Superintegrability on Three-Dimensional Riemannian and Relativistic Spaces of Constant Curvature

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Abstract. A family of classical superintegrable Hamiltonians, depending on an arbitrary radial function, which are defined on the 3D spherical, Euclidean and hyperbolic spaces as well as on the (2+1)D anti-de Sitter, Minkowskian and de Sitter spacetimes is constructed. Such systems admit three integrals of the motion (besides the Hamiltonian) which are explicitly given in terms of ambient and geodesic polar coordinates. The resulting expressions cover the six spaces in a unified way as these are parametrized by two contraction parameters that govern the curvature and the signature of the metric on each space. Next two maximally superintegrable Hamiltonians are identified within the initial superintegrable family by finding the remaining constant of the motion. The former potential is the superposition of a (curved) central harmonic oscillator with other three oscillators or centrifugal barriers (depending on each specific space), so that this generalizes the Smorodinsky–Winternitz system. The latter one is a superposition of the Kepler–Coulomb potential with another two oscillators or centrifugal barriers. As a byproduct, the Laplace–Runge–Lenz vector for these spaces is deduced. Furthermore both potentials are analysed in detail for each particular space. Some comments on their generalization to arbitrary dimension are also presented.

Key words: integrable systems; curvature; contraction; harmonic oscillator; Kepler–Coulomb; hyperbolic; de Sitter

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1 Introduction

In [14] Evans obtained a classification of classical superintegrable systems [38] on the three-dimensional (3D) Euclidean space $\mathbf{E}^3$. At this dimension he called *minimally* superintegrable systems those endowed with *three* constants of the motion besides the Hamiltonian, that is, they have one constant more than those necessary to ensure complete integrability, but one less than the necessary number to determine maximal superintegrability. Amongst the resulting potentials let us consider

$$\mathcal{U} = \mathcal{F}(r) + \frac{\beta_1}{x^2} + \frac{\beta_2}{y^2} + \frac{\beta_3}{z^2}, \qquad (1.1)$$

where $\mathcal{F}(r)$ is an arbitrary smooth function, the three β_i are arbitrary real parameters, (x, y, z) are Cartesian coordinates on $\mathbf{E}^3$, and $r = \sqrt{x^2 + y^2 + z^2}$. Thus this potential is formed by a central term with three centrifugal barriers. Next by analysing the radial function $\mathcal{F}(r)$ two relevant and well known expressions arise conveying the additional constant of the motion. These two cases then appear in the classification by Evans as *maximally* superintegrable systems as they have the maximum number of functionally independent constants of the motion, *four* ones plus the Hamiltonian. Explicitly, these are:

- The Smorodinsky–Winternitz (SW) potential [17] when $\mathcal{F}(r) = \beta_0 r^2$:

$$\mathcal{U}^{\rm SW} = \beta_0 \left(x^2 + y^2 + z^2\right) + \frac{\beta_1}{x^2} + \frac{\beta_2}{y^2} + \frac{\beta_3}{z^2}, \tag{1.2}$$

  which corresponds to the superposition of a harmonic oscillator with angular frequency $\sqrt{\beta_0}$ and the three centrifugal terms.

- And a generalized Kepler–Coulomb (GKC) potential when $\mathcal{F}(r) = -k/r$:

$$\mathcal{U}^{\rm GKC} = -\frac{k}{\sqrt{x^2 + y^2 + z^2}} + \frac{\beta_1}{x^2} + \frac{\beta_2}{y^2}, \tag{1.3}$$

  which is formed by the proper Kepler–Coulomb (KC) potential with parameter k together with *two* of the famous centrifugal terms.

Superintegrable systems on $\mathbf{E}^2$ and $\mathbf{E}^3$ [14, 35] have also been implemented on the two classical Riemannian spaces of constant curvature. In particular, some superintegrable systems on the 2D and 3D spheres, $\mathbf{S}^2$ and $\mathbf{S}^3$, have been studied in [20], on the hyperbolic plane $\mathbf{H}^2$ in [29, 30], while on $\mathbf{H}^3$ can be found in [21]. Moreover classifications of superintegrable systems on $\mathbf{S}^2$ and $\mathbf{H}^2$ have been carried out in [28, 31, 34, 39]. These results contain the corresponding (curved) harmonic oscillator [26, 36] and KC potential [43], which in arbitrary dimension correspond, in this order, to the following radial potential

$$\mathcal{F}(r) = \left\{ \begin{array}{lll} \beta_0 \tan^2 r, & \text{on} & \mathbf{S}^N; \\ \beta_0\, r^2, & \text{on} & \mathbf{E}^N; \\ \beta_0 \tanh^2 r, & \text{on} & \mathbf{H}^N. \end{array} \right. \qquad \mathcal{F}(r) = \left\{ \begin{array}{lll} -k/\tan r, & \text{on} & \mathbf{S}^N; \\ -k/r, & \text{on} & \mathbf{E}^N; \\ -k/\tanh r, & \text{on} & \mathbf{H}^N. \end{array} \right. \tag{1.4}$$

We recall that the SW system on $\mathbf{S}^N$ and $\mathbf{H}^N$ have been constructed in [9, 23] (curved harmonic oscillator plus N terms) showing that this keeps maximal superintegrability for any value of the curvature.

However, as far as we know, the construction of the GKC potential on $\mathbf{S}^N$ and $\mathbf{H}^N$ as well as which are the corresponding SW and GKC systems on the relativistic spacetimes of constant curvature is still lacking, that is, also covering the anti-de Sitter, Minkowskian and de Sitter spacetimes. The aim of this paper is to present all of these Hamiltonians on these *six* 3D spaces in a unified setting by making use of two explicit contraction parameters which determine the curvature and the signature of the metric. In this sense, the results here presented can be considered as the cornerstone for a further generalization of all of these systems to arbitrary dimension. In this respect, we would like to mention that although very recently such potentials have been deduced on the $(1+1)$D relativistic spacetimes [8, 12], this low dimension does not show the guide for a direct generalization to ND.

The structure of this paper is as follows. The next section contains the necessary basics on the Lie groups of isometries on the six spaces together with the two coordinate systems we shall deal with throughout the paper: ambient (Weierstrass) coordinates in an auxiliary linear space $\mathbb{R}^4$ and intrinsic geodesic polar (spherical) coordinates. The kinetic energy determining the geodesic motion is then studied in Section 3 by starting from the metric. The generalization of the Euclidean family (1.1) to these spaces is developed in Section 4 in such a manner that general and global expressions for the Hamiltonian and its three integrals of motion are explicitly given.

The next two sections are devoted to the study of two maximal superintegrable Hamiltonians arising in the above family by choosing in an adequate way the radial function $\mathcal{F}(r)$ (fulfilling (1.4) for the Riemannian spaces) and finding at the same time the remaining constant of the motion. In this way we obtain the generalization of the SW (1.2) and GKC (1.3) potentials for any value of the curvature and signature of the metric. Furthermore a detail description of such systems is performed on each particular space. We stress that, by following the geometrical interpretation formerly introduced in [40, 41, 42] and generalized in [8, 9, 23], the SW potential is interpreted as the superposition of a central harmonic oscillator with three non-central oscillators or centrifugal barriers according to each specific space. Likewise, the GKC system can be seen as the superposition of the KC potential with two oscillators or centrifugal barriers; in this case, we moreover deduce the corresponding Laplace–Runge–Lenz vector. Finally, some remarks and comments mainly concerning the pattern for the construction of such systems for arbitrary dimension close the paper.

2 Riemannian spaces and relativistic spacetimes

Let us consider a subset of real Lie algebras contained in the family of the Cayley–Klein orthogonal algebras [4, 18]. These can also be obtained as the $\mathbb{Z}_2\otimes\mathbb{Z}_2$ graded contractions of $so(4)$ and are denoted $so_{\kappa_1,\kappa_2}(4)$ where κ_1 and κ_2 are two real contraction parameters. The Lie brackets of $so_{\kappa_1,\kappa_2}(4)$ in the basis spanned by $\{J_{\mu\nu}\}$ where $\mu,\nu=0,1,2,3$ and $\mu<\nu$ read [4]

$$
\begin{aligned}
&[J_{12},J_{13}]=\kappa_2 J_{23}, &&[J_{12},J_{23}]=-J_{13}, &&[J_{13},J_{23}]=J_{12},\\
&[J_{12},J_{01}]=J_{02}, &&[J_{13},J_{01}]=J_{03}, &&[J_{23},J_{02}]=J_{03},\\
&[J_{12},J_{02}]=-\kappa_2 J_{01}, &&[J_{13},J_{03}]=-\kappa_2 J_{01}, &&[J_{23},J_{03}]=-J_{02},\\
&[J_{01},J_{02}]=\kappa_1 J_{12}, &&[J_{01},J_{03}]=\kappa_1 J_{13}, &&[J_{02},J_{03}]=\kappa_1\kappa_2 J_{23},\\
&[J_{01},J_{23}]=0, &&[J_{02},J_{13}]=0, &&[J_{03},J_{12}]=0.
\end{aligned}
\tag{2.1}
$$

There are two Casimir invariants

$$
\begin{aligned}
&\mathcal{C}_1=\kappa_2 J_{01}^2+J_{02}^2+J_{03}^2+\kappa_1 J_{12}^2+\kappa_1 J_{13}^2+\kappa_1\kappa_2 J_{23}^2,\\
&\mathcal{C}_2=\kappa_2 J_{01}J_{23}-J_{02}J_{13}+J_{03}J_{12},
\end{aligned}
\tag{2.2}
$$

where $\mathcal{C}_1$ is associated to the Killing–Cartan form.

Let us explain the geometrical role of the contraction parameters κ_1 and κ_2. The involutive automorphisms defined by

$$
\begin{aligned}
&\Theta_0: \quad J_{ij}\to J_{ij}, \quad J_{0i}\to -J_{0i}, \quad i=1,2,3,\\
&\Theta_{01}: \ \{J_{01},J_{23}\}\to\{J_{01},J_{23}\}, \quad \{J_{0j},J_{1j}\}\to-\{J_{0j},J_{1j}\}, \quad j=2,3,
\end{aligned}
$$

generate a $\mathbb{Z}_2\otimes\mathbb{Z}_2$-grading of $so_{\kappa_1,\kappa_2}(4)$ in such a manner that κ_1 and κ_2 are two graded contraction parameters coming from the $\mathbb{Z}_2$-grading determined by Θ_0 and Θ_{01}, respectively. By scaling the Lie generators each parameter κ_i can be reduced to either $+1$, 0 or -1; the vanishment of κ_i is equivalent to apply an Inönü–Wigner contraction.

Furthermore, these automorphisms give rise to the following Cartan decompositions:

$$
\begin{aligned}
&so_{\kappa_1,\kappa_2}(4)=\mathfrak{h}_0\oplus\mathfrak{p}_0, &&\mathfrak{h}_0=\langle J_{12},J_{13},J_{23}\rangle=so_{\kappa_2}(3), &&\mathfrak{p}_0=\langle J_{01},J_{02},J_{03}\rangle,\\
&so_{\kappa_1,\kappa_2}(4)=\mathfrak{h}_{01}\oplus\mathfrak{p}_{01}, &&\mathfrak{h}_{01}=\langle J_{01},J_{23}\rangle=so_{\kappa_1}(2)\oplus so(2), &&\mathfrak{p}_{01}=\langle J_{02},J_{03},J_{12},J_{13}\rangle.
\end{aligned}
$$

If H_0 and H_{01} denote the Lie subgroups with Lie algebras $\mathfrak{h}_0$ and $\mathfrak{h}_{01}$, we obtain two families of symmetric homogeneous spaces [22], namely the usual 3D space of points $SO_{\kappa_1,\kappa_2}(4)/H_0$

and the 4D space of lines $SO_{\kappa_1,\kappa_2}(4)/H_{01}$, which have constant curvature equal to κ_1 and κ_2, respectively.

