Symmetry, Integrability and Geometry: Methods and Applications Vol. 2 (2006), Paper 009, 5 pages

On Action Invariance under Linear Spinor-Vector Supersymmetry

Kazunari SHIMA and Motomu TSUDA

Laboratory of Physics, Saitama Institute of Technology, Okabe-machi, Saitama 369-0293, Japan
E-mail: *shima@sit.ac.jp, tsuda@sit.ac.jp*

Received October 21, 2005, in final form January 10, 2006; Published online January 24, 2006
Original article is available at http://www.emis.de/journals/SIGMA/2006/Paper009/

Abstract. We show explicitly that a free Lagrangian expressed in terms of scalar, spinor, vector and Rarita–Schwinger (RS) fields is invariant under linear supersymmetry transformations generated by a global spinor-vector parameter. A (generalized) gauge invariance of the Lagrangian for the RS field is also discussed.

Key words: spinor-vector supersymmetry; Rarita–Schwinger field

2000 Mathematics Subject Classification: 81T60

Both linear (L) [1] and nonlinear (NL) [2] supersymmetry (SUSY) are realized based on a SUSY algebra where spinor generators are introduced in addition to Poincaré generators. The relation between the L and the NL SUSY, i.e., the algebraic equivalence between various (renormalizable) spontaneously broken L supermultiplets and a NL SUSY action [2] in terms of a Nambu-Goldstone (NG) fermion has been investigated by many authors [3, 4, 5, 6].

An extension of the Volkov–Akulov (VA) model [2] of NL SUSY based on a spinor-vector generator, called the spin-3/2 SUSY, hitherto, and its NL realization in terms of a spin-3/2 NG fermion have been constructed by N.S. Baaklini [7]. From the spin-3/2 NL SUSY model, L realizations of the spin-3/2 SUSY are suggested as corresponding supermultiplets to a spin-3/2 NL SUSY action [7] through a linearization. The linearization of the spin-3/2 NL SUSY is also useful from the viewpoint towards constructing a SUSY composite unified theory based on $SO(10)$ super-Poincaré (SP) group (the superon-graviton model (SGM)) [8, 9], and it may give new insight into an analogous mechanism with the super-Higgs one [10] for high spin fields which appear in SGM (up to spin-3 fields).

Recently, we have studied the unitary representation of the spin-3/2 SUSY algebra in [7] towards the linearization of the spin-3/2 NL SUSY [11]. Since the spinor-vector generator has the role of creation and annihilation operators which raise or lower the helicity of states by 1/2 or by 3/2, the structure of the (physical) L supermultiplets induced from the spin-3/2 SUSY algebra is shown for example as

$$\left[\, \underline{1}\left(+\frac{3}{2}\right), \underline{2}(+1), \underline{1}\left(+\frac{1}{2}\right), \underline{1}(0), \underline{2}\left(-\frac{1}{2}\right), \underline{1}(-1) \,\right] + [\text{ CPT conjugate }] \tag{1}$$

for the massless case. In equation (1) $\underline{n}(\lambda)$ means the number of states n for the helicity λ. Therefore, it is expected in the above examples that the spin-3/2 L supermultiplets contain scalar, spinor, vector and Rarita–Schwinger (RS) fields as fundamental fields. In order to explicitly show that those fields constitute the spin-3/2 L supermultiplet, we have to prove an action invariance under appropriate spin-3/2 L SUSY transformations whose commutator algebras close as a representation of the Baaklini's spin-3/2 SUSY algebra. Namely, we have to determine the form of the spin-3/2 L SUSY transformations both from the action invariance and from the closure of those commutator algebra.

