $|R\rangle$. Here we note that some of the spectral degeneracies of the XYZ spin chain (the eight-vertex model) at roots of unity were first discussed by Baxter [2] (see also [3, 4]). In fact, there are such spectral degeneracies of the XYZ spin chain at roots of unity that are closely related to the sl_2 loop algebra symmetry of the XXZ spin chain [9, 10, 17, 18]. Quite interestingly, various aspects of the spectral degeneracies of the XXZ spin chain have been discussed by several authors from different viewpoints [1, 21, 22, 27, 29].

Recently, there has been some progress on the degenerate multiplicity as far as regular Bethe states in some sectors such as (2) are concerned [11]. For the XXZ spin chain at roots of unity, Fabricius and McCoy had made important observations on degenerate multiplicities of the sl_2 loop algebra, and conjectured the 'Drinfeld polynomials of Bethe ansatz eigenvectors' [14, 15, 16]. However, it was not clear whether the representation generated by a Bethe state is irreducible or not. Here we remark that a Drinfeld polynomial is defined for an irreducible finite-dimensional representation. Motivated by the previous results [14, 15, 16], an algorithm for determining the dimensions of the representation generated by a given regular Bethe state has been rigorously formulated in some sectors such as (2) and some restricted cases of evaluation parameters a_j, which will be defined in Section 5. We show rigorously [11] that every regular Bethe state in the sectors is highest weight, assuming a conjecture that the Bethe roots are continuous with respect to the q parameter at roots of unity. We evaluate the highest weight $\bar{d}_k^{\pm}$ for the Bethe state, which leads to evaluation parameters a_j. Here $\bar{d}_k^{\pm}$ will be defined in Section 4. Independently, it has been shown [8] that if the evaluation parameters of a highest weight representation of the sl_2 loop algebra are distinct, then it is irreducible. It thus follows that the 'Drinfeld polynomial of a regular Bethe state' corresponds to the standard Drinfeld polynomial defined for the irreducible representation generated by the regular Bethe state, if the evaluation parameters are distinct. The conjecture of Fabricius and McCoy has been proved at least in the sectors such as (2) and in the case of distinct evaluation parameters. Thus, the purpose of the paper is to review the main points of the complete formulation of [11] briefly with some illustrative examples. Some other aspects of the sl_2 loop algebra symmetry have also been discussed [5, 24, 30].

The contents of the paper is given as follows. In Section 2, we introduce Bethe ansatz equations and regular solutions of the Bethe ansatz equations. We define regular Bethe states. In Section 3, we discuss briefly the sl_2 loop algebra symmetry of the XXZ spin chain at roots of unity. Here, the conditions of roots of unity are specified precisely. In Section 4, we explain the Drinfeld realization of the sl_2 loop algebra, i.e. the classical analogues of the Drinfeld realization of the quantum affine group $U_q(\hat{sl}_2)$. In Section 5, we review the algorithm for determining the degenerate multiplicity of a regular Bethe state in the sectors. Here we note that Theorem 2 generalizes the $su(2)$ symmetry of the XXX spin chain shown by Takhtajan and Faddeev [28]. In Section 6 we discuss some examples of the Drinfeld polynomials explicitly.

2 Bethe ansatz equations and the transfer matrix

2.1 Regular solutions of the Bethe ansatz equations

Let us assume that a set of complex numbers, $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$ satisfy the Bethe ansatz equations at a root of unity:

$$\left(\frac{\sinh(\tilde{t}_j + \eta_0)}{\sinh(\tilde{t}_j - \eta_0)}\right)^L = \prod_{k=1;k\neq j}^{M} \frac{\sinh(\tilde{t}_j - \tilde{t}_k + 2\eta_0)}{\sinh(\tilde{t}_j - \tilde{t}_k - 2\eta_0)}, \qquad \text{for} \quad j = 1, 2, \ldots, R. \tag{3}$$

Here the parameter η is defined by the relation $q = \exp(2\eta)$, and η_0 is given by $q_0 = \exp(2\eta_0)$. If a given set of solutions of the Bethe ansatz equations are finite and distinct, we call them *regular*. We call a set of solutions of the Bethe ansatz equations *Bethe roots*.

A set of regular solutions of the Bethe ansatz equations leads to an eigenvector of the XXZ Hamiltonian. We call it a *regular Bethe state* of the XXZ spin chain or a *regular XXZ Bethe state*, briefly. Explicit expressions of regular Bethe states are derived through the algebraic Bethe ansatz method [23].