We shall make use of the former space which has a metric with a signature governed by κ_2 as diag$(+1,\kappa_2,\kappa_2)$ and we denote it

$$\mathbb{S}^3_{[\kappa_1]\kappa_2} = SO_{\kappa_1,\kappa_2}(4)/SO_{\kappa_2}(3).$$

Thus when κ_2 is positive we recover the three classical Riemannian spaces, while if this is negative we find a Lorentzian metric. In this case, there is a kinematical interpretation for the homogeneous spaces. Let P_0, P_i, K_i and J $(i=1,2)$ the usual generators of time translation, space translations, boosts and spatial rotations, respectively. Under the following identification

$$P_0 = J_{01}, \qquad P_i = J_{0\,i+1}, \qquad K_i = J_{1\,i+1}, \qquad J = J_{23}, \qquad i=1,2, \tag{2.3}$$

the three algebras with $\kappa_2 = -1/c^2 < 0$ (c is the speed of light) are the Lie algebras of the groups of motions of $(2+1)$D relativistic spacetime models. Thus the commutation relations (2.1) read now

$$\begin{aligned}
&[J,K_i] = \epsilon_{ij}K_j, \qquad [K_1,K_2] = -\frac{1}{c^2}\,J, \qquad [P_0,K_i] = -P_i, \qquad [P_i,K_j] = -\frac{1}{c^2}\,\delta_{ij}P_0,\\
&[J,P_i] = \epsilon_{ij}P_j, \qquad [P_1,P_2] = -\frac{\kappa_1}{c^2}\,J, \qquad [P_0,P_i] = \kappa_1 K_i, \qquad [P_0,J_i] = 0,
\end{aligned}\tag{2.4}$$

where ϵ_{ij} is a skew-symmetric tensor such that $\epsilon_{12}=1$. In this framework the curvature of the spacetime can be written in terms of the (time) universe radius τ as $\kappa_1 = \pm 1/\tau^2$ (which is also proportional to the cosmological constant). The Casimir invariants (2.2), $\mathcal{C}_1$ and $\mathcal{C}_2$, correspond to the energy and angular momentum of a particle in the free kinematics of the relativistic spacetime:

$$\begin{aligned}
&\mathcal{C}_1 = -\frac{1}{c^2}\,P_0^2 + P_1^2 + P_2^2 + \kappa_1\left(K_1^2+K_2^2\right) - \frac{\kappa_1}{c^2}\,J^2,\\
&\mathcal{C}_2 = -\frac{1}{c^2}\,P_0 J - P_1K_2 + P_2K_1.
\end{aligned}\tag{2.5}$$

On the other hand, if $\kappa_2 = 0$ we obtain a degenerate metric which corresponds to Newtonian spacetimes. Since our aim is to construct superintegrable systems on these homogeneous spaces, for which the kinetic energy is provided by the metric, we avoid the contraction $\kappa_2 = 0$. The resulting *six* particular spaces contained in the family $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ are displayed in Table 1.

Table 1. 3D symmetric homogeneous spaces $\mathbb{S}^3_{[\kappa_1]\kappa_2} = SO_{\kappa_1,\kappa_2}(4)/SO_{\kappa_2}(3)$ and their metric in geodesic polar coordinates according to $\kappa_1 \in \{+1,0,-1\}$ and $\kappa_2 \in \{+1,-1\}$.

3D Riemannian spaces	$(2+1)$D Relativistic spacetimes
• Spherical space $\mathbf{S}^3$	• Anti-de Sitter spacetime $\mathbf{AdS}^{2+1}$
$\mathbb{S}^3_{[+]+} = SO(4)/SO(3)$	$\mathbb{S}^3_{[+]-} = SO(2,2)/SO(2,1)$
$\mathrm{d}s^2 = \mathrm{d}r^2 + \sin^2 r\,\mathrm{d}\theta^2 + \sin^2 r \sin^2\theta\,\mathrm{d}\phi^2$	$\mathrm{d}s^2 = \mathrm{d}r^2 - \sin^2 r\,\mathrm{d}\theta^2 - \sin^2 r \sinh^2\theta\,\mathrm{d}\phi^2$
• Euclidean space $\mathbf{E}^3$	• Minkowskian spacetime $\mathbf{M}^{2+1}$
$\mathbb{S}^3_{[0]+} = ISO(3)/SO(3)$	$\mathbb{S}^3_{[0]-} = ISO(2,1)/SO(2,1)$
$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\phi^2$	$\mathrm{d}s^2 = \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2 \sinh^2\theta\,\mathrm{d}\phi^2$
• Hyperbolic space $\mathbf{H}^3$	• De Sitter spacetime $\mathbf{dS}^{2+1}$
$\mathbb{S}^3_{[-]+} = SO(3,1)/SO(3)$	$\mathbb{S}^3_{[-]-} = SO(3,1)/SO(2,1)$
$\mathrm{d}s^2 = \mathrm{d}r^2 + \sinh^2 r\,\mathrm{d}\theta^2 + \sinh^2 r \sin^2\theta\,\mathrm{d}\phi^2$	$\mathrm{d}s^2 = \mathrm{d}r^2 - \sinh^2 r\,\mathrm{d}\theta^2 - \sinh^2 r \sinh^2\theta\,\mathrm{d}\phi^2$

2.1 Vector model and ambient coordinates

The vector representation of $so_{\kappa_1,\kappa_2}(4)$ is given by the following 4×4 real matrices [4]:

$$
J_{01}=\begin{pmatrix}\cdot & -\kappa_1 & \cdot & \cdot\\ 1 & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix},\qquad
J_{12}=\begin{pmatrix}\cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & -\kappa_2 & \cdot\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix},
$$
$$
J_{02}=\begin{pmatrix}\cdot & \cdot & -\kappa_1\kappa_2 & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ 1 & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix},\qquad
J_{13}=\begin{pmatrix}\cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & -\kappa_2\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & 1 & \cdot & \cdot\end{pmatrix},
$$
$$
J_{03}=\begin{pmatrix}\cdot & \cdot & \cdot & -\kappa_1\kappa_2\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ 1 & \cdot & \cdot & \cdot\end{pmatrix},\qquad
J_{23}=\begin{pmatrix}\cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & -1\\ \cdot & \cdot & 1 & \cdot\end{pmatrix}. \tag{2.6}
$$

Their exponential provides the corresponding one-parametric subgroups of $SO_{\kappa_1,\kappa_2}(4)$:

$$
e^{xJ_{01}}=\begin{pmatrix}\mathrm{C}_{\kappa_1}(x) & -\kappa_1\,\mathrm{S}_{\kappa_1}(x) & \cdot & \cdot\\ \mathrm{S}_{\kappa_1}(x) & \mathrm{C}_{\kappa_1}(x) & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot\\ \cdot & \cdot & \cdot & 1\end{pmatrix},\qquad
e^{xJ_{12}}=\begin{pmatrix}1 & \cdot & \cdot & \cdot\\ \cdot & \mathrm{C}_{\kappa_2}(x) & -\kappa_2\,\mathrm{S}_{\kappa_2}(x) & \cdot\\ \cdot & \mathrm{S}_{\kappa_2}(x) & \mathrm{C}_{\kappa_2}(x) & \cdot\\ \cdot & \cdot & \cdot & 1\end{pmatrix},
$$
$$
e^{xJ_{02}}=\begin{pmatrix}\mathrm{C}_{\kappa_1\kappa_2}(x) & \cdot & -\kappa_1\kappa_2\,\mathrm{S}_{\kappa_1\kappa_2}(x) & \cdot\\ \cdot & 1 & \cdot & \cdot\\ \mathrm{S}_{\kappa_1\kappa_2}(x) & \cdot & \mathrm{C}_{\kappa_1\kappa_2}(x) & \cdot\\ \cdot & \cdot & \cdot & 1\end{pmatrix},\qquad
e^{xJ_{13}}=\begin{pmatrix}1 & \cdot & \cdot & \cdot\\ \cdot & \mathrm{C}_{\kappa_2}(x) & \cdot & -\kappa_2\,\mathrm{S}_{\kappa_2}(x)\\ \cdot & \cdot & 1 & \cdot\\ \cdot & \mathrm{S}_{\kappa_2}(x) & \cdot & \mathrm{C}_{\kappa_2}(x)\end{pmatrix},
$$
$$
e^{xJ_{03}}=\begin{pmatrix}\mathrm{C}_{\kappa_1\kappa_2}(x) & \cdot & \cdot & -\kappa_1\kappa_2\,\mathrm{S}_{\kappa_1\kappa_2}(x)\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot\\ \mathrm{S}_{\kappa_1\kappa_2}(x) & \cdot & \cdot & \mathrm{C}_{\kappa_1\kappa_2}(x)\end{pmatrix},\qquad
e^{xJ_{23}}=\begin{pmatrix}1 & \cdot & \cdot & \cdot\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & \cos x & -\sin x\\ \cdot & \cdot & \sin x & \cos x\end{pmatrix}, \tag{2.7}
$$

where we have introduced the κ-dependent cosine and sine functions defined by [3, 5]

$$
\mathrm{C}_\kappa(x)=\sum_{l=0}^{\infty}(-\kappa)^l\frac{x^{2l}}{(2l)!}=\begin{cases}\cos\sqrt{\kappa}\,x, & \kappa>0;\\ 1, & \kappa=0;\\ \cosh\sqrt{-\kappa}\,x, & \kappa<0,\end{cases}
$$
$$
\mathrm{S}_\kappa(x)=\sum_{l=0}^{\infty}(-\kappa)^l\frac{x^{2l+1}}{(2l+1)!}=\begin{cases}\frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}\,x, & \kappa>0;\\ x, & \kappa=0;\\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}\,x, & \kappa<0.\end{cases}
$$

Notice that $\kappa\in\{\kappa_1,\kappa_1\kappa_2,\kappa_2\}$. The tangent is defined as $\mathrm{T}_\kappa(x)=\mathrm{S}_\kappa(x)/\,\mathrm{C}_\kappa(x)$. Properties and trigonometric relations for these κ-functions, which are necessary in the further computations, can be found in [24, 25]; for instance,

$$
\mathrm{C}^2_\kappa(x)+\kappa\,\mathrm{S}^2_\kappa(x)=1,\qquad \frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{C}_\kappa(x)=-\kappa\,\mathrm{S}_\kappa(x),\qquad \frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{S}_\kappa(x)=\,\mathrm{C}_\kappa(x).
$$

Under the above matrix algebra and group representations it is verified that

$$
X^T\mathbb{I}_\kappa+\mathbb{I}_\kappa X=0,\quad X\in so_{\kappa_1,\kappa_2}(4),\qquad Y^T\mathbb{I}_\kappa Y=\mathbb{I}_\kappa,\quad Y\in SO_{\kappa_1,\kappa_2}(4),
$$

(X^T is the transpose matrix of X) with respect to the bilinear form

$$\mathbb{I}_\kappa = \mathrm{diag}(+1, \kappa_1, \kappa_1\kappa_2, \kappa_1\kappa_2).$$

Therefore $SO_{\kappa_1,\kappa_2}(4)$ is a group of isometries of $\mathbb{I}_\kappa$ acting on a linear ambient space $\mathbb{R}^4 = (x_0, x_1, x_2, x_3)$ through matrix multiplication. The origin O in $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ has ambient coordinates $O = (1, 0, 0, 0)$ and this point is invariant under the subgroup $H_0 = SO_{\kappa_2}(3) = \langle J_{12}, J_{13}, J_{23}\rangle$ (see (2.7)). The orbit of O corresponds to the homogeneous space $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ which is contained in the "sphere"

$$\Sigma \equiv x_0^2 + \kappa_1 x_1^2 + \kappa_1\kappa_2 x_2^2 + \kappa_1\kappa_2 x_3^2 = 1, \tag{2.8}$$

determined by $\mathbb{I}_\kappa$. The ambient coordinates (x_0, x_1, x_2, x_3), subjected to (2.8), are also called *Weierstrass coordinates*. The metric on $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ follows from the flat ambient metric in $\mathbb{R}^4$ divided by the curvature and restricted to Σ:

$$\mathrm{d}s^2 = \frac{1}{\kappa_1}\left(\mathrm{d}x_0^2 + \kappa_1\mathrm{d}x_1^2 + \kappa_1\kappa_2\mathrm{d}x_2^2 + \kappa_1\kappa_2\mathrm{d}x_3^2\right)\Big|_\Sigma. \tag{2.9}$$

A differential realization of $so_{\kappa_1,\kappa_2}(4)$, fulfilling (2.1), as first-order vector fields in the ambient coordinates is provided by the vector representation (2.6) and reads

$$\begin{aligned} &J_{01} = \kappa_1 x_1\partial_0 - x_0\partial_1, && J_{0j} = \kappa_1\kappa_2 x_j\partial_0 - x_0\partial_j,\\ &J_{23} = x_3\partial_2 - x_2\partial_3, && J_{1j} = \kappa_2 x_j\partial_1 - x_1\partial_j, \end{aligned} \tag{2.10}$$

where $j = 2, 3$ and $\partial_\mu = \partial/\partial x_\mu$.

2.2 Geodesic polar coordinate system

Let us consider a point Q in $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ with Weierstrass coordinates (x_0, x_1, x_2, x_3). This can be parametrized in terms of three intrinsic quantities of the space itself in different ways. We shall make use of the *geodesic polar coordinates* (r, θ, ϕ) which are defined through the following action of the one-parametric subgroups (2.7) on the origin $O = (1, 0, 0, 0)$:

$$\begin{gathered} Q(r,\theta,\phi) = \exp\{\phi J_{23}\}\exp\{\theta J_{12}\}\exp\{r J_{01}\}O,\\ \begin{pmatrix} x_0\\ x_1\\ x_2\\ x_3 \end{pmatrix} = \begin{pmatrix} \mathrm{C}_{\kappa_1}(r)\\ \mathrm{S}_{\kappa_1}(r)\,\mathrm{C}_{\kappa_2}(\theta)\\ \mathrm{S}_{\kappa_1}(r)\,\mathrm{S}_{\kappa_2}(\theta)\cos\phi\\ \mathrm{S}_{\kappa_1}(r)\,\mathrm{S}_{\kappa_2}(\theta)\sin\phi \end{pmatrix}. \end{gathered} \tag{2.11}$$

Let l_1 be a (time-like) geodesic and l_2, l_3 two other (space-like) geodesics in $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ orthogonal at O in such a manner that each translation generator J_{0i} moves the origin along l_i. Then the (physical) geometrical meaning of the coordinates (r, θ, ϕ) is as follows.