In this paper, as a first step to do these calculations we explicitly demonstrate the spin-3/2 L SUSY invariance of a free Lagrangian in terms of spin-$(0^\pm, 1/2, 1, 3/2)$ fields, and we discuss the spin-3/2 L SUSY transformations determined from the invariance of the Lagrangian. Here we just mention the relation to the so-called no-go theorem [12, 13] based upon the S-matrix arguments, i.e. the case for the S-matrix (the true vacuum) is well defined. (Note that the vacuum of NLSUSY VA model may have rich structures, for $N = 1$ VA model is equivalent to $N = 1$ LSUSY scalar supermultiplet and also to $N = 1$ LSUSY axial vector supermultiplet as we have proved.) We discuss in this paper the *global* L SUSY with spin-3/2 charges for the *free* Lagrangian, which are free from the no-go theorem, so far. Those are important preliminaries not only to find out a (spontaneously broken) LSUSY supermultiplet, which is equivalent to the NL realization of the spin-3/2 SUSY algebra [7], but also to obtain some information for linearizing the *interacting global* NL SUSY theory with spin-3/2 (NG) fields in curved spacetime (i.e., the spin-3/2 SGM) [9]. From these viewpoints we think it is worthwhile presenting the progress report along this direction.

Let us denote spin-$(0^\pm, 1/2, 1, 3/2)$ fields beside auxiliary fields as follows: namely, A and B for scalar fields, λ for a (Majorana) spinor, v_a for a $U(1)$ gauge field and λ_a for a (Majorana) RS field. For these component fields we consider a parity conserving free Lagrangian given by[1]

$$\begin{aligned} L = {} & \frac{1}{2}(\partial_a A)^2 + \frac{1}{2}(\partial_a B)^2 + \frac{i}{2}\bar\lambda \partial\!\!\!/ \lambda - \frac{1}{4}(F_{ab})^2 + \frac{i}{2}X_1 \bar\lambda^a \partial\!\!\!/ \lambda_a + \frac{i}{2}X_2(\bar\lambda^a \gamma^b \partial_a \lambda_b + \bar\lambda^a \gamma_a \partial^b \lambda_b) \\ & - \frac{1}{2}X_3 \epsilon_{abcd} \bar\lambda^a \gamma_5 \gamma^b \partial^c \lambda^d + Y_1 \bar\lambda \partial^a \lambda_a + i Y_2 \bar\lambda \sigma^{ab} \partial_a \lambda_b, \end{aligned} \tag{2}$$

where $F_{ab} = \partial_a v_b - \partial_b v_a$, and X_i (for $i = 1, 2, 3$) with $X_3 = 1 - X_1$ and Y_i (for $i = 1, 2$) are arbitrary real parameters. Note that in equation (2) the general form of the Lagrangian for the RS field λ_a is adopted, and also the derivative coupling kinetic-like terms expressed in terms of λ and λ_a, as the last two terms are introduced without the loss of generality.

Furthermore, we define spin-3/2 L SUSY transformations generated by a global (Majorana) spinor-vector parameter ζ_a as

$$\delta_Q A = i\alpha \bar\zeta^a \gamma_a \lambda + a_1 \bar\zeta^a \lambda_a + i a_2 \bar\zeta_a \sigma^{ab} \lambda_b, \tag{3}$$

$$\delta_Q B = \alpha' \bar\zeta^a \gamma_5 \gamma_a \lambda + i a'_1 \bar\zeta^a \gamma_5 \lambda_a + a'_2 \bar\zeta_a \gamma_5 \sigma^{ab} \lambda_b, \tag{4}$$

$$\delta_Q v_a = \alpha''_1 \bar\zeta_a \lambda + i\alpha''_2 \bar\zeta^b \sigma_{ab} \lambda + i a''_1 \bar\zeta_a \gamma^b \lambda_b + i a''_2 \bar\zeta^b \gamma_a \lambda_b + i a''_3 \bar\zeta^b \gamma_b \lambda_a + a''_4 \epsilon_{abcd} \bar\zeta^b \gamma_5 \gamma^c \lambda^d, \tag{5}$$

$$\begin{aligned} \delta_Q \lambda = {} & \beta_1 \zeta^a \partial_a A + i\beta_2 \sigma^{ab} \zeta_a \partial_b A + i\beta'_1 \gamma_5 \zeta^a \partial_a B + \beta'_2 \gamma_5 \sigma^{ab} \zeta_a \partial_b B \\ & + i\beta''_1 \gamma^a \zeta^b F_{ab} + \frac{1}{2}\beta''_2 \epsilon_{abcd} \gamma_5 \gamma^a \zeta^b F^{cd}, \end{aligned} \tag{6}$$

