On the Gaussian Random Matrix Ensembles with Additional Symmetry Conditions

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Abstract. The Gaussian unitary random matrix ensembles satisfying some additional symmetry conditions are considered. The effect of these conditions on the limiting normalized counting measures and correlation functions is studied.

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1 Introduction and main results

Let us consider a standard $2n \times 2n$ Gaussian Unitary Ensemble (GUE) of Hermitian random matrices W_n:

$$W_n = W_n^\dagger, \qquad (W_n)_{xy} = \frac{1}{\sqrt{2n}} \left(\xi_{xy} + i\eta_{xy}\right), \tag{1}$$

where ξ_{xy}, η_{xy}, $x, y = -n, \dots, -1, 1, \dots, n$ are i.i.d. Gaussian random variables with zero mean and variance $1/2$. Consider also the *normalized eigenvalue counting measure* (NCM) N_n of the ensemble (1), defined for any Borel set $\Delta \subset \mathbb{R}$ by the formula

$$N_n(\Delta) = \frac{\#\{\lambda_i \in \Delta\}}{2n}, \tag{2}$$

where λ_i, $i = 1, \dots, 2n$ are the eigenvalues of W_n.

Suppose now that ensemble (1) has also an additional symmetry of negative (positive) indices x and y. We consider four different cases of symmetry:

1. $(W_n)_{xy} = (W_n)_{-y-x}\,,$ (3)
2. $(W_n)_{xy} = (W_n)_{-x-y}\,,$ (4)
3. $(W_n)_{xy} = (W_n)_{-xy}\,,$ (5)
4. $(W_n)_{xy} = (W_n)_{y-x}\,.$ (6)

The Gaussian unitary ensemble and Gaussian orthogonal ensemble (GOE) was considered in numerous papers (see e.g. [4]). The Gaussian unitary ensemble with additional symmetry of type (3) was proposed in the papers [1, 3] as an approach to the weak disorder regime in the Anderson model. This ensemble was also considered in the papers [2, 9]. In all these papers ensemble (3) was called as flip matrix model and studied by some supersymmetry approach and moments method. In this paper an approach is proposed that is simpler and the same for all four cases (3)–(6). This approach is a version of technique initially proposed in [7] and developed in the papers [5, 4, 6, 8].

Using this technique we obtain the following results.

First two ensembles (3) and (4) are GOE-like.

Proposition 1. *The NCMs $N_n^{(1)}$ and $N_n^{(2)}$ of the ensembles* (3) *and* (4) *converge weakly with probability* 1 *to the semi-circle law N_{sc}*

$$N_{sc}(\mathrm{d}\lambda) = (2\pi)^{-1}\sqrt{4-\lambda^2}\chi_{[-2,2]}(\lambda)\mathrm{d}\lambda$$

and the n^{-1}-asymptotics of the correlation functions

$$F_n^{(i)}(z_1,z_2) = \mathbb{E}\left\{\left(g_n^{(i)}(z_1)-\mathbb{E}g_n^{(i)}(z_1)\right)\left(g_n^{(i)}(z_2)-\mathbb{E}g_n^{(i)}(z_2)\right)\right\}, \qquad i=1,2$$

of their Stieltjes transforms

$$g_n^{(i)}(z) = \int_{-\infty}^{\infty}\frac{N_n^{(i)}(\mathrm{d}\lambda)}{\lambda - z}, \qquad \operatorname{Im} z>0, \quad i=1,2$$

coincide with corresponding $2n\times 2n$-GOE asymptotic $(2n)^{-2}\,S(z_1,z_2)$ [4]*:*

$$\begin{aligned} F_n^{(i)}(z_1,z_2) &= (2n)^{-2}\,S(z_1,z_2)+o\left(n^{-2}\right), \\ S(z_1,z_2) &= \frac{2}{\left(1-f_{sc}^2(z_1)\right)\left(1-f_{sc}^2(z_2)\right)}\left(\frac{f_{sc}(z_1)-f_{sc}(z_2)}{z_1-z_2}\right)^2, \end{aligned} \tag{7}$$

where

$$f_{sc}(z) = \int_{-\infty}^{\infty}\frac{N_{sc}(\mathrm{d}\lambda)}{\lambda - z}, \qquad \operatorname{Im} z>0$$

is the Stieltjes transform of the semi-circle law N_{sc}.

The fourth ensemble (6) is GUE-like:

Proposition 2. *The NCM $N_n^{(4)}$ of the ensemble* (6) *converges weakly with probability* 1 *to the semi-circle law N_{sc} and the n^{-1}-asymptotic of the correlation function of its Stieltjes transform coincides with* (7) *divided by* 2 *(i.e. GUE asymptotic).*

As for the third ensemble, the additional symmetry produces new limiting NCM and correlation function:

Theorem 1. *The NCM $N_n^{(3)}$ of the ensemble* (5) *converges weakly with probability* 1 *to the limiting non-random measure N*

$$N(d\lambda) = \frac{1}{4}\delta(\lambda)d\lambda + \frac{1}{4\pi}\sqrt{6-(\lambda^2-\lambda^{-2})}\chi_{[-\lambda_+,-\lambda_-]\cup[\lambda_-,\lambda_+]}(\lambda)d\lambda, \tag{8}$$

where $\lambda_\pm = \sqrt{3\pm 2\sqrt{2}}$ and the n^{-1}-asymptotic of the correlation function of its Stieltjes transform is given by the formula

$$\begin{aligned} F_n^{(3)}(z_1,z_2) &= (2n)^{-2}\,C(z_1,z_2)+o\left(n^{-2}\right), \\ C(z_1,z_2) &= \left(2\frac{f^2(z_1)+f^2(z_2)}{f(z_1)f(z_2)(z_1-z_2)^2}+\frac{z_2 f(z_2)+z_1 f(z_1)}{2z_1^2 z_2^2 f(z_1)f(z_2)}\right)\prod_{k=1,2}\left(z_k+z_k^{-1}+4f(z_k)\right)^{-1}, \end{aligned} \tag{9}$$

where $f(z)$ is Stieltjes transform of the limiting measure N.