- The radial coordinate r is the distance between Q and O measured along the (time-like) geodesic l that joins both points. In the curved Riemannian spaces with $\kappa_1 = \pm 1/R^2$, r has dimensions of *length*, $[r] = [R]$; notice however that the dimensionless coordinate r/R is usually taken instead of r, and so the former is considered as an ordinary angle (see, e.g., [27]). In the relativistic spacetimes with $\kappa_1 = \pm 1/\tau^2$, r has dimensions of a time-like length, that is, $[r] = [\tau]$.
- The coordinate θ is an ordinary angle in the three Riemannian spaces ($\kappa_2 = +1$) and this parametrizes the orientation of l with respect to l_1, whilst θ corresponds to a rapidity in the spacetimes ($\kappa_2 = -1/c^2$) with dimensions $[\theta] = [c]$.

- Finally, ϕ is an ordinary angle for the six spaces that determines the orientation of l with respect to the reference flag spanned by l_1 and l_2, that is, the 2-plane $l_1 l_2$.

In the Riemannian spaces (r, θ, ϕ) parametrize the complete space, while in the spacetimes these only cover the time-like region (in ambient coordinates this is $x_2^2 + x_3^2 \le x_1^2$) limited by the light-cone on which $\theta \to \infty$. The flat contraction $\kappa_1 = 0$ gives rise to the usual spherical coordinates in the Euclidean space ($\kappa_2 = 1$).

By introducing the parametrization (2.11) in the metric written in terms of ambient coordinates (2.9) we obtain that

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \kappa_2 \, \mathrm{S}^2_{\kappa_1}(r) \left(\mathrm{d}\theta^2 + \mathrm{S}^2_{\kappa_2}(\theta)\mathrm{d}\phi^2\right), \tag{2.12}$$

which is particularized in Table 1 to each space. From it we compute the Levi-Civita connection Γ^k_{ij}, the Riemann R^i_{jkl} and Ricci R_{ij} tensors [13]. Their nonzero components are given by

$$\begin{aligned}
&\Gamma^\theta_{\theta r} = \Gamma^\phi_{\phi r} = 1/\,\mathrm{T}_{\kappa_1}(r), \qquad \Gamma^\phi_{\phi\theta} = 1/\,\mathrm{T}_{\kappa_2}(\theta), \qquad \Gamma^r_{\theta\theta} = -\kappa_2 \, \mathrm{S}_{\kappa_1}(r) \, \mathrm{C}_{\kappa_1}(r), \\
&\Gamma^r_{\phi\phi} = -\kappa_2 \, \mathrm{S}_{\kappa_1}(r) \, \mathrm{C}_{\kappa_1}(r) \, \mathrm{S}^2_{\kappa_2}(\theta), \qquad \Gamma^\theta_{\phi\phi} = -\,\mathrm{S}_{\kappa_2}(\theta) \, \mathrm{C}_{\kappa_2}(\theta), \\
&R^r_{\theta r\theta} = R^\phi_{\theta\phi\theta} = \kappa_1\kappa_2 \, \mathrm{S}^2_{\kappa_1}(r), \qquad R^r_{\phi r\phi} = R^\theta_{\phi\theta\phi} = \kappa_1\kappa_2 \, \mathrm{S}^2_{\kappa_1}(r) \, \mathrm{S}^2_{\kappa_2}(\theta), \qquad R^\theta_{r\theta r} = R^\phi_{r\phi r} = \kappa_1, \\
&R_{rr} = 2\kappa_1, \qquad R_{\theta\theta} = 2\kappa_1\kappa_2 \, \mathrm{S}^2_{\kappa_1}(r), \qquad R_{\phi\phi} = 2\kappa_1\kappa_2 \, \mathrm{S}^2_{\kappa_1}(r) \, \mathrm{S}^2_{\kappa_2}(\theta).
\end{aligned} \tag{2.13}$$

Therefore all the sectional curvatures turn out to be constant $K_{ij} = \kappa_1$, while the scalar curvature reads $K = 6\kappa_1$.

3 Geodesic motion

The metric (2.12) can be read as the kinetic energy of a particle written in terms of the velocities $(\dot r, \dot\theta, \dot\phi)$, that is, the Lagrangian of the geodesic motion on the space $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ given by

$$\mathcal{T} = \frac{1}{2}\big(\dot r^2 + \kappa_2 \, \mathrm{S}^2_{\kappa_1}(r)\big(\dot\theta^2 + \mathrm{S}^2_{\kappa_2}(\theta)\dot\phi^2\big)\big). \tag{3.1}$$

Then the canonical momenta (p_r, p_θ, p_ϕ) are obtained through $p = \partial\mathcal{T}/\partial\dot q$ ($\dot q = \dot r, \dot\theta, \dot\phi$), namely,

$$\begin{aligned}
&p_r = \dot r, \\
&p_\theta = \kappa_2 \, \mathrm{S}^2_{\kappa_1}(r)\dot\theta, \\
&p_\phi = \kappa_2 \, \mathrm{S}^2_{\kappa_1}(r) \, \mathrm{S}^2_{\kappa_2}(\theta)\dot\phi,
\end{aligned} \tag{3.2}$$

so that the free Hamiltonian in the geodesic polar phase space $(q; p) = (r, \theta, \phi; p_r, p_\theta, p_\phi)$ with respect to the canonical Lie–Poisson bracket,

$$\{f, g\} = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\right), \tag{3.3}$$

turns out to be

$$\mathcal{T} = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{\kappa_2 \, \mathrm{S}^2_{\kappa_1}(r)} + \frac{p_\phi^2}{\kappa_2 \, \mathrm{S}^2_{\kappa_1}(r) \, \mathrm{S}^2_{\kappa_2}(\theta)}\right). \tag{3.4}$$

Note that the connection (2.13) would allow one to write the geodesic equations whose solution would correspond to the geodesic motion associated with $\mathcal{T}$ (see [8] for the 2D case).

Now we proceed to deduce a phase space realization of the Lie generators of $so_{\kappa_1,\kappa_2}(4)$. In Weierstrass coordinates x_μ and momenta p_μ this comes from the vector fields (2.10) through the replacement $\partial_\mu \to -p_\mu$:

$$
\begin{aligned}
&J_{01} = x_0 p_1 - \kappa_1 x_1 p_0, \qquad && J_{0j} = x_0 p_j - \kappa_1\kappa_2 x_j p_0,\\
&J_{23} = x_2 p_3 - x_3 p_2, && J_{1j} = x_1 p_j - \kappa_2 x_j p_1.
\end{aligned}
\tag{3.5}
$$

The metric (2.9) can also be understood as the kinetic energy in the ambient velocities $\dot{x}_\mu$ so that the momenta p_μ are ($j = 2, 3$):

$$
p_0 = \dot{x}_0/\kappa_1, \qquad p_1 = \dot{x}_1, \qquad p_j = \kappa_2 \dot{x}_j. \tag{3.6}
$$

Next if we compute the velocities $\dot{x}_i$ in the parametrization (2.11) and introduce the momenta (3.2) and (3.6) we obtain the relationship between the ambient momenta and the geodesic polar ones:

$$
\begin{aligned}
p_0 &= -\,\mathrm{S}_{\kappa_1}(r)\, p_r,\\
p_1 &= \mathrm{C}_{\kappa_1}(r)\,\mathrm{C}_{\kappa_2}(\theta)\, p_r - \frac{\mathrm{S}_{\kappa_2}(\theta)}{\mathrm{S}_{\kappa_1}(r)}\, p_\theta,\\
p_2 &= \kappa_2\, \mathrm{C}_{\kappa_1}(r)\,\mathrm{S}_{\kappa_2}(\theta) \cos\phi\, p_r + \frac{\mathrm{C}_{\kappa_2}(\theta)\cos\phi}{\mathrm{S}_{\kappa_1}(r)}\, p_\theta - \frac{\sin\phi}{\mathrm{S}_{\kappa_1}(r)\,\mathrm{S}_{\kappa_2}(\theta)}\, p_\phi,\\
p_3 &= \kappa_2\, \mathrm{C}_{\kappa_1}(r)\,\mathrm{S}_{\kappa_2}(\theta) \sin\phi\, p_r + \frac{\mathrm{C}_{\kappa_2}(\theta)\sin\phi}{\mathrm{S}_{\kappa_1}(r)}\, p_\theta + \frac{\cos\phi}{\mathrm{S}_{\kappa_1}(r)\,\mathrm{S}_{\kappa_2}(\theta)}\, p_\phi.
\end{aligned}
$$

Hence the generators (3.5) in geodesic polar coordinates and momenta turn out to be

$$
\begin{aligned}
J_{01} &= \mathrm{C}_{\kappa_2}(\theta)\, p_r - \frac{\mathrm{S}_{\kappa_2}(\theta)}{\mathrm{T}_{\kappa_1}(r)}\, p_\theta,\\
J_{02} &= \kappa_2\, \mathrm{S}_{\kappa_2}(\theta)\cos\phi\, p_r + \frac{\mathrm{C}_{\kappa_2}(\theta)\cos\phi}{\mathrm{T}_{\kappa_1}(r)}\, p_\theta - \frac{\sin\phi}{\mathrm{T}_{\kappa_1}(r)\,\mathrm{S}_{\kappa_2}(\theta)}\, p_\phi,\\
J_{03} &= \kappa_2\, \mathrm{S}_{\kappa_2}(\theta)\sin\phi\, p_r + \frac{\mathrm{C}_{\kappa_2}(\theta)\sin\phi}{\mathrm{T}_{\kappa_1}(r)}\, p_\theta + \frac{\cos\phi}{\mathrm{T}_{\kappa_1}(r)\,\mathrm{S}_{\kappa_2}(\theta)}\, p_\phi,\\
J_{12} &= \cos\phi\, p_\theta - \frac{\sin\phi}{\mathrm{T}_{\kappa_2}(\theta)}\, p_\phi,\\
J_{13} &= \sin\phi\, p_\theta + \frac{\cos\phi}{\mathrm{T}_{\kappa_2}(\theta)}\, p_\phi,\\
J_{23} &= p_\phi.
\end{aligned}
\tag{3.7}
$$

By direct computations it can be proven the following statement.

Proposition 1. *The generators* (3.7) *fulfil the commutation relations* (2.1) *with respect to the Lie–Poisson bracket* (3.3) *and all of them Poisson commute with* $\mathcal{T}$ (3.4)*.*

In this respect, notice that, under (3.7), the kinetic energy is related with the Casimir $\mathcal{C}_1$ (2.2) by $2\kappa_2\mathcal{T} = \mathcal{C}_1$, while the second Casimir $\mathcal{C}_2$ vanishes.

The realization of the generators (3.7) is particularized for each specific space and Poisson–Lie algebra contained in $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ and $so_{\kappa_1,\kappa_2}(4)$ in Table 2. In order to present the simplest expressions, hereafter we shall set in all the tables $\kappa_1 \in \{+1, 0, -1\}$ and $\kappa_2 \in \{+1, -1\}$, which corresponds to deal with units $R = \tau = c = 1$.

Table 2. Phase space realization of the generators of $so_{\kappa_1,\kappa_2}(4)$ in canonical geodesic polar coordinates and momenta $(r,\theta,\phi;p_r,p_\theta,p_\phi)$ on each space $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ with $\kappa_1\in\{+1,0,-1\}$ and $\kappa_2\in\{+1,-1\}$.