2.2 Transfer matrix of the six-vertex model

We now introduce L operators for the XXZ spin chain. Let V_n be two-dimensional vector spaces for $n = 0, 1, \ldots, L$. We define an operator-valued matrix $L_n(z)$ by

$$L_n(z) = \begin{pmatrix} L_n(z)^1_1 & L_n(z)^1_2 \\ L_n(z)^2_1 & L_n(z)^2_2 \end{pmatrix} = \begin{pmatrix} \sinh(z\, I_n + \eta\sigma_n^z) & \sinh 2\eta\, \sigma_n^- \\ \sinh 2\eta\, \sigma_n^+ & \sinh(z\, I_n - \eta\sigma_n^z) \end{pmatrix}.$$

Here $L_n(z)$ is a matrix acting on the auxiliary vector space V_0, where I_n and σ_n^a $(a = z, \pm)$ are operators acting on the nth vector space V_n. The symbol I denotes the two-by-two identity matrix, $\sigma^\pm$ denote $\sigma^+ = E_{12}$ and $\sigma^- = E_{21}$, and σ^x, σ^y, σ^z are the Pauli matrices.

We define the monodromy matrix T by the product:

$$T(z) = L_L(z) \cdots L_2(z) L_1(z).$$

Here the matrix elements of $T(z)$ are given by

$$T(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}.$$

We define the transfer matrix of the six vertex model $\tau_{6V}(z)$ by the following trace:

$$\tau_{6V}(z) = \operatorname{tr} T(z) = A(z) + D(z).$$

We call the transfer matrix *homogeneous*. It is invariant under lattice translation.

3 The sl_2 loop algebra symmetry at roots of unity

We shall show the sl_2 loop algebra symmetry of the XXZ spin chain at roots of unity in some sectors. We shall discuss two cases with even L and odd L.

3.1 Roots of unity conditions

Let us explicitly formulate roots of unity conditions as follows.

Definition 1 (Roots of unity conditions). We say that q_0 is a root of unity with $q_0^{2N} = 1$, if one of the three conditions hold: (i) q_0 is a primitive Nth root of unity with N odd ($q_0^N = 1$); (ii) q_0 is a primitive $2N$th root of unity with N odd ($q_0^N = -1$); (iii) q_0 is a primitive $2N$th root of unity with N even ($q_0^N = -1$). We call the cases (i) and (iii) type I, the case (ii) type II.

In the case of $S^Z \equiv 0 \pmod N$ we consider all the three conditions of roots of unity. However, in the case of $S^Z \equiv N/2 \pmod N$ with N odd, we consider only the condition (i) of roots of unity, i.e. q_0 is a primitive Nth root of unity with N odd ($q_0^N = 1$).

3.2 (Anti-)commutation relations at roots of unity

We show the sl_2 loop algebra symmetry of the XXZ spin chain in the following two sectors: (a) in the sector $S^Z \equiv 0 \pmod N$ where q_0 is a root of unity with $q_0^{2N} = 1$, as specified in Definition 1; (b) in the sector $S^Z \equiv N/2 \pmod N$ with N odd where q_0 is a primitive Nth root of unity.

Let us assume that there exists a set of regular solutions of Bethe ansatz equations (3) with R down-spins, i.e. R regular Bethe roots. We also assume that the lattice size L, the number of regular Bethe roots R and the integer N satisfy the following relation:

$$L - 2R = nN, \qquad n \in \mathbb{Z}. \tag{4}$$

If n is even, the regular Bethe state $|R\rangle$ is in the sector $S^Z \equiv 0 \pmod N$, while if n is odd and N is also odd, then it is in the sector $S^Z \equiv N/2 \pmod N$. Here we recall that the symbol $|R\rangle$ denotes the regular Bethe state constructed from the given R regular Bethe roots.