This result is somewhat unexpected for the Hermitian Gaussian random matrix ensemble with the rather large number (of the order n^2) of independent random parameters. But it shows how much the additional symmetry may affect the asymptotic behavior of the eigenvalues.

2 The limiting NCMs

In this section we consider the limiting normalized countable measures of the ensembles (3)–(6).

In that follows we use the notations

$$G(z) = (W_n - z)^{-1}, \qquad \hat{g}(z) = \frac{1}{2n}\sum_{j=-n}^{n} G_{j-j}(z),$$

$$g(z) = \frac{1}{2n}\operatorname{Tr} G(z) = \sum_{j=-n}^{n} G_{jj}(z),$$

and $\langle\cdot\rangle$ to denote the average over GUE. We also use the resolvent identity

$$G(z) = -z^{-1}I + z^{-1}W_n G(z)$$

and the Novikov–Furutsu formula for the complex Gaussian random variable $\zeta = \xi + i\eta$ with zero mean and variance 1, and for the continuously differentiable function $q(x, \overline{x})$

$$\mathbb{E}\zeta q(\zeta, \overline{\zeta}) = \mathbb{E}\frac{\partial}{\partial\overline{\zeta}} q(\zeta, \overline{\zeta}), \tag{10}$$

where $\frac{\partial}{\partial\overline{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial\xi} + i\frac{\partial}{\partial\eta}\right)$.

We will perform our calculations in parallel for all four ensembles. First, let us observe that properties (3)–(5) are valid not only for the matrices of ensembles (3)–(5) but for their powers and hence also for their resolvents. Indeed, using induction by m and the symmetry of summing index we obtain:

1. $$\left(W_n^{m+1}\right)_{jk} = \sum_{l=-n}^{n} (W_n)_{jl}\left(W_n^{m+1}\right)_{lk} = \sum_{l=-n}^{n} (W_n)_{j-l}\left(W_n^{m+1}\right)_{-lk}$$
$$= \sum_{l=-n}^{n} (W_n)_{l-j}\left(W_n^{m+1}\right)_{-kl} = \left(W_n^{m+1}\right)_{-k-j},$$

Thus, $G_{jk}(z) = G_{-k-j}(z)$.

2. $$\left(W_n^{m+1}\right)_{jk} = \sum_{l=-n}^{n} (W_n)_{jl}\left(W_n^{m+1}\right)_{lk} = \sum_{l=-n}^{n} (W_n)_{j-l}\left(W_n^{m+1}\right)_{-lk}$$
$$= \sum_{l=-n}^{n} (W_n)_{-jl}\left(W_n^{m+1}\right)_{l-k} = \left(W_n^{m+1}\right)_{-j-k},$$

Thus, $G_{jk}(z) = G_{-j-k}(z)$.

3. $$\left(W_n^{m+1}\right)_{jk} = \sum_{l=-n}^{n} (W_n)_{jl}\left(W_n^{m+1}\right)_{lk}$$
$$= \sum_{l=-n}^{n} (W_n)_{-jl}\left(W_n^{m+1}\right)_{lk} = \left(W_n^{m+1}\right)_{-jk}.$$

Thus,

$$G_{jk}(z) = G_{-jk}(z) - z^{-1}(\delta_{jk} - \delta_{-jk}) \tag{11}$$

and, hence,

$$g(z)=\hat g(z)-z^{-1},\qquad \text{where}\qquad \hat g(z)=\sum_{r=-n}^{n}G_{r-r}(z).$$

Unfortunately, there is no any such property for the fourth ensemble.

Now using the resolvent identity for the average $\langle G_{pq}(z)\rangle$, relation (10) and formula for the derivative of the resolvent

$$G'(z)\cdot X=-G(z)XG(z), \tag{12}$$

we obtain

$$\begin{aligned}\langle G_{pq}(z)\rangle &= -z^{-1}\delta_{pq}+z^{-1}\left\langle (W_nG(z))_{pq}\right\rangle\\ &= -z^{-1}\delta_{pq}+z^{-1}\frac{1}{\sqrt{2n}}\sum_{r=-n}^{n}\left\langle\frac{1}{2}\left(\frac{\partial}{\partial\xi_{pr}}+i\frac{\partial}{\partial\eta_{pr}}\right)G_{rq}(z)\right\rangle\\ &= -z^{-1}\delta_{pq}+z^{-1}\frac{1}{\sqrt{2n}}\sum_{r,j,k=-n}^{n}\left\langle G_{rj}(z)\left(W_n'\right)_{jk}G_{kq}(z)\right\rangle,\end{aligned}$$

where $W_n'=\frac{1}{2}\left(\frac{\partial}{\partial\xi_{pr}}+i\frac{\partial}{\partial\eta_{pr}}\right)W_n$. Now we calculate W_n' for all four ensembles:

$$\begin{aligned}
1.\quad W_n' &= \frac{1}{2\sqrt{2n}}\begin{pmatrix}\delta_{jp}\delta_{kr}+\delta_{jr}\delta_{kp}+\delta_{j-r}\delta_{k-p}+\delta_{j-p}\delta_{k-r}\\ -\delta_{jp}\delta_{kr}+\delta_{jr}\delta_{kp}-\delta_{j-r}\delta_{k-p}+\delta_{j-p}\delta_{k-r}\end{pmatrix}\\
&= \frac{1}{\sqrt{2n}}\left(\delta_{jr}\delta_{kp}+\delta_{j-p}\delta_{k-r}\right);\\
2.\quad W_n' &= \frac{1}{2\sqrt{2n}}\begin{pmatrix}\delta_{jp}\delta_{kr}+\delta_{jr}\delta_{kp}+\delta_{j-p}\delta_{k-r}+\delta_{j-r}\delta_{k-p}\\ -\delta_{jp}\delta_{kr}+\delta_{jr}\delta_{kp}-\delta_{j-p}\delta_{k-r}+\delta_{j-r}\delta_{k-p}\end{pmatrix}\\
&= \frac{1}{\sqrt{2n}}\left(\delta_{jr}\delta_{kp}+\delta_{j-r}\delta_{k-p}\right);\\
3.\quad W_n' &= \frac{1}{2\sqrt{2n}}\begin{pmatrix}\delta_{jp}\delta_{kr}+\delta_{jr}\delta_{kp}+\delta_{j-p}\delta_{kr}+\delta_{jr}\delta_{k-p}\\ -\delta_{jp}\delta_{kr}+\delta_{jr}\delta_{kp}-\delta_{j-p}\delta_{kr}+\delta_{jr}\delta_{k-p}\end{pmatrix}\\
&= \frac{1}{\sqrt{2n}}\left(\delta_{jr}\delta_{kp}+\delta_{jr}\delta_{k-p}\right);\\
4.\quad W_n' &= \frac{1}{2\sqrt{2n}}\begin{pmatrix}\delta_{jp}\delta_{kr}+\delta_{jr}\delta_{kp}+\delta_{jr}\delta_{k-p}+\delta_{j-p}\delta_{kr}\\ -\delta_{jp}\delta_{kr}+\delta_{jr}\delta_{kp}-\delta_{jr}\delta_{k-p}+\delta_{j-p}\delta_{kr}\end{pmatrix}\\
&= \frac{1}{\sqrt{2n}}\left(\delta_{jr}\delta_{kp}+\delta_{j-p}\delta_{kr}\right).
\end{aligned}$$

Using these formulas, we obtain the following relations:

$$\begin{aligned}
&1.\quad \langle G_{pq}(z)\rangle=-z^{-1}\delta_{pq}-z^{-1}\langle g(z)G_{pq}(z)\rangle-z^{-1}\left\langle\frac{1}{2n}\sum_{r=-n}^{n}G_{r-p}(z)G_{-rq}(z)\right\rangle;\\
&2.\quad \langle G_{pq}(z)\rangle=-z^{-1}\delta_{pq}-z^{-1}\langle g(z)G_{pq}(z)\rangle-z^{-1}\langle\hat g(z)G_{-pq}(z)\rangle;\\
&3.\quad \langle G_{pq}(z)\rangle=-z^{-1}\delta_{pq}-z^{-1}\langle g(z)G_{pq}(z)\rangle-z^{-1}\langle g(z)G_{-pq}(z)\rangle;\\
&4.\quad \langle G_{pq}(z)\rangle=-z^{-1}\delta_{pq}-z^{-1}\langle g(z)G_{pq}(z)\rangle-z^{-1}\left\langle\frac{1}{2n}\sum_{r=-n}^{n}G_{r-p}(z)G_{rq}(z)\right\rangle.
\end{aligned}\tag{13}$$

Now we put $p=q$ in all four cases and $p=-q$ another time in the second case, and apply $\frac{1}{2n}\sum\limits_{p=-n}^{n}$. Thus, using also the additional symmetries of the resolvents of ensembles (3)–(5), we obtain:

1. $$\langle g(z)\rangle = -z^{-1}\left(1+\langle g(z)\rangle^2\right) - z^{-1}\left[\frac{1}{2n}\left\langle \frac{1}{2n}\mathrm{Tr}G^2(z)\right\rangle + \langle g^\circ(z)g(z)\rangle\right],$$

where $g^\circ(z) = g(z) - \langle g(z)\rangle$;

2. $$\langle g(z)\rangle = -z^{-1}\left(1+\langle g(z)\rangle^2 + \left\langle \hat g^2(z)\right\rangle\right) - z^{-1}\left[\langle g^\circ(z)g(z)\rangle + \langle \hat g^\circ(z)\hat g(z)\rangle\right],$$
$$\langle \hat g(z)\rangle = -2z^{-1}\langle g(z)\rangle\langle \hat g(z)\rangle - 2z^{-1}\langle g^\circ(z)\hat g(z)\rangle;$$

3. $$\langle g(z)\rangle = -z^{-1}\left(1+\langle g(z)\rangle^2 + \langle g(z)\rangle\langle \hat g(z)\rangle\right) - z^{-1}\left[\langle g^\circ(z)g(z)\rangle + \langle \hat g^\circ(z)\hat g(z)\rangle\right],$$
$$\hat g(z) = g(z) + z^{-1}; \tag{14}$$

4. $$\langle g(z)\rangle = -z^{-1}\left(1+\langle g(z)\rangle^2\right) - z^{-1}\left[\frac{1}{2n}\left\langle \frac{1}{2n}\,\mathrm{Tr}\,P(z)G(z)\right\rangle + \langle g^\circ(z)g(z)\rangle\right],$$

where matrix $P(z)$ is defined by $P_{xy}(z) = G_{y-x}(z)$.