$$\begin{aligned} \delta_Q \lambda_a = {} & i b_1 \gamma_a \zeta^b \partial_b A + i b_2 \gamma^b \zeta_a \partial_b A + i b_3 \gamma^b \zeta_b \partial_a A + b_4 \epsilon_{abcd} \gamma_5 \gamma^b \zeta^c \partial^d A \\ & + b'_1 \gamma_5 \gamma_a \zeta^b \partial_b B + b'_2 \gamma_5 \gamma^b \zeta_a \partial_b B + b'_3 \gamma_5 \gamma^b \zeta_b \partial_a B + i b'_4 \epsilon_{abcd} \gamma^b \zeta^c \partial^d B \\ & + b''_1 \zeta^b F_{ab} + i b''_2 \sigma_a{}^b \zeta^c F_{bc} + \frac{i}{2} b''_3 \sigma^{bc} \zeta_a F_{bc} + i b''_4 \sigma^{bc} \zeta_b F_{ac} + \frac{i}{2} b''_5 \epsilon_{abcd} \gamma_5 \zeta^b F^{cd}, \end{aligned} \tag{7}$$

where the α, α', α''_i (for $i = 1, 2$), β_i (for $i = 1, 2$), β'_i (for $i = 1, 2$), β''_i (for $i = 1, 2$), a_i (for $i = 1, 2$), a'_i (for $i = 1, 2$), a''_i (for $i = 1, \ldots, 4$), b_i (for $i = 1, \ldots, 4$), b'_i (for $i = 1, \ldots, 4$) and b''_i (for $i = 1, \ldots, 5$) are also arbitrary real parameters. The values of those parameters in equation (2) and in equations from (3) to (7) are determined from conditions for the spin-3/2 SUSY invariance of the Lagrangian (2) as is shown below.

Application of the spin-3/2 SUSY transformations to equation (2) (3) to (7) gives various terms as

$$\delta_Q L = F_1(\bar\zeta^a \lambda_a \Box A, \bar\zeta^a \lambda^b \partial_a \partial_b A, \bar\zeta^a \sigma_{ab} \lambda^b \Box A, \bar\zeta^a \sigma^{bc} \lambda_c \partial_a \partial_b A, \bar\zeta^a \sigma_{ab} \lambda_c \partial^b \partial^c A)$$

[1] Minkowski spacetime indices are denoted by $a, b, \ldots = 0, 1, 2, 3$. The Minkowski spacetime metric is $\frac{1}{2}\{\gamma^a, \gamma^b\} = \eta^{ab} = (+, -, -, -)$ and $\sigma^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]$.

$$
\begin{aligned}
&+ F_2(\bar\zeta^a\gamma_5\lambda_a\Box B, \bar\zeta^a\gamma_5\lambda^b\partial_a\partial_b B, \bar\zeta^a\gamma_5\sigma_{ab}\lambda^b\Box B, \bar\zeta^a\gamma_5\sigma^{bc}\lambda_c\partial_a\partial_b B, \bar\zeta^a\gamma_5\sigma_{ab}\lambda_c\partial^b\partial^c B) \\
&+ F_3(\bar\zeta^a\gamma_a\lambda\Box A, \bar\zeta^a\gamma^b\lambda\partial_a\partial_b A) + F_4(\bar\zeta^a\gamma_5\gamma_a\lambda\Box B, \bar\zeta^a\gamma_5\gamma^b\lambda\partial_a\partial_b B) \\
&+ F_5(\bar\zeta^a\gamma^b\lambda_b\partial^c F_{ac}, \bar\zeta^a\gamma^b\lambda_a\partial^c F_{bc}, \bar\zeta^a\gamma_a\lambda^b\partial^c F_{bc}, \bar\zeta^a\gamma^b\lambda^c\partial_c F_{ab}, \bar\zeta^a\gamma^b\lambda^c\partial_a F_{bc}, \\
&\qquad \epsilon^{abcd}\bar\zeta^e\gamma_5\gamma_a\lambda_b\partial_e F_{cd}, \epsilon^{abcd}\bar\zeta_a\gamma_5\gamma^e\lambda_b\partial_e F_{cd}, \epsilon^{abcd}\bar\zeta_a\gamma_5\gamma_b\lambda^e\partial_e F_{cd}) \\
&+ F_6(\bar\zeta^a\lambda\partial^b F_{ab}, \bar\zeta^a\sigma_{ab}\lambda\partial_c F^{bc}, \bar\zeta^a\sigma^{bc}\lambda\partial_b F_{ac}) + \text{ [tot. der. terms]}, \qquad (8)
\end{aligned}
$$