It has been shown [13] that operators $S^{\pm(N)}$ and $T^{\pm(N)}$ (anti-)commute with the transfer matrix of the six-vertex model $\tau_{6V}(v)$ in the sector $S^Z \equiv 0 \pmod N$ at q_0 with $q_0^{2N} = 1$

$$S^{\pm(N)}\, \tau_{6V}(z) = q_0^N \tau_{6V}(z)\, S^{\pm(N)}, \qquad T^{\pm(N)}\, \tau_{6V}(z) = q_0^N \tau_{6V}(z)\, T^{\pm(N)}. \tag{5}$$

Furthermore, it is also shown [11] that in the sector $S^Z \equiv N/2 \pmod N$ when N is odd and q_0 satisfies $q_0^N = 1$, operators $S^{\pm(N)}$ and $T^{\pm(N)}$ (anti-)commute with the transfer matrix of the six-vertex model $\tau_{6V}(v)$.

From the (anti-)commutation relations (5) it follows that the operators $S^{\pm(N)}$ and $T^{\pm(N)}$ commute with the XXZ Hamiltonian in the sector $S^Z \equiv 0 \pmod N$ when q_0 satisfies $q_0^{2N} = 1$, and in the sector $S^Z \equiv N/2$ when N is odd and $q_0^N = 1$. Here we recall that the XXZ Hamiltonian H_{XXZ} is given by the logarithmic derivative of the (homogeneous) transfer matrix $\tau_{6V}(v)$.

3.3 The algebra generated by $S^{\pm(N)}$ and $T^{\pm(N)}$

Let us discuss the algebra generated by the operators [13], $S^{\pm(N)}$ and $T^{\pm(N)}$. When q_0 is of type I, we have the following identification [13]:

$$\begin{aligned} & E_0^+ = T^{-(N)}, \qquad E_0^- = T^{+(N)}, \qquad E_1^+ = S^{+(N)}, \qquad E_1^- = S^{-(N)}, \\ & -H_0 = H_1 = \frac{2}{N} S^Z. \end{aligned} \tag{6}$$

When q is of type II, we have the following [9]:

$$E_0^+ = \sqrt{-1}\, T^{-(N)}, \qquad E_0^- = \sqrt{-1}\, T^{+(N)}, \qquad E_1^+ = \sqrt{-1}\, S^{+(N)},$$

$$E_1^- = \sqrt{-1}\, S^{-(N)}, \qquad -H_0 = H_1 = \frac{2}{N} S^Z. \tag{7}$$

Here $\sqrt{-1}$ denotes the square root of -1 (cf. (A.13) of [9]; see also [24]). The operators $E_j^{\pm}$ and H_j for $j = 0, 1$, are the Chevalley generators of the affine Lie algebra $\hat{sl}_2$. In fact, the operators $E_j^{\pm}$, H_j for $j = 0, 1$, satisfy the defining relations [20] of the sl_2 loop algebra [13]:

$$H_0 + H_1 = 0, \qquad [H_i, E_j^{\pm}] = \pm a_{ij} E_j^{\pm}, \qquad i, j = 0, 1, \tag{8}$$

$$[E_i^+, E_j^-] = \delta_{ij} H_j, \qquad i, j = 0, 1, \tag{9}$$

$$[E_i^{\pm}, [E_i^{\pm}, [E_i^{\pm}, E_j^{\pm}]]] = 0, \qquad i, j = 0, 1, \quad i \neq j. \tag{10}$$

Here, the Cartan matrix (a_{ij}) of $A_1^{(1)}$ is defined by

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The Serre relations (10) hold if q_0 is a primitive $2N$th root of unity, or a primitive Nth root of unity with N odd [13]. We derive it through the higher order quantum Serre relations due to Lusztig [26]. The Cartan relations (8) hold for generic q. The relation (9) holds for the identification (6) when q_0 is a root of unity of type I, and for the identification (7) when q_0 is a root if unity of type II.

In the sector $S^Z \equiv 0 \pmod N$ we have the commutation relation [13]:

$$\left[S^{+(N)}, S^{-(N)}\right] = (-1)^{N-1} q^N \, \frac{2}{N} S^Z.$$

Here the sign factor $(-1)^{N-1} q^N$ is given by 1 or -1 when q is a root of unity of type I or II, respectively. In the case of the sector $S^Z \equiv N/2 \pmod N$ with N odd and q_0 a primitive Nth root of unity, we have the following commutation relation:

$$\left[S^{+(N)}, S^{-(N)}\right] = \frac{2}{N} S^Z.$$

3.4 Some remarks on quantum groups at roots of unity

Let the symbol $U_q^{\mathrm{res}}(g)$ denote the algebra generated by the q-divided powers of the Chevalley generators of a Lie algebra g [7]. The correspondence of the algebra $U_{q_0}^{\mathrm{res}}(g)$ at a root of unity, q_0, to the Lie algebra $U(g)$ was obtained essentially through the machinery introduced by Lusztig [25, 26] both for finite-dimensional simple Lie algebras and infinite-dimensional affine Lie algebras. In fact, by using the higher order quantum Serre relations [26], it has been shown that the affine Lie algebra $U(\hat{sl}_2)$ is generated by the operators such as $S^{\pm(N)}$ at roots of unity. However, in the case of the affine Lie algebras $\hat{g}$, the highest weight conditions for the Drinfeld generators are different from those for the Chevalley generators. Through the highest weight vectors of the Drinfeld generators, finite-dimensional representations were discussed by Chari and Pressley for $U_{q_0}^{\mathrm{res}}(\hat{g})$ [7].

4 The Drinfeld realization of the sl_2 loop algebra

Finite-dimensional representations of the sl_2 loop algebra, $U(L(sl_2))$, are derived by taking the classical analogues of the Drinfeld realization of the quantum sl_2 loop algebra, $U_q(L(sl_2))$ [6, 7]. The classical analogues of the Drinfeld generators, $\bar{x}_k^{\pm}$ and $\bar{h}_k$ $(k \in \mathbb{Z})$, satisfy the defining relations in the following:

$$[\bar{h}_j, \bar{x}_k^{\pm}] = \pm 2 \bar{x}_{j+k}^{\pm}, \qquad [\bar{x}_j^+, \bar{x}_k^-] = \bar{h}_{j+k}, \qquad \text{for } j, k \in \mathbb{Z}.$$

Here $[\bar{h}_j, \bar{h}_k] = 0$ and $[\bar{x}_j^{\pm}, \bar{x}_k^{\pm}] = 0$ for $j, k \in \mathbb{Z}$.

Let us now define highest weight vectors. In a representation of $U(L(sl_2))$, a vector Ω is called *a highest weight vector* if Ω is annihilated by generators $\bar{x}_k^+$ for all integers k and such that Ω is a simultaneous eigenvector of every generator of the Cartan subalgebra, $\bar{h}_k$ $(k \in \mathbb{Z})$ [6, 7]:

$$\bar{x}_k^+ \Omega = 0, \qquad \text{for } k \in \mathbb{Z}, \tag{11}$$

$$\bar{h}_k \Omega = \bar{d}_k^+ \Omega, \qquad \bar{h}_{-k} \Omega = \bar{d}_{-k}^- \Omega, \qquad \text{for } k \in \mathbb{Z}_{\geq 0}. \tag{12}$$

We call a representation of $U(L(sl_2))$ *highest weight* if it is generated by a highest weight vector. The set of the complex numbers $\bar{d}_k^{\pm}$ given in (12) is called *the highest weight*. It is shown [6] that every finite-dimensional irreducible representation is highest weight. To a finite-dimensional irreducible representation V we associate a unique polynomial through the highest weight $\bar{d}_k^{\pm}$ [6]. We call it the Drinfeld polynomial. Here the degree r is given by the weight $\bar{d}_0^{\pm}$.

It is easy to see that the highest weight vector of a finite-dimensional irreducible representation V is a simultaneous eigenvector of operators $(\bar{x}_0^+)^k (\bar{x}_1^-)^k/(k!)^2$ for $k > 0$, and the Drinfeld polynomial of the representation V has another expression as follows [11]

$$P(u) = \sum_{k=0}^{r} \lambda_k (-u)^k, \tag{13}$$

where λ_k denote the eigenvalues of operators $(\bar{x}_0^+)^k (\bar{x}_1^-)^k/(k!)^2$. It is noted that the author learned the expression of the Drinfeld polynomial (13) from Jimbo [19] (see also [14, 16]).