3D Riemannian spaces	$(2+1)$D Relativistic spacetimes
• Spherical space $\mathbb{S}^3_{[+]+}\equiv\mathbf{S}^3$: $\quad so(4)$	• Anti-de Sitter spacetime $\mathbb{S}^3_{[+]-}\equiv\mathbf{AdS}^{2+1}$: $\quad so(2,2)$
$J_{01}=\cos\theta\, p_r-\dfrac{\sin\theta}{\tan r}\,p_\theta$	$J_{01}=\cosh\theta\, p_r-\dfrac{\sinh\theta}{\tan r}\,p_\theta$
$J_{02}=\sin\theta\cos\phi\, p_r+\dfrac{\cos\theta\cos\phi}{\tan r}\,p_\theta-\dfrac{\sin\phi\, p_\phi}{\tan r\sin\theta}$	$J_{02}=-\sinh\theta\cos\phi\, p_r+\dfrac{\cosh\theta\cos\phi}{\tan r}\,p_\theta-\dfrac{\sin\phi\, p_\phi}{\tan r\sinh\theta}$
$J_{03}=\sin\theta\sin\phi\, p_r+\dfrac{\cos\theta\sin\phi}{\tan r}\,p_\theta+\dfrac{\cos\phi\, p_\phi}{\tan r\sin\theta}$	$J_{03}=-\sinh\theta\sin\phi\, p_r+\dfrac{\cosh\theta\sin\phi}{\tan r}\,p_\theta+\dfrac{\cos\phi\, p_\phi}{\tan r\sinh\theta}$
$J_{12}=\cos\phi\, p_\theta-\dfrac{\sin\phi}{\tan\theta}\,p_\phi$	$J_{12}=\cos\phi\, p_\theta-\dfrac{\sin\phi}{\tanh\theta}\,p_\phi$
$J_{13}=\sin\phi\, p_\theta+\dfrac{\cos\phi}{\tan\theta}\,p_\phi$	$J_{13}=\sin\phi\, p_\theta+\dfrac{\cos\phi}{\tanh\theta}\,p_\phi$
$J_{23}=p_\phi$	$J_{23}=p_\phi$
• Euclidean space $\mathbb{S}^3_{[0]+}\equiv\mathbf{E}^3$: $\quad iso(3)$	• Minkowskian spacetime $\mathbb{S}^3_{[0]-}\equiv\mathbf{M}^{2+1}$: $\quad iso(2,1)$
$J_{01}=\cos\theta\, p_r-\dfrac{\sin\theta}{r}\,p_\theta$	$J_{01}=\cosh\theta\, p_r-\dfrac{\sinh\theta}{r}\,p_\theta$
$J_{02}=\sin\theta\cos\phi\, p_r+\dfrac{\cos\theta\cos\phi}{r}\,p_\theta-\dfrac{\sin\phi}{r\sin\theta}\,p_\phi$	$J_{02}=-\sinh\theta\cos\phi\, p_r+\dfrac{\cosh\theta\cos\phi}{r}\,p_\theta-\dfrac{\sin\phi}{r\sinh\theta}\,p_\phi$
$J_{03}=\sin\theta\sin\phi\, p_r+\dfrac{\cos\theta\sin\phi}{r}\,p_\theta+\dfrac{\cos\phi}{r\sin\theta}\,p_\phi$	$J_{03}=-\sinh\theta\sin\phi\, p_r+\dfrac{\cosh\theta\sin\phi}{r}\,p_\theta+\dfrac{\cos\phi}{r\sinh\theta}\,p_\phi$
$J_{12}=\cos\phi\, p_\theta-\dfrac{\sin\phi}{\tan\theta}\,p_\phi$	$J_{12}=\cos\phi\, p_\theta-\dfrac{\sin\phi}{\tanh\theta}\,p_\phi$
$J_{13}=\sin\phi\, p_\theta+\dfrac{\cos\phi}{\tan\theta}\,p_\phi$	$J_{13}=\sin\phi\, p_\theta+\dfrac{\cos\phi}{\tanh\theta}\,p_\phi$
$J_{23}=p_\phi$	$J_{23}=p_\phi$
• Hyperbolic space $\mathbb{S}^3_{[-]+}\equiv\mathbf{H}^3$: $\quad so(3,1)$	• De Sitter spacetime $\mathbb{S}^3_{[-]-}\equiv\mathbf{dS}^{2+1}$: $\quad so(3,1)$
$J_{01}=\cos\theta\, p_r-\dfrac{\sin\theta}{\tanh r}\,p_\theta$	$J_{01}=\cosh\theta\, p_r-\dfrac{\sinh\theta}{\tanh r}\,p_\theta$
$J_{02}=\sin\theta\cos\phi\, p_r+\dfrac{\cos\theta\cos\phi}{\tanh r}\,p_\theta-\dfrac{\sin\phi\, p_\phi}{\tanh r\sin\theta}$	$J_{02}=-\sinh\theta\cos\phi\, p_r+\dfrac{\cosh\theta\cos\phi}{\tanh r}\,p_\theta-\dfrac{\sin\phi\, p_\phi}{\tanh r\sinh\theta}$
$J_{03}=\sin\theta\sin\phi\, p_r+\dfrac{\cos\theta\sin\phi}{\tanh r}\,p_\theta+\dfrac{\cos\phi\, p_\phi}{\tanh r\sin\theta}$	$J_{03}=-\sinh\theta\sin\phi\, p_r+\dfrac{\cosh\theta\sin\phi}{\tanh r}\,p_\theta+\dfrac{\cos\phi\, p_\phi}{\tanh r\sinh\theta}$
$J_{12}=\cos\phi\, p_\theta-\dfrac{\sin\phi}{\tan\theta}\,p_\phi$	$J_{12}=\cos\phi\, p_\theta-\dfrac{\sin\phi}{\tanh\theta}\,p_\phi$
$J_{13}=\sin\phi\, p_\theta+\dfrac{\cos\phi}{\tan\theta}\,p_\phi$	$J_{13}=\sin\phi\, p_\theta+\dfrac{\cos\phi}{\tanh\theta}\,p_\phi$
$J_{23}=p_\phi$	$J_{23}=p_\phi$

4 Superintegrable potentials

Now if we look for superintegrable potentials $\mathcal{U}(q)=\mathcal{U}(r,\theta,\phi)$ which generalize the Euclidean one (1.1) to the space $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ we find

$$\begin{aligned}\mathcal{U}&=\mathcal{F}'(x_0)+\frac{\beta_1}{x_1^2}+\frac{\beta_2}{x_2^2}+\frac{\beta_3}{x_3^2}\\ &=\mathcal{F}(r)+\frac{1}{\mathrm{S}^2_{\kappa_1}(r)}\left(\frac{\beta_1}{\mathrm{C}^2_{\kappa_2}(\theta)}+\frac{\beta_2}{\mathrm{S}^2_{\kappa_2}(\theta)\cos^2\phi}+\frac{\beta_3}{\mathrm{S}^2_{\kappa_2}(\theta)\sin^2\phi}\right),\end{aligned}\tag{4.1}$$

where $\mathcal{F}'(\,\mathrm{C}_{\kappa_1}(r))\equiv\mathcal{F}(r)$ is an arbitrary smooth function and β_i are arbitrary real constants. As in $\mathbf{E}^3$, the three β_i-terms can be interpreted on the six spaces in a common way as "centrifugal barriers"; for some particular curved spaces these may admit an alternative interpretation as non-central harmonic oscillators. These facts will be explained in detail in the next section.

The resulting Hamiltonian $\mathcal{H}=\mathcal{T}+\mathcal{U}$, with kinetic energy (3.4) and potential (4.1), has three integrals of the motion quadratic in the momenta which are associated with the (Lorentz) rotation generators ($j=2,3$):

$$I_{1j}=J_{1j}^2+2\beta_1\kappa_2^2\frac{x_j^2}{x_1^2}+2\beta_j\kappa_2\frac{x_1^2}{x_j^2},\qquad I_{23}=J_{23}^2+2\beta_2\kappa_2\frac{x_3^2}{x_2^2}+2\beta_3\kappa_2\frac{x_2^2}{x_3^2}, \tag{4.2}$$

which in the geodesic polar phase space explicitly read

$$\begin{aligned}
I_{12}&=\left(\cos\phi\, p_\theta-\frac{\sin\phi}{\mathrm{T}_{\kappa_2}(\theta)}\,p_\phi\right)^2+2\beta_1\kappa_2^2\,\mathrm{T}^2_{\kappa_2}(\theta)\cos^2\phi+\frac{2\beta_2\kappa_2}{\mathrm{T}^2_{\kappa_2}(\theta)\cos^2\phi},\\
I_{13}&=\left(\sin\phi\, p_\theta+\frac{\cos\phi}{\mathrm{T}_{\kappa_2}(\theta)}\,p_\phi\right)^2+2\beta_1\kappa_2^2\,\mathrm{T}^2_{\kappa_2}(\theta)\sin^2\phi+\frac{2\beta_3\kappa_2}{\mathrm{T}^2_{\kappa_2}(\theta)\sin^2\phi},\\
I_{23}&=p_\phi^2+2\beta_2\kappa_2\tan^2\phi+\frac{2\beta_3\kappa_2}{\tan^2\phi}.
\end{aligned} \tag{4.3}$$

These constants of the motion do not Poisson commute with each other. In order to find quantities in involution we define another integral from the above set:

$$\begin{aligned}
I_{123}&=I_{12}+I_{13}+\kappa_2 I_{23}+2\kappa_2(\beta_1+\kappa_2\beta_2+\kappa_2\beta_3)\\
&=p_\theta^2+\frac{p_\phi^2}{\mathrm{S}^2_{\kappa_2}(\theta)}+\frac{2\beta_1\kappa_2}{\mathrm{C}^2_{\kappa_2}(\theta)}+\frac{2\beta_2\kappa_2}{\mathrm{S}^2_{\kappa_2}(\theta)\cos^2\phi}+\frac{2\beta_3\kappa_2}{\mathrm{S}^2_{\kappa_2}(\theta)\sin^2\phi},
\end{aligned} \tag{4.4}$$

which is related with the Casimir of the rotation subalgebra $\mathfrak{h}_0=so_{\kappa_2}(3)$.

Superintegrability of $\mathcal{H}$ is then characterized by:

Proposition 2. (i) *The three functions $\{I_{12},I_{123},\mathcal{H}\}$ are mutually in involution. The same holds for the set $\{I_{23},I_{123},\mathcal{H}\}$.*

(ii) *The four functions $\{I_{12},I_{23},I_{123},\mathcal{H}\}$ are functionally independent, thus $\mathcal{H}$ is a superintegrable Hamiltonian.*

These results, which can be checked directly, are displayed in Table 3 for each particular space arising within $\mathbb{S}^3_{[\kappa_1]\kappa_2}$. Notice that the integrals $\{I_{12},I_{23},I_{123}\}$ do depend on κ_2 and $(\theta,\phi;p_\theta,p_\phi)$ but neither on the curvature κ_1 nor on (r,p_r), so these are the same for each set of three spaces with the same signature.

A straightforward consequence of the complete integrability determined by $\{I_{23},I_{123},\mathcal{H}\}$ is that $\mathcal{H}$ is separable and we obtain three equations, each of them depending on a canonical pair (q_i,p_i):

$$\begin{aligned}
I_{23}(\phi,p_\phi)&=p_\phi^2+2\beta_2\kappa_2\tan^2\phi+\frac{2\beta_3\kappa_2}{\tan^2\phi},\\
I_{123}(\theta,p_\theta)&=p_\theta^2+\frac{2\beta_1\kappa_2}{\mathrm{C}^2_{\kappa_2}(\theta)}+\frac{1}{\mathrm{S}^2_{\kappa_2}(\theta)}\left(I_{23}+2\kappa_2(\beta_2+\beta_3)\right),\\
\mathcal{H}(r,p_r)&=\frac{1}{2}\,p_r^2+\mathcal{F}(r)+\frac{1}{2\kappa_2\,\mathrm{S}^2_{\kappa_1}(r)}\,I_{123},
\end{aligned} \tag{4.5}$$

and $\mathcal{H}$ is so reduced to a 1D radial system.

Therefore there remains *one* constant of the motion to obtain *maximal superintegrability* so that we shall say that $\mathcal{H}$ is a *quasi-maximally superintegrable* Hamiltonian. In the next sections we study two specific choices for the arbitrary radial function $\mathcal{F}(r)$ that lead to an additional integral thus providing maximally superintegrable potentials. The resulting systems are generalizations of the (curved) harmonic oscillator and KC potentials with additional terms (dependending on the β_i).

Table 3. Superintegrable Hamiltonian $\mathcal{H}=\mathcal{T}+\mathcal{U}$ and its three constants of the motion $\{I_{12},I_{23},I_{123}\}$ for the six spaces $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ with $\kappa_1\in\{+1,0,-1\}$ and $\kappa_2\in\{+1,-1\}$.