where we have used the relation, $\partial_a F_{bc}+\partial_c F_{ab}+\partial_b F_{ca}=0$. Therefore, the conditions for $\delta_Q L = 0$ (up to total derivative terms) are as follows; namely, the vanishing conditions of coefficients for the each kind of the terms in equation (8) are

$$
\begin{aligned}
&a_1 + X_1 b_2 + X_2 b_3 + 2X_3 b_4 - \frac{1}{4}Y_2\beta_2 = 0, \\
&(X_1+5X_2)b_1 + 2X_2 b_2 + (X_1+X_2)b_3 - 2X_3 b_4 + Y_1\beta_1 + \frac{1}{4}Y_2\beta_2 = 0, \\
&a_2 + 2X_3 b_2 - 2X_2 b_3 + 2(X_1-X_3)b_4 + \frac{1}{2}Y_2\beta_2 = 0, \\
&(X_1-X_2-2X_3)b_1 - (X_2+X_3)b_2 - (X_1-X_3)b_4 + \frac{1}{2}Y_2\left(\beta_1 - \frac{1}{2}\beta_2\right) = 0, \\
&(X_2-X_3)b_2 - (X_1+X_2)b_3 - (X_1+2X_2-X_3)b_4 - \frac{1}{2}\left(Y_1 + \frac{1}{2}Y_2\right)\beta_2 = 0 \qquad (9)
\end{aligned}
$$

for the terms in F_1,

$$
\begin{aligned}
&a'_1 + X_1 b'_2 + X_2 b'_3 - 2X_3 b'_4 + \frac{1}{4}Y_2\beta'_2 = 0, \\
&(X_1+5X_2)b'_1 + 2X_2 b'_2 + (X_1+X_2)b'_3 + 2X_3 b'_4 + Y_1\beta'_1 - \frac{1}{4}Y_2\beta'_2 = 0, \\
&-a'_2 + 2X_3 b'_2 - 2X_2 b'_3 - 2(X_1-X_3)b'_4 - \frac{1}{2}Y_2\beta'_2 = 0, \\
&(X_1-X_2-2X_3)b'_1 - (X_2+X_3)b'_2 + (X_1-X_3)b'_4 + \frac{1}{2}Y_2\left(\beta'_1 + \frac{1}{2}\beta'_2\right) = 0, \\
&(X_2-X_3)b'_2 - (X_1+X_2)b'_3 + (X_1+2X_2-X_3)b'_4 + \frac{1}{2}\left(Y_1 + \frac{1}{2}Y_2\right)\beta'_2 = 0 \qquad (10)
\end{aligned}
$$

for the terms in F_2,

$$
\begin{aligned}
&2\alpha + \beta_2 + Y_2 b_2 + 2Y_1 b_3 - 2Y_2 b_4 = 0, \\
&2\beta_1 - \beta_2 + (2Y_1 - 3Y_2)b_1 + (2Y_1 - Y_2)b_2 + 2Y_2 b_4 = 0 \qquad (11)
\end{aligned}
$$

for the terms in F_3,

$$
\begin{aligned}
&-2\alpha' - \beta'_2 + Y_2 b'_2 + 2Y_1 b'_3 + 2Y_2 b'_4 = 0, \\
&2\beta'_1 + \beta'_2 + (2Y_1 - 3Y_2)b'_1 + (2Y_1 - Y_2)b'_2 - 2Y_2 b'_4 = 0 \qquad (12)
\end{aligned}
$$

for the terms in F_4,

$$
\begin{aligned}
&2a''_1 - 2X_2 b''_1 - (X_1 - 2X_3)b''_2 - X_3 b''_3 - 2X_3 b''_4 - Y_2\beta''_1 = 0, \\
&2a''_2 + X_1 b''_3 + (X_2+X_3)b''_4 + 2X_3 b''_5 + Y_2\beta''_2 = 0, \\
&2a''_3 + X_3 b''_3 + (X_1-X_2)b''_4 - 2X_3 b''_5 - Y_2\beta''_2 = 0, \\
&2(X_1+X_2)b''_1 - (X_1+3X_2-2X_3)b''_2 + (X_2-X_3)b''_3 + (X_2-X_3)b''_4 - (2Y_1+Y_2)\beta''_1 = 0,
\end{aligned}
$$