In the appendix we prove that the variances of random variables $g(z)$ in all cases above are of the order $O(n^{-2})$ uniformly in z for some compact in $C_\pm$ (as well as the variance of $\hat g(z)$ in the second case). Besides, using Schwartz inequality for the matrix scalar product $(A,B) = \mathrm{Tr}\,AB$, we obtain

$$\left|\frac{1}{2n}\,\mathrm{Tr}\,P(z)G(z)\right| \le \left(\frac{1}{2n}\,\mathrm{Tr}\,P(z)P^\dagger(z)\right)^{1/2}\left(\frac{1}{2n}\,\mathrm{Tr}G(z)G^\dagger(z)\right)^{1/2} \le \frac{1}{|\mathrm{Im}\,z|^2},$$
$$\left|\frac{1}{2n}\,\mathrm{Tr}\,G^2(z)\right| \le \frac{1}{|\mathrm{Im}\,z|^2}.$$

Thus, all terms in square brackets in all four cases are at least of the order $O(n^{-1})$. Hence, in the first and in the fourth cases we obtain the following limiting equation:

$$f(z) = -z^{-1}\left(1+f^2(z)\right), \tag{15}$$

which is the equation for $f_{sc}(z)$ — the Stieltjes transform of the semi-circle Law.

Besides, since

$$g(z) = -z_{pq}^{-1} - z^{-1}\frac{1}{2n}\,\mathrm{Tr}\,\left(W_nG(z)\right),$$
$$\left|\frac{1}{2n}\,\mathrm{Tr}\,\left(W_nG(z)\right)\right| \le \frac{1}{|\mathrm{Im}\,z|}\left(\frac{1}{2n}\,\mathrm{Tr}\,W_nW_n^\dagger\right)^{1/2},$$
$$\left\langle\left(\frac{1}{2n}\,\mathrm{Tr}\,W_nW_n^\dagger\right)^{1/2}\right\rangle \le \left\langle\frac{1}{2n}\,\mathrm{Tr}\,W_nW_n^\dagger\right\rangle^{1/2} \le 1,$$

then for all z with e.g. $|\mathrm{Im}\,z| \ge 3$ uniformly in n we have in all cases

$$\left|1+2z^{-1}\langle g(z)\rangle\right| > \frac{1}{2}. \tag{16}$$

Thus, in the second case $\langle \hat g(z)\rangle$ of the order $O(n^{-2})$:

$$\langle \hat g(z)\rangle = -2z^{-1}\left(1+2z^{-1}\langle g(z)\rangle\right)^{-1}\langle g^\circ(z)\hat g(z)\rangle.$$

Hence, the second case lead to the same limiting equation (15).

As for the third case, it leads to the following equation

$$f(z) = -z^{-1}\left(1 + 2f^2(z) + z^{-1}f(z)\right). \tag{17}$$

Its solution in the class of Nevanlinna functions is the Stieltjes transform of the measure (8).

The convergence with probability one in all four cases follows from the bounds for the variances in the section bellow and the Borel–Cantelli lemma.

3 The correlation functions

As in the previous section, we perform our calculations in parallel for all four ensembles.

Using the resolvent identity for the average $\langle g^\circ(z_1)G_{pq}(z_2)\rangle$, relations (10) and (12), we obtain

$$\begin{aligned}\langle g^\circ(z_1)G_{pq}(z_2)\rangle = z_2^{-1}\frac{1}{\sqrt{2n}}\sum_{r,j,k=-n}^{n}\left\langle g^\circ(z_1)G_{rj}(z_2)\left(W_n'\right)_{jk}G_{kq}(z_2)\right\rangle \\ + z_2^{-1}\frac{1}{(2n)^{3/2}}\sum_{l.r,j,k=-n}^{n}\left\langle G_{lj}(z_1)\left(W_n'\right)_{jk}G_{kl}(z_1)G_{rq}(z_2)\right\rangle.\end{aligned}$$

Substituting in this relation the value of W_n' in all four cases and using the symmetries of the resolvents, we obtain

$$\begin{aligned}
1.\quad \langle g^\circ(z_1)G_{pq}(z_2)\rangle &= -z_2^{-1}\langle g^\circ(z_1)g(z_2)G_{pq}(z_2)\rangle - z_2^{-1}\left\langle g^\circ(z_1)\frac{1}{2n}G_{pq}^2(z_2)\right\rangle \\
&\quad - z_2^{-1}\frac{1}{(2n)^2}\left(\left\langle\left(G^2(z_1)G(z_2)\right)_{pq}\right\rangle + \left\langle\left(G(z_2)G^2(z_1)\right)_{-q-p}\right\rangle\right); \\
2.\quad \langle g^\circ(z_1)G_{pq}(z_2)\rangle &= -z_2^{-1}\langle g^\circ(z_1)g(z_2)G_{pq}(z_2)\rangle - z_2^{-1}\langle g^\circ(z_1)\hat{g}(z_2)G_{-pq}(z_2)\rangle \\
&\quad - z_2^{-1}\frac{1}{(2n)^2}\left(\left\langle\left(G^2(z_1)G(z_2)\right)_{pq}\right\rangle + \left\langle\left(G(z_2)G^2(z_1)\right)_{-p-q}\right\rangle\right); \\
3.\quad \langle g^\circ(z_1)G_{pq}(z_2)\rangle &= -z_2^{-1}\langle g^\circ(z_1)g(z_2)G_{pq}(z_2)\rangle - z_2^{-1}\langle g^\circ(z_1)g(z_2)G_{-pq}(z_2)\rangle \\
&\quad - z_2^{-1}\frac{1}{(2n)^2}\left(\left\langle\left(G^2(z_1)G(z_2)\right)_{pq}\right\rangle + \left\langle\left(G^2(z_1)G(z_2)\right)_{-pq}\right\rangle\right); \\
4.\quad \langle g^\circ(z_1)G_{pq}(z_2)\rangle &= -z_2^{-1}\langle g^\circ(z_1)g(z_2)G_{pq}(z_2)\rangle - z_2^{-1}\left\langle g^\circ(z_1)\frac{1}{2n}\sum_{r=-n}^{n}G_{r-p}(z_2)G_{rq}(z_2)\right\rangle \\
&\quad - z_2^{-1}\frac{1}{(2n)^2}\left(\left\langle\left(G^2(z_1)G(z_2)\right)_{pq}\right\rangle + \left\langle\sum_{r=-n}^{n}\left(G^2(z_1)\right)_{r-p}G_{rq}(z_2)\right\rangle\right).
\end{aligned}$$