5 Algorithm for evaluating the degenerate multiplicity

5.1 A useful theorem on the sl_2 loop algebra

Let Ω be a highest weight vector and V the representation generated by Ω. Here V is not necessarily irreducible. Suppose that V is finite-dimensional and $\bar{h}_0 \Omega = r\,\Omega$. We define a polynomial $P_\Omega(u)$ by the relation (13) with λ_k. We show that the roots of the polynomial $P_\Omega(u)$ are nonzero and finite, and the degree of $P_\Omega(u)$ is given by r [11]. Let us factorize $P_\Omega(u)$ as

$$P_\Omega(u) = \prod_{k=1}^{s} (1 - a_k u)^{m_k},$$

where $a_1, a_2, \ldots, a_s$ are distinct, and their multiplicities are given by $m_1, m_2, \ldots, m_s$, respectively. Then, we call a_j the *evaluation parameters* of Ω. Here we note that r is given by the sum: $r = m_1 + \cdots + m_s$. If all the multiplicities are given by 1, i.e. $m_j = 1$ for $j = 1, 2, \ldots, s$, we say that evaluation parameters $a_1, a_2, \ldots, a_r$ are distinct. In the case of distinct evaluation parameters (i.e. $m_j = 1$ for all j), we have the following theorem [8]:

Theorem 1. *Every finite-dimensional highest weight representation of the sl_2 loop algebra with distinct evaluation parameters $a_1, a_2, \ldots, a_r$ is irreducible. Furthermore, it has dimensions 2^r.*

Theorem 1 plays an important role in connecting the polynomial $P_\Omega(u)$ with an irreducible finite-dimensional representation. In fact, if V has distinct evaluation parameters, then it is irreducible, and the polynomial $P_\Omega(u)$ is equivalent to the Drinfeld polynomial of V. Moreover, we rigorously obtain the dimensions of V.

Theorem 1 has been shown in [8] by using the fact that the specialized irreducible modules for the quantum algebra are quotients of the Weyl module. Here we note that theorem 1 is also derived by constructing explicitly a basis of a highest weight representation with distinct evaluation parameters [12].

5.2 Regular Bethe states as highest weight vectors

For the XXZ spin chain at roots of unity, Fabricius and McCoy made important observations on the highest weight conjecture [14, 15, 16]. Motivated by them, we discuss the following:

Theorem 2. (i) *Every regular Bethe state $|R\rangle$ in the sector $S^Z \equiv 0 \pmod N$ at q_0 is a highest weight vector of the sl_2 loop algebra. Here q_0 is a root of unity with $q_0^{2N} = 1$, as specified in Definition* 1. (ii) *Every regular Bethe state $|R\rangle$ in the sector $S^Z \equiv N/2 \pmod N$ at q_0 is a highest weight vector of the sl_2 loop algebra. Here N is odd and q_0 is a primitive Nth root of unity.*

The Theorem 2 is proved in [11] by assuming the conjecture that for a given regular Bethe state in the sector $S^Z \equiv 0 \pmod N$ (or $S^Z \equiv N/2 \pmod N$) the set of solutions of the Bethe ansatz equations are continuous with respect to the parameter q at the root of unity q_0. In some cases such as $R = 0$ or 1, the conjecture is trivial.

By the method of the algebraic Bethe ansatz, we derive the following relations [11]:

$$\begin{aligned}
&S^{+(N)}\,|R\rangle = T^{+(N)}\,|R\rangle = 0,\\
&\left(S^{+(N)}\right)^k \left(T^{-(N)}\right)^k/(k!)^2\,|R\rangle = \mathcal{Z}_k^{+}\,|R\rangle \qquad \text{for } k \in \mathbb{Z}_{\geq 0},\\
&\left(T^{+(N)}\right)^k \left(S^{-(N)}\right)^k/(k!)^2\,|R\rangle = \mathcal{Z}_k^{-}\,|R\rangle \qquad \text{for } k \in \mathbb{Z}_{\geq 0}.
\end{aligned} \tag{14}$$

Here, the operators $S^{\pm(N)}$, $T^{+(N)}$, $T^{-(N)}$ and $2S^Z/N$ satisfy the same defining relations of the sl_2 loop algebra as generators $\bar{x}_0^{\pm}$, $\bar{x}_{-1}^{+}$, $\bar{x}_1^{-}$ and $\bar{h}_0$, respectively, and hence the relations (14) correspond to (11) and (12).

In equations (14) eigenvalues $\mathcal{Z}_k^{\pm}$ are explicitly evaluated as follows

$$\mathcal{Z}_k^{+} = (-1)^{kN}\tilde{\chi}_{kN}^{+}, \qquad \mathcal{Z}_k^{-} = (-1)^{kN}\tilde{\chi}_{kN}^{-}.$$

Here the $\tilde{\chi}_m^{\pm}$ have been defined by the coefficients of the following expansion with respect to small x:

$$\frac{\phi(x)}{\tilde{F}^{\pm}(xq_0)\tilde{F}^{\pm}(xq_0^{-1})} = \sum_{j=0}^{\infty} \tilde{\chi}_j^{\pm}\, x^j,$$

where $\phi(x) = (1-x)^L$ and $\tilde{F}^{\pm}(x) = \prod_{j=1}^{R}(1 - x\exp(\pm 2\tilde{t}_j))$.