3D Riemannian spaces

• Spherical space $\mathbb{S}^3_{[+]+}\equiv\mathbf{S}^3$

$$\mathcal{H}=\frac{1}{2}\left(p_r^2+\frac{p_\theta^2}{\sin^2 r}+\frac{p_\phi^2}{\sin^2 r\sin^2\theta}\right)+\mathcal{F}(r)+\frac{1}{\sin^2 r}\left(\frac{\beta_1}{\cos^2\theta}+\frac{\beta_2}{\sin^2\theta\cos^2\phi}+\frac{\beta_3}{\sin^2\theta\sin^2\phi}\right)$$

• Euclidean space $\mathbb{S}^3_{[0]+}\equiv\mathbf{E}^3$

$$\mathcal{H}=\frac{1}{2}\left(p_r^2+\frac{p_\theta^2}{r^2}+\frac{p_\phi^2}{r^2\sin^2\theta}\right)+\mathcal{F}(r)+\frac{1}{r^2}\left(\frac{\beta_1}{\cos^2\theta}+\frac{\beta_2}{\sin^2\theta\cos^2\phi}+\frac{\beta_3}{\sin^2\theta\sin^2\phi}\right)$$

• Hyperbolic space $\mathbb{S}^3_{[-]+}\equiv\mathbf{H}^3$

$$\mathcal{H}=\frac{1}{2}\left(p_r^2+\frac{p_\theta^2}{\sinh^2 r}+\frac{p_\phi^2}{\sinh^2 r\sin^2\theta}\right)+\mathcal{F}(r)+\frac{1}{\sinh^2 r}\left(\frac{\beta_1}{\cos^2\theta}+\frac{\beta_2}{\sin^2\theta\cos^2\phi}+\frac{\beta_3}{\sin^2\theta\sin^2\phi}\right)$$

$$I_{12}=\left(\cos\phi\,p_\theta-\frac{\sin\phi}{\tan\theta}\,p_\phi\right)^2+2\beta_1\tan^2\theta\cos^2\phi+\frac{2\beta_2}{\tan^2\theta\cos^2\phi}$$

$$I_{23}=p_\phi^2+2\beta_2\tan^2\phi+\frac{2\beta_3}{\tan^2\phi}$$

$$I_{123}=p_\theta^2+\frac{p_\phi^2}{\sin^2\theta}+\frac{2\beta_1}{\cos^2\theta}+\frac{2\beta_2}{\sin^2\theta\cos^2\phi}+\frac{2\beta_3}{\sin^2\theta\sin^2\phi}$$

$(2+1)$D Relativistic spacetimes

• Anti-de Sitter spacetime $\mathbb{S}^3_{[+]-}\equiv\mathbf{AdS}^{2+1}$

$$\mathcal{H}=\frac{1}{2}\left(p_r^2-\frac{p_\theta^2}{\sin^2 r}-\frac{p_\phi^2}{\sin^2 r\sinh^2\theta}\right)+\mathcal{F}(r)+\frac{1}{\sin^2 r}\left(\frac{\beta_1}{\cosh^2\theta}+\frac{\beta_2}{\sinh^2\theta\cos^2\phi}+\frac{\beta_3}{\sinh^2\theta\sin^2\phi}\right)$$

• Minkowskian spacetime $\mathbb{S}^3_{[0]-}\equiv\mathbf{M}^{2+1}$

$$\mathcal{H}=\frac{1}{2}\left(p_r^2-\frac{p_\theta^2}{r^2}-\frac{p_\phi^2}{r^2\sinh^2\theta}\right)+\mathcal{F}(r)+\frac{1}{r^2}\left(\frac{\beta_1}{\cosh^2\theta}+\frac{\beta_2}{\sinh^2\theta\cos^2\phi}+\frac{\beta_3}{\sinh^2\theta\sin^2\phi}\right)$$

• De Sitter spacetime $\mathbb{S}^3_{[-]-}\equiv\mathbf{dS}^{2+1}$

$$\mathcal{H}=\frac{1}{2}\left(p_r^2-\frac{p_\theta^2}{\sinh^2 r}-\frac{p_\phi^2}{\sinh^2 r\sinh^2\theta}\right)+\mathcal{F}(r)+\frac{1}{\sinh^2 r}\left(\frac{\beta_1}{\cosh^2\theta}+\frac{\beta_2}{\sinh^2\theta\cos^2\phi}+\frac{\beta_3}{\sinh^2\theta\sin^2\phi}\right)$$

$$I_{12}=\left(\cos\phi\,p_\theta-\frac{\sin\phi}{\tanh\theta}\,p_\phi\right)^2+2\beta_1\tanh^2\theta\cos^2\phi-\frac{2\beta_2}{\tanh^2\theta\cos^2\phi}$$

$$I_{23}=p_\phi^2-2\beta_2\tan^2\phi-\frac{2\beta_3}{\tan^2\phi}$$

$$I_{123}=p_\theta^2+\frac{p_\phi^2}{\sinh^2\theta}-\frac{2\beta_1}{\cosh^2\theta}-\frac{2\beta_2}{\sinh^2\theta\cos^2\phi}-\frac{2\beta_3}{\sinh^2\theta\sin^2\phi}$$

5 Harmonic oscillator potential

If we like to extend the (curved) harmonic oscillator potential (1.4) to our six spaces, we have to consider the following choice for the arbitrary function appearing in (4.1):

$$\mathcal{F}'(x_0)=\beta_0\left(\frac{1-x_0^2}{\kappa_1 x_0^2}\right)=\beta_0\left(\frac{x_1^2+\kappa_2x_2^2+\kappa_2x_3^2}{x_0^2}\right),\qquad \mathcal{F}(r)=\beta_0\,\mathrm{T}^2_{\kappa_1}(r), \tag{5.1}$$

where β_0 is an arbitrary real parameter. When the complete Hamiltonian is considered we obtain the generalization of 3D SW system (1.2), $\mathcal{H}^{\mathrm{SW}}=\mathcal{T}+\mathcal{U}^{\mathrm{SW}}$, to the space $\mathbb{S}^3_{[\kappa_1]\kappa_2}$, namely

$$\mathcal{U}^{\mathrm{SW}}=\beta_0\,\mathrm{T}^2_{\kappa_1}(r)+\frac{1}{\mathrm{S}^2_{\kappa_1}(r)}\left(\frac{\beta_1}{\mathrm{C}^2_{\kappa_2}(\theta)}+\frac{\beta_2}{\mathrm{S}^2_{\kappa_2}(\theta)\cos^2\phi}+\frac{\beta_3}{\mathrm{S}^2_{\kappa_2}(\theta)\sin^2\phi}\right). \tag{5.2}$$

As we already mentioned in the introduction, the proper SW Hamiltonian arises in the (flat) Euclidean space [15, 16, 17, 19], here written in polar coordinates, which is formed by an isotropic harmonic oscillator with angular frequency $\omega = \sqrt{\beta_0}$ together with three centrifugal barriers associated with the β_i's. Different constructions of the SW system on the (curved) spherical and hyperbolic spaces can be found in [9, 20, 23, 28, 30, 31, 39]. More recently such a potential has also been deduced and analysed in the $(1+1)$D relativistic spacetimes of constant curvature in [8, 12] as well as in 2D spaces of variable curvature in [8].

In our case, any of the translation generators J_{0i} (3.7) provides a constant of the motion quadratic in the momenta in the form ($j = 2, 3$):

$$
\begin{aligned}
&I_{01} = J_{01}^2 + 2\beta_0 \frac{x_1^2}{x_0^2} + 2\beta_1 \frac{x_1^2}{x_0^2},\\
&I_{0j} = J_{0j}^2 + 2\beta_0 \kappa_2^2 \frac{x_j^2}{x_0^2} + 2\beta_j \kappa_2 \frac{x_0^2}{x_j^2},
\end{aligned}
\tag{5.3}
$$

that is,

$$
\begin{aligned}
&I_{01} = J_{01}^2 + 2\beta_0 \, \mathrm{T}_{\kappa_1}^2(r)\, \mathrm{C}_{\kappa_2}^2(\theta) + \frac{2\beta_1}{\mathrm{T}_{\kappa_1}^2(r)\, \mathrm{C}_{\kappa_2}^2(\theta)},\\
&I_{02} = J_{02}^2 + 2\beta_0 \kappa_2^2 \, \mathrm{T}_{\kappa_1}^2(r)\, \mathrm{S}_{\kappa_2}^2(\theta)\cos^2\phi + \frac{2\beta_2\kappa_2}{\mathrm{T}_{\kappa_1}^2(r)\, \mathrm{S}_{\kappa_2}^2(\theta)\cos^2\phi},\\
&I_{03} = J_{03}^2 + 2\beta_0 \kappa_2^2 \, \mathrm{T}_{\kappa_1}^2(r)\, \mathrm{S}_{\kappa_2}^2(\theta)\sin^2\phi + \frac{2\beta_3\kappa_2}{\mathrm{T}_{\kappa_1}^2(r)\, \mathrm{S}_{\kappa_2}^2(\theta)\sin^2\phi}.
\end{aligned}
\tag{5.4}
$$

Obviously, the seven integrals of the motion $\{I_{01}, I_{02}, I_{03}, I_{12}, I_{23}, I_{123}, \mathcal{H}^{\mathrm{SW}}\}$ cannot be functionally independent. One constraint for them is given by

$$
2\kappa_2 \mathcal{H}^{\mathrm{SW}} = \kappa_2 I_{01} + I_{02} + I_{03} + \kappa_1 I_{123},
$$

which reminds the aforementioned relation for the geodesic motion $2\kappa_2 \mathcal{T} = \mathcal{C}_1$. Note also that

$$
\{I_{01}, I_{23}\} = \{I_{02}, I_{13}\} = \{I_{03}, I_{12}\} = 0.
$$

The final result concerning the superintegrability of $\mathcal{H}^{\mathrm{SW}}$ is established by:

Proposition 3. (i) *Each function I_{0i}* (5.4) ($i = 1, 2, 3$) *Poisson commutes with $\mathcal{H}^{\mathrm{SW}}$.*

(ii) *The five functions $\{I_{0i}, I_{12}, I_{23}, I_{123}, \mathcal{H}^{\mathrm{SW}}\}$, where i is fixed, are functionally independent, thus $\mathcal{H}^{\mathrm{SW}}$ is a maximally superintegrable Hamiltonian.*

The Hamiltonian $\mathcal{H}^{\mathrm{SW}}$ and the additional constant of the motion I_{01} (that ensures maximal superintegrability) are presented for each particular space contained in $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ in Table 4.

5.1 Description of the SW potential

The 2D version of the potential $\mathcal{U}^{\mathrm{SW}}$ (5.2) on $\mathbf{S}^2$ has been interpreted in [40, 41, 42] as a superposition of three spherical oscillators; the interpretation for arbitrary dimension on $\mathbf{S}^N$ and $\mathbf{H}^N$ has been presented in [9, 23]. Furthermore, a detail description on this potential on $\mathbf{AdS}^{1+1}$, $\mathbf{M}^{1+1}$ and $\mathbf{dS}^{1+1}$ was recently performed in [8]. In what follows we analyse the (physical) geometrical role of the 3D potential (5.2) on each particular space $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ thus generalizing all of the mentioned 2D results.

Table 4. Maximally superintegrable Smorodinsky–Winternitz Hamiltonian $\mathcal{H}^{\rm SW} = \mathcal{T} + \mathcal{U}^{\rm SW}$ and the additional constant of the motion I_{01} to the set $\{I_{12}, I_{23}, I_{123}\}$ for the six spaces $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ with the same conventions given in Table 3.

3D Riemannian spaces

• Spherical space $\mathbf{S}^3$

$$\mathcal{H}^{\rm SW} = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{\sin^2 r} + \frac{p_\phi^2}{\sin^2 r \sin^2\theta}\right) + \beta_0 \tan^2 r + \frac{1}{\sin^2 r}\left(\frac{\beta_1}{\cos^2\theta} + \frac{\beta_2}{\sin^2\theta\cos^2\phi} + \frac{\beta_3}{\sin^2\theta\sin^2\phi}\right)$$

$$I_{01} = \left(\cos\theta\, p_r - \frac{\sin\theta}{\tan r}\, p_\theta\right)^2 + 2\beta_0 \tan^2 r \cos^2\theta + \frac{2\beta_1}{\tan^2 r\cos^2\theta}$$

• Euclidean space $\mathbf{E}^3$

$$\mathcal{H}^{\rm SW} = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + \beta_0 r^2 + \frac{1}{r^2}\left(\frac{\beta_1}{\cos^2\theta} + \frac{\beta_2}{\sin^2\theta\cos^2\phi} + \frac{\beta_3}{\sin^2\theta\sin^2\phi}\right)$$

$$I_{01} = \left(\cos\theta\, p_r - \frac{\sin\theta}{r}\, p_\theta\right)^2 + 2\beta_0 r^2 \cos^2\theta + \frac{2\beta_1}{r^2\cos^2\theta}$$

• Hyperbolic space $\mathbf{H}^3$

$$\mathcal{H}^{\rm SW} = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{\sinh^2 r} + \frac{p_\phi^2}{\sinh^2 r \sin^2\theta}\right) + \beta_0 \tanh^2 r + \frac{1}{\sinh^2 r}\left(\frac{\beta_1}{\cos^2\theta} + \frac{\beta_2}{\sin^2\theta\cos^2\phi} + \frac{\beta_3}{\sin^2\theta\sin^2\phi}\right)$$

$$I_{01} = \left(\cos\theta\, p_r - \frac{\sin\theta}{\tanh r}\, p_\theta\right)^2 + 2\beta_0 \tanh^2 r \cos^2\theta + \frac{2\beta_1}{\tanh^2 r\cos^2\theta}$$