Then we put $p = q$ in all four cases and $p = -q$ another time in the second case, and apply $\frac{1}{2n}\sum_{p=-n}^{n}$ and obtain

$$\begin{aligned}1.\quad \langle g^\circ(z_1)g(z_2)\rangle &= -2z_2^{-1}\langle g(z_2)\rangle\langle g^\circ(z_1)g(z_2)\rangle \\ &\quad - z_2^{-1}\frac{2}{(2n)^2}\left\langle\frac{1}{2n}\operatorname{Tr}G^2(z_1)G(z_2)\right\rangle + r_{1,n},\end{aligned} \tag{18}$$

where

$$r_{1,n} = -z_2^{-1}\left(\left\langle g^\circ(z_1)\left(g^\circ(z_2)\right)^2\right\rangle + \frac{1}{2n}\left\langle g^\circ(z_1)\frac{1}{2n}\operatorname{Tr}G^2(z_2)\right\rangle\right); \tag{19}$$

2. $$\begin{aligned}\langle g^\circ(z_1)g(z_2)\rangle &= -2z_2^{-1}\left(\langle g(z_2)\rangle\langle g^\circ(z_1)g(z_2)\rangle + \langle\hat g(z_2)\rangle\langle g^\circ(z_1)\hat g(z_2)\rangle\right)\\ &\quad - z_2^{-1}\frac{2}{(2n)^2}\left\langle\frac{1}{2n}\operatorname{Tr}G^2(z_1)G(z_2)\right\rangle + r_{2,n},\\ \langle\hat g^\circ(z_1)\hat g(z_2)\rangle &= -2z_2^{-1}\left(\langle g(z_2)\rangle\langle\hat g^\circ(z_1)\hat g(z_2)\rangle + \langle\hat g(z_2)\rangle\langle g^\circ(z_1)\hat g(z_2)\rangle\right)\\ &\quad - z_2^{-1}\frac{2}{(2n)^2}\left\langle\frac{1}{2n}\operatorname{Tr}G^2(z_1)G(z_2)\right\rangle + r_{3,n},\end{aligned}$$

where

$$r_{2,n} = -z_2^{-1}\left(\left\langle g^\circ(z_1)\left(g^\circ(z_2)\right)^2\right\rangle + \left\langle g^\circ(z_1)\left(\hat g^\circ(z_2)\right)^2\right\rangle\right),$$
$$r_{3,n} = -2z_2^{-1}\langle\hat g^\circ(z_1)\hat g^\circ(z_2)g^\circ(z_2)\rangle;$$

3. $$\begin{aligned}\langle g^\circ(z_1)g(z_2)\rangle &= -4z_2^{-1}\langle g(z_2)\rangle\langle g^\circ(z_1)g(z_2)\rangle - z_2^{-2}\langle g^\circ(z_1)g(z_2)\rangle\\ &\quad - z_2^{-1}\frac{1}{(2n)^2}\left\langle\frac{1}{2n}\operatorname{Tr}G^2(z_1)G(z_2)\right\rangle\\ &\quad - z_2^{-1}\frac{1}{(2n)^2}\left\langle\frac{1}{2n}\sum_{p=-n}^{n}\left(G^2(z_1)G(z_2)\right)_{-pp}\right\rangle + r_{4,n},\end{aligned}$$

where

$$r_{4,n} = -2z_2^{-1}\left\langle g^\circ(z_1)\left(g^\circ(z_2)\right)^2\right\rangle;$$

4. $$\begin{aligned}\langle g^\circ(z_1)g(z_2)\rangle &= -2z_2^{-1}\langle g(z_2)\rangle\langle g^\circ(z_1)g(z_2)\rangle\\ &\quad - z_2^{-1}\frac{1}{(2n)^2}\left\langle\frac{1}{2n}\operatorname{Tr}G^2(z_1)G(z_2)\right\rangle + r_{5,n},\end{aligned}$$

where

$$\begin{aligned}r_{5,n} &= -z_2^{-1}\frac{1}{2n}\left\langle g^\circ(z_1)\frac{1}{2n}\operatorname{Tr}P(z_2)G(z_2)\right\rangle - z_2^{-1}\left\langle g^\circ(z_1)\left(g^\circ(z_2)\right)^2\right\rangle\\ &\quad - z_2^{-1}\frac{1}{(2n)^2}\left\langle\frac{1}{2n}\sum_{r,p=-n}^{n}\left(G^2(z_1)\right)_{r-p}G_{rp}(z_2)\right\rangle. \qquad (20)\end{aligned}$$

As we show in the appendix, all $r_{j,n}$, $j=1,\dots,5$ are of the order $o(n^{-2})$. Thus, as one can easily show, all correlation functions $F(z_1,z_2)=\langle g^\circ(z_1)g(z_2)\rangle$ above are of the order $O(n^{-2})$. Moreover, since $\langle\hat g(z_2)\rangle$ is of the order $O(n^{-2})$ in the second case, its easy to see that cases one and two lead to the same relation for $F(z_1,z_2)$

$$F(z_1,z_2) = -2z_2^{-1}\langle g(z_2)\rangle F(z_1,z_2) - z_2^{-1}\frac{2}{(2n)^2}\left\langle\frac{1}{2n}\operatorname{Tr}G^2(z_1)G(z_2)\right\rangle + o\left(n^{-2}\right). \qquad (21)$$