5.3 Drinfeld polynomials of regular Bethe states and the degenerate multiplicity

Let $|R\rangle$ be a regular Bethe state at q_0 in one of the sectors specified in Theorem 2. The Drinfeld polynomial of the regular Bethe state $|R\rangle$ is explicitly derived by putting $\lambda_k = (-1)^{kN}\tilde{\chi}_{kN}^{+}$ into equation (13). Here, the coefficients $\tilde{\chi}_{kN}^{\pm}$ are explicitly evaluated as [11]

$$\tilde{\chi}_{kN}^{\pm} = \sum_{n=0}^{\min(L,kN)} (-1)^n \binom{L}{n} \sum_{n_1+\cdots+n_R = kN-n} e^{\pm\sum_{j=1}^{R} 2n_j\tilde{t}_j} \prod_{j=1}^{R} [n_j+1]_{q_0}. \tag{15}$$

Here $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ and the sum is taken over all nonnegative integers $n_1, n_2, \ldots, n_R$ satisfying $n_1 + \cdots + n_R = kN - n$: when $R = 0$, n is given by $n = kN$.

For the case of distinct evaluation parameters, we obtain the algorithm for the degeneracy of a regular Bethe state as follows.

Corollary 1. *Let $|R\rangle$ be a regular Bethe state such as specified in Theorem* 2. *If the Drinfeld polynomial of the representation V generated by $|R\rangle$ gives evaluation parameters a_j with multiplicities $m_j = 1$ for $j = 1, 2, \ldots, r$, then we have $\dim V = 2^r$, where $r = (L - 2R)/N$.*

6 Examples of Drinfeld polynomials of Bethe states

6.1 The vacuum state with even L

We now calculate the Drinfeld polynomial $P(u)$ for the the vacuum state $|0\rangle$ where $L=6$ and $N=3$ with $q_0^3=1$. When N is odd and $q_0^N=1$, we have $\lambda_k^+ = \mathcal{Z}_k^+ = (-1)^k \tilde{\chi}_{kN}^+$. From the formula (15) we have

$$\tilde{\chi}_3^+ = (-1)^3 \begin{pmatrix} 6 \\ 3 \end{pmatrix} = -20, \qquad \tilde{\chi}_6^+ = (-1)^6 \begin{pmatrix} 6 \\ 6 \end{pmatrix} = 1.$$

Thus we have $\lambda_1^+ = 20$ and $\lambda_2^+ = 1$, and the Drinfeld polynomial is given by

$$P(u) = 1 - 20u + u^2.$$

Here, the evaluation parameters are given by

$$a_1, a_2 = 10 \pm 3\sqrt{11}.$$

We note that the two evaluation parameters are distinct, and the degree of P is two, i.e. $r=2$, $m_1=m_2=1$. Therefore, the degenerate multiplicity is given by $2^2=4$.

6.2 The vacuum state with odd L

Let us calculate the Drinfeld polynomial $P(u)$ for the odd L case. We consider the the vacuum state $|0\rangle$ where $L=9$ and $N=3$ with $q_0^3=1$. The vacuum state $|0\rangle$ is in the sector $S^Z \equiv 3/2 \pmod 3$, since $S^Z = 9/2 = 3/2+3$.

From the formula (15) we have

$$\tilde{\chi}_3^+ = (-1)^3 \begin{pmatrix} 9 \\ 3 \end{pmatrix} = -84, \qquad \tilde{\chi}_6^+ = (-1)^6 \begin{pmatrix} 9 \\ 6 \end{pmatrix} = 84, \qquad \tilde{\chi}_9^+ = (-1)^9 \begin{pmatrix} 9 \\ 9 \end{pmatrix} = -1.$$

Thus we have $\lambda_1^+ = 84$, $\lambda_2^+ = 84$, and $\lambda_3^+ = 1$. The Drinfeld polynomial is given by

$$P(u) = 1 - 84u + 84u^2 - u^3.$$

Here, the evaluation parameters are given by

$$a_1, a_2 = \frac{1}{2}\left(83 \pm 9\sqrt{85}\right).$$

We note that the three evaluation parameters are distinct, and the degree of P is three, i.e. $r=3$, $m_1=m_2=m_3=1$. Therefore, the degenerate multiplicity is given by $2^3=8$.