$$\begin{aligned}
&2X_1b_1'' + (X_2 + 2X_3)b_2'' - (X_2 + X_3)b_3'' - X_1b_4'' + 2X_3b_5'' - Y_2\beta_1'' + Y_2\beta_2'' = 0,\\
&2a_4'' + 2X_3b_1'' - (X_1 - X_2 - X_3)b_2'' + X_2b_3'' + X_2b_4'' + Y_2\beta_1'' = 0,\\
&2a_4'' - X_3b_3'' + (X_2 + X_3)b_4'' - 2X_1b_5'' + Y_2\beta_2'' = 0,\\
&2a_4'' - X_2b_3'' + (X_1 + 3X_2)b_4'' + 2X_2b_5'' - 2Y_1\beta_2'' = 0
\end{aligned} \tag{13}$$

for the terms in F_5, and

$$\begin{aligned}
&4(\alpha_1'' - \beta_1'') + 4Y_1b_1'' + 3Y_2b_2'' - Y_2b_3'' - Y_2b_4'' = 0,\\
&2(\alpha_2'' - 2\beta_2'') + Y_2b_3'' + (2Y_1 + Y_2)b_4'' - 2Y_2b_5'' = 0,\\
&4(\beta_1'' - \beta_2'') + 2Y_2b_1'' + 2(Y_1 - Y_2)b_2'' - (2Y_1 - Y_2)b_3'' - Y_2b_4'' - 2Y_2b_5'' = 0
\end{aligned} \tag{14}$$

for the terms in F_6. Up to the above arguments we can easily observe the existence of the nontrivial solutions for equations from (9) to (14).

Note that if we choose tentatively the arbitrary parameters as

$$\begin{aligned}
&a_1' = a_1, \qquad a_2' = -a_2, \qquad b_i' = b_i \quad (\text{for } i = 1,2,3), \qquad b_4' = -b_4,\\
&\alpha' = -\alpha, \qquad \beta_1' = \beta_1, \qquad \beta_2' = -\beta_2,
\end{aligned} \tag{15}$$

then the conditions in equation (10) and (12) are equal to those in equation (9) and (11), respectively. We can find solutions of the parameters which satisfy the conditions (9) to (14) with equation (15) , i.e., it can be shown that the Lagrangian (2) is invariant under the spin-3/2 SUSY transformation (3) to (7).

Here we notice a special example of solutions for the conditions (9) to (14), which is given by $X_1 = X_2 = Y_1 = Y_2 = 0$ (it automatically gives $X_3 = 0$ as is understood from the second equation in equation (9)). This example means that the RS field does not contribute to equation (2), and then the free Lagrangian for the spin-$(0^\pm, 1/2, 1)$ fields is spin-3/2 SUSY invariant under $\beta_2 = 2\beta_1 = -2\alpha$ $(\beta_2' = -2\beta_1' = -2\alpha')$ and $\beta_2'' = \beta_1'' = (1/2)\alpha_2'' = \alpha_1''$. (However, in this case commutator algebras for the spin-3/2 SUSY transformations (3) to (6) do not close as a spin-3/2 SUSY representation of the Baaklini's type [7].)

Let us also discuss on the invariance of the Lagrangian (2) under the gauge transformation of the RS field. We define the (generalized) gauge transformation of λ_a generated by a local spinor parameter ϵ as

$$\delta_g\lambda_a = p\partial_a\epsilon + iq\sigma_{ab}\partial^b\epsilon, \tag{16}$$

where p and q are arbitrary (real) parameters. The variation of equation (2) with respect to equation (16) becomes

$$\begin{aligned}
\delta_g L = {}& i\left\{(X_1 + X_2)p - \frac{1}{2}(X_1 + 3X_2 - 2X_3)q\right\}\bar\lambda^a \partial\!\!\!/ \partial_a\epsilon + i\left\{X_2p + \frac{1}{2}(X_1 - 2X_3)q\right\}\bar\lambda^a\gamma_a\Box\epsilon\\
&+ \left(Y_1p + \frac{3}{4}Y_2q\right)\bar\lambda\Box\epsilon + [\text{ tot. der. terms }].
\end{aligned} \tag{17}$$