As to the case four, it leads to

$$F(z_1,z_2) = -2z_2^{-1}\langle g(z_2)\rangle F(z_1,z_2) - z_2^{-1}\frac{1}{(2n)^2}\left\langle\frac{1}{2n}\operatorname{Tr}G^2(z_1)G(z_2)\right\rangle + o\left(n^{-2}\right). \qquad (22)$$

Besides, due to the resolvent identity we have

$$\frac{1}{2n}\operatorname{Tr}G^2(z_1)G(z_2) = \frac{1}{z_1-z_2}\left(\frac{1}{2n}\operatorname{Tr}G^2(z_1) - \frac{g(z_1)-g(z_2)}{z_1-z_2}\right). \qquad (23)$$

In addition, as we show in the appendix, in these cases

$$\left\langle \frac{1}{2n} \operatorname{Tr} G^2(z) \right\rangle = \frac{\langle g(z)\rangle}{1-\langle g(z)\rangle^2} + O\left(n^{-1}\right). \tag{24}$$

Thus, substituting in the relations (21), (22) the expressions (23), (24) and using the equation (15) for the limit of $\langle g(z)\rangle$, we obtain in the cases one and two the GOE correlator asymptotic (7) and in the case four the twice less GUE asymptotic.

To treat the third case we use (11) and obtain that

$$\frac{1}{2n}\sum_{p=-n}^{n}\left(G^2(z_1)G(z_2)\right)_{-pp} = \frac{1}{2n}\mathrm{Tr}G^2(z_1)G(z_2) + \frac{1}{z_1^2 z_2}.$$

This gives the following relation for $F(z_1, z_2)$

$$\begin{aligned} F(z_1,z_2) &= \left(-4\frac{\langle g(z_2)\rangle}{z_2} - z_2^{-2}\right) F(z_1,z_2) \\ &\quad - \frac{z_2^{-1}}{(2n)^2}\left(\frac{1}{z_1^2 z_2} + \frac{1}{z_1-z_2}\left\langle \frac{1}{2n}\operatorname{Tr} G^2(z_1) - \frac{g(z_1)-g(z_2)}{z_1-z_2}\right\rangle\right) + o\left(n^{-2}\right). \end{aligned} \tag{25}$$

We show also in the appendix that in this case

$$\left\langle \frac{1}{2n} \operatorname{Tr} G^2(z) \right\rangle = -\frac{\langle g(z)\rangle}{z}\frac{1-z^{-2}}{1+\langle g(z)\rangle z^{-1} + z^{-2}} + o\left(n^{-2}\right).$$

Substituting this relation in (25) we obtain

$$\begin{aligned} F(z_1,z_2) &= \frac{1}{n^2}\left(-\frac{\frac{1}{(z_1z_2)^2} + \frac{2}{z_1}\frac{f(z_1)-f(z_2)}{(z_1-z_2)^2}}{1+z_1^{-2}+4z_1^{-1}f(z_1)}\right. \\ &\quad \left. - 2\frac{1-z_2^{-2}}{z_1-z_2}\frac{f(z_2)}{z_1z_2}\prod_{k=1,2}\left(1+z_k^{-2}+4\frac{f(z_k)}{z_k}\right)\right) + o\left(n^{-2}\right). \end{aligned}$$

Then, using the equation (17), we rewrite this relation in the form (9).

4 Conclusion

The purpose of this paper was to answer the question: "Can the additional symmetry properties influence on the asymptotic behavior of eigenvalue distribution of GUE?" The negative answer for the three cases of additional symmetry is not surprising, as these symmetries leave the number of independent random parameters of the order n^2. The effect when in one case the additional symmetry essentially changes the limiting eigenvalue counting measure is very unexpected, especially the appearance of the gap in the support of limiting NCM. Unfortunately, the physical application of this effect is unknown to the author, though one of the other considered ensembles (flip matrix model) was used as an approach to weak coupling regime of the Anderson model.

A Appendix

Proposition 3. *The variance* $v = \left\langle |g^\circ(z)|^2 \right\rangle$ *is of the order* $O(n^{-2})$ *in all four cases, and the terms* $r_{j,n}$, $j = 1, \ldots, 5$ *are of the order* $o(n^{-2})$.

Proof. First we proof that the variance is of the order $O(n^{-2})$ in all four cases. Indeed, in the first case, using (18) with $z_2 = \overline{z_1} = z$, we obtain

$$v(1 + 2z^{-1} \langle g(z) \rangle) = -z_2^{-1} \frac{2}{(2n)^2} \left\langle \frac{1}{2n} \operatorname{Tr} G^2(z_1) G(z_2) \right\rangle + r_{1,n}.$$

Besides, using the Schwartz inequality we obtain from (19)

$$r_{1,n} \le |z|^{-1} \left(\frac{1}{|\operatorname{Im} z|} v + \frac{1}{2n\, |\operatorname{Im} z|^2} v^{1/2} \right).$$

Thus, due to the bounds (16) and

$$\left| \frac{1}{2n} \operatorname{Tr} G^2(z_1) G(z_2) \right| \le \frac{1}{|\operatorname{Im} z|^3}, \tag{26}$$

we have for $|\operatorname{Im} z| \ge 3$ the inequality

$$v \le \frac{2}{9\,(2n)^2} + \frac{1}{2n} v^{1/2},$$

which leads to $v = O(n^{-2})$. For the other cases the proofs are analogous.