(2 + 1)D Relativistic spacetimes

• Anti-de Sitter spacetime $\mathbf{AdS}^{2+1}$

$$\mathcal{H}^{\rm SW} = \frac{1}{2}\left(p_r^2 - \frac{p_\theta^2}{\sin^2 r} - \frac{p_\phi^2}{\sin^2 r \sinh^2\theta}\right) + \beta_0 \tan^2 r + \frac{1}{\sin^2 r}\left(\frac{\beta_1}{\cosh^2\theta} + \frac{\beta_2}{\sinh^2\theta\cos^2\phi} + \frac{\beta_3}{\sinh^2\theta\sin^2\phi}\right)$$

$$I_{01} = \left(\cosh\theta\, p_r - \frac{\sinh\theta}{\tan r}\, p_\theta\right)^2 + 2\beta_0 \tan^2 r \cosh^2\theta + \frac{2\beta_1}{\tan^2 r\cosh^2\theta}$$

• Minkowskian spacetime $\mathbf{M}^{2+1}$

$$\mathcal{H}^{\rm SW} = \frac{1}{2}\left(p_r^2 - \frac{p_\theta^2}{r^2} - \frac{p_\phi^2}{r^2 \sinh^2\theta}\right) + \beta_0 r^2 + \frac{1}{r^2}\left(\frac{\beta_1}{\cosh^2\theta} + \frac{\beta_2}{\sinh^2\theta\cos^2\phi} + \frac{\beta_3}{\sinh^2\theta\sin^2\phi}\right)$$

$$I_{01} = \left(\cosh\theta\, p_r - \frac{\sinh\theta}{r}\, p_\theta\right)^2 + 2\beta_0 r^2 \cosh^2\theta + \frac{2\beta_1}{r^2\cosh^2\theta}$$

• De Sitter spacetime $\mathbf{dS}^{2+1}$

$$\mathcal{H}^{\rm SW} = \frac{1}{2}\left(p_r^2 - \frac{p_\theta^2}{\sinh^2 r} - \frac{p_\phi^2}{\sinh^2 r \sinh^2\theta}\right) + \beta_0 \tanh^2 r + \frac{1}{\sinh^2 r}\left(\frac{\beta_1}{\cosh^2\theta} + \frac{\beta_2}{\sinh^2\theta\cos^2\phi} + \frac{\beta_3}{\sinh^2\theta\sin^2\phi}\right)$$

$$I_{01} = \left(\cosh\theta\, p_r - \frac{\sinh\theta}{\tanh r}\, p_\theta\right)^2 + 2\beta_0 \tanh^2 r \cosh^2\theta + \frac{2\beta_1}{\tanh^2 r\cosh^2\theta}$$

Consider the (time-like) geodesic l_1 and the two (space-like) geodesics l_2, l_3 in $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ orthogonal at the origin O and the generic point $Q(r,\theta,\phi)$ as given in Subsection 2.2. Next let Q_{ij} $(i,j=1,2,3;\ i<j)$ be the intersection point of the reference flag spanned by l_i and l_j (the 2-plane l_il_j) with its orthogonal geodesic through Q. Hence we introduce the (time-like) geodesic distance $x = QQ_{23}$ and the two (space-like) distances $y = QQ_{13}$, $z = QQ_{12}$. Finally, let Q_1 be the intersection point of l_1 with its orthogonal (space-like) geodesic l_1' through Q for which $h = QQ_1$ is the (space-like) distance measured along l_1'. Now by applying trigonometry [24] on the orthogonal triangles OQQ_1 (with inner angle θ), OQQ_{23} (with external angle θ), $Q_{13}QQ_1$ (with external angle ϕ) and $Q_{12}QQ_1$ (with inner angle ϕ), we find that

$$\begin{aligned} OQQ_1:&\quad \mathrm{S}_{\kappa_1\kappa_2}(h) = \mathrm{S}_{\kappa_1}(r)\,\mathrm{S}_{\kappa_2}(\theta),\\ OQQ_{23}:&\quad \mathrm{S}_{\kappa_1}(x) = \mathrm{S}_{\kappa_1}(r)\,\mathrm{C}_{\kappa_2}(\theta), \end{aligned}$$

$$
\begin{aligned}
&Q_{13}QQ_1: \quad \mathrm{S}_{\kappa_1\kappa_2}(y) = \mathrm{S}_{\kappa_1\kappa_2}(h)\cos\phi,\\
&Q_{12}QQ_1: \quad \mathrm{S}_{\kappa_1\kappa_2}(z) = \mathrm{S}_{\kappa_1\kappa_2}(h)\sin\phi.
\end{aligned}
$$

Hence the ambient coordinates x_i (2.11) can be expressed as

$$
\begin{aligned}
x_1 &= \mathrm{S}_{\kappa_1}(r)\,\mathrm{C}_{\kappa_2}(\theta) = \mathrm{S}_{\kappa_1}(x),\\
x_2 &= \mathrm{S}_{\kappa_1}(r)\,\mathrm{S}_{\kappa_2}(\theta)\cos\phi = \mathrm{S}_{\kappa_1\kappa_2}(y),\\
x_3 &= \mathrm{S}_{\kappa_1}(r)\,\mathrm{S}_{\kappa_2}(\theta)\sin\phi = \mathrm{S}_{\kappa_1\kappa_2}(z),
\end{aligned} \tag{5.5}
$$

so that the SW potential (5.2) can be rewritten as

$$
\mathcal{U}^{\mathrm{SW}} = \beta_0\,\mathrm{T}^2_{\kappa_1}(r) + \frac{\beta_1}{\mathrm{S}^2_{\kappa_1}(x)} + \frac{\beta_2}{\mathrm{S}^2_{\kappa_1\kappa_2}(y)} + \frac{\beta_3}{\mathrm{S}^2_{\kappa_1\kappa_2}(z)}, \tag{5.6}
$$

which allows for a unified interpretation on the six spaces:

- The β_0-term is a *central* harmonic oscillator, that is, the Higgs oscillator [26] with center at the origin O.
- The three β_i-terms ($i = 1, 2, 3$) are "centrifugal barriers".

Furthermore, the β_i-potentials can be interpreted as non-central oscillators in some particular spaces that we proceed to describe by considering the simplest values for $\kappa_i \in \{\pm 1\}$.

5.1.1 Spherical space $\mathbf{S}^3$

Let O_i be the points placed along the basic geodesics l_i ($i = 1, 2, 3$) which are a quadrant apart from the origin O, that is, each two points taken from the set $\{O, O_i\}$ are mutually separated a distance $\frac{\pi}{2}$ (if $\kappa_1 = 1/R^2$, a quadrant is $\pi/(2\sqrt{\kappa_1}) = R\pi/2$). In fact, each O_i is the intersection point between the geodesic l_i and the axis x_i of the ambient space. If we denote by r_i the distance between Q and O_i measured along the geodesic joining both points then

$$
r_1 + x = r_2 + y = r_3 + z = \frac{\pi}{2},
$$

which means that each set of three points $\{O_1QQ_{23}\}$, $\{O_2QQ_{13}\}$ and $\{O_3QQ_{12}\}$ lie on the same geodesic. Thus

$$
x_1 = \sin x = \cos r_1, \qquad x_2 = \sin y = \cos r_2, \qquad x_3 = \sin z = \cos r_3,
$$

so that the SW potential (5.2) on $\mathbf{S}^3$ can be expressed in two manners

$$
\mathcal{U}^{\mathrm{SW}} = \beta_0 \tan^2 r + \frac{\beta_1}{\sin^2 x} + \frac{\beta_2}{\sin^2 y} + \frac{\beta_3}{\sin^2 z} \tag{5.7}
$$

$$
= \beta_0 \tan^2 r + \sum_{i=1}^{3} \left(\beta_i \tan^2 r_i + \beta_i\right), \tag{5.8}
$$

which show a superposition of the central spherical oscillator with center at O either with three spherical centrifugal barriers, or with three spherical oscillators with centers placed at O_i [9, 23].

5.1.2 Hyperbolic space $\mathbf{H}^3$

The analogous points to the previous "centers" O_i would be beyond the "actual" hyperbolic space and so placed in the exterior ("ideal") region of $\mathbf{H}^3$. The SW potential can only written in the form (5.6):

$$
\mathcal{U}^{\mathrm{SW}} = \beta_0 \tanh^2 r + \frac{\beta_1}{\sinh^2 x} + \frac{\beta_2}{\sinh^2 y} + \frac{\beta_3}{\sinh^2 z}, \tag{5.9}
$$

giving rise to the superposition of a central hyperbolic oscillator with three hyperbolic centrifugal barriers [23].

5.1.3 Euclidean space $\mathbf{E}^3$

The contraction $\kappa_1 \to 0$ ($R \to \infty$) of the SW potential on $\mathbf{S}^3$ and $\mathbf{H}^3$ can be applied on both expressions (5.7) and (5.9) reducing to

$$\mathcal{U}^{\mathrm{SW}} = \beta_0 r^2 + \frac{\beta_1}{x^2} + \frac{\beta_2}{y^2} + \frac{\beta_3}{z^2}, \tag{5.10}$$

which is just the proper SW potential (1.2) formed by the flat harmonic oscillator with three centrifugal barriers; in this case (x, y, z) are Cartesian coordinates on $\mathbf{E}^3$ and $r^2 = x^2 + y^2 + z^2$. This contraction cannot be performed on $\mathbf{S}^3$ when the potential is written in the form (5.8); notice that if $\kappa_1 \to 0$ the points $O_i \to \infty$.

5.1.4 Anti-de Sitter spacetime $\mathbf{AdS}^{2+1}$

We consider the intersection point O_1 between the time-like geodesic l_1 and the axis x_1 of the ambient space, which is at a time-like distance $\frac{\pi}{2}$ from the origin O [8]. If r_1 denotes the time-like distance QO_1, then $r_1 + x = \frac{\pi}{2}$. Therefore the SW potential becomes

$$\begin{aligned}\mathcal{U}^{\mathrm{SW}} &= \beta_0 \tan^2 r + \frac{\beta_1}{\sin^2 x} + \frac{\beta_2}{\sinh^2 y} + \frac{\beta_3}{\sinh^2 z} &(5.11)\\ &= \beta_0 \tan^2 r + \beta_1 \tan^2 r_1 + \frac{\beta_2}{\sinh^2 y} + \frac{\beta_3}{\sinh^2 z} + \beta_1. &(5.12)\end{aligned}$$

The former expression corresponds to the superposition of a time-like (spherical) oscillator centered at O with a time-like (spherical) centrifugal potential and two space-like (hyperbolic) ones. Under the latter form, the time-like centrifugal term is transformed into another spherical oscillator now with center at O_1.

5.1.5 De Sitter spacetime $\mathbf{dS}^{2+1}$

Recall that $\mathbf{AdS}^{2+1}$ and $\mathbf{dS}^{2+1}$ are related through an interchange between time-like lines and space-like ones; the former are compact (circular) on $\mathbf{AdS}^{2+1}$ and non-compact (hyperbolic) on $\mathbf{dS}^{2+1}$, while the latter are non-compact on $\mathbf{AdS}^{2+1}$ but compact on $\mathbf{dS}^{2+1}$.

So, we consider the intersection point O_j ($j = 2, 3$) between the space-like geodesic l_j and the axis x_j which is at a space-like distance $\frac{\pi}{2}$ from O, so that r_j is the space-like distance QO_j verifying $r_2 + y = r_3 + z = \frac{\pi}{2}$ [8]. Hence the SW potential can be rewritten as

$$\begin{aligned}\mathcal{U}^{\mathrm{SW}} &= \beta_0 \tanh^2 r + \frac{\beta_1}{\sinh^2 x} + \frac{\beta_2}{\sin^2 y} + \frac{\beta_3}{\sin^2 z} &(5.13)\\ &= \beta_0 \tanh^2 r + \frac{\beta_1}{\sinh^2 x} + \beta_2 \tan^2 r_2 + \beta_3 \tan^2 r_3 + \beta_2 + \beta_3. &(5.14)\end{aligned}$$

In this way, we find the superposition of a central time-like (hyperbolic) oscillator with a time-like (hyperbolic) centrifugal barrier, and either with two other space-like (spherical) centrifugal barriers or with two space-like (spherical) oscillators centered at O_j.

5.1.6 Minkowskian spacetime $\mathbf{M}^{2+1}$

Finally, the contraction $\kappa_1 \to 0$ ($\tau \to \infty$) of (5.11) and (5.13) gives

$$\mathcal{U}^{\mathrm{SW}} = \beta_0 r^2 + \frac{\beta_1}{x^2} + \frac{\beta_2}{y^2} + \frac{\beta_3}{z^2}, \tag{5.15}$$

which is formed by a time-like harmonic oscillator $\beta_0 r^2$, one time-like centrifugal barrier β_1/x^2 together with two space-like ones β_2/y^2, β_3/z^2. The coordinates (x,y,z) are the usual time and space ones such that $r^2 = x^2 - y^2 - z^2$. On the contrary, the expressions (5.12) and (5.14) are not well defined when $\kappa_1 \to 0$ since the points O_1 and O_j go to infinity.