6.3 The regular Bethe state with one down-spin ($R=1$)

For the case of $R=1$, the Bethe ansatz equations at generic q are given by

$$\left(\frac{\sinh(t_j+\eta)}{\sinh(t_j-\eta)}\right)^L = 1 \qquad \text{for } j=0,1,\ldots,L-1.$$

We solve the Bethe ansatz equations in terms of variable $\exp(2t_j)$ as follows

$$\exp(2t_j) = \frac{1-\omega_j q}{q-\omega_j} \qquad \text{for } j=0,1,\ldots,L-1, \tag{16}$$

where ω_j denotes an Lth root of unity: $\omega_j = \exp\left(2\pi\sqrt{-1}j/L\right)$, for $j=0,1,\ldots,L-1$.

Let us assume that the regular Bethe state with one down-spin is in the sector $S^Z \equiv 0 \pmod N$ and q_0 be a root of unity with $q_0^{2N} = 1$, or in the sector $S^Z \equiv N/2 \pmod N$ where q_0 is a primitive Nth root of unity with N odd. Then we have from (15)

$$\tilde{\chi}^+_{kN} = \sum_{\ell=0}^{\min(nN+2,kN)} (-1)^j [kN+1-\ell]_{q_0} \begin{pmatrix} nN+2 \\ \ell \end{pmatrix} \left(\frac{1-\omega_j q_0}{q_0 - \omega_j} \right)^{kN-\ell}.$$

Here we have from (4) that $L = nN + 2$ when $R = 1$.

Let us consider the case of $N = 3$, and $q_0 = \exp(\pm 2\pi\sqrt{-1}/3)$. Here $L = 8$. The regular Bethe state with rapidity $\tilde{t}_2$ (the case of $j = 2$ in equation (16)) has the Drinfeld polynomial in the following:

$$P(u) = 1 - 13\left(2 - \sqrt{3}\right) u + \left(7 - 4\sqrt{3}\right) u^2,$$

where the evaluation parameters a_1 and a_2 are given by

$$a_1, a_2 = \frac{1}{2}\left(13 \pm \sqrt{165}\right)\left(2 - \sqrt{3}\right).$$

The dimensions of the highest weight representation generated by the Bethe state are therefore given by $2^2 = 4$.

Acknowledgements

The author would like to thank K. Fabricius and B.M. McCoy for useful discussions. He would also like to thank the organizers for their kind invitation to the Sixth International Conference "Symmetry in Nonlinear Mathematical Physics", June 20–26, 2005, Institute of Mathematics, Kyiv, Ukraine. He is grateful to many participants of the conference for useful comments and discussions. This work is partially supported by the Grant-in-Aid (No. 17540351).

[1] Alcaraz F.C., Grimm U., Rittenberg V., The XXZ Heisenberg chain, conformal invariance and the operator content of $c < 1$ systems, *Nucl. Phys. B*, 1989, V.316, 735–768.

[2] Baxter R.J., Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. I. Some fundamental eigenvectors, *Ann. Phys.*, 1973, V.76, 1–24; II. Equivalence to a generalized Ice-type lattice model, *Ann. Phys.*, 1973, V.76, 25–47; III. Eigenvectors of the transfer matrix and Hamiltonian, *Ann. Phys.*, 1973, V.76, 48–71.

[3] Baxter R.J., Completeness of the Bethe ansatz for the six and eight vertex models, *J. Statist. Phys.*, 2002, V.108, 1–48; cond-mat/0111188.

[4] Baxter R.J., The six and eight-vertex models revisited, *J. Statist. Phys.*, 2004, V.116, 43–66; cond-mat/0403138.

[5] Braak D., Andrei N., On the spectrum of the XXZ-chain at roots of unity, *J. Statist. Phys.*, 2001, V.105, 677–709; cond-mat/0106593.

[6] Chari V., Pressley A., Quantum affine algebras, *Comm. Math. Phys.*, 1991, V.142, 261–283.

[7] Chari V., Pressley A., Quantum affine algebras at roots of unity, *Represent. Theory*, 1997, V.1, 280–328; q-alg/9609031.