To prove $r_{1,n} = o(n^{-2})$ for

$$r_{1,n} = -z_2^{-1} \left(\left\langle g^\circ(z_1)\, (g^\circ(z_2))^2 \right\rangle + \frac{1}{2n} \left\langle g^\circ(z_1) \frac{1}{2n} \operatorname{Tr} G^2(z_2) \right\rangle \right),$$

we rewrite the second term in the parentheses as

$$\frac{1}{2n} \left\langle g^\circ(z_1) \frac{1}{2n} \operatorname{Tr} G^2(z_2) \right\rangle = \frac{1}{2n} \left\langle g^\circ(z_1) \frac{1}{2n} \operatorname{Tr} \frac{\partial}{\partial z_2} G(z_2) \right\rangle = \frac{1}{2n} \frac{\partial}{\partial z_2} \langle g^\circ(z_1) g^\circ(z_2) \rangle .$$

Since the value $\langle g^\circ(z_1) g^\circ(z_2) \rangle$ is analytical and uniformly in n bounded for $|\operatorname{Im} z_{1,2}| \ge 3$, and since, due to the Schwartz inequality $|\langle g^\circ(z_1) g^\circ(z_2) \rangle| \le v = O(n^{-2})$, its derivative on z_2 is also of the order $O(n^{-2})$ and hence the second term is of the order $O(n^{-3})$.

To prove that the first term is $o(n^{-2})$ let us consider

$$\left\langle |g^\circ(z)|^4 \right\rangle = \left\langle (g^\circ(z_1) g^\circ(z_2))^2 \right\rangle = \langle R^\circ g(z_2) \rangle , \qquad R \equiv (g^\circ(z_1))^2\, g^\circ(z_2), \qquad z_1 = \overline{z_2} = z.$$

Then, using the resolvent identity for the average $\langle R^\circ G_{pq}(z_2) \rangle$, relations (10) and (12), we obtain

$$\begin{aligned}
\langle R^\circ G_{pq}(z_2) \rangle = {} & z_2^{-1} \frac{1}{\sqrt{2n}} \sum_{r,j,k=-n}^{n} \left\langle R^\circ G_{rj}(z_2) \left(W_n' \right)_{jk} G_{kq}(z_2) \right\rangle \\
& + z_2^{-1} \frac{2}{(2n)^{3/2}} \sum_{l.r,j,k=-n}^{n} \left\langle g^\circ(z_1) g^\circ(z_2) G_{lj}(z_1) \left(W_n' \right)_{jk} G_{kl}(z_1) G_{rq}(z_2) \right\rangle \\
& + z_2^{-1} \frac{1}{(2n)^{3/2}} \sum_{l.r,j,k=-n}^{n} \left\langle (g^\circ(z_1))^2\, G_{lj}(z_2) \left(W_n' \right)_{jk} G_{kl}(z_2) G_{rq}(z_2) \right\rangle .
\end{aligned}$$

Substituting in this relation the value of W_n' and using the symmetry of the resolvent we obtain

$$\langle R^\circ G_{pq}(z_2) \rangle = -z_2^{-1} \langle R^\circ g(z_2) G_{pq}(z_2) \rangle - z_2^{-1} \left\langle R^\circ \frac{1}{2n} G_{pq}^2(z_2) \right\rangle$$

$$
\begin{aligned}
&- z_2^{-1}\frac{2}{(2n)^2}\left\langle g^\circ(z_1)g^\circ(z_2)\left(\left(G^2(z_1)G(z_2)\right)_{pq}+\left(G(z_2)G^2(z_1)\right)_{-q-p}\right)\right\rangle\\
&- z_2^{-1}\frac{2}{(2n)^2}\left\langle\left(g^\circ(z_1)\right)^2 G^3_{pq}(z_2)\right\rangle .
\end{aligned}
$$

Then we put $p=q$ in all four cases and $p=-q$ another time in the second case, and apply $\frac{1}{2n}\sum\limits_{p=-n}^{n}$ and obtain

$$
\begin{aligned}
\left\langle|g^\circ(z)|^4\right\rangle = \langle R^\circ g(z_2)\rangle &= -z_2^{-1}\left\langle R^\circ g^2(z_2)\right\rangle - z_2^{-1}\frac{1}{2n}\left\langle R^\circ\frac{1}{2n}\operatorname{Tr}G^2(z_2)\right\rangle\\
&- z_2^{-1}\frac{4}{(2n)^2}\left\langle g^\circ(z_1)g^\circ(z_2)\frac{1}{2n}\operatorname{Tr}\left(G^2(z_1)G(z_2)\right)\right\rangle\\
&- z_2^{-1}\frac{2}{(2n)^2}\left\langle\left(g^\circ(z_1)\right)^2\frac{1}{2n}\operatorname{Tr}G^3(z_2)\right\rangle .
\end{aligned}
$$

Using this relation, the bounds (26) and

$$
\begin{aligned}
&\left|\left\langle R^\circ g^2(z_2)\right\rangle\right| = \left|\langle R^\circ g^\circ(z_2)g(z_2)\rangle + \langle R^\circ g(z_2)\rangle\langle g(z_2)\rangle\right| \le 2\frac{\left\langle|g^\circ(z)|^4\right\rangle}{|\operatorname{Im} z|},\\
&\left|\left\langle R^\circ\frac{1}{2n}\operatorname{Tr}G^2(z_2)\right\rangle\right| \le \frac{v}{|\operatorname{Im} z|^3},
\end{aligned}
$$

we obtain that for $|\operatorname{Im} z|\ge 3$ $\left\langle|g^\circ(z)|^4\right\rangle = O(n^{-3})$. Thus, due to the Schwartz inequality the term $\left\langle g^\circ(z_1)\left(g^\circ(z_2)\right)^2\right\rangle$ is of the order $O(n^{-5/2})$ and, hence, $r_{1,n}$ is of the same order.