6 Kepler–Coulomb potential

The generalization of the KC potential (1.4) to the space $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ is achieved by choosing

$$\mathcal{F}'(x_0) = -k\,\frac{x_0}{\sqrt{(1-x_0^2)/\kappa_1}} = -k\,\frac{x_0}{\sqrt{x_1^2+\kappa_2 x_2^2+\kappa_2 x_3^2}}, \qquad \mathcal{F}(r) = -\frac{k}{\mathrm{T}_{\kappa_1}(r)}, \tag{6.1}$$

where k is an arbitrary real parameter. As it already happens in $\mathbf{E}^3$ [14], it is not possible to add the three potential terms depending on the β_i's keeping at the same time maximal superintegrability; so that, at least, one of them must vanish. Consequently, we find, in principle, three possible generalizations of the Euclidean potential (1.3) to $\mathbb{S}^3_{[\kappa_1]\kappa_2}$:

$$\begin{aligned}
\mathcal{U}_1^{\mathrm{GKC}} &= -\frac{k}{\mathrm{T}_{\kappa_1}(r)} + \frac{1}{\mathrm{S}^2_{\kappa_1}(r)\,\mathrm{S}^2_{\kappa_2}(\theta)}\left(\frac{\beta_2}{\cos^2\phi}+\frac{\beta_3}{\sin^2\phi}\right),\\
\mathcal{U}_2^{\mathrm{GKC}} &= -\frac{k}{\mathrm{T}_{\kappa_1}(r)} + \frac{1}{\mathrm{S}^2_{\kappa_1}(r)}\left(\frac{\beta_1}{\mathrm{C}^2_{\kappa_2}(\theta)}+\frac{\beta_3}{\mathrm{S}^2_{\kappa_2}(\theta)\sin^2\phi}\right),\\
\mathcal{U}_3^{\mathrm{GKC}} &= -\frac{k}{\mathrm{T}_{\kappa_1}(r)} + \frac{1}{\mathrm{S}^2_{\kappa_1}(r)}\left(\frac{\beta_1}{\mathrm{C}^2_{\kappa_2}(\theta)}+\frac{\beta_2}{\mathrm{S}^2_{\kappa_2}(\theta)\cos^2\phi}\right).
\end{aligned}\tag{6.2}$$

Thus each potential $\mathcal{U}_i^{\mathrm{GKC}}$ contains the proper KC k-term [11, 12, 28, 31, 32, 33, 37, 39, 43] together with two additional β_i-terms, which can further be interpreted as centrifugal barriers or non-central oscillators; for each of them there is an additional constant of the motion given by $(i=1,2,3)$:

$$L_i = \sum_{l=1;l\neq i}^{3} J_{0l}J_{li} + k\,\frac{\kappa_2 x_i}{\sqrt{x_1^2+\kappa_2 x_2^2+\kappa_2 x_3^2}} - 2\kappa_2 \sum_{l=1;l\neq i}^{3} \beta_l \frac{x_0 x_i}{x_l^2}, \tag{6.3}$$

where $J_{li} = -J_{il}$ for $i<l$. In terms of the geodesic polar phase space these integrals read

$$\begin{aligned}
L_1 &= -J_{02}J_{12} - J_{03}J_{13} + k\,\kappa_2\,\mathrm{C}_{\kappa_2}(\theta) - \frac{2\kappa_2\,\mathrm{C}_{\kappa_2}(\theta)}{\mathrm{T}_{\kappa_1}(r)\,\mathrm{S}^2_{\kappa_2}(\theta)}\left(\frac{\beta_2}{\cos^2\phi}+\frac{\beta_3}{\sin^2\phi}\right),\\
L_2 &= J_{01}J_{12} - J_{03}J_{23} + k\,\kappa_2\,\mathrm{S}_{\kappa_2}(\theta)\cos\phi - \frac{2\kappa_2\cos\phi}{\mathrm{T}_{\kappa_1}(r)}\left(\frac{\beta_1\,\mathrm{S}_{\kappa_2}(\theta)}{\mathrm{C}^2_{\kappa_2}(\theta)}+\frac{\beta_3}{\mathrm{S}_{\kappa_2}(\theta)\sin^2\phi}\right),\\
L_3 &= J_{01}J_{13} + J_{02}J_{23} + k\,\kappa_2\,\mathrm{S}_{\kappa_2}(\theta)\sin\phi - \frac{2\kappa_2\sin\phi}{\mathrm{T}_{\kappa_1}(r)}\left(\frac{\beta_1\,\mathrm{S}_{\kappa_2}(\theta)}{\mathrm{C}^2_{\kappa_2}(\theta)}+\frac{\beta_2}{\mathrm{S}_{\kappa_2}(\theta)\cos^2\phi}\right).
\end{aligned}\tag{6.4}$$

The superintegrability of *each* Hamiltonian $\mathcal{H}_i^{\mathrm{GKC}} = \mathcal{T} + \mathcal{U}_i^{\mathrm{GKC}}$ $(i=1,2,3)$ is determined by:

Proposition 4. (i) *The function L_i (6.4) Poisson commutes with $\mathcal{H}_i^{\mathrm{GKC}}$.*

(ii) *The five functions $\{L_i, I_{12}, I_{23}, I_{123}, \mathcal{H}_i^{\mathrm{GKC}}\}$ are functionally independent, thus $\mathcal{H}_i^{\mathrm{GKC}}$ is a maximally superintegrable Hamiltonian.*

6.1 The Laplace–Runge–Lenz vector

When another β_j is taken equal to zero in a given potential $\mathcal{U}_i^{\mathrm{GKC}}$ $(j \neq i)$, the function L_j is also a constant of the motion. Therefore when all the $\beta_j = 0$, the three functions (6.4) are constants of the motion for the GKC potential which reduces in this case to the proper KC system. This is summed up in the following statements.

Proposition 5. *Let one $\beta_j = 0$ in the Hamiltonian $\mathcal{H}_i^{\mathrm{GKC}} = \mathcal{T} + \mathcal{U}_i^{\mathrm{GKC}}$ $(i = 1, 2, 3)$ with $j \neq i$.*
(i) *The two functions L_i, L_j Poisson commute with $\mathcal{H}_i^{\mathrm{GKC}}$.*
(ii) *The functions $\{I_{12}, I_{23}, I_{123}, \mathcal{H}_i^{\mathrm{GKC}}\}$ together with either L_i or L_j are functionally independent.*

Proposition 6. *Let the three $\beta_i = 0$, then:*
(i) *The three GKC potentials reduce to its common k-term, $\mathcal{U}_i^{\mathrm{GKC}} \equiv \mathcal{U}^{\mathrm{KC}} = -k/\,\mathrm{T}_{\kappa_1}(r)$, which is the (curved) KC potential on $\mathbb{S}^3_{[\kappa_1]\kappa_2}$.*
(ii) *The three functions*

$$
\begin{aligned}
L_1 &= -J_{02}J_{12} - J_{03}J_{13} + k\,\kappa_2\,\mathrm{C}_{\kappa_2}(\theta),\\
L_2 &= J_{01}J_{12} - J_{03}J_{23} + k\,\kappa_2\,\mathrm{S}_{\kappa_2}(\theta)\cos\phi,\\
L_3 &= J_{01}J_{13} + J_{02}J_{23} + k\,\kappa_2\,\mathrm{S}_{\kappa_2}(\theta)\sin\phi,
\end{aligned}
\tag{6.5}
$$

Poisson commutes with $\mathcal{H}^{\mathrm{KC}} = \mathcal{T} + \mathcal{U}^{\mathrm{KC}}$ and these are the components of the Laplace–Runge–Lenz vector on $\mathbb{S}^3_{[\kappa_1]\kappa_2}$.
(iii) *The functions $\{I_{12}, I_{23}, I_{123}, \mathcal{H}_i^{\mathrm{GKC}}\}$ together with any of the components L_i are functionally independent.*

On the other hand, equivalence amongst the Hamiltonians $\mathcal{H}_i^{\mathrm{GKC}}$ comes from their interpretation on $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ that we proceed to study. We shall show that the three potentials (6.2) are equivalent on the Riemannian spaces (take $i = 3$), meanwhile we can distinguish two different potentials on the spacetimes (take $i = 1, 3$). Thus we display in Table 5 the corresponding non-equivalent GKC potentials together with the additional constant of the motion (6.4).

6.2 Description of the GKC potential

In Subsection 5.1 we have interpreted each of the β_i-terms appearing within the SW potential either as a non-central oscillator or as a centrifugal barrier according to the particular space under consideration. This, in turn, means that each potential (6.2) is a superposition of the KC potential with either oscillators or centrifugal barriers. The latter interpretation arises directly by introducing the distances (x, y, z) (5.5) and this holds simultaneously for the six spaces:

$$
\begin{aligned}
\mathcal{U}_1^{\mathrm{GKC}} &= -\frac{k}{\mathrm{T}_{\kappa_1}(r)} + \frac{\beta_2}{\mathrm{S}^2_{\kappa_1\kappa_2}(y)} + \frac{\beta_3}{\mathrm{S}^2_{\kappa_1\kappa_2}(z)},\\
\mathcal{U}_2^{\mathrm{GKC}} &= -\frac{k}{\mathrm{T}_{\kappa_1}(r)} + \frac{\beta_1}{\mathrm{S}^2_{\kappa_1}(x)} + \frac{\beta_3}{\mathrm{S}^2_{\kappa_1\kappa_2}(z)},\\
\mathcal{U}_3^{\mathrm{GKC}} &= -\frac{k}{\mathrm{T}_{\kappa_1}(r)} + \frac{\beta_1}{\mathrm{S}^2_{\kappa_1}(x)} + \frac{\beta_2}{\mathrm{S}^2_{\kappa_1\kappa_2}(y)}.
\end{aligned}
\tag{6.6}
$$

These expressions clearly show that some $\mathcal{H}_i^{\mathrm{GKC}}$ are equivalent according to the value of κ_2, that is, the signature of the metric, so that we analyze the two possibilities separately.

Table 5. Maximally superintegrable generalized Kepler–Coulomb potential $\mathcal{U}_i^{\rm GKC}$, such that $\mathcal{H}_i^{\rm GKC} = \mathcal{T} + \mathcal{U}_i^{\rm GKC}$, and the additional constant of the motion L_i to the set $\{I_{12}, I_{23}, I_{123}\}$ for $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ with the same conventions given in Table 3 ($i = 3$ for the Riemannian spaces and $i = 3, 1$ for the spacetimes).

3D Riemannian spaces

• Spherical space $\mathbf{S}^3$

$$\mathcal{U}_3^{\rm GKC} = -\frac{k}{\tan r} + \frac{1}{\sin^2 r}\left(\frac{\beta_1}{\cos^2\theta} + \frac{\beta_2}{\sin^2\theta\cos^2\phi}\right)$$

$$L_3 = J_{01}J_{13} + J_{02}J_{23} + k\sin\theta\sin\phi - \frac{2\sin\phi}{\tan r}\left(\frac{\beta_1\sin\theta}{\cos^2\theta} + \frac{\beta_2}{\sin\theta\cos^2\phi}\right)$$

• Euclidean space $\mathbf{E}^3$

$$\mathcal{U}_3^{\rm GKC} = -\frac{k}{r} + \frac{1}{r^2}\left(\frac{\beta_1}{\cos^2\theta} + \frac{\beta_2}{\sin^2\theta\cos^2\phi}\right)$$

$$L_3 = J_{01}J_{13} + J_{02}J_{23} + k\sin\theta\sin\phi - \frac{2\sin\phi}{r}\left(\frac{\beta_1\sin\theta}{\cos^2\theta} + \frac{\beta_2}{\sin\theta\cos^2\phi}\right)$$

• Hyperbolic space $\mathbf{H}^3$

$$\mathcal{U}_3^{\rm GKC} = -\frac{k}{\tanh r} + \frac{1}{\sinh^2 r}\left(\frac{\beta_1}{\cos^2\theta} + \frac{\beta_2}{\sin^2\theta\cos^2\phi}\right)$$

$$L_3 = J_{01}J_{13} + J_{02}J_{23} + k\sin\theta\sin\phi - \frac{2\sin\phi}{\tanh r}\left(\frac{\beta_1\sin\theta}{\cos^2\theta} + \frac{\beta_2}{\sin\theta\cos^2\phi}\right)$$