From equation (17) the conditions for $\delta_g L = 0$ (up to total derivative terms) are read as

$$\begin{aligned}
&X_1^2 + 3X_2^2 + 2X_1X_2 - 4X_2X_3 - 2X_1X_3 = 0,\\
&4Y_1\ p + 3Y_2\ q = 0.
\end{aligned} \tag{18}$$

Therefore, the Lagrangian for λ_a in equation (2) is invariant under equation (16) for arbitrary values of X_i, Y_i, p and q which satisfy equation (18).

It can be also shown that the Lagrangian (2) is invariant under both the spin-3/2 SUSY transformations (3) to (7) and the gauge transformation (16). We need further investigations on the closure of commutator algebras for equations from (3) to (7) as a representation of the spin-3/2 SUSY algebra in [7].

[1] Wess J., Zumino B., A Lagrangian model invariant under supergauge transformations, *Phys. Lett. B*, 1974, V.49, 52–75.

[2] Volkov D.V., Akulov V.P., Possible universal neutrino interaction, *JETP Lett.*, 1972, V.16, 438–440.
Volkov D.V., Akulov V.P., Is the neutrino a goldstone particle?, *Phys. Lett. B*, 1973, V.46, 109–110.

[3] Ivanov E.A., Kapustnikov A.A., Relation between linear and nonlinear realizations of supersymmetry, JINR Report No. E2-10765, Dubna, 1977 (unpublished).
Ivanov E.A., Kapustnikov A.A., General relationship between linear and nonlinear realisations of supersymmetry, *J. Phys. A: Math. Gen.*, 1978, V.11, 2375–2384.
Ivanov E.A., Kapustnikov A.A., The non-linear realization structure of models with spontaneously broken supersymmetry, *J. Phys. G*, 1982, V.8, 167–191.

[4] Roček M., Linearizing the Volkov–Akulov model, *Phys. Rev. Lett.*, 1978, V.41, 451–453.

[5] Uematsu T., Zachos C.K., Structure of phenomenological Lagrangians for broken supersymmetry, *Nucl. Phys. B*, 1982, V.201, 250–268.

[6] Shima K., Tanii Y., Tsuda M., On linearization of $N = 1$ nonlinear supersymmetry, *Phys. Lett. B*, 2002, V.525, 183–188; hep-th/0110102.
Shima K., Tanii Y., Tsuda M., Linearizing $N = 2$ nonlinear supersymmetry, *Phys. Lett. B*, 2002, V.546, 162–166; hep-th/0205178.

[7] Baaklini N.S., New superalgebra and Goldstone spin-3/2 particle, *Phys. Lett. B*, 1977, V.67, 335–336.

[8] Shima K., $SO(N)$ supergravity and unification of all fundamental forces, *Z. Phys. C*, 1983, V.18, 25–30.
Shima K., Superon-quintet and graviton model for supersymmetric spacetime and matter, *Eur. Phys. J. C*, 1999, V.7, 341–345.
Shima K., Supersymmetric structure of space-time and matter: superon-graviton model, *Phys. Lett. B*, 2001, V.501, 237–244.

[9] Shima K., Tsuda M., On gravitational interaction of spin 3/2 Nambu–Goldstone fermion, *Phys. Lett. B*, 2001, V.521, 67–70; hep-th/0012235.

[10] Deser S., Zumino B., Broken supersymmetry and supergravity, *Phys. Rev. Lett.*, 1977, V.38, 1433–1436.

[11] Shima K., Tsuda M., On supersymmetry algebra based on a spinor-vector generator, *Phys. Lett. B*, 2005, V.628, 171–175; hep-th/0507288.

[12] Coleman S., Mandula J., All possible symmetries of the S matrix, *Phys. Rev.*, 1967, V.159, 1251–1256.

[13] Haag R., Lopuszanski J., Sohnius M., All possible generators of supersymmetries of the S matrix, *Nucl. Phys. B*, 1975, V.88, 257–274.