The cases of the terms $r_{j,n}$, $j=2,\dots,5$ can be treated analogously, with exception for the last term of $r_{5,n}$. The last term of (20)

$$
\frac{1}{(2n)^2}\left\langle\frac{1}{2n}\sum_{r,p=-n}^{n}\left(G^2(z_1)\right)_{r-p}G_{rp}(z_2)\right\rangle
$$

can be treated as follows.

First, observe that in the case four $\langle\hat g(z)\rangle = o(n^{-1})$. Indeed, using (13) with $q=-p$, we obtain

$$
\langle\hat g(z)\rangle = -z^{-1}\langle g(z)\rangle\langle\hat g(z)\rangle - z^{-1}\langle g^\circ(z)\hat g(z)\rangle - z^{-1}\frac{1}{2n}\left\langle\frac{1}{2n}\operatorname{Tr}\left(G(z)G^T(z)\right)\right\rangle ,
$$

where G^T is transpose of G. Due to the the Schwartz inequality for the trace, the last term in r.h.s. of this relation is of the order $O(n^{-1})$. Since the variance of $g(z)$ is of the order $O(n^{-2})$, the second term is at least of the order $O(n^{-1})$ (in fact it is of the order $O(n^{-2})$, since, as one can show, the variance of $\hat g(z)$ is of the same order). Thus, $\langle\hat g(z)\rangle$ is of the order $O(n^{-1})$. Its easy to show in the same way that

$$
\left\langle\hat h(z)\right\rangle = \left\langle\frac{1}{2n}\sum_{j=-n}^{n}\left(G^2(z)\right)_{j-j}\right\rangle
$$

is also of the order $O(n^{-1})$ and its variance is of the order $O(n^{-2})$.

Now, using the resolvent identity for the average of

$$
\Phi = \frac{1}{2n}\sum_{p,q=-n}^{n}\left(G^2(z_1)\right)_{p-q}G_{pq}(z_2),
$$

relations (10) and (12), we obtain

$$\langle \Phi \rangle = -z_2^{-1} \left\langle \hat{h}(z_1) \right\rangle - z_2^{-1} \frac{1}{(2n)^{3/2}} \left\langle \sum_{p,q,r,j,k=-n}^{n} \left(G^2(z_1)\right)_{p-q} G_{rj}(z_2) \left(W_n'\right)_{jk} G_{kq}(z_2) \right\rangle$$
$$- z_2^{-1} \frac{1}{(2n)^{3/2}} \left\langle \sum_{p,q,r,j,k,m=-n}^{n} G_{pj}(z_1) \left(W_n'\right)_{jk} G(z_1)_{km} G_{m-q}(z_1) G_{rq}(z_2) \right\rangle$$
$$- z_2^{-1} \frac{1}{(2n)^{3/2}} \left\langle \sum_{p,q,r,j,k,m=-n}^{n} G_{pm}(z_1) G(z_1)_{mj} \left(W_n'\right)_{jk} G_{k-q}(z_1) G_{rq}(z_2) \right\rangle .$$

Substituting in this relation the value of W_n', we obtain

$$\langle \Phi \rangle = -z_2^{-1} \left\langle \hat{h}(z_1) \right\rangle - z_2^{-1} \langle g(z_2)\Phi \rangle - z_2^{-1} \frac{1}{2n} \left\langle \frac{1}{2n} \sum_{p,q,r=-n}^{n} \left(G^2(z_1)\right)_{p-q} G_{r-p}(z_2) G_{rq}(z_2) \right\rangle$$
$$- z_2^{-1} \frac{1}{2n} \left\langle \frac{1}{2n} \sum_{p,q=-n}^{n} \left(G(z_1)G(z_2)\right)_{pq} G^2{}_{p-q}(z_1) \right\rangle$$
$$- z_2^{-1} \frac{1}{2n} \left\langle \frac{1}{2n} \sum_{p,q=-n}^{n} \left(G^2(z_1)G(z_2)\right)_{pq} G_{p-q}(z_1) \right\rangle - z_2^{-1} \langle \hat{g}(z_1)\Phi \rangle - z_2^{-1} \left\langle \hat{h}(z_1)\Phi \right\rangle .$$

The first term of the r.h.s. is of the order $O(n^{-1})$, the last five terms are of the same order, because of the Schwartz inequality and of the bounds for the variances of $\hat{g}(z_1)$ and $\hat{h}(z_1)$. The second term we rewrite as follows

$$-z_2^{-1} \langle g(z_2)\Phi \rangle = -z_2^{-1} \langle g(z_2) \rangle \langle \Phi \rangle - z_2^{-1} \langle g^{\circ}(z_2)\Phi \rangle ,$$

where due to the Schwartz inequality the last term is also of the order $O(n^{-1})$. Thus, we conclude that $\langle \Phi \rangle$ is of the order $O(n^{-1})$ and, hence, the last term of $r_{5,n}$ is of the order $O(n^{-3})$. ■

Proposition 4. *In the third case we have*

$$\left\langle \frac{1}{2n} \operatorname{Tr} G^2(z) \right\rangle = -\frac{\langle g(z) \rangle}{z} \frac{1 - z^{-2}}{1 + \langle g(z) \rangle z^{-1} + z^{-2}} + o\left(n^{-2}\right) .$$

Proof. Indeed, we have

$$\left\langle \frac{1}{2n} \operatorname{Tr} G^2(z) \right\rangle = \left\langle \frac{1}{2n} \operatorname{Tr} \frac{d}{dz} G(z) \right\rangle = \frac{d}{dz} \langle g(z) \rangle .$$

Hence, we can just take the derivative of the identity (14) for $\langle g(z) \rangle$. Since all terms of the order $O(n^{-2})$ in square brackets in (14) are analytical and uniformly bounded for $|\operatorname{Im} z_{1,2}| \geq 3$, they remain of the same order. Thus, we obtain the relation needed. ■

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