(2 + 1)D Relativistic spacetimes

• Anti-de Sitter spacetime $\mathbf{AdS}^{2+1}$

$$\mathcal{U}_3^{\rm GKC} = -\frac{k}{\tan r} + \frac{1}{\sin^2 r}\left(\frac{\beta_1}{\cosh^2\theta} + \frac{\beta_2}{\sinh^2\theta\cos^2\phi}\right)$$

$$L_3 = J_{01}J_{13} + J_{02}J_{23} - k\sinh\theta\sin\phi + \frac{2\sin\phi}{\tan r}\left(\frac{\beta_1\sinh\theta}{\cosh^2\theta} + \frac{\beta_2}{\sinh\theta\cos^2\phi}\right)$$

$$\mathcal{U}_1^{\rm GKC} = -\frac{k}{\tan r} + \frac{1}{\sin^2 r\sinh^2\theta}\left(\frac{\beta_2}{\cos^2\phi} + \frac{\beta_3}{\sin^2\phi}\right)$$

$$L_1 = -J_{02}J_{12} - J_{03}J_{13} - k\cosh\theta + \frac{2\cosh\theta}{\tan r\sinh^2\theta}\left(\frac{\beta_2}{\cos^2\phi} + \frac{\beta_3}{\sin^2\phi}\right)$$

• Minkowskian spacetime $\mathbf{M}^{2+1}$

$$\mathcal{U}_3^{\rm GKC} = -\frac{k}{r} + \frac{1}{r^2}\left(\frac{\beta_1}{\cosh^2\theta} + \frac{\beta_2}{\sinh^2\theta\cos^2\phi}\right)$$

$$L_3 = J_{01}J_{13} + J_{02}J_{23} - k\sinh\theta\sin\phi + \frac{2\sin\phi}{r}\left(\frac{\beta_1\sinh\theta}{\cosh^2\theta} + \frac{\beta_2}{\sinh\theta\cos^2\phi}\right)$$

$$\mathcal{U}_1^{\rm GKC} = -\frac{k}{r} + \frac{1}{r^2\sinh^2\theta}\left(\frac{\beta_2}{\cos^2\phi} + \frac{\beta_3}{\sin^2\phi}\right)$$

$$L_1 = -J_{02}J_{12} - J_{03}J_{13} - k\cosh\theta + \frac{2\cosh\theta}{r\sinh^2\theta}\left(\frac{\beta_2}{\cos^2\phi} + \frac{\beta_3}{\sin^2\phi}\right)$$

• De Sitter spacetime $\mathbf{dS}^{2+1}$

$$\mathcal{U}_3^{\rm GKC} = -\frac{k}{\tanh r} + \frac{1}{\sinh^2 r}\left(\frac{\beta_1}{\cosh^2\theta} + \frac{\beta_2}{\sinh^2\theta\cos^2\phi}\right)$$

$$L_3 = J_{01}J_{13} + J_{02}J_{23} - k\sinh\theta\sin\phi + \frac{2\sin\phi}{\tanh r}\left(\frac{\beta_1\sinh\theta}{\cosh^2\theta} + \frac{\beta_2}{\sinh\theta\cos^2\phi}\right)$$

$$\mathcal{U}_1^{\rm GKC} = -\frac{k}{\tanh r} + \frac{1}{\sinh^2 r\sinh^2\theta}\left(\frac{\beta_2}{\cos^2\phi} + \frac{\beta_3}{\sin^2\phi}\right)$$

$$L_1 = -J_{02}J_{12} - J_{03}J_{13} - k\cosh\theta + \frac{2\cosh\theta}{\tanh r\sinh^2\theta}\left(\frac{\beta_2}{\cos^2\phi} + \frac{\beta_3}{\sin^2\phi}\right)$$

6.2.1 Riemannian spaces

When $\kappa_2 = +1$ the three distances (x, y, z) are completely equivalent, and their "label" in the trigonometric functions is always κ_1 (recall that in this case both θ and ϕ are ordinary angles). Hence the three Hamiltonians $\mathcal{H}_i^{\rm GKC}$ are also equivalent and we only consider a unique potential, say $\mathcal{U}_3^{\rm GKC}$ with constant of the motion L_3. On the spherical space $\mathbf{S}^3$ both β_1, β_2 terms can alternatively be expressed as non-central oscillators as commented in Subsection 5.1.1, meanwhile on $\mathbf{E}^3$ and $\mathbf{H}^3$ these only can be interpreted as centrifugal barriers. In this way we find the following expressions for each space:

$$\begin{aligned}
&\mathbf{S}^3: \quad \mathcal{U}_3^{\rm GKC} = -\frac{k}{\tan r} + \frac{\beta_1}{\sin^2 x} + \frac{\beta_2}{\sin^2 y} = -\frac{k}{\tan r} + \beta_1 \tan^2 r_1 + \beta_2 \tan^2 r_2 + \beta_1 + \beta_2;\\
&\mathbf{E}^3: \quad \mathcal{U}_3^{\rm GKC} = -\frac{k}{r} + \frac{\beta_1}{x^2} + \frac{\beta_2}{y^2};\\
&\mathbf{H}^3: \quad \mathcal{U}_3^{\rm GKC} = -\frac{k}{\tanh r} + \frac{\beta_1}{\sinh^2 x} + \frac{\beta_2}{\sinh^2 y}.
\end{aligned}$$

When all the $\beta_i = 0$ we obtain the components of the Laplace–Runge–Lenz vector (6.5) for the three Riemannian spaces:

$$\begin{aligned}
&L_1 = -J_{02}J_{12} - J_{03}J_{13} + k\cos\theta,\\
&L_2 = J_{01}J_{12} - J_{03}J_{23} + k\sin\theta\cos\phi,\\
&L_3 = J_{01}J_{13} + J_{02}J_{23} + k\sin\theta\sin\phi,
\end{aligned}$$

where the difference for each particular space comes from the translations J_{0i} (3.7) that do depend on the curvature κ_1.

6.2.2 Relativistic spacetimes

On the contrary, if $\kappa_2 = -1$ (in units $c = 1$), only the two space-like distances y and z are equivalent while x is a time-like distance (ϕ is also an angle for the three spacetimes but θ is a rapidity). Thus $\mathcal{U}_2^{\rm GKC} \simeq \mathcal{U}_3^{\rm GKC}$ containing a time-like centrifugal barrier and another space-like one, while $\mathcal{U}_1^{\rm GKC}$ defines a different potential with two space-like centrifugal barriers for the three spacetimes. By taking into account the results given in Subsection 5.1, these potentials show different superpositions of the KC potential with non-central harmonic oscillators and centrifugal barriers on $\mathbf{AdS}^{2+1}$ and $\mathbf{dS}^{2+1}$. The explicit expressions on each spacetime turn out to be

$$\begin{aligned}
&\mathbf{AdS}^{2+1}: \quad \mathcal{U}_3^{\rm GKC} = -\frac{k}{\tan r} + \frac{\beta_1}{\sin^2 x} + \frac{\beta_2}{\sinh^2 y} = -\frac{k}{\tan r} + \beta_1 \tan^2 r_1 + \frac{\beta_2}{\sinh^2 y} + \beta_1,\\
&\qquad\qquad \mathcal{U}_1^{\rm GKC} = -\frac{k}{\tan r} + \frac{\beta_2}{\sinh^2 y} + \frac{\beta_3}{\sinh^2 z};\\
&\mathbf{M}^{2+1}: \quad \mathcal{U}_3^{\rm GKC} = -\frac{k}{r} + \frac{\beta_1}{x^2} + \frac{\beta_2}{y^2},\\
&\qquad\qquad \mathcal{U}_1^{\rm GKC} = -\frac{k}{r} + \frac{\beta_2}{y^2} + \frac{\beta_3}{z^2};\\
&\mathbf{dS}^{2+1}: \quad \mathcal{U}_3^{\rm GKC} = -\frac{k}{\tanh r} + \frac{\beta_1}{\sinh^2 x} + \frac{\beta_2}{\sin^2 y} = -\frac{k}{\tanh r} + \frac{\beta_1}{\sinh^2 x} + \beta_2 \tan^2 r_2 + \beta_2,\\
&\qquad\qquad \mathcal{U}_1^{\rm GKC} = -\frac{k}{\tanh r} + \frac{\beta_2}{\sin^2 y} + \frac{\beta_3}{\sin^2 z}\\
&\qquad\qquad\qquad = -\frac{k}{\tanh r} + \beta_2 \tan^2 r_2 + \beta_3 \tan^2 r_3 + \beta_2 + \beta_3.
\end{aligned}$$

The components of the Laplace–Runge–Lenz vector (6.5) (for $\beta_i = 0$) written in terms of the kinematical generators (2.3) are

$$\begin{aligned}
&L_1 = -P_1K_1 - P_2K_2 - k\cosh\theta, \qquad L_2 = P_0K_1 - P_2J - k\sinh\theta\cos\phi,\\
&L_3 = P_0K_2 + P_1J - k\sinh\theta\sin\phi.
\end{aligned}$$

7 Concluding remarks

We have achieved the generalization of the 3D Euclidean superintegrable family (1.1) as well as the maximally superintegrable SW (1.2) and GKC (1.3) potentials to the space $\mathbb{S}^3_{[\kappa_1]\kappa_2}$ by applying a unified approach which makes use of a built-in scheme of contractions. Furthermore the results so obtained have been described and interpreted on each particular space and have also been displayed along the paper in tabular form. Thus we have explicitly shown that (maximal) superintegrability is preserved for any value of the curvature and for either a Riemannian or Lorentzian metric. Notice that on the complex sphere (see e.g. [31]) and on the ambient space $\mathbb{R}^4$ these two maximally superintegrable Hamiltonians read

$$\mathcal{H}^{\rm SW} = \sum_{\mu=0}^{3}\left(\frac{1}{2}p_\mu^2 + \frac{\beta_\mu}{x_\mu^2}\right) - \beta_0, \qquad \mathcal{H}_3^{\rm GKC} = \frac{1}{2}\sum_{\mu=0}^{3} p_\mu^2 - \frac{k\,x_0}{\sqrt{x_1^2+x_2^2+x_3^2}} + \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2},$$

where $\sum_{\mu=0}^{3} x_\mu^2 = 1$. Therefore the Hamiltonians here studied can be regarded as different real forms coming from known complex superintegrable systems through graded contractions, that is, by introducing the parameters κ_1 and κ_2.

As far as the superintegrable potential $\mathcal{U}$ (4.1) is concerned, we recall that in this 3D case, we have one constant of the motion (besides de Hamiltonian) more than the two ones that ensure its complete integrability, but one integral less than the four ones that determine maximal superintegrability. By taking into account the former point of view one may claim that $\mathcal{U}$ is *minimally* (or weak) superintegrable, while from the latter, $\mathcal{U}$ would be *quasi-maximally* superintegrable. Our opinion is that when the corresponding Hamiltonian $\mathcal{H} = \mathcal{T} + \mathcal{U}$ is constructed on the ND spaces $\mathbb{S}^N_{[\kappa_1]\kappa_2}$, each of the $N(N-1)$ generators J_{ij} $(i,j = 1,\dots,N;\ i<j)$ of the (Lorentz) rotation subalgebra $so_{\kappa_2}(N)$ would provide a constant of the motion I_{ij} of the type (4.2). Next, by following [9, 23], two subsets of $N-1$ constants of the motion, $Q_{(l)}$ and $Q^{(l)}$, should be deduced from the initial set of $N(N-1)$ integrals as:

$$Q^{(l)} = \sum_{i,j=1}^{l} I_{ij}, \qquad Q_{(l)} = \sum_{i,j=N-l+1}^{N} I_{ij}, \qquad l = 2,\dots,N, \tag{7.7}$$

where $Q^{(N)} \equiv Q_{(N)}$. In this way the complete integrability of $\mathcal{H}$ would be characterized by either the N constants of the motion $\{Q^{(l)},\mathcal{H}\}$ or by $\{Q_{(l)},\mathcal{H}\}$. The quasi-maximal superintegrability would be provided by the $2N-2$ functions

$$\{Q^{(2)}, Q^{(3)}, \dots, Q^{(N)} \equiv Q_{(N)}, \dots, Q_{(3)}, Q_{(2)}, \mathcal{H}\}.$$

The corresponding SW potential on $\mathbb{S}^N_{[\kappa_1]\kappa_2}$ would be obtained by taking the same $\mathcal{F}(r)$ as in (5.1) and the remaining constant of the motion would come from one of the translation generators J_{0i} in the form I_{0i} (5.3). Likewise, a set of N GKC potentials could be constructed by starting from the radial function (6.1) and then taking $N-1$ centrifugal terms for each of them as in (6.2); in this case the additional constant of the motion L_i would be of the form (6.3).

We stress that this scheme of the possible ND generalization of all the 3D results here presented (currently in progress) relies on the fact that the potential $\mathcal{U}$ can be endowed with a coalgebra symmetry [10]. This indeed allowed us to obtain the integrals (7.7) for the ND SW system on the three Riemannian spaces in [9, 23] by starting from the quantum deformation of the Euclidean SW system introduced in [1, 2]. Furthermore, quantum deformations have been shown [6, 7] to give rise to Riemannian and relativistic spaces of non-constant curvature on which SW- and KC-type potentials can be considered [8].

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