[8] Chari V., Pressley A., Weyl modules for classical and quantum affine algebras, *Represent. Theory*, 2001, V.5, 191–223; math.QA/0004174.

[9] Deguchi T., Construction of some missing eigenvectors of the XYZ spin chain at the discrete coupling constants and the exponentially large spectral degeneracy of the transfer matrix, *J. Phys. A: Math. Gen.*, 2002, V.35, 879–895; cond-mat/0109078.

[10] Deguchi T., The 8V CSOS model and the sl_2 loop algebra symmetry of the six-vertex model at roots of unity, *Internat. J. Modern Phys. B*, 2002, V.16, 1899–1905; cond-mat/0110121.

[11] Deguchi T., XXZ Bethe states as highest weight vectors of the sl_2 loop algebra at roots of unity, cond-mat/0503564.

[12] Deguchi T., The six-vertex model at roots of unity and some highest weight representations of the sl_2 loop algebra, in preparation (to be submitted to the Proceedings of RAQIS'05, Annecy, France).

[13] Deguchi T., Fabricius K., McCoy B.M., The sl_2 loop algebra symmetry of the six-vertex model at roots of unity, *J. Statist. Phys.*, 2001, V.102, 701–736; cond-mat/9912141.

[14] Fabricius K., McCoy B.M., Bethe's equation is incomplete for the XXZ model at roots of unity, *J. Statist. Phys.*, 2001, V.103, 647–678; cond-mat/0009279.

[15] Fabricius K., McCoy B.M., Completing Bethe's equations at roots of unity, *J. Statist. Phys.*, 2001, V.104, 573–587; cond-mat/0012501.

[16] Fabricius K., McCoy B.M., Evaluation parameters and Bethe roots for the six-vertex model at roots of unity, *Progress in Mathematical Physics*, Vol. 23 (MathPhys Odyssey 2001), Editors M. Kashiwara and T. Miwa, Boston, Birkhäuser, 2002, 119–144; cond-mat/0108057.

[17] Fabricius K., McCoy B.M., New developments in the eight-vertex model, *J. Statist Phys.*, 2003, V.111, 323–337; cond-mat/0207177.
Fabricius K., McCoy B.M., Functional equations and fusion matrices for the eight-vertex model, *Publ. Res. Inst. Math. Sci.*, 2004, V.40, 905–932; cond-mat/0311122.

[18] Fabricius K., McCoy B.M., New developments in the eight-vertex model II. Chains of odd length, cond-mat/0410113.

[19] Jimbo M., Private communication, July 2004.

[20] Kac V., Infinite dimensional Lie algebras, Cambridge, Cambridge University Press, 1990.

[21] Korepanov I.G., Hidden symmetries in the 6-vertex model of statistical physics, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 1994, V.215, 163–177 (English transl.: *J. Math. Sci. (New York)*, 1997, V.85, 1661–1670); hep-th/9410066.

[22] Korepanov I.G., Vacuum curves of the $\mathcal{L}$-operators related to the six-vertex model, *St. Petersburg Math. J.*, 1995, V.6, 349–364.

[23] Korepin V.E., Bogoliubov N.M., Izergin A.G., Quantum inverse scattering method and correlation functions, Cambridge, Cambridge University Press, 1993.

[24] Korff C., McCoy B.M., Loop symmetry of integrable vertex models at roots of unity, *Nucl. Phys. B*, 2001, V.618, 551–569; hep-th/0104120.

[25] Lusztig G., Modular representations and quantum groups, *Contemp. Math.*, 1989, V.82, 59–77.

[26] Lusztig G., Introduction to quantum groups, Boston, Birkhäuser, 1993.

[27] Pasquier V., Saleur H., Common structures between finite systems and conformal field theories through quantum groups, *Nucl. Phys. B*, 1990, V.330, 523–556.

[28] Takhtajan L., Faddeev L., Spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model, *J. Sov. Math.*, 1984, V.24, 241–267.

[29] Tarasov V.O., Cyclic monodromy matrices for the R-matrix of the six-vertex model and the chiral Potts model with fixed spin boundary conditions, in Infinite Analysis, Part A, B (Kyoto, 1991), *Adv. Ser. Math. Phys.*, Vol. 16, River Edge, NJ, World Sci. Publishing, 1992, 963–975.

[30] Tarasov V.O., On the Bethe vectors for the XXZ model at roots of unity, math.QA/0306032.